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On excess filtration on the Steenrod algebra

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In this note, we study some properties of the filtration of the Steenrod algebra defined from the excess of admissible monomials. We give several conditions on a cocommutative graded Hopf algebra A^* which enable us to develop the theory of unstable A^*–modules.

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Introduction

The theory of unstable modules over the Steenrod algebra has been developed by many researchers and has various geometric applications. (See Schwartz [6] and its references.) It was so successful that it might be interesting to consider the structure of the Steenrod algebra which enable us to define the notion of unstable modules. Let us call the filtration on the Steenrod algebra defined from the excess of admissible monomials the excess filtration. (See Definition 1.7 below.) We note that this filtration plays an essential role in developing the theory of unstable modules.

The aim of this note is to give several conditions on filtered graded Hopf algebra A^* which allows us to deal the theory of unstable A^*–modules axiomatically. In the first and second sections, we study properties of the excess filtration on the Steenrod algebra $\mathcal{A}_p$. In Section 3, we propose nine conditions on a decreasing filtration on a cocommutative graded Hopf algebra A^* over a field which may suffice to develop the theory of unstable modules. We also verify several facts (eg Proposition 3.12, Lemma 3.13, Proposition 3.14) which are known to hold for the case of the Steenrod algebra. To give an example of a filtered Hopf algebra other than the Steenrod algebra, we consider the group scheme defined from the unipotent matrix groups in Section 4. We embed the group scheme represented by the dual Steenrod algebra as a closed subscheme of infinite dimensional unipotent group scheme represented by a certain Hopf algebra $A_{(p)*}$ which has a filtration satisfying the dual of first six conditions given in Section 3. We observe that this filtration induces the filtration on the mod p dual Steenrod algebra $\mathcal{A}_{p*}$ which is the dual of the excess filtration. In Appendix A we show that the affine group scheme represented by $\mathcal{A}_{p*}$ is naturally equivalent to a $\mathbb{F}_p$–group functor which assigns to an $\mathbb{F}_p$–algebra R^* a certain subgroup of the strict isomorphisms of the additive formal group law over $R^*[\varepsilon]/(\varepsilon^2)$ in Section 4.

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1 Basic properties of excess filtration

We denote by $\mathcal{A}_p$ the mod p Steenrod algebra and by $\mathcal{A}_{p*}$ its dual. Let Seq be the set of all infinite sequences $(i_1, i_2, \ldots, i_n, \ldots)$ of non-negative integers such that $i_n = 0$ for all but finite number of n. Let Seq^o be a subset of Seq consisting of sequences $(i_1, i_2, \ldots, i_n, \ldots)$ such that $i_k = 0, 1$ if k is odd. If $i_n = 0$ for $n > N$, we denote $(i_1, i_2, \ldots, i_n, \ldots)$ by $(i_1, i_2, \ldots, i_N)$.

Definition 1.1 (Steenrod–Epstein [7]) For $I = (\varepsilon_0, i_1, \varepsilon_1, \ldots, i_n, \varepsilon_n) \in \mathrm{Seq}^o$ and an odd prime p, we put

$$d_p(I) = 2(p-1)\sum_{s=1}^{n} i_s + \sum_{s=0}^{n} \varepsilon_s, \qquad e_p(I) = \sum_{s=0}^{n} \varepsilon_s + 2\sum_{s=1}^{n}(i_s - p i_{s+1} - \varepsilon_s).$$

For $J = (j_1, j_2, \ldots, j_n) \in \mathrm{Seq}$, we put

$$d_2(J) = \sum_{s=1}^{n} j_s, \quad e_2(J) = \sum_{s=1}^{n}(j_s - 2j_{s+1}).$$

Then

$$\begin{aligned} \wp^I &= \beta^{\varepsilon_0}\wp^{i_1}\beta^{\varepsilon_1}\wp^{i_2}\beta^{\varepsilon_2}\cdots\wp^{i_n}\beta^{\varepsilon_n} \in \mathcal{A}_p^{d_p(I)} \\ \text{and} \quad Sq^J &= Sq^{j_1}Sq^{j_2}\ldots Sq^{j_n} \in \mathcal{A}_2^{d_2(J)}. \end{aligned}$$

We call $d_p(I)$ the degree of I and $e_p(I)$ the excess of I.

Proposition 1.2 *Suppose* $I = (\varepsilon_0, i_1, \varepsilon_1, \ldots, i_n, \varepsilon_n, \ldots) \in \mathrm{Seq}^o$ *and*

$$J = (j_1, j_2, \ldots, j_n, \ldots) \in \mathrm{Seq}.$$

(1) $e_p(I) = 2pi_1 + 2\varepsilon_0 - d_p(I)$ *if* p *is an odd prime,*
$e_2(J) = 2j_1 - d_2(J)$.

(2) $e_p(I) \le 2i_1 + \varepsilon_0$ *and the equality holds if and only if* $I = (\varepsilon_0, i_1)$ *for an odd prime* p.
$e_2(J) \le j_1$ *and the equality holds if and only if* $J = (j_1)$.

(3) $d_p(I) \ge (p-1)e_p(I) - \varepsilon_0(p-2)$ *and the equality holds if and only if* $I = (\varepsilon_0, i_1)$ *for an odd prime* p.
$d_2(J) \ge e_2(J)$ *and the equality holds if and only if* $J = (j_1)$.

Proof (1) These are direct consequences of the definitions of $d_p(I)$ and $e_p(I)$.

(2) For an odd prime p, $e_p(I) = 2i_1 + \varepsilon_0 - 2(p-1)\sum_{s=2}^n i_s - \sum_{s=1}^n \varepsilon_s \leqq 2i_1 + \varepsilon_0$. If $p = 2$, $e_2(J) = j_1 - \sum_{s=2}^n j_s \leqq j_1$.

(3) If p is an odd prime, $e_p(I) \le 2i_1 + \varepsilon_0$ is equivalent to $e_p(I) - 2pi_1 - 2\varepsilon_0 \le -(p-1)e_p(I) + \varepsilon_0(p-2)$. Then the assertion follows from (1).

The proof of the case $p = 2$ is similar. □

Corollary 1.3 *Let j be a fixed non-negative integer, $\varepsilon = 0, 1$ and $I \in \mathrm{Seq}^o$, $J \in \mathrm{Seq}$.*

1. *Suppose that p is an odd prime. If $e_p(I) \ge 2j + \varepsilon$, then $d_p(I) \ge 2j(p-1) + \varepsilon$ and the equality holds if and only if $I = (\varepsilon, j)$.*
2. *If $e_2(J) \ge j$, then $d_2(J) \ge j$ and the equality holds if and only if $J = (j)$.*

Proof (1) Assume that $e_p(I) \ge 2j$ and $d_p(I) \le 2j(p-1) - 1$. By Proposition 1.2,

$$2i_1 + \varepsilon_0 - 2pi_1 - \varepsilon_0 \ge e_p(I) - 2pi_1 - \varepsilon_0 = -d_p(I) \ge -2j(p-1) + 1.$$

Hence $2j(p-1) \ge 2i_1(p-1) + 1$, which implies $j \ge i_1 + 1$.

Then $2i_1 + \varepsilon_0 \ge e_p(I) \ge 2j \ge 2i_1 + 2$ but this contradicts $\varepsilon_0 \le 1$. Therefore $d_p(I) \ge 2j(p-1)$.

Suppose $e_p(I) \ge 2j$ and $d_p(I) = 2j(p-1)$. Since $d_p(I) \ge 2i_1(p-1)$, we have $j \ge i_1$. On the other hand, since $2j \le e_p(I) \le 2i_1 + \varepsilon_0$, we have $j \le i_1$. Hence $j = i_1$ and this implies $i_s = 0$ for $s \ge 2$ and $\varepsilon_s = 0$ for $s \ge 0$.

Assume $e_p(I) \ge 2j + 1$. By Proposition 1.2,

$$d_p(I) \ge (p-1)e_p(I) - \varepsilon_0(p-2) \ge (p-1)(2j+1) - p + 2 = 2j(p-1) + 1.$$

Suppose that $e_p(I) \ge 2j + 1$ and $d_p(I) = 2j(p-1) + 1$. We have

$$d_p(I) \ge (p-1)e_p(I) - \varepsilon_0(p-2) \ge 2j(p-1) + 1 = d_p(I).$$

Hence I is of the form (ε_0, i_1) by Proposition 1.2. Then $d_p(I) = 2i_1(p-1) + \varepsilon_0$ which equals to $2j(p-1) + 1$. Therefore $I = (1, j)$.

(2) The proof is similar as above. □

Definition 1.4 (Steenrod–Epstein [7]) We say $I = (\varepsilon_0, i_1, \varepsilon_1, \dots, i_n, \varepsilon_n, \dots) \in \mathrm{Seq}^o$ is $(p-)$admissible if p is an odd prime and $i_s \ge pi_{s+1} + \varepsilon_s$ for $s = 1, 2, \dots$. For $p = 2$, we say that $I = (i_1, i_2, \dots, i_n, \dots) \in \mathrm{Seq}$ is (2–)admissible if $i_s \ge 2i_{s+1}$ for $s = 1, 2, \dots$. We denote by Seq_p the subset of Seq consisting of p–admissible sequences.

We quote the following fundamental results for later use.

Theorem 1.5 (Steenrod–Epstein [7])

(1) *If $I \in \mathrm{Seq}^o$, $\wp^I = \sum_{k=1}^{l} c_k \wp^{I_k}$ for some $c_k \in \mathbb{F}_p$ and $I_k \in \mathrm{Seq}_p$ such that $e_p(I_k) \geq e_p(I)$ for all $k = 1, 2, \ldots, l$.*
Similarly, if $I \in \mathrm{Seq}$, $Sq^I = \sum_{k=1}^{l} c_k Sq^{I_k}$ for some $c_k \in \mathbb{F}_2$ and $I_k \in \mathrm{Seq}_2$ such that $e_2(I_k) \geq e_2(I)$ for all $k = 1, 2, \ldots, l$.

(2) *$\{\wp^I \mid I \in \mathrm{Seq}_p\}$ is a basis of $\mathcal{A}_p$ if p is an odd prime and $\{Sq^I \mid I \in \mathrm{Seq}_2\}$ is a basis of $\mathcal{A}_2$.*

Let $\tau_n \in (\mathcal{A}_{p*})^{2p^n-1}$, $\xi_n \in (\mathcal{A}_{p*})^{2p^n-2}$ and $\zeta_n \in (\mathcal{A}_{2*})^{2^n-1}$ be the elements given by Milnor [5]. Recall that $\mathcal{A}_{p*} = E(\tau_0, \tau_1, \ldots) \otimes \mathbb{F}_p[\xi_1, \xi_2, \ldots]$ if $p \neq 2$, and $\mathcal{A}_{2*} = \mathbb{F}_2[\zeta_1, \zeta_2, \ldots]$.

Let Seq^b be a subset of Seq consisting of all sequences $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n, \ldots)$ such that $\varepsilon_n = 0, 1$ for all $n = 0, 1, \ldots$.

For $E = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_m) \in \mathrm{Seq}^b$ and $R = (r_1, r_2, \ldots, r_n) \in \mathrm{Seq}$, we put

$$\tau(E) = \tau_0^{\varepsilon_0}\tau_1^{\varepsilon_1}\cdots\tau_m^{\varepsilon_m}, \quad \xi(R) = \xi_1^{r_1}\xi_2^{r_2}\cdots\xi_n^{r_n} \quad \text{and} \quad \zeta(R) = \zeta_1^{r_1}\zeta_2^{r_2}\cdots\zeta_n^{r_n}$$

as in [5]. Then, the Milnor basis is defined as follows.

Definition 1.6 (Milnor [5]) We denote by $\wp(S)$ the dual of $\xi(S)$ with respect to the basis $\{\tau(E)\xi(R) | E \in \mathrm{Seq}^b, R \in \mathrm{Seq}\}$ is of $\mathcal{A}_{p*}$ if $p \neq 2$ and by $Sq(S)$ the dual of $\zeta(S)$ with respect to the basis $\{\zeta(R) | R \in \mathrm{Seq}\}$ of $\mathcal{A}_{2*}$. If p is odd, let Q_n be the dual of τ_n with respect to the basis $\{\tau(E)\xi(R) | E \in \mathrm{Seq}^b, R \in \mathrm{Seq}\}$. Put $Q(E) = Q_0^{\varepsilon_0} Q_1^{\varepsilon_1} \cdots Q_n^{\varepsilon_n}$ for $E = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n) \in \mathrm{Seq}^b$.

