On the Stiefel–Whitney classes of the representations associated with Spin(15)

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We determine the Stiefel–Whitney classes of the second exterior representation and the spin representation of Spin(15), which are useful to calculate the mod 2 cohomology of the classifying space of the exceptional Lie group E_8.

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1 Introduction

The study of the cohomology of the classifying space of the Lie groups has a long history; in particular, among the exceptional Lie groups G_2, F_4, E_6, E_7, E_8, Borel [4] first determined the algebra structure of the mod 2 cohomology of the classifying spaces BG_2 and BF_4 using the Serre spectral sequence. In these cases, it is well known that the numbers of generators as an algebra over the Steenrod algebra are 1 and 2 respectively. Kono, Shimada and the first author [7; 8] studied the mod 2 cohomology of BE_6 and BE_7, using the Rothenberg–Steenrod spectral sequence $\{E_r\}$ such that

$$E_2 = \mathrm{Cotor}_A(\mathbb{Z}/2, \mathbb{Z}/2) \Longrightarrow H^*(BE_i; \mathbb{Z}/2),$$

where $A = H^*(E_i; \mathbb{Z}/2)$ for $i = 6, 7$. Especially, Toda [11] announced that the numbers of generators as an algebra over the Steenrod algebra are 2 for both BE_6 and BE_7. Obviously, the first generators are of degree 4 in the cases of BG_2, BF_4, BE_6 and BE_7. It is possible to represent the second generators as the characteristic classes $w_{16}(\rho_4)$, $c_{16}(\rho_6)$ and $p_{16}(\rho_7)$ of some representations

$$\rho_4\colon F_4 \longrightarrow SO(26),$$
$$\rho_6\colon E_6 \longrightarrow SU(27),$$
$$\rho_7\colon E_7 \longrightarrow Sp(28),$$

in the case of BF_4, BE_6 and BE_7 respectively (see Adams [2, Corollaries 8.1, 8.3 and 8.2] for the representations).

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Little is known about the structure of mod 2 cohomology of the classifying space BE_8 of the exceptional Lie group E_8. However, it is quite natural to conjecture that the number of generators as an algebra over the Steenrod algebra is 2.

Conjecture 1.1 *The mod 2 cohomology of the classifying space BE_8 has two generators as an algebra over the Steenrod algebra; the first one is of degree 4, and the second one is the 128–th Stiefel–Whitney class $w_{128}(\rho_8)$ of the adjoint representation*

$$\rho_8 = \mathrm{Ad}_{E_8}\colon E_8 \longrightarrow SO(248).$$

(For the adjoint representation Ad_{E_8}, see Adams [2, Chapters 6 and 7].)

Based on the computation in Mori [9], we conjecture more precisely that the number of generators as an algebra (not as an algebra over the Steenrod algebra) is 33 and the action of the Steenrod square is expressed in the following diagram:

$$\begin{tikzcd}
4 \arrow[r, "\mathrm{Sq}^2"] & 6 \arrow[r, "\mathrm{Sq}^4"] \arrow[d, "\mathrm{Sq}^1"'] & 10 \arrow[r, "\mathrm{Sq}^8"] \arrow[d, "\mathrm{Sq}^1"'] & 18 \arrow[r, "\mathrm{Sq}^{16}"] \arrow[d, "\mathrm{Sq}^1"'] & 34 \arrow[r, "\mathrm{Sq}^{32}"] \arrow[d, "\mathrm{Sq}^1"'] & 66 \arrow[r, "\mathrm{Sq}^{64}"] \arrow[d, "\mathrm{Sq}^1"'] & 130 \arrow[d, "\mathrm{Sq}^1"'] & 128 \arrow[d, "\mathrm{Sq}^{64}"'] \\
 & 7 & 11 \arrow[d, "\mathrm{Sq}^2"'] & 19 \arrow[d, "\mathrm{Sq}^2"'] & 35 \arrow[d, "\mathrm{Sq}^2"'] & 67 \arrow[d, "\mathrm{Sq}^2"'] & 131 \arrow[d, "\mathrm{Sq}^2"'] & 192 \arrow[d, "\mathrm{Sq}^{32}"'] \\
 & & 13 & 21 \arrow[d, "\mathrm{Sq}^4"'] & 37 \arrow[d, "\mathrm{Sq}^4"'] & 69 \arrow[d, "\mathrm{Sq}^4"'] & 133 \arrow[d, "\mathrm{Sq}^4"'] & 224 \arrow[d, "\mathrm{Sq}^{16}"'] \\
 & & & 25 & 41 \arrow[d, "\mathrm{Sq}^8"'] & 73 \arrow[d, "\mathrm{Sq}^8"'] & 137 \arrow[d, "\mathrm{Sq}^8"'] & 240 \arrow[d, "\mathrm{Sq}^8"'] \\
 & & & & 49 & 81 \arrow[d, "\mathrm{Sq}^{16}"'] & 145 \arrow[d, "\mathrm{Sq}^{16}"'] & 248 \\
 & & & & & 97 & 161 \arrow[d, "\mathrm{Sq}^{32}"'] & \\
 & & & & & & 193 &
\end{tikzcd}$$

where the numbers in the diagram indicate the degrees of the generators.

Our main tool to prove the conjecture on $H^*(BE_8;\mathbb{Z}/2)$ is the Rothenberg–Steenrod spectral sequence.

Our tactics to prove the conjecture may be stated as follows; firstly we calculate $E_2 = \mathrm{Cotor}_A(\mathbb{Z}/2, \mathbb{Z}/2)$ for $A = H^*(E_8; \mathbb{Z}/2)$. We claim that generators are exactly given by the elements of degree indicated in the above diagram; we will determine it in the forthcoming paper (see Mori [9]), and secondly we show that these generators are detected by the images of the Steenrod squares of the generator of degree 4 and by the images of the Steenrod squares of the generator of degree 128. Observe that the remaining generators except those in the right column in the above diagram come from the mod 2 cohomology of the Eilenberg–Mac Lane space $K(\mathbb{Z}, 4)$.

