Bocksteins and the nilpotent filtration on the cohomology of spaces

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N Kuhn has given several conjectures on the special features satisfied by the singular cohomology of topological spaces with coefficients in a finite prime field, as modules over the Steenrod algebra [4]. The so-called *realization conjecture* was solved in special cases in [4] and in complete generality by L Schwartz [9]. The more general *strong realization conjecture* has been settled at the prime 2, as a consequence of the work of L Schwartz [10] and the subsequent work of F-X Dehon and the author [1]. We are here interested in the even more general *unbounded strong realization conjecture*. We prove that it holds at the prime 2 for the class of spaces whose cohomology has a trivial Bockstein action in high degrees.

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1 Introduction

The singular cohomology of a topological space with coefficients in a finite prime field is naturally endowed with the structure of an unstable algebra over the Steenrod algebra. That is, a graded ring structure with a compatible action of the Steenrod algebra; see Schwartz [8, page 21].

An unstable module isomorphic to the cohomology of some space is termed *topologically realizable*. N Kuhn's conjectures [4] claim that realizable unstable modules have rather special algebraic features. Namely, these conjectures tell us that the action of the Steenrod algebra on the cohomology of a topological space ought to be *either very big or very small*.

The first of these conjectures [4, Realization Conjecture, page 321] was settled by L Schwartz [9, Theorem 0.1] and says that the singular cohomology of a space X with coefficients in a finite prime field is finitely generated as a module over the Steenrod algebra if and only if it is finite dimensional as a (graded) vector space. In other words, the cohomology is nontrivial in finitely many degrees, and is a finite dimensional vector space in these degrees.

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The more general *strong realization conjecture* [4, page 324] was settled at the prime 2 by L Schwartz [10] under some finiteness assumptions later removed by the work of F-X Dehon and the author in [1].

Let us explain briefly the content of the *strong realization conjecture*. Lannes' T functor is the endofunctor of the category of unstable modules which is left adjoint to tensoring with the cohomology of the infinite real projective space $\mathrm{H}^*\mathrm{B}(\mathbb{Z}/2\mathbb{Z})$. There is a reduced version $\overline{\mathrm{T}}$ of this functor which is left adjoint to tensoring with the *reduced* cohomology of the infinite real projective space. Let $\mathcal{U}_d$ be the full subcategory of unstable modules annihilated by $\overline{\mathrm{T}}^{d+1}$, the reduced Lannes' functor iterated $(d+1)$ times. The subcategory $\mathcal{U}_0$ happens to be the subcategory of *locally finite* modules [8; 10], ie the full subcategory of unstable modules such that all monogenic submodules are finite dimensional over the ground field. The *strong realization conjecture* says that if the singular cohomology of a space X with coefficients in a finite prime field is in $\mathcal{U}_d$ for some d, then it is in $\mathcal{U}_0$.

From now on, we turn our attention to the even more general *unbounded strong realization conjecture* (described in Section 2.3), which we show to hold at the prime 2 for the class of spaces having a trivial action of Bocksteins in high degrees.

Remark 1.1 In the following, H^*X always means the modulo 2 singular cohomology of the space X. Also, all unstable modules are modules over the modulo 2 Steenrod algebra.

The main result

We denote by $\mathcal{U}$ the category of unstable modules over the modulo 2 Steenrod algebra. Every object M of $\mathcal{U}$ is equipped with a natural decreasing filtration, the so-called *nilpotent filtration* [4; 10]:

$$M = M_0 \supset M_1 \supset \dots M_s \supset M_{s+1} \supset \dots$$

This filtration is defined in the following way. An unstable module is called s–nilpotent if it belongs to the smallest full abelian subcategory of unstable modules containing s–th suspensions, and stable under extensions and filtered colimits. The s–th step M_s of the nilpotent filtration of an unstable module M is its largest s–nilpotent submodule.

For each s, the subquotients M_s/M_{s+1} of the nilpotent filtration of M are of the form $\Sigma^s \mathrm{R}_s M$ where $\mathrm{R}_s M$ is a reduced module (see Section 2.1).

Our main result is the following.

Theorem 1.2 *Let X be a topological space and let H^*X be its cohomology modulo 2. Assume furthermore that H^*X has a trivial action of the Bockstein operator in high degrees and that H^*X is not locally finite. The module $\mathrm{R}_t\overline{\mathrm{H}}^*X$ cannot be locally finite for all integers $t \geq 0$, so let s be the smallest t such that $\mathrm{R}_t\overline{\mathrm{H}}^*X$ is not locally finite. Then the unstable module $\mathrm{R}_s\overline{\mathrm{H}}^*X$ does not belong to $\mathcal{U}_d$ for any integer d.*

The first assertion of the theorem follows from Lemma 2.7.

So we get in particular that the *unbounded strong realization conjecture* (to be explained in Section 2.3) holds for the class of spaces such that the Bockstein acts trivially in high degrees:

Theorem 1.3 *Let M be an unstable module such that for all s, the module R_sM is in some $\mathcal{U}_{d(s)}$. Suppose moreover that the Bockstein acts trivially on M in high degrees. If M is topologically realizable then M is locally finite.*

In this statement, the number $d(s)$ is not supposed to be bounded with s; this explains the term *unbounded* for the conjecture. Let us explain briefly how Theorem 1.3 follows from Theorem 1.2. Let M be a topologically realizable unstable module M such that the module R_sM is in some $\mathcal{U}_{d(s)}$ for all s and such that M has a trivial action of Bocksteins in high degrees. Suppose now, contradicting Theorem 1.3, that M is not locally finite. From Lemma 2.7, we know that some R_sM is not locally finite. Assume s is the smallest integer having this property. On the one hand, the hypotheses of Theorem 1.2 are fulfilled and R_sM is not in $\mathcal{U}_d$ for any d. But on the other hand, we had assumed the module R_sM to be in some $\mathcal{U}_{d(s)}$ for all s. This is a contradiction.

One might compare Theorem 1.3 to [4, Theorem 0.1, Theorem 0.3] in the seminal article of N Kuhn, where he proves the *realization conjecture* under the same hypothesis on Bocksteins as ours. The method he uses relies on secondary operations and does a priori not apply to the more general setting of the *unbounded strong realization conjecture*. We realized actually that the method of L Schwartz applies in our situation precisely in trying to extend (unsuccessfully) secondary operation technology to the more general realization conjectures.

Assume the unbounded conjecture is true in general (see Section 2.3). If the cohomology ring $\overline{\mathrm{H}}^*X$ of a space X is not locally constant, then for some integer s, the reduced module $\mathrm{R}_s\overline{\mathrm{H}}^*X$ does not belong $\mathcal{U}_d$ for any integer d. L Schwartz has provided precise conjectures [10, Conjecture 0.2, Conjecture 0.3] about the value of the smallest such s in special cases. Our main theorem says that in the case of the vanishing of Bocksteins in high degrees, the smallest s such that $R_s\overline{\mathrm{H}}^*X$ is not locally finite is also the smallest s such that $R_s\overline{\mathrm{H}}^*X$ does not belong to $\mathcal{U}_d$ for any integer d.

Example 0.11 of [4, page 326] is very useful in order to understand our result. Let Y be the s–th bar filtration of $\mathrm{B}\mathbb{C}\mathrm{P}^\infty$. Then Y is a space with nilpotent cohomology (see Section 2.1) such that for $1 \le t < s$, the module $\mathrm{R}_t\overline{\mathrm{H}}^*Y$ is in $\mathcal{U}_t$, but $\mathrm{R}_s\overline{\mathrm{H}}^*Y$ is not in $\mathcal{U}_d$ for any finite d. Our result shows that all cohomology classes which reduce nontrivially in $\mathrm{R}_1\overline{\mathrm{H}}^*Y$ have a nonzero Bockstein.