Definition 1.7 Let $F_i\mathcal{A}_p$ be the subspace of $\mathcal{A}_p$ spanned by

$$\left\{\wp^I \middle| I \in \mathrm{Seq}_p, e_p(I) \geq i\right\} \text{ if } p \neq 2, \left\{Sq^I \middle| I \in \mathrm{Seq}_2, e_2(I) \geq i\right\} \text{ if } p = 2.$$

Thus we have an decreasing filtration $\mathfrak{F}_p = (F_i\mathcal{A}_p)_{i \in \mathbb{Z}}$ on $\mathcal{A}_p$. We call $\mathfrak{F}_p$ the excess filtration.

Clearly, $\mathfrak{F}_p$ satisfies the following.

(1) (E1) $F_i\mathcal{A}_p = \mathcal{A}_p$ if $i \leq 0$.

(2) (E2) $\bigcap_{i \in \mathbb{Z}} F_i\mathcal{A}_p = \{0\}$.

The next result is a direct consequence of Theorem 1.5.

Proposition 1.8 *For $I \in \mathrm{Seq}^o$, $\wp^I \in F_i\mathcal{A}_p$ if p is an odd prime and $e_p(I) \geq i$. For $I \in \mathrm{Seq}$, $Sq^I \in F_i\mathcal{A}_2$ if $e_2(I) \geq i$.*

The following properties of the excess filtration are a sort of "folklore".

Proposition 1.9 *Let $\mu\colon \mathcal{A}_p \otimes \mathcal{A}_p \to \mathcal{A}_p$ and $\delta\colon \mathcal{A}_p \to \mathcal{A}_p \otimes \mathcal{A}_p$ be the product and the coproduct of $\mathcal{A}_p$. Then $\mathfrak{F}_p$ satisfies the following conditions.*

(1) *(E3) $F_i\mathcal{A}_p$ are left ideals of $\mathcal{A}_p$ for $i \in \mathbb{Z}$.*

(2) *(E4) $\mu(F_i\mathcal{A}_p \otimes \mathcal{A}_p^j) \subset F_{i-j}\mathcal{A}_p$ for $i, j \in \mathbb{Z}$.*

(3) *(E5) $\delta(F_i\mathcal{A}_p) \subset \sum_{j+k=i} F_j\mathcal{A}_p \otimes F_k\mathcal{A}_p$ for $i \in \mathbb{Z}$.*

Proof Let $I = (\varepsilon_0, i_1, \varepsilon_1, \dots, i_n, \varepsilon_n)$ be a sequence belonging to Seq_p such that $e_p(I) \geq i$ and $i_n \geq 1$ if $n \geq 1$.

If $\varepsilon_0 = 1$, then $\beta\wp^I = 0$. If $\varepsilon_0 = 0$, the excess of $(1, i_1, \varepsilon_1, \dots, i_n, \varepsilon_n)$ is bigger than $e_p(I)$. Hence $\beta\wp^I \in F_i\mathcal{A}_p$. If $j \geq pi_1 + \varepsilon_0$, then $(0, j, \varepsilon_0, i_1, \varepsilon_1, \dots, i_n, \varepsilon_n)$ is admissible and its excess is not less than $e_p(I)$. Hence $\wp^j\wp^I \in F_i\mathcal{A}_p$ in this case. Suppose $1 \leq j < pi_1 + \varepsilon_0$. Then, by Theorem 1.5, $\wp^j\wp^I$ is a linear combination of $\wp^J$'s such that $e_p(J) \geq e_p(I)$ and $J \in \mathrm{Seq}_p$. Thus $\mathfrak{F}_p$ satisfies (E3).

If $\varepsilon_n = 1$, then $\wp^I\beta = 0$. If $\varepsilon_n = 0$, then $e_p(\varepsilon_0, i_1, \varepsilon_1, \dots, i_n, 1) = e_p(I) - 1$. Hence $\wp^I\beta \in F_{i-1}\mathcal{A}_p$ by Proposition 1.8. If $n \geq 1$, then $e_p(\varepsilon_0, i_1, \varepsilon_1, \dots, i_n, \varepsilon_n, j) = 2pi_1 + 2\varepsilon_0 - d_p(I) - 2j(p-1) = e_p(I) - 2j(p-1) \geq i - 2j(p-1)$. Hence $\wp^I\wp^j \in F_{i-2j(p-1)}\mathcal{A}_p$ by Proposition 1.8. It is clear that $\wp^I\wp^j \in F_{i-2j(p-1)}\mathcal{A}_p$ if $n = 0$. Thus $\mathfrak{F}_p$ satisfies (E4).

By the Cartan formula, $\delta(\wp^I) = \sum_{J+L=I} \wp^J \otimes \wp^L$. Put $J = (\alpha_0, j_1, \dots)$, $L = (\beta_0, l_1, \dots)$. Then, $e_p(J) + e_p(L) = 2p(j_1 + l_1) + 2(\alpha_0 + \beta_0) - d_p(J) - d_p(L) = 2pi_1 + 2\varepsilon_0 - d_p(I) = e_p(I)$. Hence (E5) follows from Proposition 1.8. □

Consider the dual filtration $\mathfrak{F}_p^* = (F_i\mathcal{A}_{p*})_{i\in\mathbb{Z}}$ on $\mathcal{A}_{p*}$, that is, $(F_i\mathcal{A}_{p*})^n$ is the kernel of

$$\kappa_{i+1}^*\colon (\mathcal{A}_{p*})^n = \mathrm{Hom}(\mathcal{A}_p^n, \mathbb{F}_p) \to \mathrm{Hom}((F_{i+1}\mathcal{A}_p)^n, \mathbb{F}_p),$$

where $\kappa_i\colon F_i\mathcal{A}_p \to \mathcal{A}_p$ is the inclusion map. The following is the dual of (E5) of Proposition 1.9.

Proposition 1.10 *Let $\delta^*\colon \mathcal{A}_{p*} \otimes \mathcal{A}_{p*} \to \mathcal{A}_{p*}$ be the product of $\mathcal{A}_{p*}$. Then $\mathfrak{F}_{p*}$ satisfies the following.*

(1) *(E5*) $\delta^*(F_j\mathcal{A}_{p*} \otimes F_k\mathcal{A}_{p*}) \subset F_{j+k}\mathcal{A}_{p*}$ for $j, k \in \mathbb{Z}$.*

We set $J_n = (0, p^{n-1}, 0, p^{n-2}, \dots, 0, 1, 0)$ and $J'_n = (0, p^{n-1}, 0, p^{n-2}, \dots, 0, 1, 1)$ for an odd prime p, $K_n = (2^{n-1}, 2^{n-2}, \dots, 2, 1,)$. Then, J_n's and J'_n's are admissible and $d_p(J_n) = 2p^n - 2$, $d_p(J'_n) = 2p^n - 1$, $e_p(J_n) = 2$, $e_p(J'_n) = 1$ if p is an odd prime, $d_2(K_n) = 2^n - 1$, $e_2(K_n) = 1$.

For $R = (r_1, r_2, \dots, r_n, \dots) \in \mathrm{Seq}$, we put $|R| = \sum_{i\geq 1} r_i$.

Proposition 1.11 *$\tau(E)\xi(R) \in F_{|E|+2|R|}\mathcal{A}_{p*} - F_{|E|+2|E|-1}\mathcal{A}_{p*}$ for $R \in \mathrm{Seq}$ and $E \in \mathrm{Seq}^b$, if p is an odd prime. $\zeta(R) \in F_{|R|}\mathcal{A}_{2*} - F_{|R|-1}\mathcal{A}_{2*}$ for $R \in \mathrm{Seq}$.*

Proof Since $e_p(J'_n) = 1$ and $e_p(J_n) = 2$, it follows from Milnor [5, Lemma 8] that $\tau_i \in F_1\mathcal{A}_{p*}$ and $\xi_i \in F_2\mathcal{A}_{p*}$. Similarly, since $e_2(K_n) = 1$, we have $\zeta_i \in F_1\mathcal{A}_{2*}$. Hence, by Proposition 1.10, we have $\tau(E)\xi(R) \in F_{|E|+2|R|}\mathcal{A}_{p*}$ if $p \neq 2$, $\zeta(R) \in F_{|R|}\mathcal{A}_{2*}$. On the other hand, it follows from Lemma 1.13 and [5, Lemma 8] that $\tau(E)\xi(R) \notin F_{|E|+2|E|-1}\mathcal{A}_{p*}$ and $\zeta(R) \notin F_{|R|-1}\mathcal{A}_{2*}$. □

We define the maps $\varrho_p\colon \mathrm{Seq}^o \to \mathrm{Seq}_p$ and $\varrho_2\colon \mathrm{Seq} \to \mathrm{Seq}_2$ as follows. For $J = (\varepsilon_0, j_1, \varepsilon_1, \dots, j_n, \varepsilon_n) \in \mathrm{Seq}^o$, put

$$i_s = \sum_{k=s}^{n} (\varepsilon_k + j_k) p^{k-s} \quad (s = 1, 2, \dots, n)$$

and $\varrho_p(J) = (\varepsilon_0, i_1, \varepsilon_1, \dots, i_n, \varepsilon_n)$.

If $p = 2$, for $J = (j_1, j_2, \dots, j_n) \in \mathrm{Seq}$, put

$$i_s = \sum_{k=s}^{n} j_k 2^{k-s} \quad (s = 1, 2, \dots, n)$$

and $\varrho_2(J) = (i_1, i_2, \dots, i_n)$.

The following Lemmas are straightforward.

Lemma 1.12 *ϱ_p is bijective and its inverse ϱ_p^{-1} is given as follows. If p is an odd prime, $\varrho_p^{-1}(\varepsilon_0, i_1, \varepsilon_1, \dots, i_n, \varepsilon_n) = (\varepsilon_0, j_1, \varepsilon_1, \dots, j_n, \varepsilon_n)$, where $j_s = i_s - p i_{s+1} - \varepsilon_s$ (for $s = 1, 2, \dots, n-1$), and $j_n = i_n - \varepsilon_n$.*

Similarly, $\varrho_2^{-1}(i_1, i_2, \dots, i_n) = (j_1, j_2, \dots, j_n)$, where $j_s = i_s - 2i_{s+1}$ (for $s = 1, 2, \dots, n-1$), and $j_n = i_n$.

Lemma 1.13 *If $p \neq 2$, for $J = (\varepsilon_0, j_1, \varepsilon_1, \dots, j_n, \varepsilon_n) \in \mathrm{Seq}^o$, we have*

$$d_p(\varrho_p(J)) = \sum_{k=1}^{n} 2j_k(p^k - 1) + \sum_{k=0}^{n} \varepsilon_k(2p^k - 1) \text{ and } e_p(\varrho_p(J)) = 2\sum_{k=1}^{n} j_k + \sum_{k=0}^{n} \varepsilon_k.$$

If $p = 2,\ J = (j_1, j_2, \dots, j_n) \in \mathrm{Seq}$, *we have*

$$d_2(\varrho_2(J)) = \sum_{k=1}^{n} j_k(2^k - 1) \text{ and } e_2(\varrho_2(J)) = \sum_{k=1}^{n} j_k.$$

Proposition 1.14 $\{\tau(E)\xi(R) | E \in \mathrm{Seq}^b,\ R \in \mathrm{Seq},\ |E| + 2|R| \le i\}$ *is a basis of* $F_i\mathcal{A}_{p*}$ *and* $\{\zeta(R) | R \in \mathrm{Seq},\ |R| \le i\}$ *is a basis of* $F_i\mathcal{A}_{2*}$.

Proof Since $(F_i\mathcal{A}_{p*})^n$ is isomorphic to $\mathrm{Hom}^*((\mathcal{A}_p/F_{i+1}\mathcal{A}_p)^n, \mathbb{F}_p)$, we have

$$\dim(F_i\mathcal{A}_{p*})^n = \dim(\mathcal{A}_p/F_{i+1}\mathcal{A}_p)^n = \dim \mathcal{A}_p^n - \dim(F_{i+1}\mathcal{A}_p)^n.$$

Suppose p is odd. By Theorem 1.5 (2), Lemma 1.12 and Lemma 1.13, $\dim \mathcal{A}_p^n$ is the number of elements of a subset S_n of Seq^o defined by

$$S_n = \left\{(\varepsilon_0, j_1, \varepsilon_1, \dots, j_n, \varepsilon_n, \dots) \in \mathrm{Seq}^o \,\middle|\, \sum_{k\ge 0}(2p^k - 1)\varepsilon_k + \sum_{k\ge 1} 2(p^k - 1) j_k = n\right\}$$

and $\dim(F_{i+1}\mathcal{A}_p)^n$ is the number of elements of

$$\left\{(\varepsilon_0, j_1, \varepsilon_1, \dots, j_n, \varepsilon_n, \dots) \in S_n \,\middle|\, \sum_{k\ge 0}\varepsilon_k + \sum_{k\ge 1} 2j_k \ge i + 1\right\}.$$

Hence $\dim(F_i\mathcal{A}_{p*})^n$ is the number of elements of

$$\left\{(\varepsilon_0, j_1, \varepsilon_1, \dots, j_n, \varepsilon_n, \dots) \in S_n \,\middle|\, \sum_{k\ge 0}\varepsilon_k + \sum_{k\ge 1} 2j_k \le i\right\},$$

which coincides with the number of elements of

$$\{\tau(E)\xi(R) |\ E \in \mathrm{Seq}^b,\ R \in \mathrm{Seq},\ |E| + 2|R| \le i\}.$$

Therefore the assertion follows from Proposition 1.11. The proof for the case $p = 2$ is similar. □

The following is shown by Kraines [1] but is also a direct consequence of Milnor [5, Theorem 4a], Proposition 1.11 and Proposition 1.14.