In order to prove the conjecture on the mod 2 cohomology of BE_8, we need to find a Lie group G and a homomorphism to E_8 such that the structure of $H^*(BG; \mathbb{Z}/2)$ as an algebra over the Steenrod algebra is known and that the induced homomorphism $H^n(BE_8; \mathbb{Z}/2) \to H^n(BG; \mathbb{Z}/2)$ is monic for $n \leq N$, where N is sufficiently large. Using the homomorphism $G \to E_8$ with the above properties, we consider the Stiefel–Whitney class of the induced representation

$$G \longrightarrow E_8 \xrightarrow{\mathrm{Ad}_{E_8}} SO(248).$$

The natural inclusion map of the semi-spinor group $Ss(16) \subset E_8$ might seem to be the best homomorphism among others, since the homomorphism

$$H^*(BE_8; \mathbb{Z}/2) \longrightarrow H^*(BSs(16); \mathbb{Z}/2)$$

is a monomorphism. However, calculating $H^*(BSs(16); \mathbb{Z}/2)$ seems to be as difficult as calculating $H^*(BE_8; \mathbb{Z}/2)$, since the Hopf algebra structure of $H^*(Ss(16); \mathbb{Z}/2)$ is similar to that of $H^*(E_8; \mathbb{Z}/2)$. The next candidate is the spinor group $\mathrm{Spin}(16)$ which is the universal covering of $Ss(16)$ with the obvious homomorphism

$$\mathrm{Spin}(16) \to Ss(16) \to E_8.$$

According to Adams [2], the Lie algebra $L(E_8)$ of type E_8 can be constructed as the direct sum $L(\mathrm{Spin}(16)) \oplus \Delta^+_{16}$ with some Lie algebra structure, where $L(\mathrm{Spin}(16))$ is the Lie algebra of type $\mathrm{Spin}(16)$ and Δ^+_{16} is the spin representation of $\mathrm{Spin}(16)$:

$$\Delta^+_{16}\colon \mathrm{Spin}(16) \longrightarrow SO(128).$$

Note that the Lie algebra $L(\mathrm{Spin}(n))$ of type $\mathrm{Spin}(n)$ is isomorphic to the second exterior representation

$$\lambda^2_n\colon \mathrm{Spin}(n) \longrightarrow SO(n) \longrightarrow SO\left(\tbinom{n}{2}\right)$$

as a representation of $\mathrm{Spin}(n)$. Thus the induced representation $\mathrm{Spin}(16) \to Ss(16) \to E_8 \to SO(248)$ is the direct sum $\lambda^2_{16} \oplus \Delta^+_{16}$. The more appropriate homomorphism than $\mathrm{Spin}(16) \to E_8$ is the composition map of the natural maps

$$\mathrm{Spin}(15) \longrightarrow \mathrm{Spin}(16) \longrightarrow Ss(16) \longrightarrow E_8,$$

since the image of $H^*(BSs(16); \mathbb{Z}/2) \to H^*(B\,\mathrm{Spin}(16); \mathbb{Z}/2)$ is isomorphic to that of $H^*(BSs(16); \mathbb{Z}/2) \to H^*(B\,\mathrm{Spin}(15); \mathbb{Z}/2)$ and since $\mathrm{Spin}(15)$ is smaller than $\mathrm{Spin}(16)$. Observe that the induced representation

$$\mathrm{Spin}(15) \longrightarrow \mathrm{Spin}(16) \longrightarrow Ss(16) \longrightarrow E_8 \xrightarrow{\mathrm{Ad}_{E_8}} SO(248)$$

is a direct sum of the first exterior representation (or the projection map)

$$\lambda_{15}^1\colon \mathrm{Spin}(15) \longrightarrow SO(15),$$

the second exterior representation

$$\lambda_{15}^2\colon \mathrm{Spin}(15) \longrightarrow SO(105),$$

and the spin representation

$$\Delta_{15}\colon \mathrm{Spin}(15) \longrightarrow SO(128),$$

since we have $f_{15}^*\lambda_{16}^2 = \lambda_{15}^1 \oplus \lambda_{15}^2$ and $f_{15}^*\Delta_{16}^+ = \Delta_{15}$, where $f_{15}\colon \mathrm{Spin}(15) \to \mathrm{Spin}(16)$ is the natural inclusion map. Quillen's theorem (cf Theorem 2.4) states that the 128–th Stiefel–Whitney class $w_{128}(\Delta_{15})$ of the spin representation Δ_{15} is a member of a system of generators of $H^*(B\,\mathrm{Spin}(15); \mathbb{Z}/2)$ as an algebra over the Steenrod algebra. Our calculation asserts (Theorem 5.1) that

$$w_{128}(\lambda_{15}^1 \oplus \lambda_{15}^2 \oplus \Delta_{15}) \equiv w_{128}(\Delta_{15}) \text{ mod decomposables},$$

which implies (Corollary 5.2) that the 128–th Stiefel–Whitney class $w_{128}(\mathrm{Ad}_{E_8})$ can be chosen as a member of a system of generators of $H^*(BE_8; \mathbb{Z}/2)$ as an algebra over the Steenrod algebra.

The paper is organized as follows. In Section 2, we prepare some results on representations which will be needed for later use. In Section 3, we calculate the Stiefel–Whitney classes of the second exterior representation λ_{15}^2 of $\mathrm{Spin}(15)$. In Section 4, we calculate the Stiefel–Whitney classes of the spin representation Δ_{15} of $\mathrm{Spin}(15)$. One can derive the total characteristic classes $w(\rho_4)$, $c(\rho_6)$, $p(\rho_7)$ from the results (4–1), (4–2), (4–4) respectively. In Section 5, we calculate the Stiefel–Whitney classes of the representation $\lambda_{15}^1 \oplus \lambda_{15}^2 \oplus \Delta_{15}$ of $\mathrm{Spin}(15)$ which is the induced representation from the adjoint representation of E_8.

Our main results are Theorem 5.1 and Corollary 5.2; the latter result assures the existence of the first generator in the right column in the previous diagram. The former result will be used to show in the forthcoming paper that the action of the Steenrod squares on this generator of degree 128, and hence on those of degrees 192, 224, 240, 248, by using the Wu formula. That is, the generators in the right column can be represented by the 128–th, 192–nd, 224–th, 240–th and 248–th Stiefel–Whitney classes of the adjoint representation respectively.

It is our pleasure to acknowledge that the present paper is motivated by the calculation in Mori [9]. Most of the calculations were performed by programs using GAP which is a system for computation in discrete abstract algebra. We thank Shingo Okuyama and Yuriko Sambe who advised us about programming of the calculations.

2 Preliminary

In this section, we recall the mod 2 cohomology of the classifying spaces of $O(n)$, $SO(n)$ and $\mathrm{Spin}(n)$ as well as the Stiefel–Whitney classes of representations.