This example shows that in general, if $\overline{\mathrm{H}}^*X$ is not locally finite,

- the smallest value s of t such that $\mathrm{R}_t\overline{\mathrm{H}}^*X$ is not is not in $\mathcal{U}_d$ for any integer d can be arbitrary high,
- the unstable modules $\mathrm{R}_t\overline{\mathrm{H}}^*X$ for $1 \le t < s$ may be nonlocally finite.

To prove Theorem 1.2, we shall use, as in [1] the theory of profinite spaces to be free of any finiteness hypotheses. The Theorem 1.2 is a consequence of the more general:

Theorem 1.4 *Let X be a* profinite space *and let H^*X be its continuous cohomology modulo 2. Assume furthermore that H^*X has a trivial action of the Bockstein operator in high degrees and that H^*X is not locally finite. The module $\mathrm{R}_t\overline{\mathrm{H}}^*X$ cannot be locally finite for all integers $t \ge 0$, and we let s be the smallest t such that $\mathrm{R}_t\overline{\mathrm{H}}^*X$ is not locally finite. Then the unstable module $\mathrm{R}_s\overline{\mathrm{H}}^*X$ does not belong to $\mathcal{U}_d$ for any integer d.*

Theorem 1.4 implies Theorem 1.2 because the cohomology of a space is naturally isomorphic to that of its profinite completion (which is a profinite space) as an unstable algebra [1, page 404, Section 2.3]. Namely, suppose X is a space such that $\mathrm{R}_s\overline{\mathrm{H}}^*X$ is finite for each s and such that the Bockstein operator is zero in high degrees. Then the same holds for the cohomology of the profinite completion of X. Hence, Theorem 1.4 implies Theorem 1.2.

In the following, the word space means profinite space. Hence, cohomology means continuous cohomology, etc. What we need from the theory of profinite spaces is strictly parallel to that of ordinary spaces. All the constructions on profinite spaces we will use are explained in detail in [1]. They behave in the same way as the usual constructions on spaces in the topological context. That's why the reader should not worry too much about profinite spaces in a first reading. From a philosophical viewpoint, profinite spaces are a replacement for usual spaces, where all our tools work without any restriction.

The setting of profinite spaces is crucial in the proofs, for otherwise the tools we use (Lannes' functor, Eilenberg–Moore spectral sequence) would not work.

2 Reformulations of the unbounded strong realization conjecture

2.1 Lannes' functor and the nilpotent filtration

The nilpotent filtration is briefly defined below Remark 1.1.

We begin by recalling an important property of the nilpotent filtration: any unstable module is complete with respect to its nilpotent filtration. This means that the natural map $M \longrightarrow \lim_s M/M_s$ is an isomorphism. This can be seen from the fact that for each s, the module M_s is $(s-1)$–connected.

We say that an unstable module is *reduced* if the operator

$$\mathrm{Sq}_0\colon M \longrightarrow M,\ m \longmapsto \mathrm{Sq}^{|m|}m$$

is injective. If M is the underlying module of some unstable algebra, then M is reduced if and only this algebra has no nilpotent elements, because in any unstable algebra M, besides the *Cartan formula* which says that

$$\mathrm{Sq}^n(xy) = \sum_{i+j=n} (\mathrm{Sq}^i x)\ (\mathrm{Sq}^j y), \quad \text{for all } x, y \in M,$$

we have the following other compatibly relation between the product and the Steenrod squares:

$$\mathrm{Sq}_0 m = \mathrm{Sq}^{|m|}m = m^2 \quad \text{for all } m \in M.$$

In other words, the higher Steenrod square acting nontrivially, coincides with the Frobenius operator of the algebra.

For each s, the subquotients M_s/M_{s+1} of the nilpotent filtration of M are of the form $\Sigma^s \mathrm{R}_s M$ where $\mathrm{R}_s M$ is a reduced module.

On the other hand, an unstable module M can be seen to be 1–nilpotent (or simply nilpotent, for short) if and only if the operator Sq_0 is locally nilpotent. This means that for all $m \in M$ there is a t (depending a priori on m) such that

$$(\mathrm{Sq}_0)^t m = 0.$$

An unstable module such that $M = M_s$ is called *at least s–nilpotent*. A 1–nilpotent module is simply called *nilpotent*. An element of an unstable module is s–nilpotent provided it spans a s–nilpotent submodule.

An important feature of the nilpotent filtration is its compatibility with tensor products: the tensor product of an s–nilpotent module with a t–nilpotent module is $(s+t)$–nilpotent.

The functor $\overline{\mathrm{T}}$ commutes with the nilpotent filtration in the following sense (see [4, Proposition 2.5, page 331]):

Proposition 2.1 *Let M be any unstable module and let*

$$M = M_0 \supset M_1 \supset \ldots M_s \supset M_{s+1} \supset \ldots$$

be the nilpotent filtration of M. Then the induced filtration of $\overline{\mathrm{T}}M$

$$\overline{\mathrm{T}}M = \overline{\mathrm{T}}M_0 \supset \overline{\mathrm{T}}M_1 \supset \ldots \overline{\mathrm{T}}M_s \supset \overline{\mathrm{T}}M_{s+1} \supset \ldots$$

is the nilpotent filtration of $\overline{\mathrm{T}}M$, ie for all s,

$$\overline{\mathrm{T}}(M_s) = (\overline{\mathrm{T}}M)_s.$$

As a consequence, by exactness and commutation of $\overline{\mathrm{T}}$ with suspensions, we have a sequence of equalities and natural isomorphisms

$$\begin{aligned}\Sigma^s \mathrm{R}_s \overline{\mathrm{T}}M &= (\overline{\mathrm{T}}M)_s/(\overline{\mathrm{T}}M)_{s+1} \\ &= \overline{\mathrm{T}}(M_s)/\overline{\mathrm{T}}(M_{s+1}) \cong \overline{\mathrm{T}}(M_s/M_{s+1}) = \overline{\mathrm{T}}\Sigma^s \mathrm{R}_s M \cong \Sigma^s \overline{\mathrm{T}}\mathrm{R}_s M.\end{aligned}$$

That is, the functors $\overline{\mathrm{T}}$ and R_s commute for all s, up to natural isomorphisms.

2.2 Weight and the Krull filtration

Let n be an integer. Let $n = \sum_{i=1}^{\ell} 2^{n_i}$ be the binary expansion of n. We attach to n the integer $\alpha(n) = \ell$.

Definition 2.2 Let M be a *reduced* unstable module. We say that M is of weight at most t if M is trivial in all degrees ℓ such that $\alpha(\ell) > t$. The weight $w(M)$ of M is the integer (maybe infinite) such that M is of weight at most $w(M)$ but not $w(M) - 1$.

To understand the definition, we give the following examples.

Example 2.3 Let $\mathrm{F}(1)$ be the unstable submodule generated by the nonzero degree one class in $\mathrm{H}^*\mathrm{B}(\mathbb{Z}/2\mathbb{Z}) = \mathbb{F}_2[u]$. It is exactly the submodule of primitive elements of the Hopf algebra $\mathrm{H}^*\mathrm{B}(\mathbb{Z}/2\mathbb{Z})$. A graded $\mathbb{F}_2$–basis for $\mathrm{F}(1)$ is given by the elements $\{u^{2^i}\}_{i\in\mathbb{N}}$. So $\mathrm{F}(1)$ is zero in degrees ℓ such that $\alpha(\ell)$ is strictly more that one. Hence the weight $w(\mathrm{F}(1))$ equals 1.

Example 2.4 It is easy to see that $w(\mathrm{F}(1)^{\otimes n}) = n$.

Example 2.5 The reduced cohomology ring $\overline{\mathrm{H}}^* B(\mathbb{Z}/2\mathbb{Z}) = \mathbb{F}_2[u]$ is of infinite weight.

A reduced module is of weight zero if and only if it is concentrated in degree zero. In this case, we say that M is *constant*. For a reduced module, one readily checks that being *constant* and *locally finite* are equivalent notions.

More generally, the notion of weight and Krull filtration coincide for reduced modules, as shown by the following proposition.