Proposition 1.15 (Kraines [1])

(1) $Q(E)\wp(R) \in F_{|E|+2|R|}\mathcal{A}_p - F_{|E|+2|R|+1}\mathcal{A}_p$ *for* $R \in \mathrm{Seq}$ *and* $E \in \mathrm{Seq}^b$ *if* p *is an odd prime.* $Sq(R) \in F_{|R|}\mathcal{A}_2 - F_{|R|+1}\mathcal{A}_2$ *for* $R \in \mathrm{Seq}$.

(2) $\{Q(E)\wp(R) |\ E \in \mathrm{Seq}^b,\ R \in \mathrm{Seq},\ |E| + 2|R| \ge i\}$ *is a basis of* $F_i\mathcal{A}_p$ *for an odd prime* p. $\{Sq(R) |\ R \in \mathrm{Seq},\ |R| \ge i\}$ *is a basis of* $F_i\mathcal{A}_2$.

We set $E_i^j\mathcal{A}_p = (F_i\mathcal{A}_p)^j/(F_{i+1}\mathcal{A}_p)^j$.

Proposition 1.16 *Let i be a non-negative integer and $\varepsilon = 0$ or 1.*

(1) $(F_{2i+\varepsilon}\mathcal{A}_p)^k = \{0\}$ *for* $k < 2i(p-1)+\varepsilon$.

(2) *If p is an odd prime, $(F_{2i+\varepsilon}\mathcal{A}_p)^{2i(p-1)+\varepsilon}$ is a one dimensional vector space spanned by $\beta^\varepsilon\wp^i$. $(F_i\mathcal{A}_2)^i$ is a one dimensional vector space spanned by Sq^i.*

(3) $E_i^j\mathcal{A}_p = \{0\}$ *if* $i+j \not\equiv 0, 2$ *modulo* $2p$.

Proof (1) and (2) are direct consequences of Corollary 1.3.

Suppose that $E = (\varepsilon_0, \varepsilon_1, \dots) \in \mathrm{Seq}^b$, and $R = (r_1, r_2, \dots) \in \mathrm{Seq}$ satisfy $|E| + 2|R| = i$ and $Q(E)\wp(R) \in \mathcal{A}_p^j$. Then

$$i + j = \sum_{s\geq 0} 2\varepsilon_s p^s + \sum_{t\geq 1} 2r_t p^t \equiv 2\varepsilon_0 \text{ modulo } 2p.$$

Thus (3) follows from Proposition 1.15. □

2 More on excess filtration

For $R = (r_1, r_2, \dots, r_n, \dots) \in \mathrm{Seq}$, put $s(R) = (0, r_1, r_2, \dots, r_n, \dots)$.

If some entry of R is not a non-negative integer, we put $\wp(R) = 0$. We regard Seq as a monoid with componentwise addition, then $\mathbf{0} = (0, 0, \dots, 0, \dots)$ is the unit of Seq. Let E_n be an element of Seq^b such that the nth entry is 1 and other entries are all 0. (We put $E_0 = \mathbf{0}$.)

Lemma 2.1

(1) *If $\varepsilon = 0, 1$, $|E| + 2|R| \leq 2i - j + 1$ and $Q(E)\wp(R) \in \mathcal{A}_p^j$,*

$$\beta^\varepsilon\wp^i Q(E)\wp(R) \equiv Q(\varepsilon E_1 + s(E))\wp\big(\big(i - \tfrac{1}{2}(|E| + j) - |R|\big)E_1 + s(R)\big)$$

modulo $F_{2i-j+\varepsilon+1}\mathcal{A}_p$ for an odd prime p.

(2) *If $|R| \leq i - j$ and $Sq(R) \in \mathcal{A}_2^j$,*

$$Sq^i Sq(R) \equiv Sq((i - j - |R|)E_1 + s(R)) \text{ modulo } F_{i-j+1}\mathcal{A}_2.$$

Proof Let $E=(\varepsilon_0,\varepsilon_1,\varepsilon_2,\dots)\in \mathrm{Seq}^b$ and $R=(r_1,r_2,\dots)\in \mathrm{Seq}$. We put $Q(E)=Q_{n_1}Q_{n_2}\cdots Q_{n_k}$ $(0\le n_1<n_2<\cdots<n_k)$. By Milnor [5, Theorem 4a, 4b], we have

$$\beta^\epsilon \wp^i Q_{n_1}Q_{n_2}\cdots Q_{n_k}=\sum_{e_t=0,1}\beta^\epsilon Q_{n_1+e_1}Q_{n_2+e_2}\cdots\beta^\epsilon Q_{n_k+e_k}\wp^{i-\sum_{t=1}^k e_t p^{n_t}}$$

and

$$\wp^m\wp(R)=\sum_{\sum_{s\ge 0}a_s p^s=m}\ \prod_{s\ge 1}\binom{r_s-a_s+a_{s-1}}{a_{s-1}}\wp(r_1-a_1+a_0,r_2-a_2+a_1,\dots).$$

Thus $\beta^\epsilon\wp^i Q_{n_1}Q_{n_2}\cdots Q_{n_k}\wp(R)$ is a linear combination of the Milnor basis

$$Q_0^\epsilon Q_{n_1+e_1}Q_{n_2+e_2}\cdots Q_{n_k+e_k}\wp(r_1-a_1+a_0,r_2-a_2+a_1,\dots)$$

for $e_1,e_2,\dots,e_k=0,1$ and non-negative integers $a_0,a_1,a_2,\dots$ satisfying

$$a_0=i-\sum_{t=1}^k e_t p^{n_t}-\sum_{s\ge 1}a_s p^s \text{ and } a_s\le r_s \text{ for } s=1,2,\dots.$$

Suppose that sequences of non-negative integers $e_1,e_2,\dots,e_k$ and $a_0,a_1,a_2,\dots$ satisfy $e_j=0$ or 1, $a_s\le r_s$ and $a_0=i-\sum_{t=1}^k e_t p^{n_t}-\sum_{s\ge 1}a_s p^s$. We note that

$$a_0\ge i-\sum_{t=1}^k p^{n_t}-\sum_{s\ge 1}r_s p^s. \tag{1}$$

Let F be a sequence of integers such that $Q^F=Q_0^\epsilon Q_{n_1+e_1}Q_{n_2+e_2}\cdots Q_{n_k+e_k}$ and put $S=(r_1-a_1+a_0,r_2-a_2+a_1,\dots)$.

Assume that $|E|+2|R|\le 2i-j+1$ and $Q(E)\wp(R)\in\mathcal{A}_p^j$. Since

$$j=\sum_{s\ge 0}\varepsilon_s(2p^s-1)+\sum_{t\ge 1}2r_t(p^t-1)=2\sum_{s\ge 0}\varepsilon_s p^s+2\sum_{t\ge 1}r_t p^t-|E|-2|R|,$$

we have

$$\sum_{t=1}^k p^{n_t}+\sum_{t\ge 1}r_t p^t=\sum_{s\ge 0}\varepsilon_s p^s+\sum_{t\ge 1}r_t p^t=\tfrac12(|E|+2|R|+j)\le i+\tfrac12.$$

Hence the right hand side of (1) is non-negative and a_0 takes the minimum value

$$i-\sum_{t=1}^k p^{n_t}-\sum_{s\ge 1}r_s p^s=i-\tfrac12(|E|+j)-|R|$$

if and only if $a_s = r_s$ for $s = 1, 2, \dots$ and $e_1 = e_2 = \cdots = e_k = 1$. In this case, $F = (\varepsilon, \varepsilon_0, \varepsilon_1, \varepsilon_2, \dots)$ and $S = \left(i - \frac{1}{2}(|E| + j) - |R|, r_1, r_2, \dots\right)$. Therefore

$$|F| + 2|S| = |E| + 2|R| + \varepsilon + 2\left(i - \tfrac{1}{2}(|E| + j) - |R|\right) = 2i - j + \varepsilon$$

and the result follows. Proof for the case $p = 2$ is similar. □

Put $E_i^* \mathcal{A}_p = \sum_{j \in \mathbb{Z}} E_i^j \mathcal{A}_p$. Since the excess filtration $\mathfrak{F}_p$ satisfies (E3) and (E4) of Proposition 1.9, $E_i^* \mathcal{A}_p$ is a left $\mathcal{A}_p$–module and the product map $\mu\colon \mathcal{A}_p \otimes \mathcal{A}_p \to \mathcal{A}_p$ induces the following maps.

$$\mu_i\colon \mathcal{A}_p \otimes E_i^* \mathcal{A}_p \to E_i^* \mathcal{A}_p, \quad \widetilde{\mu}_i^{k,j}\colon E_i^k \mathcal{A}_p \otimes (\mathcal{A}_p / F_{i-j-1} \mathcal{A}_p)^j \to E_{i-j}^{k+j} \mathcal{A}_p.$$

Theorem 2.2 *For non-negative integer i, j and $\varepsilon = 0, 1$, the following map is an isomorphism.*

$$\widetilde{\mu}_{2i+\varepsilon}^{2i(p-1)+\varepsilon, j}\colon E_{2i+\varepsilon}^{2i(p-1)+\varepsilon} \mathcal{A}_p \otimes (\mathcal{A}_p / F_{2i-j+\varepsilon+1} \mathcal{A}_p)^j \to E_{2i-j+\varepsilon}^{2i(p-1)+j+\varepsilon} \mathcal{A}_p.$$

Proof Suppose $Q^F \wp(S) \in \mathcal{A}_p^{2i(p-1)+j+\varepsilon}$ and $|F| + 2|S| = 2i - j + \varepsilon$ for $F = (\lambda_0, \lambda_1, \lambda_2, \dots)$ and $S = (s_1, s_2, \dots)$. Then,

$$\sum_{k \ge 0} \lambda_k + 2 \sum_{k \ge 1} s_k = 2i - j + \varepsilon \tag{2}$$

$$\sum_{k \ge 0} \lambda_k (2p^k - 1) + 2 \sum_{k \ge 1} s_k (p^k - 1) = 2i(p-1) + j + \varepsilon. \tag{3}$$

Hence

$$\sum_{k \ge 0} \lambda_k p^k + \sum_{k \ge 1} s_k p^k = ip + \varepsilon, \tag{4}$$

and this implies $\lambda_0 = \varepsilon$. We put $E = (\lambda_1, \lambda_2, \dots)$ and $R = (s_2, s_3, \dots)$. By (2) above, we have $s_1 = i - \frac{1}{2}(|E| + j) - |R|$. Therefore, $\beta^\varepsilon \wp^i Q(E) \wp(R) \equiv Q^F \wp(S)$ modulo $F_{2i-j+\varepsilon+1} \mathcal{A}_p$ by Lemma 2.1. This shows that $\widetilde{\mu}_{2i+\varepsilon}^{2i(p-1)+\varepsilon, j}$ is surjective. It is clear from Lemma 2.1 that $\widetilde{\mu}_{2i+\varepsilon}^{2i(p-1)+\varepsilon, j}$ is injective. The proof for the case $p = 2$ is similar. □

For $R = (r_1, r_2, \dots, r_n, \dots) \in \mathrm{Seq}$ and a non-zero integer p, we say that p divides R if $p | r_i$ for all $i \ge 1$ and denote this by $p | R$ and by $p \nmid R$ otherwise. Put $\frac{1}{p} R = \left(\frac{r_1}{p}, \frac{r_2}{p}, \dots, \frac{r_n}{p}, \dots\right)$ if $p | R$.

Lemma 2.3 *Let p be an odd prime. For $E \in \mathrm{Seq}^b$, $R \in \mathrm{Seq}$ and $j \ge 0$, the following congruences hold.*

(1) *If* $|R| \le pj$ *and* $p|R$,

$$\wp(R)\wp^j \equiv \wp\big(\big(j - \tfrac{1}{p}|R|\big)E_1 + s\big(\tfrac{1}{p}R\big)\big) \text{ modulo } F_{2j+1}\mathcal{A}_p.$$

If $E \neq \mathbf{0}$ or $|R| > pj$ *or* $p\not|R$, $Q(E)\wp(R)\wp^j \in F_{2j+1}\mathcal{A}_p$.