Let H_n be the subgroup of $O(n)$ consisting of the diagonal matrices, which is isomorphic to $(\mathbb{Z}/2)^n$. Let $i_n\colon H_n \to O(n)$ be the natural inclusion map. Let $W = N(H_n)/H_n$ be the Weyl group of $O(n)$, where $N(H_n)$ is the normalizer of H_n. As is well known, W is isomorphic to the n–th symmetric group Σ_n. The mod 2 cohomology of $BO(n)$ is a polynomial algebra whose generators are defined as the invariants under the action of the Weyl group:

$$\begin{aligned} H^*(BO(n);\mathbb{Z}/2) &= H^*(B(\mathbb{Z}/2)^n;\mathbb{Z}/2)^W \\ &= \mathbb{Z}/2[t_1, t_2, \ldots, t_n]^W \\ &= \mathbb{Z}/2[w_1, w_2, \ldots, w_n], \end{aligned}$$

where $\{t_j : 1 \le j \le n\}$ is a basis of $H^1(B(\mathbb{Z}/2)^n;\mathbb{Z}/2)$ and w_i, the i–th elementary symmetric polynomial of t_j, is called the i–th Stiefel–Whitney class. Similarly, the mod 2 cohomology of $BSO(n)$ is a polynomial algebra generated by w_i for $2 \le i \le n$:

$$H^*(BSO(n);\mathbb{Z}/2) = \mathbb{Z}/2[w_2, w_3, \ldots, w_n].$$

Let G be a compact Lie group and ρ an n–dimensional real representation $G \to O(n)$. Since a homomorphism induces a map between their classifying spaces

$$B\rho\colon BG \longrightarrow BO(n)$$

uniquely up to homotopy, we obtain a homomorphism between their mod 2 cohomologies:

$$B\rho^*\colon H^*(BO(n);\mathbb{Z}/2) \longrightarrow H^*(BG;\mathbb{Z}/2).$$

We denote $B\rho^*(w_i)$ simply by $w_i(\rho)$. For a representation $\rho\colon G \to SO(n)$, we also denote by $w_i(\rho)$ the induced element. One of the important properties of the Stiefel–Whitney class is the Whitney product formula:

$$w_k(\iota_{m,n}) = \sum_{i+j=k} w'_i \times w''_j, \tag{2–1}$$

where $\iota_{m,n}\colon O(m)\times O(n) \to O(m+n)$ is the obvious map, and w'_i and w''_i are the i–th Stiefel–Whitney classes of $H^*(BO(m);\mathbb{Z}/2)$ and $H^*(BO(n);\mathbb{Z}/2)$ respectively.

The action of the Steenrod square on the Stiefel–Whitney classes is given by the Wu formula:

$$\mathrm{Sq}^j\, w_i = \sum_{k=0}^{j} \binom{i-k-1}{j-k} w_{i+j-k} w_k \quad (0 \le j \le i).$$

Then, using it, one can easily see that generators of $H^*(BO(n);\mathbb{Z}/2)$ as an algebra over the Steenrod algebra are given by w_{2^k} for $1 \le 2^k \le n$.

Now we recall a result due to Borel–Hirzebruch [5]. Let H be an elementary abelian 2–subgroup of G, and $i\colon H \to G$ the inclusion map. Suppose that there is a representation $\rho\colon G \to O(n)$ satisfying the following commutative diagram:

$$\begin{tikzcd} H \arrow[r, "\overline{\rho}"] \arrow[d, "i"'] & H_n \arrow[d, "i_n"] \\ G \arrow[r, "\rho"'] & O(n), \end{tikzcd}$$

where $\overline{\rho} = \rho|_H$.

Proposition 2.1 (Borel–Hirzebruch [5]) *There holds*

$$Bi^*(w(\rho)) = \prod_{i=1}^{n} (1 + B\overline{\rho}^*(t_i)).$$

Moreover, if $G = O(m)$ *and* $H = H_m$, *then* $Bi^*(w(\rho))$ *is a symmetric polynomial of* t'_i *for* $1 \le i \le m$, *where* $t'_i \in H^1(BH_m;\mathbb{Z}/2)$.

Thus one can calculate $w(\rho)$, if $B\overline{\rho}^*(t_i)$ is calculable.

To state Quillen's result [10] concerning $H^*(B\,\mathrm{Spin}(n);\mathbb{Z}/2)$, we define h_n as follows:

$$h_n = \begin{cases} (n-2)/2, & n \equiv 0 \mod 8, \\ (n-1)/2, & n \equiv 1,7 \mod 8, \\ n/2, & n \equiv 2,4,6 \mod 8, \\ (n+1)/2, & n \equiv 3,5 \mod 8. \end{cases}$$

We often denote h_n by h if there is no confusion. Let $J \subset H^*(BSO(n);\mathbb{Z}/2)$ be the ideal generated by the elements

$$w_2,\quad \mathrm{Sq}^1 w_2,\quad \mathrm{Sq}^2\,\mathrm{Sq}^1 w_2,\quad \dots,\quad \mathrm{Sq}^{2^{h-2}}\,\mathrm{Sq}^{2^{h-3}}\cdots\mathrm{Sq}^2\,\mathrm{Sq}^1 w_2.$$

As is well known (see for example Adams [2]), $\mathrm{Spin}(n)$ has the spin representations:

$$\begin{aligned}
&\Delta^{\pm}_{8m}\colon \mathrm{Spin}(8m) \longrightarrow SO(2^{4m-1}),\\
&\Delta_{8m+1}\colon \mathrm{Spin}(8m+1) \longrightarrow SO(2^{4m}),\\
&\Delta^{\pm}_{8m+2}\colon \mathrm{Spin}(8m+2) \longrightarrow SU(2^{4m}),\\
&\Delta_{8m+3}\colon \mathrm{Spin}(8m+3) \longrightarrow Sp(2^{4m}),\\
&\Delta^{\pm}_{8m+4}\colon \mathrm{Spin}(8m+4) \longrightarrow Sp(2^{4m}),\\
&\Delta_{8m+5}\colon \mathrm{Spin}(8m+5) \longrightarrow Sp(2^{4m+1}),\\
&\Delta^{\pm}_{8m+6}\colon \mathrm{Spin}(8m+6) \longrightarrow SU(2^{4m+2}),\\
&\Delta_{8m+7}\colon \mathrm{Spin}(8m+7) \longrightarrow SO(2^{4m+3}),
\end{aligned}$$

where $\Delta^{\pm}$ means that there are two representations Δ^{+} and Δ^{-}.

Notation 2.2 (1) For a real representation $\rho\colon G \to SO(n)$, we denote by $\rho_{\mathbb{C}}$ a complex representation

$$G \xrightarrow{\rho} SO(n) \longrightarrow SU(n).$$

(2) For a quaternionic representation $\rho\colon G \to Sp(n)$, we denote by $\rho_{\mathbb{C}}$ a complex representation

$$G \xrightarrow{\rho} Sp(n) \longrightarrow SU(2n).$$

(3) For a complex representation $\rho\colon G \to SU(n)$, we denote by $\rho_{\mathbb{R}}$ a real representation

$$G \xrightarrow{\rho} SU(n) \longrightarrow SO(2n).$$

(4) For a quaternionic representation $\rho\colon G \to Sp(n)$, we denote by $\rho_{\mathbb{R}}$ a real representation

$$G \xrightarrow{\rho} Sp(n) \longrightarrow SU(2n) \longrightarrow SO(4n).$$

The following remark is well known (see for example Adams [1]).