Proposition 2.6 (Franjou and Schwartz [2]) *A reduced unstable module M is in $\mathcal{U}_n$ if and only if its weight $w(M)$ is less or equal to n.*

In particular, this implies that a reduced module M is in $\mathcal{U}_n$ if and only if $\overline{\mathrm{T}}^n M \neq 0$ and $\overline{\mathrm{T}}^{n+1} M = 0$. This proposition is an important tool for us, as we wish to consider the Krull filtration of the subquotients of the nilpotent filtration of certain unstable modules, and these subquotients are precisely reduced modules.

2.3 The unbounded realization conjecture

We can state the *unbounded strong realization conjecture* [4, page 326] in a slightly modified form.

Unbounded strong realization conjecture *Let M be an unstable module such that $\mathrm{R}_s M$ is of finite weight for each s. If M is topologically realizable, then the module $\mathrm{R}_s M$ is constant for all s.*

The original conjecture of N Kuhn is not stated in terms of weight, but in terms of polynomial degree of functors [4, pages 325–326]. This deserves a short explanation. Let $\mathcal{N}il$ be the full subcategory of $\sqcap$ of *nilpotent* unstable modules. One can form the quotient category $\mathcal{U}/\mathcal{N}il$. It is known by Henn, Lannes and Schwartz [3] that $\mathcal{U}/\mathcal{N}il$ is equivalent to the full subcategory $\mathcal{F}_\omega$ of *analytic functors* of the category $\mathcal{F}$, where $\mathcal{F}$ is the category of functors from finite dimensional $\mathbb{F}_2$–vector spaces to all $\mathbb{F}_2$–vector spaces (with natural transformations as morphisms). In the category $\mathcal{F}$, one has a notion of polynomial functor of degree n.

Let $q\colon \mathcal{U} \longrightarrow \mathcal{F}_\omega$ denote the quotient functor $\mathcal{U} \longrightarrow \mathcal{U}/\mathcal{N}il$ composed with the equivalence of categories $\mathcal{U}/\mathcal{N}il \cong \mathcal{F}_\omega$.

The point is that a reduced unstable module is of weight n if and only if $q(M)$ is polynomial of degree n.

We shall underline the proof of the fact that the strong realization conjecture is a consequence of the unbounded strong realization conjecture. It relies on the following lemma.

Lemma 2.7 *An unstable module M is in $\mathcal{U}_n$ if and only if $R_s M$ is in $\mathcal{U}_n$ for all s.*

Proof Suppose M is in $\mathcal{U}_n$. As $\mathcal{U}_n$ is a Serre subcategory (ie abelian and stable under extensions [10]), the modules M_s and $M_s/M_{s+1} = \Sigma^s R_s M$ are in $\mathcal{U}_n$ for each s. But the functor $\overline{\mathrm{T}}$ commutes with suspensions and (more generally) with the nilpotent filtration (Proposition 2.1), so $R_s M$ is also in $\mathcal{U}_n$.

Conversely, if $R_s M$ is in $\mathcal{U}_n$ for all s, by exactness of $\overline{\mathrm{T}}$ it follows that M/M_s (recall that the nilpotent filtration is decreasing) is in $\mathcal{U}_n$ for each s. In other words,

$$\overline{\mathrm{T}}^{n+1}(M)/(\overline{\mathrm{T}}^{n+1}(M))_s = \overline{\mathrm{T}}^{n+1}(M)/\overline{\mathrm{T}}^{n+1}(M_s) \cong \overline{\mathrm{T}}^{n+1}(M/M_s) = 0$$

for each s. But $\overline{\mathrm{T}}^{n+1}(M)$ is complete with respect to its nilpotent filtration, hence

$$\overline{\mathrm{T}}^{n+1}(M) = 0.$$

It follows that M is in $\mathcal{U}_n$. □

Now suppose we have an unstable module M which is realizable and is in $\mathcal{U}_n$, ie such that $\overline{\mathrm{T}}^{n+1} M = 0$. By the preceding lemma, the module $\mathrm{R}_s M$ is also in $\mathcal{U}_n$. But an unstable module is of finite weight n if and only if it is in $\mathcal{U}_n$.

So, the unbounded strong realization conjecture implies that $\mathrm{R}_s M$ is constant for $s \geq 0$. Now, for a reduced module, being constant and being in $\mathcal{U}_0$ are the same thing. Hence, by the lemma, the module M is in $\sqcap_0$ and so the strong realization conjecture holds for M.

Another consequence of Lemma 2.7 is to give another form of the unbounded strong realization conjecture:

Unbounded strong realization conjecture *Let M be an unstable module such that $\mathrm{R}_s M$ is of finite weight for each s. If M is topologically realizable, then M is locally finite.*

This reformulation shows that Theorem 1.3 states a particular case of the *unbounded strong realization conjecture*.

3 Proof of Theorem 1.4

3.1 Notations and summary of the proof

It is not difficult to see that by replacing *cohomology* by *reduced cohomology* in Theorem 1.2, one gets an equivalent statement. We will therefore work from now on with reduced cohomology.

The proof of Theorem 1.4 is by contradiction. We want to prove that there exists no profinite space X such that

(i) the cohomology of X is not locally constant and for the lowest d such that $\mathrm{R}_d\overline{\mathrm{H}}^*X$ is nonconstant, the module $\mathrm{R}_s\overline{\mathrm{H}}^*X$ is of finite weight,

(ii) the action of the Bockstein is trivial in high degrees in $\overline{\mathrm{H}}^*X$.

To this end, we refine the proof that was used in [8; 9; 1]. Let us recall how it goes.

Suppose that a profinite space X satisfying the above conditions exists. Let d be the minimal integer s such that $\mathrm{R}_s\overline{\mathrm{H}}^*X$ is nonconstant. Necessarily by [4, Proposition 0.8, Corollary 0.9], d is nonzero. According to the discussion at the beginning of Section 7.2 in [1], we can suppose that $\overline{\mathrm{H}}^*X$ is d–nilpotent, and as connected as necessary (the point here is that exchanging X with the quotient of X by some skeleton provides a new space with the same properties, but with higher connectivity).

We define for $0 \leq \ell \leq d$,

$$X_\ell = \Omega^{d-\ell}X$$

so that $X_d = X$ and $X_0 = \Omega^d X$.

It follows from the hypotheses that $\mathrm{R}_d\overline{\mathrm{H}}^*X$ is of finite weight $f > 0$. We use *Kuhn's reduction* in the framework of profinite spaces [1, Section 7.1] to lower the weight until $f = 1$. This is done in Section 3.2. This is the step that uses the technology of Lannes' T functor.

We construct a family $(\alpha_{i,d})_{i\geq\kappa}$ of classes in $\overline{\mathrm{H}}^*X$ satisfying a certain set of conditions $(\mathcal{H}_d)$. We follow these classes for $d \geq \ell \geq 0$ in the cohomology of the iterated loop spaces $\Omega^{d-\ell}X$: the classes $(\alpha_{i,\ell})_{i\geq\kappa}$ induced in $\Omega^{d-\ell}X$ through iterated evaluation map

$$\Sigma\Omega Z \longrightarrow Z$$

satisfy a similar set of conditions $(\mathcal{H}_\ell)$. This is done in Section 3.3 and Section 3.4. The properties of the Eilenberg–Moore spectral sequence are there heavily used.

The set of conditions $(\mathcal{H}_1)$ implies that the cup square of $\alpha_{i,1}$ is trivial for large i (see Section 3.5). This is precisely the point where the hypothesis on the action of Bocksteins is needed. We show finally in Section 3.6, following ideas of [10; 1] that the cup square of $\alpha_{i,0}$ is trivial for large i. Since the set of conditions $(\mathcal{H}_0)$ says in particular that the cup square of $\alpha_{i,0}$ is nontrivial for large i, this gives a contradiction.