(2) *If* $|R| \le pj+1$ *and* $p|R$,

$$\wp(R)\beta\wp^j \equiv \beta\wp\big(\big(j - \tfrac{1}{p}|R|\big)E_1 + s\big(\tfrac{1}{p}R\big)\big) \text{ modulo } F_{2j+2}\mathcal{A}_p.$$

If $|R| \le pj+1$ *and* $p|R - E_n$ *for some* $n \ge 1$,

$$\wp(R)\beta\wp^j \equiv Q_n\wp\big(\big(j - \tfrac{1}{p}(|R|-1)\big)E_1 + s\big(\tfrac{1}{p}(R-E_n)\big)\big) \text{ modulo } F_{2j+2}\mathcal{A}_p.$$

If $E \neq \mathbf{0}$ or $|R| > pj+1$ *or* $p\not|R - E_n$ *for any* $n \ge 0$, $Q(E)\wp(R)\beta\wp^j \in F_{2j+2}\mathcal{A}_p$.

Proof (1) By Milnor [5, Theorem 4b], we have

$$\wp(R)\wp^j = \sum_{x_0+x_1+\cdots=j}\prod_{k\ge 0}\binom{r_k - px_k + x_{k-1}}{x_{k-1}} \wp(r_1 - px_1 + x_0, r_2 - px_2 + x_1, \dots),$$

for $R = (r_1, r_2, \dots)$. Since $\binom{r_k - px_k + x_{k-1}}{x_{k-1}} = 0$ if $r_k < px_k$, the summation of the right hand side of the above is taken over non-negative integers $x_0, x_1, \dots$ satisfying $x_0 + x_1 + \cdots = j$ and $px_k \le r_k$ for all $k = 1, 2, \dots$.

Hence $p(j - x_0) = p(x_1 + x_2 + \cdots) \le |R|$ and $p(j - x_0) = |R|$ holds if and only if $px_k = r_k$ for all $k = 1, 2, \dots$.

Put

$$S = (r_1 - px_1 + x_0, \dots, r_k - px_k + x_{k-1}, \dots),$$

then $|S| = |R| - p(j - x_0) + j \ge j$ and $|S| = j$ hold if and only if $p|R$, $|R| \le pj$ and $S = \big(j - \frac{1}{p}|R|\big)E_1 + s\big(\frac{1}{p}R\big)$.

Therefore $Q(E)\wp(R)\wp^j \in F_{2j+1}\mathcal{A}_p$ unless $E = \mathbf{0}$, $|R| \le pj$ and $p|R$.

(2) Since $\wp(R)\beta = \sum_{n\ge 0} Q_n\wp(R - E_n)$ by Milnor [5, Theorem 4a], the result follows from (1). □

In the case $p = 2$, a similar result holds.

Lemma 2.4 *For $R \in \mathrm{Seq}$ and $j \geq 0$, the following congruences hold.*

If $|R| \leq j$ and $2|R$,

$$Sq(R)Sq^j \equiv Sq\big(\big(j - \tfrac{1}{2}|R|\big)E_1 + s\big(\tfrac{1}{2}R\big)\big) \text{ modulo } F_{j+1}\mathcal{A}_2.$$

If $|R| > j$ or $2\nmid R$, $Sq(R)Sq^j \in F_{j+1}\mathcal{A}_2$.

Lemma 2.5 *Let p be an odd prime, $R \in \mathrm{Seq}$ and $j \geq 0$. If $\wp(R) \in \mathcal{A}_p^k$. then, the following congruences hold.*

(1) *If $|R| \leq pj$ and $p|R$,*

$$\wp(R)\wp^j \equiv \wp^{j+\frac{k}{2p}}\wp\big(\tfrac{1}{p}R\big) \text{ modulo } F_{2j+1}\mathcal{A}_p.$$

(2) *If $|R| \leq pj+1$ and $p|R$,*

$$\wp(R)\beta\wp^j \equiv \beta\wp^{j+\frac{k}{2p}}\wp\big(\tfrac{1}{p}R\big) \text{ modulo } F_{2j+2}\mathcal{A}_p,$$

if $|R| \leq pj+1$ and $p|R-E_n$ for some $n \geq 1$,

$$\wp(R)\beta\wp^j \equiv \wp^{j+\frac{k+2}{2p}}Q_{n-1}\wp\big(\tfrac{1}{p}(R-E_n)\big) \text{ modulo } F_{2j+2}\mathcal{A}_p.$$

Proof The first congruence and is a direct consequence of Lemma 2.1 and Lemma 2.3. Suppose $p|R-E_n$ and $|R| \leq pj+1$. By Milnor [5, Theorem 4a], Lemma 2.1 and Lemma 2.3,

$$\begin{aligned}
&\wp^{j+\frac{k+2}{2p}}Q_{n-1}\wp\big(\tfrac{1}{p}(R-E_n)\big)\\
&\qquad = Q_{n-1}\wp^{j+\frac{k+2}{2p}}\wp\big(\tfrac{1}{p}(R-E_n)\big) + Q_n\wp^{j+\frac{k+2}{2p}-p^{n-1}}\wp\big(\tfrac{1}{p}(R-E_n)\big)\\
&\qquad \equiv Q_n\wp\big(\big(j-\tfrac{1}{p}(|R|-1)\big)E_1 + s\big(\tfrac{1}{p}R\big)\big) \text{ modulo } F_{2j+2}\mathcal{A}_p\\
&\qquad \equiv \wp(R)\beta\wp^j \text{ modulo } F_{2j+2}\mathcal{A}_p.
\end{aligned}$$

We also obtain $\wp(R)\beta\wp^j \equiv \beta\wp^{j+\frac{k}{2p}}\wp\big(\tfrac{1}{p}R\big)$ if $|R| \leq pj+1$ and $p|R$ from Lemma 2.1 and Lemma 2.3. □

Lemma 2.6 *For $R \in \mathrm{Seq}$ and $j \geq 0$, if $|R| \leq j$, $2|R$ and $Sq(R) \in \mathcal{A}_2^k$,*

$$Sq(R)Sq^j \equiv Sq^{j+\frac{k}{2}}Sq\big(\tfrac{1}{2}R\big) \text{ modulo } F_{j+1}\mathcal{A}_2.$$

For non-negative integers i, j and $\varepsilon = 0, 1$, put $\kappa = \varepsilon$ if j is even and $\kappa = 1-\varepsilon$ if j is odd. Let $\gamma_{i,j,\varepsilon}$ be the composition of maps

$$\mu_{2i-j+\varepsilon}\colon \mathcal{A}_p^{pj-(p-2)(\varepsilon-\kappa)} \otimes E_{2i-j+\varepsilon}^{(2i-j+\varepsilon-\kappa)(p-1)+\kappa}\mathcal{A}_p \to E_{2i-j+\varepsilon}^{2i(p-1)+j+\varepsilon}\mathcal{A}_p$$

and

$$(\tilde{\mu}_{2i+\varepsilon}^{2i(p-1)+\varepsilon,j})^{-1}\colon E_{2i-j+\varepsilon}^{2i(p-1)+j+\varepsilon}\mathcal{A}_p \to E_{2i+\varepsilon}^{2i(p-1)+\varepsilon}\mathcal{A}_p \otimes (\mathcal{A}_p/F_{2i-j+\varepsilon+1}\mathcal{A}_p)^j.$$

Let us denote by $\rho\colon \mathcal{A}_p^{pj} \to \mathcal{A}_p^j$ the pth root map, that is, the dual of pth power map $\mathcal{A}_{p*}^j \to \mathcal{A}_{p*}^{pj}$, $x \mapsto x^p$. By Milnor [5, Lemma 9], we have

$$\rho(Q(E)\wp(R)) = \begin{cases} \wp\left(\frac{1}{p}R\right) & E = \mathbf{0},\ p|R \\ 0 & \text{otherwise.} \end{cases}$$

Let $\pi_i\colon \mathcal{A}_p \to \mathcal{A}_p/F_i\mathcal{A}_p$ be the quotient map. Put $g_{2i+\varepsilon} = \pi_{2i+\varepsilon+1}(\beta^\varepsilon\wp^i)$, then, $g_{2i+\varepsilon}$ generates $E_{2i+\varepsilon}^{2i(p-1)+\varepsilon}\mathcal{A}_p$. The next result is a direct consequence of Lemma 2.3 and Lemma 2.5.

Proposition 2.7 *Let i, j, k be non-negative integers, $\varepsilon = 0, 1$ and p an odd prime.*

(1) $\gamma_{i+j,2j,\varepsilon}\colon \mathcal{A}_p^{2jp} \otimes E_{2i+\varepsilon}^{2i(p-1)+\varepsilon}\mathcal{A}_p \to E_{2(i+j)+\varepsilon}^{2(i+j)(p-1)+\varepsilon}\mathcal{A}_p \otimes (\mathcal{A}_p/F_{2i+1}\mathcal{A}_p)^{2j}$ *maps* $\theta \otimes g_{2i+\varepsilon} \in \mathcal{A}_p^{2jp} \otimes E_{2i+\varepsilon}^{2i(p-1)+\varepsilon}\mathcal{A}_p$ *to* $g_{2i+2j+\varepsilon} \otimes \pi_{2i+1}\rho(\theta)$.

(2) $\gamma_{i+j,2j+1,1}\colon \mathcal{A}_p^{2jp+2} \otimes E_{2i}^{2i(p-1)}\mathcal{A}_p \to E_{2(i+j)+1}^{2(i+j)(p-1)+1}\mathcal{A}_p \otimes (\mathcal{A}_p/F_{2i+1}\mathcal{A}_p)^{2j+1}$ *is a trivial map.*

(3) $\gamma_{i+j,2j-1,0}\colon \mathcal{A}_p^{2jp-2} \otimes E_{2i+1}^{2i(p-1)+1}\mathcal{A}_p \to E_{2(i+j)}^{2(i+j)(p-1)}\mathcal{A}_p \otimes (\mathcal{A}_p/F_{2i+2}\mathcal{A}_p)^{2j-1}$ *maps* $(F_{kp+2}\mathcal{A}_p)^{2jp-2} \otimes E_{2i+1}^{2i(p-1)+1}\mathcal{A}_p$ *into*

$$E_{2(i+j)}^{2(i+j)(p-1)}\mathcal{A}_p \otimes (F_{k+1}\mathcal{A}_p/F_{2i+2}\mathcal{A}_p)^{2j-1}.$$

For $p = 2$, we have the following Proposition.

Proposition 2.8 *Let i, j be non-negative integers.*

$$\gamma_{i,j,\varepsilon}\colon \mathcal{A}_2^{2j} \otimes E_{2i-j+\varepsilon}^{2i-j+\varepsilon}\mathcal{A}_2 \to E_{2i+\varepsilon}^{2i+\varepsilon}\mathcal{A}_2 \otimes (\mathcal{A}_2/F_{2i-j+\varepsilon+1}\mathcal{A}_2)^j$$

maps $\theta \otimes g_{2i-j+\varepsilon} \in \mathcal{A}_2^{2j} \otimes E_{2i+\varepsilon}^{2i+\varepsilon}\mathcal{A}_2$ *to* $g_{2i+\varepsilon} \otimes \pi_{2i-j+\varepsilon+1}\rho(\theta)$.

3 Filtered Hopf algebra

We denote by $\mathcal{E}^*$ the category of graded vector spaces over a field K and linear maps preserving degrees. We also denote by $\mathcal{E}$ the category of (ungraded) vector spaces over K. For $n \in \mathbb{Z}$, define functors $\Sigma^n\colon \mathcal{E}^* \to \mathcal{E}^*$, $\epsilon_n\colon \mathcal{E}^* \to \mathcal{E}$ and $\iota_n\colon \mathcal{E} \to \mathcal{E}^*$ as follows.

$$(\Sigma^n V^*)^i = V^{i-n}, \quad (\Sigma^n f)^i = f^{i-n}, \quad \epsilon_n(V^*) = V^n, \quad \epsilon_n(f) = f^n,$$

for an object V^* and morphism f of $\mathcal{E}^*$.

$$\iota_n(W)^k = \begin{cases} W & k = n \\ 0 & k \neq n \end{cases}, \qquad \iota_n(g)^k = \begin{cases} g & k = n \\ 0 & k \neq n, \end{cases}$$

for an object W and morphism g of $\mathcal{E}$.