Remark 2.3 For real representations ρ and σ, $\rho_{\mathbb{C}} \cong \sigma_{\mathbb{C}}$ if and only if $\rho \cong \sigma$. For quaternionic representations ρ and σ, $\rho_{\mathbb{C}} \cong \sigma_{\mathbb{C}}$ if and only if $\rho \cong \sigma$.

We sometimes denote by $\rho_{\mathbb{R}}$ a real representation ρ, and denote by $\rho_{\mathbb{C}}$ a complex representation ρ, by abuse of the notations. It is well known that $w(\rho_{\mathbb{R}}) = c(\rho)$ if ρ is a complex representation. It is also known that $w(\rho_{\mathbb{R}}) = c(\rho_{\mathbb{C}}) = p(\rho)$ if ρ is a quaternionic representation. Note that, if $n = 4m+2$, the representations of $(\Delta_n^+)_{\mathbb{R}}$ and $(\Delta_n^-)_{\mathbb{R}}$ are isomorphic, since Δ_n^+ and Δ_n^- are conjugate to each other. We denote $w_{2^h}((\Delta_n)_{\mathbb{R}})$ and $w_{2^h}((\Delta_n^+)_{\mathbb{R}})$ simply by u_{2^h} for n odd and n even respectively.

Now the results due to Quillen [10] can be summarized as follows:

Theorem 2.4 (1) *The algebra structure of the* mod 2 *cohomology of* $B\,\mathrm{Spin}(n)$ *is given by*

$$H^*(B\,\mathrm{Spin}(n);\mathbb{Z}/2) \cong H^*(BSO(n);\mathbb{Z}/2)/J \otimes \mathbb{Z}/2[u_{2^h}].$$

(2) *The nonzero Stiefel–Whitney classes of the spin representation are those of degrees* 2^h *and* $2^h - 2^i$ *for* $r \le i \le h$, *where*

$$r = \begin{cases} 0 & n \equiv 0,1,7 \mod 8, \\ 1 & n \equiv 2,6 \mod 8, \\ 2 & n \equiv 3,4,5 \mod 8. \end{cases}$$

In order to choose a Gröbner basis of the ideal of J (see for example Cox, Little and O'Shea [6]), we need to introduce a total order to the set of monomials in $H^*(BSO(n);\mathbb{Z}/2)$. Let $a = w_2^{j_2} w_3^{j_3} \cdots w_n^{j_n}$ and $b = w_2^{k_2} w_3^{k_3} \cdots w_n^{k_n}$ be monomials in $H^*(BSO(n);\mathbb{Z}/2)$. Then we define a total order as follows:

$$a < b \iff \begin{cases} \deg a < \deg b, \text{ or} \\ \deg a = \deg b,\ j_2 = k_2,\ \ldots,\ j_{i-1} = k_{i-1},\ j_i > k_i. \end{cases}$$

For the projection $\lambda^1\colon \mathrm{Spin}(n) \to SO(n)$, denote $w_i(\lambda^1)$ simply by y_i, since we need to distinguish the elements of $H^*(BSO(n);\mathbb{Z}/2)$ from those of $H^*(B\,\mathrm{Spin}(n);\mathbb{Z}/2)$. We exclude the elements y_2, y_3, y_5 and y_9 in the generators of $H^*(B\,\mathrm{Spin}(n);\mathbb{Z}/2)$, since $\mathrm{Sq}^1\, y_2 = y_3$, $\mathrm{Sq}^2\, y_3 = y_5 + y_2 y_3$ and $\mathrm{Sq}^4\, y_5 = y_9 + y_2 y_7 + y_3 y_6 + y_4 y_5$, that is, they are 0 in $H^*(B\,\mathrm{Spin}(n); \mathbb{Z}/2)$. Now we can describe the algebra structure of $H^*(B\,\mathrm{Spin}(n); \mathbb{Z}/2)$ for $3 \le n \le 15$ explicitly using the reduced Gröbner basis $\{R_i\}$ as follows:

$$H^*(B\,\mathrm{Spin}(3); \mathbb{Z}/2) = \mathbb{Z}/2[u_4];$$
$$H^*(B\,\mathrm{Spin}(4); \mathbb{Z}/2) = \mathbb{Z}/2[y_4, u_4];$$
$$H^*(B\,\mathrm{Spin}(5); \mathbb{Z}/2) = \mathbb{Z}/2[y_4, u_8];$$
$$H^*(B\,\mathrm{Spin}(6); \mathbb{Z}/2) = \mathbb{Z}/2[y_4, y_6, u_8];$$
$$H^*(B\,\mathrm{Spin}(7); \mathbb{Z}/2) = \mathbb{Z}/2[y_4, y_6, y_7, u_8];$$
$$H^*(B\,\mathrm{Spin}(8); \mathbb{Z}/2) = \mathbb{Z}/2[y_4, y_6, y_7, y_8, u_8];$$
$$H^*(B\,\mathrm{Spin}(9); \mathbb{Z}/2) = \mathbb{Z}/2[y_4, y_6, y_7, y_8, u_{16}];$$
$$H^*(B\,\mathrm{Spin}(10); \mathbb{Z}/2) = \mathbb{Z}/2[y_4, y_6, y_7, y_8, y_{10}, u_{32}]/(R_1),$$
$$R_1 = y_7 y_{10};$$
$$H^*(B\,\mathrm{Spin}(11); \mathbb{Z}/2) = \mathbb{Z}/2[y_4, y_6, y_7, y_8, y_{10}, y_{11}, u_{64}]/(R_1, R_2),$$
$$R_1 = y_7 y_{10} + y_6 y_{11},$$
$$R_2 = y_{11}^3 + y_7^2 y_8 y_{11} + y_4 y_7 y_{11}^2;$$
$$H^*(B\,\mathrm{Spin}(12); \mathbb{Z}/2) = \mathbb{Z}/2[y_4, y_6, y_7, y_8, y_{10}, y_{11}, y_{12}, u_{64}]/(R_1, R_2),$$
$$R_1 = y_7 y_{10} + y_6 y_{11},$$
$$R_2 = y_{11}^3 + y_7^2 y_8 y_{11} + y_7^3 y_{12} + y_4 y_7 y_{11}^2;$$
$$H^*(B\,\mathrm{Spin}(13); \mathbb{Z}/2) = \mathbb{Z}/2[y_4, y_6, y_7, y_8, y_{10}, y_{11}, y_{12}, y_{13}, u_{128}]/(R_1, R_2, R_3),$$
$$R_1 = y_7 y_{10} + y_6 y_{11} + y_4 y_{13},$$
$$\begin{aligned} R_2 = {} & y_{11}^3 + y_{10}^2 y_{13} + y_7 y_{13}^2 + y_7^2 y_8 y_{11} + y_7^3 y_{12} + y_6 y_7^2 y_{13} + y_6^2 y_8 y_{13} + y_4 y_7 y_{11}^2 \\ & + y_4 y_6 y_{10} y_{13} + y_4^2 y_{12} y_{13}, \end{aligned}$$
$$\begin{aligned} R_3 = {} & y_{13}^5 + y_{10}^3 y_{11}^2 y_{13} + y_{10}^4 y_{12} y_{13} + y_8 y_{10}^2 y_{11} y_{13}^2 + y_7 y_8 y_{11} y_{13}^3 + y_7^2 y_8^2 y_{11}^2 y_{13} \\ & + y_7^3 y_8 y_{11} y_{12} y_{13} + y_7^4 y_{12}^2 y_{13} + y_7^4 y_8^3 y_{13} + y_6 y_{10}^2 y_{13}^3 + y_6 y_7^2 y_8 y_{11} y_{13}^2 \\ & + y_6^2 y_8^2 y_{11} y_{13}^2 + y_6^2 y_7^2 y_{13}^3 + y_6^3 y_{11}^2 y_{12} y_{13} + y_6^3 y_{10} y_{11} y_{13}^2 + y_6^3 y_8 y_{13}^3 \\ & + y_6^3 y_7 y_8^2 y_{11} y_{13} + y_6^3 y_7^2 y_8 y_{12} y_{13} + y_6^4 y_8^2 y_{12} y_{13} + y_6^4 y_7 y_8 y_{13}^2 + y_4 y_{11}^2 y_{13}^3 \\ & + y_4 y_7^2 y_{11}^2 y_{12} y_{13} + y_4 y_7^3 y_8^2 y_{11} y_{13} + y_4 y_6 y_7 y_{11}^2 y_{13}^2 + y_4 y_6^3 y_7 y_{11} y_{12} y_{13} \\ & + y_4 y_6^4 y_{11} y_{13}^2 + y_4^2 y_{10} y_{11}^2 y_{12} y_{13} + y_4^2 y_{10}^2 y_{11} y_{13}^2 + y_4^2 y_8 y_{11} y_{12} y_{13}^2 \\ & + y_4^2 y_7 y_{11} y_{13}^3 + y_4^2 y_7^3 y_{11} y_{12} y_{13} + y_4^2 y_6 y_7 y_8 y_{11} y_{12} y_{13} + y_4^2 y_6 y_7^2 y_{12}^2 y_{13} \end{aligned}$$