An attentive reader may have noticed the method used here is very similar to that of [10; 1]. There are of course variations here, due the different situation. These are essentially

- we need to see that the hypothesis on Bocksteins carries over the Kuhn reduction (Section 3.2),
- the behaviour of the classes $(\alpha_{i,d})_{i\geq\kappa}$ is easier to analyse than in [10; 1], because the set of hypotheses $(\mathcal{H}_\ell)$ is smaller,
- we need on the other hand the slightly sharper statements on weight settled in Section 3.3,
- the last step explained in Section 3.6 is essentially the same as in [10; 1], but in these sources, no clear statement we could rely on is made, and the situation is also slightly different. We find it therefore useful to give full details in Section 3.6.

3.2 Kuhn's reduction with trivial Bocksteins

Let Y be a profinite space. Let RY be the Bousfield–Kan functorial fibrant replacement of Y [7] (see also [1, Section 2.4]). We denote by ΔY the homotopy cofiber (in the homotopical algebra of profinite spaces) of the natural map

$$Y \longrightarrow \mathrm{Map}(B(\mathbb{Z}/2\mathbb{Z}), RY).$$

Let $f \geq 1$ be the weight of $\mathrm{R}_d\overline{\mathrm{H}}^* X$. We consider the space $\Delta^{f-1}X$.

Lemma 3.1 *The space $\Delta^{f-1}X$ satisfies*

(i) *the unstable module $\mathrm{R}_d\overline{\mathrm{H}}^*\Delta^{f-1}X$ is of weight 1,*

(ii) *the action of the Bockstein is trivial in high degrees in $\overline{\mathrm{H}}^*\Delta^{f-1}X$.*

Proof It follows from [1, Section 5] that

$$\overline{\mathrm{T}\mathrm{H}}^* X \cong \overline{\mathrm{H}}^*\Delta X$$

as unstable modules.

As the nilpotent filtration commutes with $\overline{\mathrm{T}}$, it follows that for all s and t

$$\overline{\mathrm{T}}^s\mathrm{R}_t\overline{\mathrm{H}}^* X \cong \mathrm{R}_t\overline{\mathrm{T}}^s\overline{\mathrm{H}}^* X.$$

On the other hand, we know that M is of weight k if and only if

$$\overline{\mathrm{T}}^{k+1}M = 0 \quad \text{and} \quad \overline{\mathrm{T}}^{k}M \neq 0.$$

We only need to prove that the action of the Bockstein is also trivial in high degrees in $\overline{\mathrm{T}}^{f-1}\overline{\mathrm{H}}^* X \cong \overline{\mathrm{H}}^*\Delta^{f-1}X$. But this is a consequence of Corollary A.2. □

3.3 Weight watchers

We rely in this section and also in the last section on the properties of the Eilenberg–Moore spectral sequence for profinite spaces. We therefore recall the basic properties that will be used. Full details of its construction are given in [1, Section 4].

Let X be a pointed profinite space. Then there is a natural second quadrant spectral sequence $\{(E_r^{-s,t}, d_r^{s,t}), s,t \geq 0\}_{r\geq 1}$, converging to the cohomology of the loop space ΩX, compatible with product and Steenrod operations. This means that for all $s \geq 0$ and $r \geq 2$, the graded vector space is an unstable module $E_r^{-s,*}$. The differential

$$d_r\colon E_r^{-s-r,*} \longrightarrow \Sigma^{r-1} E_r^{-s,*}$$

is linear with respect to the action of the Steenrod algebra. The cohomology of the profinite loop space ΩX has a natural filtration by unstable submodules

$$0 = \mathrm{F}_0\bar{\mathrm{H}}^*\Omega X \subset \mathrm{F}_{-1}\bar{\mathrm{H}}^*\Omega X \subset \mathrm{F}_{-2}\bar{\mathrm{H}}^*\Omega X \subset \ldots \subset \mathrm{F}_{-s}\bar{\mathrm{H}}^*\Omega X \subset \ldots \subset \bar{\mathrm{H}}^*\Omega X$$

such that
$$E_\infty^{-s,*} \cong \Sigma^s(\mathrm{F}_{-s}\bar{\mathrm{H}}^*\Omega X/\mathrm{F}_{-s+1}\bar{\mathrm{H}}^*\Omega X).$$

This filtration converges to the cohomology of ΩX

$$\bigcup\nolimits_{i\in\mathbb{N}} \mathrm{F}_{-i}\bar{\mathrm{H}}^*\Omega X = \bar{\mathrm{H}}^*\Omega X.$$

The spectral sequence carries products (in the most usual sense), and these products converge to the cup product on $\bar{\mathrm{H}}^*\Omega X$.

The E_1–term is given by the bar construction (see Mac Lane [6]) and in particular $E_1^{-s,t} = (\mathrm{H}^*X)^{\otimes s}$. The product on the E_1–term is given by the shuffle product [6] and the Steenrod module structure is the canonical one. Thus the E_2–term is given by

$$E_2^{-s,t} = \mathrm{Tor}_{H^*X}^{-s,t}(\mathbb{F}_2,\mathbb{F}_2).$$

No finiteness hypotheses are needed here to analyse the E_2–term as a Tor group because we use the profinite setting [1].

With the help of the Eilenberg–Moore spectral sequence, we will prove the following lemma.

Lemma 3.2 *For $1 \leq \ell \leq d$, the module $\mathrm{R}_\ell\bar{\mathrm{H}}^*X_\ell$ has weight one.*

Proof If $d = 1$ the lemma is clearly true from the hypotheses, otherwise we prove Lemma 3.2 by induction on:

Lemma 3.3 *Let Y be a profinite space such that $\overline{\mathrm{H}}^*Y$ is h–nilpotent, $h \geq 2$. Then $\mathrm{R}_{h-1}\overline{\mathrm{H}}^*\Omega Y$ and $\mathrm{R}_h Y$ have the same weight.*

Proof of Lemma 3.3 We use the Eilenberg–Moore spectral sequence which calculates $\overline{\mathrm{H}}^*\Omega Y$ from $\overline{\mathrm{H}}^*Y$. Its $E_2^{-s,*}$–term is a subquotient of $\overline{\mathrm{H}}^*Y^{\otimes s}$, which is hs–nilpotent (because of the compatibility of tensor products with nilpotency, see Section 2.1). Because the subcategory of t–nilpotent modules is a Serre subcategory (ie abelian and stable under extensions), it happens that $E_\infty^{-s,*}$ is also sh–nilpotent.

Let $\{\mathrm{F}_{-s}\overline{\mathrm{H}}^*\Omega Y\}_{s\in\mathbb{N}}$ be the Eilenberg–Moore filtration, whose associated graded is the abutment of the Eilenberg–Moore spectral sequence. We have

$$E_\infty^{-s,*} = \Sigma^s(\mathrm{F}_{-s}/\mathrm{F}_{-s+1})\overline{\mathrm{H}}^*\Omega Y$$

as unstable modules, hence $(\mathrm{F}_s/\mathrm{F}_{s-1})\overline{\mathrm{H}}^*\Omega Y$ is $(hs-s)$–nilpotent.

Because the Eilenberg–Moore filtration is convergent, and s–nilpotent modules form a Serre subcategory stable under filtered colimits, we have that $\overline{\mathrm{H}}^*\Omega Y/\mathrm{F}_s\overline{\mathrm{H}}^*\Omega Y$ is at least $(hs-s)$–nilpotent.

We recall the following result [1, Corollary A.3].

Proposition 3.4 *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of unstable modules and p, q, s three nonnegative integers. Suppose that $\mathrm{R}_s A$ is in $\mathcal{U}_p$ and that $\mathrm{R}_s C$ is in $\mathcal{U}_q$; then $\mathrm{R}_s B$ is in $\mathcal{U}_{\max\{p,q\}}$.*

Applying this result to the short exact sequence

$$\mathrm{F}_{-1}\overline{\mathrm{H}}^*\Omega Y \longrightarrow \overline{\mathrm{H}}^*\Omega Y \longrightarrow \overline{\mathrm{H}}^*\Omega Y/\mathrm{F}_{-1}\overline{\mathrm{H}}^*\Omega Y$$

we easily get that $\mathrm{R}_{h-1}\overline{\mathrm{H}}^*\Omega Y$ and $\mathrm{R}_{h-1}\mathrm{F}_{-1}\overline{\mathrm{H}}^*\Omega Y$ have the same weight.