Proposition 3.1 *ι_n is a right and left adjoint of ϵ_n.*

Proof Define natural transformations $u_n\colon \mathrm{id}_{\mathcal{E}^*} \to \iota_n\epsilon_n$, $\overline{u}_n\colon \epsilon_n\iota_n \to \mathrm{id}_{\mathcal{E}}$, $\overline{c}_n\colon \mathrm{id}_{\mathcal{E}} \to \epsilon_n\iota_n$ and $c_n\colon \iota_n\epsilon_n \to \mathrm{id}_{\mathcal{E}^*}$ as follows. For $V^* \in \mathrm{Ob}\ \mathcal{E}^*$,

$$u_{n\,V^*}(x) = \begin{cases} x & x \in V^n \\ 0 & x \in V^k, k \neq n \end{cases}, \qquad c_{n\,V^*}(x) = x \quad (x \in V^n).$$

For $U \in \mathrm{Ob}\ \mathcal{E}$, $\overline{u}_{n\,U}(y) = y$ $(y \in (\epsilon_n\iota_n(U))^n = U)$, $\overline{c}_{n\,U}(y) = y$ $(y \in U)$. Clearly, $c_{n\,V^*}\colon \iota_n\epsilon_n(V^*) \to V^*$ is an inclusion map and $\overline{u}_{n\,U}\colon \epsilon_n\iota_n(U) \to U$ and $\overline{c}_{n\,U}\colon U \to \epsilon_n\iota_n(U)$ can be regarded as identity maps. Then, u_n and $\overline{u}_n$ are the unit and the counit of the adjunction $\epsilon_n \vdash \iota_n$ respectively, and $\overline{c}_n$ and c_n are the unit and the counit of the adjunction $\iota_n \vdash \epsilon_n$ respectively. □

Let A^* be a graded Hopf algebra over K with an decreasing filtration $\mathfrak{F} = (F_iA^*)_{i\in\mathbb{Z}}$ of subspaces of A^*. The notion of unstable A^*–module is defined as follows.

Definition 3.2 A left A^*–module M^* with structure map $\alpha\colon A^* \otimes M^* \to M^*$ is called an unstable A^*–module with respect to $\mathfrak{F}$ if $\alpha(F_{n+1}A^* \otimes M^n) = \{0\}$ for $n \in \mathbb{Z}$. We denote by $\mathcal{UM}(A^*)$ the full subcategory of the category of left A^*–modules consisting of unstable A^*–modules.

We are going to give conditions on $\mathfrak{F}$ which suffices to develop a theory of unstable A^*–modules. The following is the first one.

Condition 3.3

(1) (E1) $F_iA^* = A^*$ if $i \leq 0$.

(2) (E2) $\bigcap_{i\in\mathbb{Z}} F_iA^* = \{0\}$.

Note that if $\mathfrak{F}$ satisfies (E1) and V^* is an unstable A^*–module, $V^n = \{0\}$ for $n < 0$. The next one comes from Proposition 1.9.

Condition 3.4 Let us denote by $\mu\colon A^* \otimes A^* \to A^*$ and $\delta : A^* \to A^* \otimes A^*$ the product and the coproduct of A^*, respectively. For an decreasing filtration $\mathfrak{F} = (F_iA^*)_{i\in\mathbb{Z}}$ of subspaces of A^*, we consider the following conditions.

(1) (E3) $F_i A^*$'s are left ideals of A^* for $i \in \mathbb{Z}$.

(2) (E4) $\mu(F_i A^* \otimes A^j) \subset F_{i-j} A^*$ for $i, j \in \mathbb{Z}$.

(3) (E5) $\delta(F_i A^*) \subset \sum_{j+k=i} F_j A^* \otimes F_k A^*$ for $i \in \mathbb{Z}$.

We remark that if $\mathfrak{F}$ satisfies (E3) of Condition 3.4 and $\Sigma^n(A^*/F_{n+1}A^*)$ is an unstable A^*–module, then $\mathfrak{F}$ satisfies (E4) of Condition 3.4. It is easy to verify the following fact.

Proposition 3.5 *Let A^* be a graded Hopf algebra over K with decreasing filtration $\mathfrak{F}$. Suppose that $\mathfrak{F}$ satisfies the condition (E5) in Condition 3.4. If V^* and W^* are unstable A^*–modules with respect to $\mathfrak{F}$, then so is $V^* \otimes W^*$.*

Let us denote by $\mathcal{O}\colon \mathcal{UM}(A^*) \to \mathcal{E}^*$ the forgetful functor. Suppose that $\mathfrak{F}$ satisfies (E3) and (E4) of Condition 3.4. Define a functor $\mathcal{F}\colon \mathcal{E}^* \to \mathcal{UM}(A^*)$ by

$$\mathcal{F}(V^*) = \sum_{n\in\mathbb{Z}} A^*/F_{n+1}A^* \otimes V^n \qquad \text{and} \qquad \mathcal{F}(f) = \sum_{n\in\mathbb{Z}} \mathrm{id}_{A^*/F_{n+1}A^*} \otimes f^n.$$

For an object M^* of $\mathcal{UM}(A^*)$, let $\alpha_n\colon A^*/F_{n+1}A^* \otimes M^n \to M^*$ ($n \in \mathbb{Z}$) be the maps induced by the structure map $\alpha\colon A^* \otimes M^* \to M^*$. These maps induce $\varepsilon_{M^*}\colon \mathcal{FO}(M^*) \to M^*$.

Let 1_n be the class of $1 \in A^0$ in $A^*/F_{n+1}A^*$. For an object V^* of $\mathcal{E}^*$, define a map $\eta_{V^*}\colon V^* \to \mathcal{OF}(V^*)$ by $\eta_{V^*}(x) = \sum_{n\in\mathbb{Z}} 1_n \otimes u_{n\,V^*}(x)$ for $x \in V^*$.

Proposition 3.6 *$\mathcal{F}$ is a left adjoint of $\mathcal{O}$.*

Proof It can be easily verified that $\eta\colon \mathrm{id}_{\mathcal{E}} \to \mathcal{OF}$ (resp. $\varepsilon\colon \mathcal{FO} \to \mathrm{id}_{\mathcal{UM}(A^*)}$) is the unit (resp. counit) of the adjunction $\mathcal{F} \vdash \mathcal{O}$. □

Remark 3.7

(1) As a special case of the above result, we see that $\mathcal{F}(\Sigma^n K) = \Sigma^n A^*/F_{n+1}A^*$ represents a functor $\epsilon_n\mathcal{O}\colon \mathcal{UM}(A^*) \to \mathcal{E}$. Thus we can verify the fact that a functor $G\colon \mathcal{UM}(A^*)^{op} \to \mathcal{E}$ is representable if G is right exact and preserves direct sums (Lannes–Zarati [2]).

(2) The above result implies that $\mathcal{UM}(A^*)$ has enough projectives and we can construct the bar resolutions (MacLane [4]) in $\mathcal{UM}(A^*)$ and that, if $\mathcal{F}$ also satisfies (E5) and L^* is an unstable A^*–module of finite type, the left adjoint to the functor $M^* \mapsto M^* \otimes L^*$ exists.

Put $E_i^j A^* = (F_i A^*)^j / (F_{i+1} A^*)^j$ and $E_i^* A^* = \sum_{j \in \mathbb{Z}} E_i^j A^*$. If $\mathfrak{F}$ satisfies (E3) of Condition 3.4, $E_i^* A^*$ is a left A^*–module. If $\mathfrak{F}$ satisfies (E4) of Condition 3.4, the product map μ: $A^* \otimes A^* \to A^*$ induces $\overline{\mu}_i^{k,j}$: $E_i^k A^* \otimes A^j \to E_{i-j}^{k+j} A^*$. Consider a bigraded vector space $E_*^* A^* = \sum_{i \in \mathbb{Z}} E_i^* A^*$. Then $E_*^* A^*$ has a structure of a right A^*–module given by $\overline{\mu}_i^{k,j}$'s. Suppose $\mathfrak{F}$ satisfies both (E3) and (E4), then $\overline{\mu}_i^{k,j}$ induces $\widetilde{\mu}_i^{k,j}$: $E_i^k A^* \otimes (A^* / F_{i-j+1} A^*)^j \to E_{i-j}^{k+j} A^*$. We can regard $\widetilde{\mu}_i^{*,j}$ as a map $E_i^* A^* \otimes \iota_j \epsilon_j (A^* / F_{i-j+1} A^*) \to E_{i-j}^* A^*$ in $\mathcal{E}^*$.

Proposition 1.16 and Theorem 2.2 suggests the following conditions.

Condition 3.8 Let A^* be an algebra over a field K of characteristic p with an decreasing filtration $\mathfrak{F} = (F_i A^*)_{i \in \mathbb{Z}}$.

(1) (E6) $E_{2i+\varepsilon}^k A^* = \{0\}$ $(i, k \in \mathbb{Z},\ \varepsilon = 0, 1)$ holds if $k < 2i(p-1) + \varepsilon$ or $2i + \varepsilon + k \not\equiv 0, 2$ modulo $2p$.

(2) (E7) $\dim E_{2i+\varepsilon}^{2i(p-1)+\varepsilon} A^* = 1$ for $i \geq 0$, $\varepsilon = 0, 1$.

(3) (E8) For non-negative integers i, j and $\varepsilon = 0, 1$, the map

$$\widetilde{\mu}_{2i+\varepsilon}^{2i(p-1)+\varepsilon, j}\colon E_{2i+\varepsilon}^{2i(p-1)+\varepsilon} A^* \otimes (A^* / F_{2i-j+\varepsilon+1} A^*)^j \to E_{2i-j+\varepsilon}^{2i(p-1)+j+\varepsilon} A^*$$

is an isomorphism.

Remark 3.9 Since $E_j^{2i(p-1)+\varepsilon} A^* = \{0\}$ if $j > 2i + \varepsilon$ by (E6), we have $\dim (F_{2i+\varepsilon} A^*)^{2i(p-1)+\varepsilon} = 1$ for $i \geq 0$ and $\varepsilon = 0, 1$ by (E2) and (E7). We also have $(F_{2i+\varepsilon} A^*)^k = \{0\}$ if $k < 2i(p-1) + \varepsilon$.

We assume that $\mathfrak{F}$ satisfies (E1), (E2), (E3), (E4), (E6), (E7) and (E8) for the rest of this section.

Proposition 3.10 *A left A^*–module M^* with structure map α: $A^* \otimes M^* \to M^*$ is unstable if and only if $\alpha((F_{2i+\varepsilon} A^*)^{2i(p-1)+\varepsilon} \otimes M^k) = \{0\}$ for any $i \in \mathbb{Z}$, $\varepsilon = 0, 1$ such that $k < 2i + \varepsilon$.*

Proof Suppose $\alpha((F_{2i+\varepsilon} A^*)^{2i(p-1)+\varepsilon} \otimes M^k) = \{0\}$ for any $i \in \mathbb{Z}$, $\varepsilon = 0, 1$ and $k < 2i + \varepsilon$. Since (E8) implies

$$\mu((F_{2i+\varepsilon} A^*)^{2i(p-1)+\varepsilon} \otimes A^j) + (F_{2i-j+\varepsilon+1} A^*)^{2i(p-1)+j+\varepsilon} = (F_{2i-j+\varepsilon} A^*)^{2i(p-1)+j+\varepsilon},$$

we have

$$\alpha((F_{2i-j+\varepsilon+1}A^*)^{2i(p-1)+j+\varepsilon}\otimes M^{k-j})=\alpha((F_{2i-j+\varepsilon}A^*)^{2i(p-1)+j+\varepsilon}\otimes M^{k-j})$$

if $k<2i+\varepsilon$. By putting $n=k-j$, $s=2i-j+\varepsilon$ and $t=2i(p-1)+j+\varepsilon$, we see that

$$(5)\qquad \alpha((F_{s+1}A^*)^t\otimes M^n)=\alpha((F_sA^*)^t\otimes M^n)$$

holds if $s>n$ and $s+t\equiv 0,2$ modulo $2p$. Since $(F_{s+1}A^*)^t=(F_sA^*)^t$ by (E6), It follows from (5) that $\alpha((F_{n+1}A^*)^t\otimes M^n)=\alpha((F_mA^*)^t\otimes M^n)$ for any $m>n$. Since $\alpha((F_mA^*)^t\otimes M^n)=\{0\}$ for sufficiently large m by (E6) and (E2), we have $\alpha((F_{n+1}A^*)^t\otimes M^n)=\{0\}$.

The converse follows from

$$\alpha((F_{2i+\varepsilon}A^*)^{2i(p-1)+\varepsilon}\otimes M^k)\subset\alpha(F_{k+1}A^*\otimes M^k)=\{0\}. \qquad \square$$

For non-negative integers i, j and $\varepsilon=0,1$, put $\kappa=\varepsilon$ if j is even and $\kappa=1-\varepsilon$ if j is odd. Let $\gamma_{i,j,\varepsilon}$ be the composition of maps

$$\mu_{2i-j+\varepsilon}\colon A^{pj-(p-2)(\varepsilon-\kappa)}\otimes E^{(2i-j+\varepsilon-\kappa)(p-1)+\kappa}_{2i-j+\varepsilon}A^*\to E^{2i(p-1)+j+\varepsilon}_{2i-j+\varepsilon}A^*$$

and

$$\left(\tilde{\mu}^{2i(p-1)+\varepsilon,j}_{2i+\varepsilon}\right)^{-1}\colon E^{2i(p-1)+j+\varepsilon}_{2i-j+\varepsilon}A^*\to E^{2i(p-1)+\varepsilon}_{2i+\varepsilon}A^*\otimes(A^*/F_{2i-j+\varepsilon+1}A^*)^j.$$

Condition 3.11 For a real number r, let us denote by $[\![r]\!]$ the minimum integer among integers which are not less than r.