$$+y_4^2y_6y_7^2y_{11}y_{13}^2+y_4^2y_6^2y_{10}y_{11}^2y_{13}+y_4^2y_6^2y_{10}^2y_{12}y_{13}+y_4^2y_6^3y_{13}^3$$
$$+y_4^3y_6y_{11}^2y_{12}y_{13}+y_4^3y_6y_{10}y_{11}y_{13}^2+y_4^4y_{11}y_{12}y_{13}^2+y_4^4y_{12}^3y_{13};$$

$H^*(B\,\mathrm{Spin}(14);\mathbb{Z}/2) =$

$$\mathbb{Z}/2[y_4,y_6,y_7,y_8,y_{10},y_{11},y_{12},y_{13},y_{14},u_{128}]/(R_1,R_2,R_3),$$

$$R_1 = y_7y_{10}+y_6y_{11}+y_4y_{13},$$
$$R_2 = y_{11}^3+y_{10}^2y_{13}+y_7y_{13}^2+y_7^2y_8y_{11}+y_7^3y_{12}+y_6y_7^2y_{13}+y_6^2y_8y_{13}+y_6^2y_7y_{14}$$
$$+y_4y_7y_{11}^2+y_4y_6y_{10}y_{13}+y_4^2y_{12}y_{13}+y_4^2y_{11}y_{14},$$
$$R_3 = y_{13}^5+y_{10}^3y_{11}^2y_{13}+y_{10}^4y_{12}y_{13}+y_{10}^4y_{11}y_{14}+y_8y_{10}^2y_{11}y_{13}^2+y_7y_8y_{11}y_{13}^3$$
$$+y_7^2y_{11}y_{13}^2y_{14}+y_7^2y_8^2y_{11}^2y_{13}+y_7^3y_8y_{11}y_{12}y_{13}+y_7^3y_8y_{11}^2y_{14}+y_7^4y_{12}^2y_{13}$$
$$+y_7^4y_{11}y_{12}y_{14}+y_7^4y_8^3y_{13}+y_7^5y_8^2y_{14}+y_6y_{10}y_{11}^2y_{13}y_{14}+y_6y_{10}^2y_{13}^3$$
$$+y_6y_7^2y_8y_{11}y_{13}^2+y_6y_7^3y_{11}y_{13}y_{14}+y_6^2y_8^2y_{11}y_{13}^2+y_6^2y_7^2y_{13}^3+y_6^2y_7^2y_{11}y_{14}^2$$
$$+y_6^3y_{11}^2y_{12}y_{13}+y_6^3y_{10}y_{11}y_{13}^2+y_6^3y_{10}^2y_{13}y_{14}+y_6^3y_8y_{13}^3+y_6^3y_7y_8^2y_{11}y_{13}$$
$$+y_6^3y_7^2y_8y_{12}y_{13}+y_6^4y_{13}y_{14}^2+y_6^4y_8^2y_{12}y_{13}+y_6^4y_8^2y_{11}y_{14}+y_6^4y_7y_8y_{13}^2$$
$$+y_6^5y_8y_{13}y_{14}+y_4y_{11}^2y_{13}^3+y_4y_{10}y_{11}y_{13}^2y_{14}+y_4y_7^2y_{11}^2y_{12}y_{13}+y_4y_7^3y_{13}^2y_{14}$$
$$+y_4y_7^3y_8^2y_{11}y_{13}+y_4y_7^4y_8y_{11}y_{14}+y_4y_7^5y_{12}y_{14}+y_4y_6y_7y_{11}^2y_{13}^2$$
$$+y_4y_6y_7^4y_{13}y_{14}+y_4y_6^2y_{11}^2y_{13}y_{14}+y_4y_6^2y_7^2y_8y_{13}y_{14}+y_4y_6^2y_7^3y_{14}^2$$
$$+y_4y_6^3y_7y_{11}y_{12}y_{13}+y_4y_6^4y_{11}y_{13}^2+y_4y_6^4y_{10}y_{13}y_{14}+y_4^2y_{10}y_{11}^2y_{12}y_{13}$$
$$+y_4^2y_{10}^2y_{11}y_{13}^2+y_4^2y_{10}^3y_{13}y_{14}+y_4^2y_8y_{11}y_{12}y_{13}^2+y_4^2y_8y_{11}^2y_{13}y_{14}$$
$$+y_4^2y_7y_{11}y_{13}^3+y_4^2y_7y_{11}y_{12}y_{13}y_{14}+y_4^2y_7y_{11}^2y_{14}^2+y_4^2y_7^3y_{11}y_{12}y_{13}$$
$$+y_4^2y_6y_{11}y_{13}^2y_{14}+y_4^2y_6y_7y_8y_{11}y_{12}y_{13}+y_4^2y_6y_7^2y_{12}^2y_{13}+y_4^2y_6y_7^2y_{11}y_{13}^2$$
$$+y_4^2y_6^2y_{10}y_{11}^2y_{13}+y_4^2y_6^2y_{10}^2y_{12}y_{13}+y_4^2y_6^2y_{10}^2y_{11}y_{14}+y_4^2y_6^2y_8y_{10}y_{13}y_{14}$$
$$+y_4^2y_6^3y_{12}y_{13}y_{14}+y_4^2y_6^3y_{13}^3+y_4^3y_6y_{11}^2y_{12}y_{13}+y_4^3y_6y_{10}y_{11}y_{13}^2$$
$$+y_4^3y_6y_{10}^2y_{13}y_{14}+y_4^3y_6^2y_{13}y_{14}^2+y_4^3y_7y_8y_{11}y_{13}y_{14}+y_4^3y_7^2y_{11}y_{14}^2$$
$$+y_4^4y_{10}y_{12}y_{13}y_{14}+y_4^4y_{11}y_{12}y_{13}^2+y_4^4y_{11}y_{12}^2y_{14}+y_4^4y_{12}^3y_{13}+y_4^4y_8y_{13}y_{14}^2$$
$$+y_4^4y_7y_{14}^3;$$