We know that

$$\mathrm{R}_{h-1}\mathrm{F}_{-1}\overline{\mathrm{H}}^*\Omega Y \cong \mathrm{R}_h\Sigma(\mathrm{F}_{-1}\overline{\mathrm{H}}^*\Omega Y/\mathrm{F}_0\overline{\mathrm{H}}^*\Omega Y) = \mathrm{R}_h E_\infty^{-1,*}$$

and so we need to compare $\mathrm{R}_h E_\infty^{-1,*}$ and $\mathrm{R}_h\overline{\mathrm{H}}^*Y$.

But $E_\infty^{-1,*}$ is isomorphic to the quotient of $\overline{\mathrm{H}}^*Y$ by B, the union of the images of all higher differentials. The image of the differential d^r is easily seen to be at least $((r+1)(h-1)+2)$–nilpotent, by using the linearity of differentials (see [1] for more details). Hence, the union of the image of the differentials is at least $(2h-1)$–nilpotent. We have a short exact sequence:

$$B \longrightarrow \overline{\mathrm{H}}^*Y \longrightarrow E_\infty^{-1,*}$$

A new application of Proposition 3.4 gives that $\mathrm{R}_h E_\infty^{-1,*}$ and $\mathrm{R}_h\overline{\mathrm{H}}^*Y$ are of the same weight, and Lemma 3.3 follows. $\square$

Lemma 3.5 *The module* $\mathrm{R}_0\mathrm{F}_{-1}\overline{\mathrm{H}}^*X_0$ *is of weight* 1*. The module* $\mathrm{R}_0\mathrm{F}_{-2}\overline{\mathrm{H}}^*X_0$ *is of weight* 2*.*

Proof We have isomorphisms

$$\mathrm{R}_0\mathrm{F}_{-1}\overline{\mathrm{H}}^*X_0 \cong \mathrm{R}_1\Sigma(\mathrm{F}_{-1}\overline{\mathrm{H}}^*X_0)/\mathrm{F}_0\overline{\mathrm{H}}^*X_0) \cong \mathrm{R}_1E_\infty^{-1,*}.$$

The module $E_\infty^{-1,*}$ is a quotient of $\overline{\mathrm{H}}^*X_1$ by an at least 2–nilpotent submodule B.

So we have an exact sequence

$$B \longrightarrow \overline{\mathrm{H}}^*X_1 \longrightarrow E_\infty^{-1,*}.$$

By Lemma 3.2, the module $\mathrm{R}_1\overline{\mathrm{H}}^*X_1$ is of weight 1 which proves the first assertion.

The module $\mathrm{R}_0(\mathrm{F}_{-2}/\mathrm{F}_{-1})\overline{\mathrm{H}}^*X_0$ is isomorphic to $\mathrm{R}_2\Sigma^2(\mathrm{F}_{-2}/\mathrm{F}_{-1})\overline{\mathrm{H}}^*X_0 = \mathrm{R}_2E_\infty^{-2,*}$. The module $E_\infty^{-2,*}$ is a subquotient of $(\overline{\mathrm{H}}^*X_1)^{\otimes 2}$. So we have modules $B \subset C \subset (\overline{\mathrm{H}}^*X_1)^{\otimes 2}$ such that $C/B = E_\infty^{-2,*}$. The module B is the union of all the images of the differentials and C is the submodule of infinite cycles. One estimates that B is at least 3–nilpotent. Hence by [1, Corollary A.2] implies that $\mathrm{R}_2E_\infty^{-2,*}$ is isomorphic to R_2C. On the other hand the functor R_2 preserves monomorphisms [10; 1, Proposition A.1] and so $\mathrm{R}_2E_\infty^{-2,*}$ is isomorphic to some submodule of $\mathrm{R}_2((\overline{\mathrm{H}}^*X_1)^{\otimes 2}$. We finally note that

$$\mathrm{R}_2(\overline{\mathrm{H}}^*X_1)^{\otimes 2} = \oplus_{i+j=2}\mathrm{R}_i(\overline{\mathrm{H}}^*X_1)\otimes \mathrm{R}_j(\overline{\mathrm{H}}^*X_1) = \mathrm{R}_1(\overline{\mathrm{H}}^*X_1)\otimes \mathrm{R}_1(\overline{\mathrm{H}}^*X_1).$$

As $\mathrm{R}_1(\overline{\mathrm{H}}^*X_1)$ is of weight one, the module $\mathrm{R}_2(\overline{\mathrm{H}}^*X_1)^{\otimes 2}$ is of weight 2, and so are $\mathrm{R}_2E_\infty^{-2,*}$ and $\mathrm{R}_0\mathrm{F}_{-1}\overline{\mathrm{H}}^*X_0$. Using the short exact sequence

$$\mathrm{F}_{-1}\overline{\mathrm{H}}^*X_0 \longrightarrow \mathrm{F}_{-2} \longrightarrow (\mathrm{F}_{-2}\overline{\mathrm{H}}^*X_0/\mathrm{F}_{-1}) \cong \Sigma^2E_\infty^{-2,*}$$

and applying Proposition 3.4 and the preceding remarks, we find that the module $\mathrm{R}_0\mathrm{F}_{-2}\overline{\mathrm{H}}^*X_0$ is of weight 2. $\square$

3.4 Construction of classes

The next lemma is a special case of Proposition 7.2 of Dehon and the author [1]. The original statements are in Schwartz [9; 10].

Lemma 3.6 *Let M be a reduced module of weight 1. Let η be the unity of the adjunction $M \to \overline{\mathrm{T}}M \otimes \overline{\mathrm{H}}^*\mathrm{B}(\mathbb{Z}/2\mathbb{Z})$. Then η factorizes by the submodule $\overline{\mathrm{T}}M \otimes \mathrm{F}(1)$. Moreover, the kernel and cokernel of*

$$\eta\colon M \to \overline{\mathrm{T}}M \otimes \mathrm{F}(1)$$

are locally finite.

We apply this lemma to $M = \mathrm{R}_d\overline{\mathrm{H}}^*X$, which we can suppose to be of weight 1 by Lemma 3.1. Then it follows that there is a cyclic submodule of the form $\mathrm{F}(1)^{\geq 2^\xi}$ in M, generated by some $\overline{\alpha}_\xi$ of degree 2^ξ. We can suppose ξ as big as we want. So we pick up some $\kappa \geq \xi$.

We lift up $\Sigma^s\overline{\alpha}_\kappa$ to a class $\alpha_{\kappa,d}$ of degree $2^\kappa + d$ through the epimorphism $(\overline{\mathrm{H}}^*X)_s \longrightarrow \Sigma^s\mathrm{R}_s(\overline{\mathrm{H}}^*X)$, and we define recursively, for $i \geq \kappa$

$$\alpha_{i+1,d} = \mathrm{Sq}^{2^i}\alpha_{i,d}.$$

We get some classes $(\alpha_{i,d})_{i\geq\kappa}$ satisfying the following set of conditions:

$$(\mathcal{H}_d)\begin{cases}\text{the class } \alpha_{i,d} \text{ is defined for } i \geq \kappa \text{ and is of degree } 2^i + d \text{ in } \overline{\mathrm{H}}^*X,\\ \text{the class } \alpha_{i,d} \text{ reduces nontrivially in } \mathrm{R}_d(\overline{\mathrm{H}}^*X) \text{ (hence is nonzero)},\\ \text{the Bockstein acts trivially on } \alpha_{i,d},\\ \text{for } i \geq \kappa, \text{ we have } \mathrm{Sq}^{2^i}\alpha_{i,d} = \alpha_{i+1,d}.\end{cases}$$

The evaluation $\mathrm{ev}_Z\colon \Sigma\Omega Z \longrightarrow Z$ induces a morphism

$$\mathrm{ev}_Z\colon \mathrm{H}^*Z \longrightarrow \mathrm{H}^*\Sigma\Omega Z \cong \Sigma\mathrm{H}^*\Omega Z.$$

We define iteratively, for all $0 \leq \ell \leq d-1$, the classes $(\alpha_{i,\ell})_{i\geq\kappa}$ as

$$\alpha_{i,\ell} = \mathrm{ev}_{X_{l+1}}\alpha_{i,\ell+1}.$$

We prove by downward induction the following proposition.