(1) (E9) $\gamma_{i,j,\varepsilon}$ maps $(F_kA^*)^{pj-(p-2)(\varepsilon-\kappa)}\otimes E^{(2i-j+\varepsilon-\kappa)(p-1)+\kappa}_{2i-j+\varepsilon}A^*$ into

$$E^{2i(p-1)+\varepsilon}_{2i+\varepsilon}A^*\otimes\left(F_{[\![k/p]\!]}A^*/F_{2i-j+\varepsilon+1}A^*\right)^j.$$

It follows from Proposition 2.7 and Proposition 2.8 that the excess filtration $\mathfrak{F}_p$ on $\mathcal{A}_p$ satisfies the above condition.

We can construct the functor $\Phi\colon\mathcal{UM}(A^*)\to\mathcal{UM}(A^*)$ as in Li [3]. For an unstable A^*–module M^*, define an A^*–module ΦM^* as follows. Put

$$\Phi M^*=\sum_{i\in\mathbb{Z},\varepsilon=0,1}E^{2i(p-1)+\varepsilon}_{2i+\varepsilon}A^*\otimes M^{2i+\varepsilon}.$$

In other words,

$$(\Phi M^*)^k=\begin{cases}E^{2i(p-1)+\varepsilon}_{2i+\varepsilon}A^*\otimes M^{2i+\varepsilon} & k=2ip+2\varepsilon,\quad i\in\mathbb{Z},\quad \varepsilon=0,1\\ \{0\} & k\not\equiv 0,2 \text{ modulo } 2p.\end{cases}$$

We denote by $\mu_i\colon A^*\otimes E_i^*A^*\to E_i^*A^*$ the map induced by the product μ of A^*. Note that $\mu_i\colon A^j\otimes E_i^kA^*\to E_i^{j+k}A^*$ is trivial if $i+j+k\not\equiv 0,2$ modulo $2p$. Let $\alpha_{M^*}\colon A^*\otimes M^*\to M^*$ be the A^*–module structure map of M^*. Since M^* is unstable, α_{M^*} induces $\overline{\alpha}_{M^*,i}\colon A^*/F_{i-1}A^*\otimes M^i\to M^*$. We define $\alpha_{\Phi M^*}\colon \otimes\Phi M^*\to\Phi M^*$ by the following compositions:

$$\begin{aligned}
&A^{2jp}\otimes E_{2i+\varepsilon}^{2i(p-1)+\varepsilon}A^*\otimes M^{2i+\varepsilon}\\
&\qquad\xrightarrow{\gamma_{i+j,2j,\varepsilon}\otimes 1} E_{2(i+j)+\varepsilon}^{2(i+j)(p-1)+\varepsilon}A^*\otimes(A^*/F_{2i+\varepsilon+1}A^*)^{2j}\otimes M^{2i+\varepsilon}\\
&\qquad\qquad\xrightarrow{1\otimes\overline{\alpha}_{M^*,2i+\varepsilon}} E_{2(i+j)+\varepsilon}^{2(i+j)(p-1)+\varepsilon}A^*\otimes M^{2(i+j)+\varepsilon};
\end{aligned}$$

$$\begin{aligned}
&A^{2jp+2}\otimes E_{2i}^{2i(p-1)}A^*\otimes M^{2i}\\
&\qquad\xrightarrow{\gamma_{i+j,2j+1,1}\otimes 1} E_{2(i+j)+1}^{2(i+j)(p-1)+1}A^*\otimes(A^*/F_{2i+1}A^*)^{2j+1}\otimes M^{2i}\\
&\qquad\qquad\xrightarrow{1\otimes\overline{\alpha}_{M^*,2i}} E_{2(i+j)+1}^{2(i+j)(p-1)+1}A^*\otimes M^{2(i+j)+1};
\end{aligned}$$

and

$$\begin{aligned}
&A^{2jp-2}\otimes E_{2i+1}^{2i(p-1)+1}A^*\otimes M^{2i+1}\\
&\qquad\xrightarrow{\gamma_{i+j,2j-1,0}\otimes 1} E_{2(i+j)}^{2(i+j)(p-1)}A^*\otimes(A^*/F_{2i+2}A^*)^{2j-1}\otimes M^{2i+1}\\
&\qquad\qquad\xrightarrow{1\otimes\overline{\alpha}_{M^*,2i+1}} E_{2(i+j)}^{2(i+j)(p-1)}A^*\otimes M^{2(i+j)}.
\end{aligned}$$

Since $\mu(F_{2ip+2\varepsilon+1}A^*\otimes(F_{2i+\varepsilon}A^*)^{2i(p-1)+\varepsilon})\subset F_{2i+\varepsilon+1}A^*$ for $\varepsilon=0,1$ and $i\in\mathbb{Z}$ by (E4), we deduce that ΦM^* is an unstable A^*–module.

For a homomorphism $f\colon M^*\to N^*$ between unstable A^*–modules, let $\Phi f\colon \Phi M^*\to\Phi N^*$ be the map induced by $\mathrm{id}_{E_{2i+\varepsilon}^{2i(p-1)+\varepsilon}}A^*\otimes f$.

Then Φf is a homomorphism of left A^*–modules and Φ is an endofunctor of $\mathcal{UM}(A^*)$. Let

$$\lambda_{M^*}^{2ip+2\varepsilon}\colon (\Phi M^*)^{2ip+2\varepsilon}=E_{2i+\varepsilon}^{2i(p-1)+\varepsilon}A^*\otimes M^{2i+\varepsilon}\to M^{2ip+2\varepsilon}\quad (i\in\mathbb{Z},\varepsilon=0,1)$$

be the restriction of $\overline{\alpha}_{M^*,2i+\varepsilon}\colon A^*/F_{2i+\varepsilon+1}A^*\otimes M^{2i+\varepsilon}\to M^*$. Thus we have a map $\lambda_{M^*}\colon\Phi M^*\to M^*$. It is easy to verify that λ_{M^*} is a homomorphism of left A^*–modules and we have a natural transformation $\lambda\colon\Phi\to\mathrm{id}_{\mathcal{UM}(A^*)}$.

For an object V^* of $\mathcal{E}^*$, let $\rho_{V^*}\colon \mathcal{F}(V^*) \to \Sigma\mathcal{F}(\Sigma^{-1}V^*)$ be the map induced by the quotient map $A^*/F_{n+1}A^* \to A^*/F_nA^*$ and the identity maps $V^n \to V^n = (\Sigma^{-1}V^*)^{n-1}$.

Proposition 3.12 *The following is a short exact sequence.*

$$0 \longrightarrow \Phi\mathcal{F}(V^*) \xrightarrow{\lambda_{\mathcal{F}(V^*)}} \mathcal{F}(V^*) \xrightarrow{\rho_{V^*}} \Sigma\mathcal{F}(\Sigma^{-1}V^*) \longrightarrow 0$$

Proof By (E6) and (E8), $\lambda_{\mathcal{F}(V^*)}$ is an injection onto $\sum_{n\in\mathbb{Z}} F_nA^*/F_{n+1}A^* \otimes V^n$, which is the kernel of ρ_{V^*}. □

Lemma 3.13 *Let M^* be an unstable A^*–module.*

(1) $\Sigma^{-1}\operatorname{Coker}\lambda_{M^*}$ *is an unstable A^*–module.*

(2) *If $\mathfrak{F}$ satisfies (E9) in Condition 3.11, $\Sigma^{-1}\operatorname{Ker}\lambda_{M^*}$ is an unstable A^*–module.*

Proof (1) Since $(\operatorname{Im}\lambda_{M^*})^{2ip+\varepsilon} = (F_{2i+\varepsilon}A^*)^{2i(p-1)+\varepsilon}M^{2i+\varepsilon}$, we have

$$(F_{2i+\varepsilon}A^*)^{2i(p-1)+\varepsilon}(\operatorname{Coker}\lambda_{M^*})^{2i+\varepsilon} = \{0\}.$$

If $k < 2i+\varepsilon$, instability of M^* and Proposition 3.10 imply

$$(F_{2i+\varepsilon}A^*)^{2i(p-1)+\varepsilon}(\operatorname{Coker}\lambda_{M^*})^{k} = \{0\}.$$

Thus the assertion follows from Proposition 3.10.

(2) Put $N^{2i+\varepsilon} = \{x \in M^{2i+\varepsilon} \mid (F_{2i+\varepsilon}A^*)^{2i(p-1)+\varepsilon}x = \{0\}\}$. Then we have $(\operatorname{Ker}\lambda_{M^*})^{2ip+2\varepsilon} = E^{2i(p-1)+\varepsilon}_{2i+\varepsilon}A^* \otimes N^{2i+\varepsilon}$ and $(F_{2i+\varepsilon}A^*)^{2i(p-1)+\varepsilon}N^{2i+\varepsilon} = \{0\}$. By Proposition 3.10, it suffices to show

$$(F_{2j+\varepsilon'}A^*)^{2j(p-1)+\varepsilon'}(E^{2i(p-1)+\varepsilon}_{2i+\varepsilon}A^* \otimes N^{2i+\varepsilon}) = \{0\}$$

for non-negative integers i, j and $\varepsilon, \varepsilon' = 0, 1$ satisfying $2j+\varepsilon' \geq 2ip+2\varepsilon$. We may assume $2j(p-1)+\varepsilon' \equiv 0, \pm 2$ modulo $2p$, that is, $\varepsilon' = 0$ and $j \equiv 0, \pm 1$ modulo p for dimensional reason. If $j = kp$, then $k \geq i+\varepsilon$ and it follows from Condition 3.11 that

$$\begin{aligned}(F_{2kp}A^*)^{2kp(p-1)}&(E^{2i(p-1)+\varepsilon}_{2i+\varepsilon}A^* \otimes N^{2i+\varepsilon})\\ &= E^{2(i+k(p-1))(p-1)+\varepsilon}_{2(i+k(p-1))+\varepsilon}A^* \otimes ((F_{2k}A^*)^{2k(p-1)}N^{2i+\varepsilon}) = \{0\}.\end{aligned}$$

If $j = kp - 1$, then $k \geq i + 1$ and we only have to consider the case $\varepsilon = 0$ for dimensional reason. Since $(F_{2k}A^*)^{2k(p-1)-1} = \{0\}$ by (E6) and (E2), we have

$$\begin{aligned}(F_{2kp-2}A^*)^{2p(kp-k-1)+2}&(E_{2i}^{2i(p-1)}A^* \otimes N^{2i})\\ &= E_{2(i+kp-k-1)+1}^{2(i+kp-k-1)(p-1)+1}A^* \otimes ((F_{2k}A^*)^{2k(p-1)-1}N^{2i}) = \{0\}.\end{aligned}$$

If $j = kp + 1$, then $k \geq i$ and we only have to consider the case $\varepsilon = 1$ for dimensional reason. Again, using Condition 3.11 and the instability of M^*, we see

$$\begin{aligned}(F_{2kp+2}A^*)^{2p(kp-k+1)-2}&(E_{2i+1}^{2i(p-1)+1}A^* \otimes N^{2i+1})\\ &= E_{2(i+kp-k+1)+1}^{2(i+kp-k+1)(p-1)+1}A^* \otimes ((F_{2k+1}A^*)^{2k(p-1)+1}N^{2i+1}) = \{0\}.\end{aligned}$$

This completes the proof. □

Define functors $\Omega, \Omega^1 \colon \mathcal{UM}(A^*) \to \mathcal{UM}(A^*)$ by $\Omega(M^*) = \Sigma^{-1}\operatorname{Coker}\lambda_{M^*}$ and $\Omega^1(M^*) = \Sigma^{-1}\operatorname{Ker}\lambda_{M^*}$. Let us denote by $\widetilde{\eta}_{M^*} \colon M^* \to \operatorname{Coker}\lambda_{M^*} = \Sigma\Omega M^*$ the quotient map and by $\iota_{M^*} \colon \Sigma\Omega^1 M^* \to \Phi M^*$ the inclusion map. For a morphism $f \colon M^* \to N^*$ of unstable modules, let $\Omega f \colon \Omega M^* \to \Omega N^*$ and $\Omega^1 f \colon \Omega^1 M^* \to \Omega^1 N^*$ be the unique maps that make the following diagram commute.