$H^*(B\,\mathrm{Spin}(15);\mathbb{Z}/2) =$

$$\mathbb{Z}/2[y_4,y_6,y_7,y_8,y_{10},y_{11},y_{12},y_{13},y_{14},y_{15},u_{128}]/(R_1,R_2,R_3),$$

$$R_1 = y_7y_{10}+y_6y_{11}+y_4y_{13},$$

$$
\begin{aligned}
R_2 = {} & y_{11}^3+y_{10}^2y_{13}+y_7y_{13}^2+y_7^2y_8y_{11}+y_7^3y_{12}+y_6y_7^2y_{13}+y_6^2y_8y_{13}+y_6^2y_7y_{14}\\
&+y_6^3y_{15}+y_4y_7y_{11}^2+y_4y_7^2y_{15}+y_4y_6y_{10}y_{13}+y_4^2y_{12}y_{13}+y_4^2y_{11}y_{14}\\
&+y_4^2y_{10}y_{15},\\
R_3 = {} & y_{13}^5+y_{10}^3y_{11}^2y_{13}+y_{10}^4y_{12}y_{13}+y_{10}^4y_{11}y_{14}+y_{10}^5y_{15}+y_8y_{10}^2y_{11}y_{13}^2\\
&+y_7y_8y_{11}y_{13}^3+y_7^2y_{11}y_{13}^2y_{14}+y_7^2y_8^2y_{11}^2y_{13}+y_7^3y_8y_{11}y_{12}y_{13}+y_7^3y_8y_{11}^2y_{14}\\
&+y_7^4y_{12}^2y_{13}+y_7^4y_{11}y_{12}y_{14}+y_7^4y_{11}^2y_{15}+y_7^4y_8^3y_{13}+y_7^5y_{15}^2+y_7^5y_8^2y_{14}\\
&+y_7^6y_8y_{15}+y_6y_{10}y_{11}^2y_{13}y_{14}+y_6y_{10}^2y_{13}^3+y_6y_{10}^2y_{11}y_{13}y_{15}\\
&+y_6y_7y_{11}y_{13}^2y_{15}+y_6y_7^2y_8y_{11}y_{13}^2+y_6y_7^2y_8y_{11}^2y_{15}+y_6y_7^3y_{11}y_{13}y_{14}\\
&+y_6y_7^3y_{11}y_{12}y_{15}+y_6y_7^4y_8^2y_{15}+y_6^2y_8^2y_{11}y_{13}^2+y_6^2y_7^2y_{13}^3+y_6^2y_7^2y_{11}y_{14}^2\\
&+y_6^2y_7^2y_{11}y_{13}y_{15}+y_6^3y_{11}^2y_{12}y_{13}+y_6^3y_{10}y_{11}y_{13}^2+y_6^3y_{10}y_{11}^2y_{15}\\
&+y_6^3y_{10}^2y_{13}y_{14}+y_6^3y_8y_{13}^3+y_6^3y_7y_8^2y_{11}y_{13}+y_6^3y_7^2y_8y_{12}y_{13}+y_6^4y_{13}y_{14}^2\\
&+y_6^4y_8^2y_{12}y_{13}+y_6^4y_8^2y_{11}y_{14}+y_6^4y_8^2y_{10}y_{15}+y_6^4y_7y_8y_{13}^2+y_6^4y_7y_8y_{11}y_{15}\\
&+y_6^5y_8y_{13}y_{14}+y_4y_{11}^2y_{13}^3+y_4y_{10}y_{11}y_{13}^2y_{14}+y_4y_7^2y_{11}^2y_{12}y_{13}\\
&+y_4y_7^2y_8y_{11}y_{13}y_{15}+y_4y_7^3y_{13}^2y_{14}+y_4y_7^3y_{11}y_{14}y_{15}+y_4y_7^3y_8^2y_{11}y_{13}\\
&+y_4y_7^4y_8y_{11}y_{14}+y_4y_7^5y_{12}y_{14}+y_4y_7^5y_{11}y_{15}+y_4y_6y_7y_{11}^2y_{13}^2\\
&+y_4y_6y_7^2y_{13}^2y_{15}+y_4y_6y_7^2y_{11}y_{15}^2+y_4y_6y_7^3y_8y_{11}y_{15}+y_4y_6y_7^4y_{13}y_{14}\\
&+y_4y_6y_7^4y_{12}y_{15}+y_4y_6^2y_{11}^2y_{13}y_{14}+y_4y_6^2y_{10}y_{11}y_{13}y_{15}+y_4y_6^2y_7^2y_8y_{13}y_{14}\\
&+y_4y_6^2y_7^3y_{14}^2+y_4y_6^2y_7^3y_{13}y_{15}+y_4y_6^3y_7y_{11}y_{12}y_{13}+y_4y_6^4y_{11}y_{13}^2\\
&+y_4y_6^4y_{11}^2y_{15}+y_4y_6^4y_{10}y_{13}y_{14}+y_4y_6^4y_7y_{15}^2+y_4^2y_{10}y_{11}^2y_{12}y_{13}\\
&+y_4^2y_{10}^2y_{11}y_{13}^2+y_4^2y_{10}^2y_{11}^2y_{15}+y_4^2y_{10}^3y_{13}y_{14}+y_4^2y_8y_{11}y_{12}y_{13}^2\\
&+y_4^2y_8y_{11}^2y_{13}y_{14}+y_4^2y_8y_{10}y_{11}y_{13}y_{15}+y_4^2y_7y_{11}y_{13}^3+y_4^2y_7y_{11}y_{12}y_{13}y_{14}\\
&+y_4^2y_7y_{11}^2y_{14}^2+y_4^2y_7y_{11}^2y_{13}y_{15}+y_4^2y_7^3y_{11}y_{12}y_{13}+y_4^2y_6y_{11}y_{13}^2y_{14}\\
&+y_4^2y_6y_{11}y_{12}y_{13}y_{15}+y_4^2y_6y_{10}y_{13}^2y_{15}+y_4^2y_6y_{10}y_{11}y_{15}^2\\
&+y_4^2y_6y_7y_8y_{11}y_{12}y_{13}+y_4^2y_6y_7^2y_{12}^2y_{13}+y_4^2y_6y_7^2y_{11}y_{13}^2+y_4^2y_6^2y_{10}y_{11}^2y_{13}\\
&+y_4^2y_6^2y_{10}^2y_{12}y_{13}+y_4^2y_6^2y_{10}^2y_{11}y_{14}+y_4^2y_6^2y_{10}^3y_{15}+y_4^2y_6^2y_8y_{11}^2y_{15}\\
&+y_4^2y_6^2y_8y_{10}y_{13}y_{14}+y_4^2y_6^2y_7y_{11}y_{12}y_{15}+y_4^2y_6^3y_{12}y_{13}y_{14}+y_4^2y_6^3y_{13}^3\\
&+y_4^3y_{11}y_{13}y_{14}y_{15}+y_4^3y_6y_{11}^2y_{12}y_{13}+y_4^3y_6y_{10}y_{11}y_{13}^2+y_4^3y_6y_{10}y_{11}^2y_{15}\\
&+y_4^3y_6y_{10}^2y_{13}y_{14}+y_4^3y_6y_8y_{11}y_{13}y_{15}+y_4^3y_6y_7y_{12}y_{13}y_{15}+y_4^3y_6^2y_{13}y_{14}^2
\end{aligned}
$$