Proposition 3.7 *The classes $(\alpha_{i,\ell})_{i\geq\kappa}$ satisfy, for $0 \leq \ell \leq d$ and $i \geq \kappa$:*

$$(\mathcal{H}_\ell)\begin{cases}\text{the class } \alpha_{i,\ell} \text{ is of degree } 2^i + \ell \text{ in } \overline{\mathrm{H}}^*X,\\ \text{the class } \alpha_{i,\ell} \text{ reduces nontrivially in } \mathrm{R}_\ell(\overline{\mathrm{H}}^*X),\\ \text{the Bockstein acts trivially on } \alpha_{i,\ell},\\ \text{for } i \geq \kappa, \text{ we have } \mathrm{Sq}^{2^i}\alpha_{i,\ell} = \alpha_{i+1,\ell}.\end{cases}$$

Proof The assertion on the degree of $(\alpha_{i,\ell})_{i\geq\kappa}$ follows from the definitions. The second point is a consequence of the following lemma (see [1, Proposition A.4]).

Lemma 3.8 *Let Y be a profinite space such that $\overline{\mathrm{H}}^*Y$ is ℓ–nilpotent for $\ell \geq 1$. Then $\overline{\mathrm{H}}^*Y$ is $(\ell-1)$–nilpotent and the evaluation morphism induces a* monomorphism

$$\mathrm{R}_d\overline{\mathrm{H}}^*Y \hookrightarrow \mathrm{R}_d\Sigma\overline{\mathrm{H}}^*\Omega Y \cong \mathrm{R}_{d-1}\Omega Y.$$

The third and fourth points are consequences of the Steenrod algebra linearity of the evaluation morphism. Namely, it follows from the equalities

$$\begin{aligned}\Sigma(\mathrm{Sq}^1\alpha_{i,\ell-1}) &= \mathrm{Sq}^1\Sigma\alpha_{i,\ell-1}\\ &= \mathrm{Sq}^1\,\mathrm{ev}_{X_\ell}\,(\alpha_{i,\ell})\\ &= \mathrm{ev}_{X_\ell}\,(\mathrm{Sq}^1\alpha_{i,\ell})\\ &= 0\end{aligned}$$

that the Bockstein acts trivially on $\alpha_{i,\ell}$, and the equalities

$$\begin{aligned}\Sigma(\mathrm{Sq}^{2^i}\alpha_{i,\ell-1}) &= \mathrm{Sq}^{2^i}\,\Sigma\alpha_{i,\ell}\\ &= \mathrm{Sq}^{2^i}\,\mathrm{ev}_{X_\ell}(\alpha_{i,\ell})\\ &= \mathrm{ev}_{X_\ell}(\mathrm{Sq}^{2^i}\alpha_{i,\ell})\\ &= \mathrm{ev}_{X_\ell}\,(\alpha_{i+1,\ell})\\ &= \Sigma\alpha_{i+1,\ell-1}\end{aligned}$$

show how Sq^{2^i} acts on $\alpha_{i,\ell}$. □

3.5 The cup square of $\alpha_{i,1}$ is trivial

This is exactly the point where the hypothesis that Bocksteins are trivial in high degrees is used.

For $\ell = 1$, the classes $\alpha_{i,1}$ have degree $2^i + 1$, and the unstable algebra structure gives for $i \geq \kappa$,

$$\alpha_{i,1} \cup \alpha_{i,1} = \mathrm{Sq}^{2^i+1}\alpha_{i,1} = \mathrm{Sq}^1\mathrm{Sq}^{2^i}\alpha_{i,1} = \mathrm{Sq}^1\alpha_{i+1,1} = 0.$$

So to sum up the situation, we have a profinite space $X_1 = \Omega^{d-1}X$ and classes $(\alpha_{i,1})_{i\geq\kappa}$ such that for $i \geq \kappa$,

(i) the class $\alpha_{i,1}$ is of degree $2^i + 1$ in $\overline{\mathrm{H}}^*X_1$,

(ii) the class $\alpha_{i,1}$ reduces nontrivially in $\mathrm{R}_1(\overline{\mathrm{H}}^*X_1)$,

(iii) the Bockstein acts trivially on $\alpha_{i,1}$,

(iv) we have $\mathrm{Sq}^{2^i}\alpha_{i,1} = \alpha_{i+1,1}$,

(v) the cup square $\alpha_{i,1} \cup \alpha_{i,1}$ is trivial.

Suppose that we are able to prove that the same set of conditions holds for $(\alpha_{i,0})_{i \geq \kappa'}$, then we obtain the following contradiction

$$0 = \alpha_{i,0} \cup \alpha_{i,0} = \mathrm{Sq}^{2^i} \alpha_{i,0} = \alpha_{i+1,0} \neq 0.$$

So we need to prove that $\alpha_{i,0} \cup \alpha_{i,0} = 0$, for $i \geq \kappa$.

3.6 The cup square of $\alpha_{i,0}$ is trivial

We use the Eilenberg–Moore spectral sequence which relates $\overline{\mathrm{H}}^* X_1$ to $\overline{\mathrm{H}}^* X_0 = \overline{\mathrm{H}}^* \Omega X_1$. Recall that the Eilenberg–Moore spectral sequence carries products in the following way: the shuffle product $[\cdot|\cdot]$ on the E_1–term of the Eilenberg–Moore spectral sequence converges to the cup product on the E_∞–term (which means in particular that the shuffle product of infinite cycles is itself an infinite cycle).

For $i \geq \kappa$, the cup square $\alpha_{i,1} \cup \alpha_{i,1}$ is trivial. So the element $\alpha_{i,1} \otimes \alpha_{i,1} = [\alpha_{i,1}, \alpha_{i,1}]$ is a 1–cycle and defines an element of $E_2^{-1,*}$, as $\alpha_{i,1} \cup \alpha_{i,1} = d_1(\alpha_{i,1} \cup \alpha_{i,1})$. For degree reasons, the higher differentials coming from $E_2^{-1,*}$ are trivial and so, the 1–cycle $\alpha_{i,1} \otimes \alpha_{i,1}$ induces a permanent cycle, which never bounds for nilpotence reasons (see [1, Section 7.4]). Let $w_{i,\ell}$ be any element of $\overline{\mathrm{H}}^* X_0$ detected by this permanent cycle.

First step We want to compare $\mathrm{Sq}^{2^i} w_{i,0}$ to $\alpha_{i+1,0} \cup \alpha_{i,0}$. The cycle $[\alpha_{i,1} \mid \alpha_{i+1,1}] = \alpha_{i,1} \otimes \alpha_{i+1,1} + \alpha_{i+1,1} \otimes \alpha_{i,1}$ detects the cup product $\alpha_{i+1,0} \cup \alpha_{i,0}$.

By Cartan's formula, we have

$$\mathrm{Sq}^{2^i}[\alpha_{i,1} \mid \alpha_{i,1}] = \mathrm{Sq}^{2^i}(\alpha_{i,1} \otimes \alpha_{i,1}) = [\alpha_{i,1} \mid \alpha_{i+1,1}] + \sum\nolimits_{0<t\leq 2^{i-1}} [\mathrm{Sq}^t \alpha_{i,1} \mid \mathrm{Sq}^{2^i - t} \alpha_{i,1}].$$

The permanent cycle $\mathrm{Sq}^{2^i}(\alpha_{i,1} \otimes \alpha_{i,1})$ converges to $\mathrm{Sq}^{2^i} w_{i,0}$ by compatibility of the Eilenberg–Moore spectral sequence with Steenrod operations. In the same way, $[\mathrm{Sq}^t \alpha_{i,1} \mid \mathrm{Sq}^{2^i-t} \alpha_{i,1}]$ converges to $\mathrm{Sq}^t \alpha_{i,1} \cup \mathrm{Sq}^{2^i-t} \alpha_{i,1}$ for $0 \leq t \leq 2^{i-1}$.