$$\begin{tikzcd} 0 \arrow[r] & \Sigma\Omega^1 M^* \arrow[r, "\iota_{M^*}"] \arrow[d, "\Sigma\Omega^1 f"] & \Phi M^* \arrow[r, "\lambda_{M^*}"] \arrow[d, "\Phi f"] & M^* \arrow[r, "\widetilde{\eta}_{M^*}"] \arrow[d, "f"] & \Sigma\Omega M^* \arrow[r] \arrow[d, "\Sigma\Omega f"] & 0 \\ 0 \arrow[r] & \Sigma\Omega^1 N^* \arrow[r, "\iota_{N^*}"] & \Phi N^* \arrow[r, "\lambda_{N^*}"] & N^* \arrow[r, "\widetilde{\eta}_{N^*}"] & \Sigma\Omega N^* \arrow[r] & 0 \end{tikzcd}$$

Proposition 3.14 *Ω is the left adjoint of the suspension functor Σ. Ω^1 is the first left derived functor of Ω and all the higher derived functors are trivial.*

Proof We first note that $\lambda_{\Sigma M^*} \colon \Phi\Sigma M^* \to \Sigma M^*$ is trivial by the instability of M^*. Hence $\widetilde{\eta}_{\Sigma M^*} \colon \Sigma M^* \to \Sigma\Omega\Sigma M^*$ is an isomorphism. Define $\widetilde{\varepsilon}_{M^*} \colon \Omega\Sigma M^* \to M^*$ by $\widetilde{\varepsilon}_{M^*} = \Sigma^{-1}\widetilde{\eta}_{\Sigma M^*}^{-1}$. Obviously, $\Sigma\widetilde{\varepsilon}_{M^*}\widetilde{\eta}_{\Sigma M^*} = \mathrm{id}_{\Sigma M^*}$. By the naturality of λ and the definition of $\widetilde{\varepsilon}$, we have

$$\Sigma(\widetilde{\varepsilon}_{\Omega M^*}\Omega\widetilde{\eta}_{M^*})\widetilde{\eta}_{M^*} = \Sigma\widetilde{\varepsilon}_{\Omega M^*}(\Sigma\Omega\widetilde{\eta}_{M^*})\widetilde{\eta}_{M^*} = \widetilde{\eta}_{\Sigma\Omega M^*}^{-1}\widetilde{\eta}_{\Sigma\Omega M^*}\widetilde{\eta}_{M^*} = \widetilde{\eta}_{M^*}.$$

Hence $\widetilde{\varepsilon}_{\Omega M^*}\Omega\widetilde{\eta}_{M^*} = \mathrm{id}_{\Omega M^*}$ and Ω is the left adjoint of Σ.

Let

$$M^* \xleftarrow{\varepsilon_{M^*}} B_0^* \xleftarrow{\partial_1} \cdots \xleftarrow{\partial_{n-1}} B_{n-1}^* \xleftarrow{\partial_n} B_n^* \xleftarrow{\partial_{n+1}} \cdots$$

be the bar resolution of M^*. Consider chain complexes

$$B. = (B_n^*, \partial_n)_{n\in\mathbb{Z}}, \quad \Phi B. = (\Phi B_n^*, \Phi(\partial_n))_{n\in\mathbb{Z}} \text{ and } \Sigma\Omega B. = (\Sigma\Omega B_n^*, \Sigma\Omega(\partial_n))_{n\in\mathbb{Z}}.$$

We denote by $\lambda.\colon \Phi B. \to B.$ and $\eta.\colon B. \to \Sigma\Omega B.$ the chain maps given by the $\lambda_{B_n^*}$ and $\eta_{B_n^*}$, respectively. Since

$$0 \to \Phi B_n^* \xrightarrow{\lambda_{B_n^*}} B_n^* \xrightarrow{\eta_{B_n^*}} \Sigma\Omega B_n^* \to 0$$

is exact by Proposition 3.12. we have a short exact sequence of complexes

$$0 \to \Phi B. \xrightarrow{\lambda.} B. \xrightarrow{\eta.} \Sigma\Omega B. \to 0.$$

Consider the long exact sequence associated with this short exact sequence. Clearly, Φ is an exact functor. We deduce that $\Sigma H^n(\Omega B.) = H^n(\Sigma\Omega B.)$ is trivial and that there is an exact sequence

$$0 \to \Sigma H^1(\Omega B.) = H^1(\Sigma\Omega B.) \to \Phi M_* \xrightarrow{\lambda_{M^*}} M^* \xrightarrow{\eta_{M^*}} \Sigma\Omega M^* \to 0.$$

Thus $\Omega^n M^* = H^n(\Omega B.)$ is trivial if $n > 1$ and Ω^1 defined above is the first left derived functor of Ω. □

4 Unipotent group scheme

For a commutative ring k, we denote by $\mathcal{A}lg_k^*$ the category of graded k–algebras and by h_{A^*} the functor represented by an object A^* of $\mathcal{A}lg_k$. We denote by $\mathcal{G}r$ the category of groups.

For a Hopf algebra A^*, let us denote by A_* the dual Hopf algebra, that is, A_n is the dual vector space $\mathrm{Hom}_K(A^n, K)$ and $A_* = \sum_{n\in\mathbb{Z}} A_n$. We assume that A^* is finite type and that $A^n = 0$ for $n < 0$.

For a filtration $\mathfrak{F} = (F_i A^*)_{i\in\mathbb{Z}}$ of A^*, define the dual filtration $\mathfrak{F}^* = (F_i A_*)_{i\in\mathbb{Z}}$ on A_* by

$$F_i A_n = \mathrm{Ker}\left(\kappa_{i+1}\colon A_n = \mathrm{Hom}_K(A^n, K) \to \mathrm{Hom}_K(F_{i+1}A^n, K)\right)$$

Here, $\kappa_i\colon F_i A^n \to A^n$ denotes the inclusion map. Note that the dual of the dual filtration $\mathfrak{F}^*$ is identified with $\mathfrak{F}$.

We list conditions on the dual filtration.

Condition 4.1 Let $\mu^*\colon A_* \to A_* \otimes A_*$ (resp. $\delta^* : A_* \otimes A_* \to A_*$) be the coproduct (resp. product) of A_*.

(1) (E1*) $F_i A_* = \{0\}$ if $i < 0$.

(2) (E2*) $\bigcup_{i\in\mathbb{Z}} F_i A_* = A_*$.

(3) (E3*) $F_i A_*$'s are left coideals of A_* (that is, $\mu^*(F_i A_*) \subset A_* \otimes F_i A_*$) for $i \in \mathbb{Z}$.

(4) (E4*) $\mu^*(F_i A_k) \subset \sum_{j\in\mathbb{Z}} F_{j+i} A_{k-j} \otimes A_j$ for $i, j \in \mathbb{Z}$.

(5) (E5*) $\delta^*(F_j A_* \otimes F_k A_*) \subset F_{j+k} A_*$ for $j, k \in \mathbb{Z}$.

(6) (E6*) $E^k_{2i+\varepsilon} A_* = \{0\}$ $(i, k \in \mathbb{Z},\ \varepsilon = 0, 1)$ holds if $k < 2i(p-1) + \varepsilon$ or $2i + \varepsilon + k \not\equiv 0, 2$ modulo $2p$.

(7) (E7*) $\dim E^{2i(p-1)+\varepsilon}_{2i+\varepsilon} A^* = 1$ for $i \geq 0$, $\varepsilon = 0, 1$.

It is easy to verify the following fact.

Proposition 4.2 (Yamaguchi [8]) *For $l = 1, 2, 3, 4, 5, 6, 7$, $\mathfrak{F}$ satisfies the condition* (El) *if and only if $\mathfrak{F}^*$ satisfies* (El^*).

For a prime p, we define a graded Hopf algebra $A_{(p)*}$ over a prime field $\mathbb{F}_p$ as follows. As an algebra, we put

$$A_{(p)*} = E(x_{i1} \,|\, i \geq 2) \otimes \mathbb{F}_p[x_{ij} \,|\, i > j \geq 2] \text{ if } p \neq 2, \quad A_{(2)*} = \mathbb{F}_2[x_{ij} \,|\, i > j \geq 1].$$

We assign the generators x_{ij} degrees as follows.

$$\deg\, x_{ij} = \begin{cases} 2p^{i-2} - 1 & i \geq 2,\ j = 1 \\ 2p^{j-2}(p^{i-j} - 1) & i > j \geq 2 \end{cases} \qquad \text{if } p \neq 2$$

$$\deg\, x_{ij} = 2^{j-1}(2^{i-j} - 1) \qquad \text{if } p = 2.$$

Define the coproduct μ^* and the counit η^* of $A_{(p)*}$ by

$$\mu^*(x_{ij}) = x_{ij} \otimes 1 + \sum_{k=j+1}^{i-1} x_{ik} \otimes x_{kj} + 1 \otimes x_{ij}, \quad \eta^*(x_{ij}) = 0.$$

Then, $A_{(p)*}$ is a commutative Hopf algebra and its conjugation (canonical anti-automorphism) ι^* is given by

$$\iota^*(x_{ij}) = -x_{ij} - \sum_{k=j+1}^{i-1} x_{ik}\iota^*(x_{kj}).$$

Hence the affine scheme $h_{A_{(p)*}}$ represented by $A_{(p)*}$ takes its values in the category of groups, namely, $h_{A_{(p)*}}\colon \mathcal{A}lg^*_{\mathbb{F}_p} \to \mathcal{G}r$ is an affine group scheme.

Remark 4.3 For a positive integer n and a graded $\mathbb{F}_p$–algebra R^*, let $U_n(R^*)$ be a set of $n\times n$ unipotent matrices A whose (i,j)th entry a_{ij} satisfies

$$\begin{aligned} &a_{ij}\in\begin{cases} R^{2p^{i-2}-1} & i\geq 2,\ j=1\\ R^{2p^{j-2}(p^{i-j}-1)} & i>j\geq 2\end{cases} && \text{if } p\neq 2\\ &a_{ij}\in R^{2^{j-1}(2^{i-j}-1)} && \text{if } p=2\end{aligned}$$

and $a_{11}=a_{22}=\cdots=a_{nn}=1$, $a_{ij}=0$ if $i<j$. Then, $U_n(R^*)$ is a group by the multiplication of matrices. Hence we have a $\mathbb{F}_p$–group functor $U_n\colon \mathcal{A}lg^*_{\mathbb{F}_p}\to\mathcal{G}r$. On the other hand, let $A(n)_{(p)*}$ be the Hopf subalgebra of $A_{(p)*}$ generated by $\{x_{ij}\,|\,1\leq j<i\leq n\}$. For a map $f\colon A(n)_{(p)*}\to R^*$ of graded K–algebras, we denote by A_f the element of $U_n(R^*)$ whose (i,j)th component is $f(x_{ij})$ if $i>j$. Define a map $\theta_{nR^*}\colon h_{A(n)_{(p)*}}(R^*)\to U_n(R^*)$ by $\theta_{nR^*}(f)=A_f$. It is easy to verify that θ_{nR^*} is an isomorphism groups and we have a natural equivalence $\theta_n\colon h_{A(n)_{(p)*}}\to U_n$.

If $A=(a_{ij})\in U_{n+1}(R^*)$, let A' be the $n\times n$ matrix whose (i,j)th component is a_{ij}. Then $A'\in U_n(R^*)$ and we define a morphism $\pi_n\colon U_{n+1}\to U_n$ by $\pi_{nR^*}(A)=A'$. Let U_∞ be the limit of the inverse system

$$\big(U_{n+1}\xrightarrow{\pi_n}U_n\big)_{n=1,2,\dots}.$$

The morphism $\iota_n^*\colon h_{A(n+1)_{(p)*}}\to h_{A(n)_{(p)*}}$ induced by the inclusion map $\iota_n\colon A(n)_{(p)*}\to A(n+1)_{(p)*}$ satisfies $\theta_n\iota_n^*=\pi_n\theta_{n+1}$. Since $A_{(p)*}$ is the colimit of the direct system

$$\big(A(n)_{(p)*}\xrightarrow{\iota_n}A(n+1)_{(p)*}\big)_{n=1,2,\dots},$$

it follows that the θ_n induce a natural equivalence $\theta_\infty\colon h_{A_{(p)*}}\to U_\infty$. Thus, $A_{(p)*}$ represents the group scheme of "infinite dimensional unipotent matrices".

In order to relate $A_{(p)*}$ with the dual Steenrod algebra $\mathcal{A}_{p*}$, we consider representation of an affine group scheme.

Definition 4.4 Let V^* be a finite dimensional vector space over K. Define a functor $F_{V^*}\colon \mathcal{A}lg^*_K\to\mathcal{E}^*$ by $F_{V^*}(R^*)=V^*\otimes R^*$. We regard $F_{V^*}(R^*)$ as a right R^*–module.

We denote by V_n^* the graded vector space over $\mathbb{F}_p$ such that $\dim V_n^k=1$ for $k=-1,-2,\dots,-2p^i,\dots,-2p^{n-2}$ and $V_n^k=\{0\}$ otherwise if $p\neq 2$, $\dim V_n^k=1$ for $k=-1,-2,\dots,-2^i,\dots,-2^{n-1}$ and $V_n^k=\{0\}$ otherwise if $p=2$. Let v_k be a base of V_n^k for k such that $\dim V_n^k=1$.