$$
\begin{aligned}
&+y_4^3y_6^2y_{13}^2y_{15}+y_4^3y_7y_8y_{11}y_{13}y_{14}+y_4^3y_7^2y_{11}y_{14}^2+y_4^3y_7^2y_{11}y_{13}y_{15}\\
&+y_4^4y_{10}y_{12}y_{13}y_{14}+y_4^4y_{10}y_{12}^2y_{15}+y_4^4y_{11}y_{12}y_{13}^2+y_4^4y_{11}y_{12}^2y_{14}\\
&+y_4^4y_{11}^2y_{12}y_{15}+y_4^4y_{12}^3y_{13}+y_4^4y_8y_{13}y_{14}^2+y_4^4y_8y_{13}^2y_{15}+y_4^4y_8y_{11}y_{15}^2\\
&+y_4^4y_7y_{14}^3+y_4^4y_7y_{12}y_{15}^2+y_4^4y_6y_{14}^2y_{15}+y_4^4y_6y_{13}y_{15}^2+y_4^5y_{15}^3,
\end{aligned}
$$

where the first term of R_i is the leading term.

In order to calculate the Stiefel–Whitney classes of the spin representation concretely, we need some facts about the spin representations.

Notation 2.5 f_n: $\mathrm{Spin}(n) \longrightarrow \mathrm{Spin}(n+1)$ is the natural inclusion map.

As is well known (see for example Adams [2, Proposition 4.4]), we have that

$$
\begin{aligned}
f_{2k-1}^*(\Delta_{2k}^{\pm})_{\mathbb{C}} &= (\Delta_{2k-1})_{\mathbb{C}},\\
f_{2k}^*(\Delta_{2k+1})_{\mathbb{C}} &= (\Delta_{2k}^{+})_{\mathbb{C}} \oplus (\Delta_{2k}^{-})_{\mathbb{C}}.
\end{aligned}
$$

Then, using these and Remark 2.3, we have the following:

$$
\begin{aligned}
f_{8m}^*\Delta_{8m+1} &= \Delta_{8m}^{+} \oplus \Delta_{8m}^{-},\\
f_{8m+1}^*\Delta_{8m+2}^{\pm} &= (\Delta_{8m+1})_{\mathbb{C}},\\
f_{8m+2}^*(\Delta_{8m+3})_{\mathbb{C}} &= \Delta_{8m+2}^{+} \oplus \Delta_{8m+2}^{-},\\
f_{8m+3}^*\Delta_{8m+4}^{\pm} &= \Delta_{8m+3},\\
f_{8m+4}^*\Delta_{8m+5} &= \Delta_{8m+4}^{+} \oplus \Delta_{8m+4}^{-},\\
f_{8m+5}^*\Delta_{8m+6}^{\pm} &= (\Delta_{8m+5})_{\mathbb{C}},\\
f_{8m+6}^*\Delta_{8m+7} &= (\Delta_{8m+6}^{+})_{\mathbb{R}} = (\Delta_{8m+6}^{-})_{\mathbb{R}},\\
f_{8m+7}^*\Delta_{8m+8}^{\pm} &= \Delta_{8m+7}.
\end{aligned}
\tag{2–2}
$$

Notation 2.6 λ^i: $G(n) \to G(\binom{n}{i})$ is the i–th exterior representation, where $G = O, SO, U, SU$.

For the usual inclusion map

$$r_n\colon SU(n) \to SO(2n),$$

there is a covering map

$$\widetilde{r}_n\colon SU(n) \to \mathrm{Spin}(2n),$$

since $SU(n)$ is simply connected. According to Atiyah, Bott and Shapiro [3], there are isomorphisms of the representations

$$\widetilde{r}_n^*(\Delta_{2n}^+)_{\mathbb{C}} = \sum_{i=0}^{[n/2]} \lambda^{2i}, \tag{2–3}$$

$$\widetilde{r}_n^*(\Delta_{2n}^-)_{\mathbb{C}} = \sum_{i=0}^{[(n-1)/2]} \lambda^{2i+1}. \tag{2–4}$$

3 The Stiefel–Whitney classes of the second exterior representation

In this section, we calculate the Stiefel–Whitney classes of the representation

$$\lambda_{15}^2\colon \mathrm{Spin}(15) \longrightarrow SO(15) \longrightarrow O(15) \longrightarrow O(105)$$

induced from the second exterior representation $\lambda^2\colon O(15) \to O(105)$.