Therefore, the element

$$\mathrm{Sq}^{2^i} w_{i,0} - \alpha_{i,0} \cup \alpha_{i+1,0} - \sum\nolimits_{0<t\leq 2^{i-1}} \mathrm{Sq}^t \alpha_{i,0} \cup \mathrm{Sq}^{2^i-t} \alpha_{i,0}$$

is in $\mathrm{F}_{-1} \overline{\mathrm{H}}^* X_d$. This equation is homogeneous of degree $2^i + 2^{i+1}$ and $\alpha(2^i + 2^{i+1}) = 2$ (the function α is defined in the beginning of Section 2.2). But $\mathrm{R}_0 \mathrm{F}_{-1} \overline{\mathrm{H}}^* X_d$ is of weight 1, by Lemma 3.5, so

$$\mathrm{Sq}^{2^i} w_{i,0} - \alpha_{i,0} \cup \alpha_{i+1,0} - \sum\nolimits_{0<t\leq 2^{i-1}} \mathrm{Sq}^t \alpha_{i,0} \cup \mathrm{Sq}^{2^i-t} \alpha_{i,0}$$

reduces to zero in $\mathrm{R}_0\mathrm{F}_{-1}\overline{\mathrm{H}}^* X_d$. Moreover, the inclusion of $\mathrm{F}_{-1}\overline{\mathrm{H}}^* X_d$ in $\mathrm{F}_{-2}\overline{\mathrm{H}}^* X_d$ induces a monomorphism $\mathrm{R}_0\mathrm{F}_{-1}\overline{\mathrm{H}}^* X_d \longrightarrow \mathrm{R}_0\mathrm{F}_{-2}\overline{\mathrm{H}}^* X_d$ [1, Proposition A.1]. Hence

$$\mathrm{Sq}^{2^i} w_{i,0} \quad \text{and} \quad \alpha_{i+1,0} \cup \alpha_{i,0} + \sum\nolimits_{0<t\leq 2^i-1} \mathrm{Sq}^t \alpha_{i,0} \cup \mathrm{Sq}^{2^i-t} \alpha_{i,0}$$

project to *equal* elements in $\mathrm{R}_0\mathrm{F}_{-2}\overline{\mathrm{H}}^* X_d$.

We now note, that for $0<t<2^{i-1}$, the class $\mathrm{Sq}^t\alpha_{i,0}$ is in degree $t+2^i$ and $\alpha(t+2^i)=2$. Therefore $\mathrm{Sq}^t\alpha_{i,0}$ reduces to zero in $\mathrm{R}_0\mathrm{F}_{-1}\overline{\mathrm{H}}^* X_d$, which is of weight 1 by Lemma 3.5. Now, the product map

$$F_{-1}\overline{\mathrm{H}}^* X_d \otimes F_{-1}\overline{\mathrm{H}}^* X_d \longrightarrow F_{-2}\overline{\mathrm{H}}^* X_d$$

induces a map

$$(\mathrm{R}_0 F_{-1}\overline{\mathrm{H}}^* X_d) \otimes (\mathrm{R}_0 F_{-1}\overline{\mathrm{H}}^* X_d) \cong \mathrm{R}_0(F_{-1}\overline{\mathrm{H}}^* X_d \otimes F_{-1}\overline{\mathrm{H}}^* X_d) \longrightarrow \mathrm{R}_0 F_{-2}\overline{\mathrm{H}}^* X_d.$$

It follows that $\sum_{0<t\leq 2^i-1} \mathrm{Sq}^t\alpha_{i,0} \cup \mathrm{Sq}^{2^i-t}\alpha_{i,0}$ reduces to zero in $\mathrm{R}_0 F_{-2}\overline{\mathrm{H}}^* X_d$, hence the following lemma holds:

Lemma 3.9 *The elements* $\mathrm{Sq}^{2^i} w_{i,0}$ *and* $\alpha_{i+1,0}\cup\alpha_{i,0}$ *project to* equal *elements in* $\mathrm{R}_0\mathrm{F}_{-2}\overline{\mathrm{H}}^* X_d$.

The class $\alpha_{i+1,0}\cup\alpha_{i,0}$ is in degree 2^i+2^{i+1} and $\alpha(2^i+2^{i+1})=2$.

Second step We now proceed to compare $\mathrm{Sq}^{2^i}\mathrm{Sq}^{2^i} w_{i,0}$ and $\mathrm{Sq}^{2^i}(\alpha_{i+1,0}\cup\alpha_{i,0})$.

The Cartan formula gives

$$\mathrm{Sq}^{2^i}(\alpha_{i,0}\cup\alpha_{i+1,0}) = \sum\nolimits_{p+q=2^i} \mathrm{Sq}^p\alpha_{i,0}\cup\mathrm{Sq}^q\alpha_{i+1,0}$$

so that

$$\begin{aligned}\mathrm{Sq}^{2^i}(\alpha_{i,0}\cup\alpha_{i+1,0}) &= (\mathrm{Sq}^{2^i}\alpha_{i,0})\cup\alpha_{i+1,0} + \sum\nolimits_{p<2^i} \mathrm{Sq}^p\alpha_{i,0}\cup\mathrm{Sq}^{2^i-p}\alpha_{i+1,0}\\ &= \alpha_{i+1,0}\cup\alpha_{i+1,0} + \sum\nolimits_{p<2^i} \mathrm{Sq}^p\alpha_{i,0}\cup\mathrm{Sq}^{2^i-p}\alpha_{i+1,0}.\end{aligned}$$

For $p<2^i$, we have the following two cases.

- If $0<p<2^i$, then $\mathrm{Sq}^p\alpha_{i,0}$ has degree $\ell=2^i+p$ such that $\alpha(\ell)>1$. The element $\alpha_{i,0}$ is in the submodule $\mathrm{F}_{-1}\overline{\mathrm{H}}^* X_0$ by definition, thus so is $\mathrm{Sq}^p\alpha_{i,0}$. But $\mathrm{R}_0\mathrm{F}_{-1}\overline{\mathrm{H}}^* X_0$ is of weight one and this implies that $\mathrm{Sq}^p\alpha_{i,0}$ reduces to zero in $\mathrm{R}_0\mathrm{F}_{-1}\overline{\mathrm{H}}^* X_0$. In other words, the class $\mathrm{Sq}^p\alpha_{i,0}$ is nilpotent.
- If $p=0$, then $\mathrm{Sq}^{2^i-p}\alpha_{i+1,0}$ has degree ℓ such that $\alpha(\ell)>1$. The same argument shows that if $p=0$, the element $\mathrm{Sq}^{2^i-p}\alpha_{i+1,0}$ is nilpotent.

So for $p < 2^i$, either $\mathrm{Sq}^p \alpha_{i,0}$ or $\mathrm{Sq}^{2^i - p}\alpha_{i+1,0}$ is nilpotent and so is the cup product $\mathrm{Sq}^p \alpha_{i,0} \cup \mathrm{Sq}^{2^i-p}\alpha_{i+1,0}$.

It follows that $\mathrm{Sq}^{2^i}(\alpha_{i,0} \cup \alpha_{i+1,0})$ and $\alpha_{i+1,0} \cup \alpha_{i+1,0}$ project to *equal* elements in $\mathrm{R}_0\mathrm{F}_{-2}\overline{\mathrm{H}}^* X_d$. In other words the following lemma holds.

Lemma 3.10 *The classes* $\mathrm{Sq}^{2^i}\mathrm{Sq}^{2^i} w_{i,0}$ *and* $\alpha_{i+1,0} \cup \alpha_{i+1,0}$ *project to equal elements in* $\mathrm{R}_0\mathrm{F}_{-2}\overline{\mathrm{H}}^* X_d$.