Define $\alpha_{nR^*}\colon F_{V_n^*}(R^*)\times U_n(R^*)\to F_{V_n^*}(R^*)$ by

$$\alpha_{nR^*}(v_j\otimes 1,(a_{ij}))=\sum_{i=1}^{n}v_i\otimes a_{ij}$$

so that $U_n(R^*)$ acts R^*–linearly on $F_{V_n^*}(R^*)$. Hence $F_{V_n^*}$ is a right U_n–module, in other words, $\alpha_n\colon F_{V_n^*}\times U_n\to F_{V_n^*}$ is a representation of U_n on V_n^*.

Let $\varphi_n\colon V_n^*\to V_n^*\otimes A(n)_{(p)*}$ be the map defined by

$$\varphi_n(v_j)=\alpha_{nA(n)_{(p)*}}(v_j\otimes 1,(x_{ij}))=v_j\otimes 1+\sum_{i=j+1}^{n}v_i\otimes x_{ij}.$$

Here, we put $x_{jj}=1$ and $x_{ij}=0$ if $i<j$. Then, φ_n is a right comodule structure map of V_n^*. Composing the map $\mathrm{id}_{V_n^*}\otimes\kappa_n\colon V_n^*\otimes A(n)_{(p)*}\to V_n^*\otimes A_{(p)*}$ induced by the inclusion map $\kappa_n\colon A(n)_{(p)*}\hookrightarrow A_{(p)*}$ to φ_n, V_n^* is regarded as a right $A_{(p)*}$–comodule.

We change the gradings of the mod p cohomology group $H^*(X)$ of a space X by replacing $H^n(X)$ by $H^{-n}(X)$ so that the Milnor coaction $\psi_X : H^*(X)\to H^*(X)\widehat{\otimes}\mathcal{A}_{p*}$ preserves degrees. Recall that the Milnor coaction on the mod p cohomology group of $B\mathbb{Z}/p\mathbb{Z}$ is a homomorphisms of algebras given as follows.

$$\psi(t)=t\otimes 1-\sum_{k\geq 0}s^{p^k}\otimes\tau_k \text{ and } \psi(s)=\sum_{k\geq 0}s^{p^k}\otimes\xi_k \text{ if } p\neq 2,$$

where $H^*(B\mathbb{Z}/p\mathbb{Z})=E(t)\otimes\mathbb{F}_p[s]$ ($t\in H^{-1}(B\mathbb{Z}/p\mathbb{Z})$, $s\in H^{-2}(B\mathbb{Z}/p\mathbb{Z})$).

$$\psi(t)=\sum_{k\geq 0}t^{2^k}\otimes\zeta_k \text{ if } p=2,$$

where $H^*(B\mathbb{Z}/p\mathbb{Z})=\mathbb{F}_2[t]$ ($t\in H^{-1}(B\mathbb{Z}/2\mathbb{Z})$). We identify V_n^* with a subspace of the mod p cohomology group of the $(2p^{n-2}+1)$–skeleton (resp. 2^{n-1}–skeleton) of $B\mathbb{Z}/p\mathbb{Z}$ spanned by $\{t,s,s^p,\dots,s^{p^{n-2}}\}$ (resp. $\{t,t^2,\dots,t^{2^{n-1}}\}$) if $p\neq 2$ (resp. $p=2$). Put $v_1=t$ and $v_j=s^{p^{j-2}}$ $(j=2,3,\dots,n)$ if $p\neq 2$ and $v_j=t^{2^{j-1}}$ $(j=1,\dots,n)$ if $p=2$. By the above equality, we have

$$\psi(v_1)=v_1\otimes 1-\sum_{i=2}^{n}v_i\otimes\tau_{i-2},\qquad \psi(v_j)=\begin{cases}\sum_{i=j}^{n}v_i\otimes\xi_{i-j}^{p^{j-2}}\ (2\leq j\leq n) & \text{if } p\neq 2,\\ \sum_{i=j}^{n}v_i\otimes\zeta_{i-j}^{2^{j-1}} & \text{if } p=2.\end{cases}$$

Hence the map $\rho_p\colon A_{(p)*} \to \mathcal{A}_{p*}$ given by $\rho_p(x_{i1}) = -\tau_{i-2}$, $\rho_p(x_{ij}) = \xi_{i-j}^{p^{j-2}}$ $(j \geq 2)$ if $p \neq 2$ and $\rho_2(x_{ij}) = \zeta_{i-j}^{2^{j-1}}$ if $p = 2$ is a map of Hopf algebras and the composition

$$V_n^* \xrightarrow{\varphi_n} V_n^* \otimes A(n)_{(p)*} \xrightarrow{\mathrm{id} \otimes \kappa_n} V_n^* \otimes A_{(p)*} \xrightarrow{\mathrm{id} \otimes \rho_p} V_n^* \otimes \mathcal{A}_{p*}$$

coincides with the Milnor coaction (See Yamaguchi [8] for details).

Remark 4.5 Since $\rho_p(x_{s+2\,1}) = -\tau_s$, $\rho_p(x_{s+2\,2}) = \xi_s$ and $\rho_2(x_{s+1\,1}) = \zeta_s$, ρ_p is surjective. Hence the affine group scheme represented by $\mathcal{A}_{p*}$ is regarded as a closed subgroup scheme of U_∞.

The kernel of ρ_p is the ideal generated by $\left\{x_{ij} - x_{i-j+2\,2}^{p^{j-2}} \,\middle|\, i > j \geq 3\right\}$ if $p \neq 2$ and $\left\{x_{ij} - x_{i-j+1\,1}^{2^{j-1}} \,\middle|\, i > j \geq 2\right\}$ if $p = 2$.

Let $F_i A_{(p)*}$ be the subspace of $A_{(p)*}$ spanned by

$$\left\{x_{k_1 1} x_{k_2 1} \cdots x_{k_m 1} x_{i_1 j_1} x_{i_2 j_2} \cdots x_{i_n j_n} \,\middle|\, j_1, j_2, \dots, j_n \geq 2, m + 2\sum_{l=1}^{n} p^{j_l - 2} \leq i\right\},$$

if $p \neq 2$ and $F_i A_{(2)*}$ be the subspace of $A_{(2)*}$ spanned by

$$\left\{x_{i_1 j_1} x_{i_2 j_2} \cdots x_{i_n j_n} \,\middle|\, \sum_{l=1}^{n} 2^{j_l - 1} \leq i\right\}.$$

By this definition and Proposition 1.14, it is easy to verify the following assertions.

Proposition 4.6

(1) *The filtration* $\left(F_i A_{(p)*}\right)_{i \in \mathbb{Z}}$ *on* $A_{(p)*}$ *satisfies the conditions (E1*)$\sim$(E6*).*

(2) $\rho_p\left(F_i A_{(p)*}\right) = F_i \mathcal{A}_{p*}$.

It follows from Proposition 4.2 that the dual filtration $\left(F_i A_{(p)}^*\right)_{i \in \mathbb{Z}}$ on the dual Hopf algebra $A_{(p)}^*$ of $A_{(p)*}$ satisfies the conditions (E1)$\sim$(E6). Note that the Steenrod algebra $\mathcal{A}_p$ is a Hopf subalgebra of $A_{(p)}^*$.

However, $\left(F_i A_{(p)*}\right)_{i \in \mathbb{Z}}$ does not satisfy the condition (E7*). In fact, the following fact can be shown.

Proposition 4.7 *If p is an odd prime, then for $s = 0, 1, 2, \dots$ and $\varepsilon = 0, 1$,*

$$\left\{x_{21}^{\varepsilon} \prod_{j \geq 2} x_{j+1\,j}^{m_j} \,\middle|\, \sum_{j \geq 2} m_j p^{j-2} = s\right\}$$

is a basis of $E^{2s(p-1)+\varepsilon}_{2s+\varepsilon} A^*_{(p)}$. *For* $s = 0, 1, 2, \ldots,$

$$\left\{ \prod_{j\geq 1} x_{j+1\,j}^{m_j} \,\middle|\, \sum_{j\geq 2} m_j 2^{j-1} = s \right\}$$

is a basis of $E^s_s A^*_{(2)}$.

Appendix A

Here we make an observation on the group scheme represented by the dual Steenrod algebra $\mathcal{A}_{p*}$.

Let $\widetilde{\mathcal{A}}_{p*}$ be the polynomial part of $\mathcal{A}_{p*}$ (hence $\widetilde{\mathcal{A}}_{2*} = \mathcal{A}_{2*}$). G Nishida observed that $\widetilde{\mathcal{A}}_{p*}$ represents the functor $\widetilde{\Gamma}\colon \mathcal{A}lg^*_{\mathbb{F}_p} \to \mathcal{G}r$ defined by

$$\widetilde{\Gamma}(R^*) = \{ f(X) \in R^*[\![X]\!]^{-2} \mid f(X+Y) = f(X) + f(Y), \quad f(0) = 0, \quad f'(0) = 1 \},$$

that is, $\widetilde{\Gamma}(R^*)$ is the group of strict automorphisms of the additive formal group law G_a over R^*. (We regard $R^*[\![X]\!]$ as a graded ring with $\deg X = -2$.)

In fact, for a morphism $\varphi\colon \widetilde{\mathcal{A}}_{p*} \to R^*$ of graded rings, put

$$f_\varphi(X) = \sum_{i\geq 0} \varphi(\xi_i) X^{p^i} \quad (\xi_0 = 1).$$

Then, it follows from Milnor [5, Theorem 3] that the correspondence $\varphi \mapsto f_\varphi(X)$ gives a natural equivalence $h_{\widetilde{\mathcal{A}}_{p*}} \to \widetilde{\Gamma}$.

This fact also has a geometric explanation as follows. Let $\alpha\colon MU_* \to \mathbb{F}_p$ be the map that classifies the additive formal group law over $\mathbb{F}_p$. Then, the pull-back of the groupoid scheme represented by the Hopf algebroid (MU_*, MU_*MU) along $h_\alpha\colon h_{\mathbb{F}_p} \to h_{MU_*}$ is the stabilizer group scheme of the additive formal group law and it is represented by $\mathbb{F}_p \otimes_{MU_*} MU_*MU \otimes_{MU_*} \mathbb{F}_p$ (Yamaguchi [9]). Since α factors through the canonical map $MU_* \to BP_*$, $\mathbb{F}_p \otimes_{MU_*} MU_*MU \otimes_{MU_*} \mathbb{F}_p$ is isomorphic to $\mathbb{F}_p \otimes_{BP_*} BP_*BP \otimes_{BP_*} \mathbb{F}_p \cong \widetilde{\mathcal{A}}_{p*}$.

We assume that p is an odd prime below. Define a functor $\Gamma\colon \mathcal{A}lg^*_{\mathbb{F}_p} \to \mathcal{G}r$ as follows. For $R^* \in \operatorname{Ob} \mathcal{A}lg^*_{\mathbb{F}_p}$, we consider an object $R^*[\varepsilon]/(\varepsilon^2)$ ($\deg \varepsilon = -1$) of $\mathcal{A}lg^*_{\mathbb{F}_p}$. Let $\Gamma(R^*)$ be the set of automorphisms $f\colon G_a \to G_a$ over $R^*[\varepsilon]/(\varepsilon^2)$ such that $f'(0) - 1 \in (\varepsilon)$. The group structure of $\Gamma(R^*)$ is given by the composition of automorphisms. If $\varphi\colon R^* \to S^*$ is a homomorphism of graded algebras, $\Gamma(\varphi)\colon \Gamma(R^*) \to \Gamma(S^*)$ maps $f(X) = \sum_{i\geq 0}(a_i + b_i\varepsilon) X^{p^i}$ to $\sum_{i\geq 0}(\varphi(a_i) + \varphi(b_i)\varepsilon) X^{p^i}$.

Proposition A.1 *The affine group scheme $h_{\mathcal{A}_{p*}}$ represented by $\mathcal{A}_{p*}$ is isomorphic to Γ.*

Proof We define a natural transformation $\theta\colon h_{\mathcal{A}_{p*}} \to \Gamma$ as follows. For $R^* \in \mathrm{Ob}\,\mathcal{A}lg^*_{\mathbb{F}_p}$ and $\varphi \in h_{\mathcal{A}_{p*}}(R^*)$, we set $\theta^*_R(\varphi) = \sum_{i\geq 0}(\varphi(\xi_i)+\varphi(\tau_i)\varepsilon)X^{p^i}$. It follows from Milnor [5, Theorem 3] that θ is a natural transformation. We can verify easily that θ is a natural equivalence. □

Thus Γ is regarded as a closed subscheme of U_∞.

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