The decomposition of $\mathrm{Sq}^{2^i}\mathrm{Sq}^{2^i}$ [10, Lemma 5.7, page 554] implies that $\mathrm{Sq}^{2^i}\mathrm{Sq}^{2^i} w_{i,0}$ belongs to a submodule of $\mathrm{F}_{-2}\overline{\mathrm{H}}^* X_0$ generated by elements having degrees ℓ such that $\alpha(\ell) \geq 3$. But $\mathrm{R}_0\mathrm{F}_{-2}\overline{\mathrm{H}}^* X_0$ is of weight 2 by Lemma 3.5, so $\mathrm{Sq}^{2^i}\mathrm{Sq}^{2^i} w_{i,0}$ reduces to zero in $\mathrm{R}_0\mathrm{F}_{-2}\overline{\mathrm{H}}^* X_0$. Hence $\alpha_{i+1,0} \cup \alpha_{i+1,0}$ reduces to zero in $\mathrm{R}_0\mathrm{F}_{-2}\overline{\mathrm{H}}^* X_0$ for $i \geq \kappa$.

In other words, the element $\alpha_{i,0} \cup \alpha_{i,0}$ is nilpotent for $i \geq \kappa$ and thus for some t,

$$\mathrm{Sq}_0^t(\alpha_{i,0} \cup \alpha_{i,0}) = \mathrm{Sq}_0^t \alpha_{i,0} \cup \mathrm{Sq}_0^t \alpha_{i,0} = 0.$$

This completes the proof that the cup square of $\alpha_{i,0}$ is trivial.

On the other hand, we know that

$$\mathrm{Sq}_0^t \alpha_{i,0} \cup \mathrm{Sq}_0^t \alpha_{i,0} = \alpha_{i+t,0} \cup \alpha_{i+t,0} = \mathrm{Sq}^{2^{i+t}} \alpha_{i+t,0} = \alpha_{i+t+1,0} \neq 0.$$

This is a contradiction and completes the proof of the main theorem.

Appendix A Trivial Bockstein actions and Lannes' functor

The material of this section is well-known. It is already used in [4, Proposition 1.3, page 328] and first proved by M Winstead [11]. We thank gratefully J Lannes who explained us the following proof. Let M be an unstable module. The notation $M^{\geq n}$ stands for the submodule of M of elements of degrees greater than n. We say that the action of the Bockstein is *trivial in degree greater than* n if $\mathrm{Sq}^1 M^{\geq n} = 0$.

Proposition A.1 *Let* M *be an unstable module. The action of the Bockstein in* M *is trivial in degree greater than* n *if and only if the action of the Bockstein in* $\mathrm{T}M$ *is trivial in degrees greater than* n.

Because $\overline{\mathrm{T}}M$ is a submodule of $\mathrm{T}M$ we have:

Corollary A.2 *Let M be an unstable module. If the action of the Bockstein in M is trivial in degree greater than n, then the action of the Bockstein in $\overline{\mathrm{T}}M$ is also trivial in degrees greater than n.*

Before proving Proposition A.1, we recall the definition of the *double* ΦM of an unstable module M [5; 8, page 27]. The module ΦM is the unique unstable module ΦM such that

(i) the module ΦM is zero in odd degrees,

(ii) for any ℓ, $(\Phi M)^{2\ell}$ is M^{ℓ},

(iii) the natural map Φ: $\Phi M \longrightarrow M$ which maps m to $\Phi m = \mathrm{Sq}_0 m$ is linear with respect to the Steenrod algebra.

In other words
$$\mathrm{Sq}^{2\ell}\Phi m = \Phi \mathrm{Sq}^{\ell} m.$$

It is evident from the definition that the action of the Bockstein is trivial on ΦM. Conversely, we have

Lemma A.3 *Let M be an unstable module such that the action of the Bockstein is trivial in each degree. Denote by M^{odd} and M^{even} the odd and even degree parts of M as graded vector spaces. Then M splits as a module over the Steenrod algebra as*
$$M = M^{\mathrm{odd}} \oplus M^{\mathrm{even}}.$$

Proof This lemma is the consequence of the following facts:

(i) the Steenrod algebra is generated as an algebra by the squares Sq^i,

(ii) we have for any odd square the Adem relation
$$\mathrm{Sq}^{2n+1} = \mathrm{Sq}^1\mathrm{Sq}^{2n}.$$

When the action of the Bockstein is trivial, it follows that M^{odd} and M^{even} are unstable submodules and that the vector space decomposition $M = M^{\mathrm{odd}} \oplus M^{\mathrm{even}}$ is in fact a Steenrod algebra module decomposition. □

Lemma A.4 *Let M be a module such that M is zero in odd degrees. Then M is of the form ΦM_1 for a unique unstable module M_1. Let M be an unstable module such that M is zero in even degrees. Then M is of the form $M = \Sigma\Phi M_2$ for a unique module M_2.*

Proof Let us prove the first assertion. It follows from the definitions that M_1 has to be defined by $M_1^\ell = M^{2\ell}$. Furthermore, we also have no choice for the Steenrod algebra structure on M_1. It remains only to show that this actually defines an action of the Steenrod algebra, which amounts to the definition of Φ.

To prove the second assertion, we remark that for any module M concentrated in odd degrees, the operator Sq_0 is trivial. But The triviality of this operator is exactly the obstruction for algebraically desuspending an unstable module. So M is of the form $M = \Sigma M'$ for a unique M'. Now M' is concentrated in even degree, and by the first part, we have that $M' = \Phi M_2$ for a unique M_2. So, we have

$$M = \Sigma M' = \Sigma\Phi M_2. \qquad \square$$

We return to the proof of Proposition A.1.

Proof of Proposition A.1 Let M be an unstable module having trivial action of the Bockstein in degrees greater than n.

We have a short exact sequence of unstable modules

$$M^{\geq n} \longrightarrow M \longrightarrow M/M^{\geq n}.$$

By exactness of the T functor, we get an exact sequence

$$\mathrm{T}M^{\geq n} \longrightarrow \mathrm{T}M \longrightarrow \mathrm{T}(M/M^{\geq n}). \tag{1}$$

Lannes' T functor admits a natural splitting as

$$T \cong \overline{\mathrm{T}} \oplus \mathrm{Id}$$

hence the exact sequence (1) splits into two short exact sequences

$$M^{\geq n} \longrightarrow M \longrightarrow M/M^{\geq n} \quad \text{and} \quad \overline{\mathrm{T}}M^{\geq n} \longrightarrow \overline{\mathrm{T}}M \longrightarrow \overline{\mathrm{T}}M/\overline{\mathrm{T}}M^{\geq n} \cong \overline{\mathrm{T}}(M/M^{\geq n}).$$

Now $M/M^{\geq n}$ is a bounded module, so $\overline{\mathrm{T}}(M/M^{\geq n}) = 0$. On the other hand, $M^{\geq n}$ has trivial action of Bocksteins and so, by Lemma A.3,

$$M^{\geq n} = (M^{\geq n})^{\text{even}} \oplus (M^{\geq n})^{\text{odd}}.$$

Now, Lemma A.4 ensures that

$$M^{\geq n} = \Phi M_1 \oplus \Sigma\Phi M_2$$

and so $$\overline{\mathrm{T}}M^{\geq n} = \overline{\mathrm{T}}(\Phi M_1 \oplus \Sigma\Phi M_2) = (\Phi\overline{\mathrm{T}}M_1 \oplus \Sigma\Phi\overline{\mathrm{T}}M_2)$$

because the functor $\overline{\mathrm{T}}$ commutes to suspensions and to Φ.

It follows that $\overline{\mathrm{T}}M^{\geq n}$ has trivial action of Bocksteins in each degrees. Finally, $\mathrm{T}M = M \oplus \overline{\mathrm{T}}M$ has trivial action of Bocksteins in degrees greater than n.

The converse is a consequence of the aforementioned splitting of the T functor. □

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