Let ν be the nontrivial real representation of one dimension

$$\nu\colon \mathbb{Z}/2 \xrightarrow{\cong} O(1),$$

and ν_i a composition map

$$\nu_i\colon H_n \cong (\mathbb{Z}/2)^n \xrightarrow{p_i} \mathbb{Z}/2 \xrightarrow{\nu} O(1),$$

where p_i is the i–th projection map. Then it is easy to show that the induced representation

$$i_n^*\lambda^2\colon H_n \xrightarrow{i_n} O(n) \xrightarrow{\lambda^2} O\left(\binom{n}{2}\right)$$

is isomorphic to $\sum_{1\le j<k\le n} \nu_j\nu_k$. Using Proposition 2.1, we obtain

$$Bi_n^* w(\lambda^2) = \prod_{1\le j<k\le n} (1+t_j+t_k),$$

since the total Stiefel–Whitney class of the representation $\nu_j\nu_k$ is given by $1+t_j+t_k$. Then we can write

$$\varphi_n^2(w_1, w_2, \dots, w_n) = \prod_{1\le j<k\le n} (1+t_j+t_k),$$

where w_i is the i–th elementary symmetric polynomial. Since it is quite difficult to calculate φ_{15}^2 directly by using GAP because of the capacity of the computer, we need to improve the algorithm to expand symmetric polynomials in terms of the elementary symmetric polynomials, that is, to calculate φ_n^2 by induction on n. Obviously $\varphi_2^2(w_1, w_2) = 1 + w_1$. Suppose that φ_{n-1}^2 is obtained. Let $\overline{w}_i$ be the i–th elementary symmetric polynomial of t_j $(1 \le j \le n-1)$. Then we have

$$w_i = \begin{cases} \overline{w}_1 + t_n, & \text{if } i = 1, \\ \overline{w}_i + \overline{w}_{i-1} t_n & \text{if } 1 < i < n, \\ \overline{w}_{n-1} t_n & \text{if } i = n. \end{cases}$$

We define a function ψ_n^2 by

$$\psi_n^2(\overline{w}_1, \dots, \overline{w}_{n-1}, t_n) = \prod_{1 \le j < k \le n-1} (1 + t_j + t_k) \prod_{i=1}^{n-1} (1 + t_i + t_n),$$

which is equal to $\varphi_n^2(w_1, \dots, w_n)$ as a polynomial of t_i. Then we obtain

$$\psi_n^2(\overline{w}_1, \dots, \overline{w}_{n-1}, t_n) = \varphi_{n-1}^2(\overline{w}_1, \overline{w}_2, \dots, \overline{w}_{n-1}) \sum_{k=0}^{n-1} \Big(t_n^k \sum_{l=0}^{n-1-k} \binom{n-1-l}{k} \overline{w}_l \Big).$$

We define $\varphi_{n,i}^2(w_1, \dots, w_{n-1})$ for $i \ge 0$ by the equation

$$\varphi_n^2(w_1, \dots, w_n) = \sum_m \varphi_{n,m}^2(w_1, \dots, w_{n-1}) w_n^m,$$

where it is easy to see that there holds the following identity

$$\varphi_{n,0}^2(\xi_1, \dots, \xi_{n-1}) = \psi_n^2(\xi_1, \dots, \xi_{n-1}, 0),$$

as polynomials for any invariant element ξ_i. Now we calculate $\varphi_{n,m}^2$ by induction on m. Assume that $\varphi_{n,l}^2$ is obtained for $1 \le l \le m-1$. Put

$$\psi_{n,l}^2(\overline{w}_1, \dots, \overline{w}_{n-1}, t_n) = \varphi_{n,l}^2(w_1, \dots, w_{n-1})$$

and $$\chi_{n,m}^2(\overline{w}_1, \dots, \overline{w}_{n-1}, t_n) = \{\psi_n^2(\overline{w}_1, \dots, \overline{w}_{n-1}, t_n) - \sum_{l=0}^{m-1} \psi_{n,l}^2(\overline{w}_1, \dots, \overline{w}_{n-1}, t_n) \overline{w}_{n-1}^l t_n^l\} / \overline{w}_{n-1}^m t_n^m.$$

Then $$\varphi_{n,m}^2(\xi_1, \dots, \xi_{n-1}) = \chi_{n,m}^2(\xi_1, \dots, \xi_{n-1}, 0),$$

which gives $\varphi_n^2(w_1, \dots, w_n)$ by the above equality. By induction on n, namely, using the improved algorithm, we obtain the following theorem.

Theorem 3.1 *The Stiefel–Whitney classes of degree 2^i for $0 \le i \le 6$ of the induced representation $\lambda^2_{15}\colon \mathrm{Spin}(15) \to O(105)$ are given as follows:*

$$
\begin{aligned}
w_1(\lambda^2_{15}) &= 0,\\
w_2(\lambda^2_{15}) &= 0,\\
w_4(\lambda^2_{15}) &= y_4,\\
w_8(\lambda^2_{15}) &= y_8+y_4^2,\\
w_{16}(\lambda^2_{15}) &= y_8^2+y_4^2y_8+y_4y_6^2,\\
w_{32}(\lambda^2_{15}) &= y_4y_{14}^2+y_6y_{13}^2+y_8y_{12}^2+y_{10}^2y_{12}+y_{10}y_{11}^2+y_4^2y_{11}y_{13}+y_6^2y_7y_{13}\\
&\quad +y_6^2y_{10}^2+y_7^3y_{11}+y_8^4+y_4^8,\\
w_{64}(\lambda^2_{15}) &= y_4y_{15}^4+y_8y_{14}^4+y_4^2y_{14}^4+y_4y_6y_{13}^3y_{15}\\
&\quad +y_4y_7y_{12}y_{13}^2y_{15}+y_4y_8y_{11}y_{13}^2y_{15}\\
&\quad +y_4y_8y_{13}^4+y_4y_{10}^2y_{12}y_{13}y_{15}+y_4y_{10}^2y_{13}^2y_{14}+y_4y_{10}y_{11}^2y_{13}y_{15}\\
&\quad +y_4y_{10}y_{11}y_{13}^3+y_4y_{11}^3y_{12}y_{15}+y_4y_{11}^2y_{12}y_{13}^2+y_6^2y_{13}^4+y_7y_{11}^4y_{13}\\
&\quad +y_8^2y_{12}^4+y_8y_{10}^3y_{11}y_{15}+y_8y_{10}^2y_{11}^2y_{14}+y_8y_{10}y_{11}^3y_{13}+y_8y_{11}^4y_{12}\\
&\quad +y_{10}^4y_{11}y_{13}+y_{10}^2y_{11}^4+y_4^3y_{10}y_{12}y_{15}^2+y_4^3y_{10}y_{13}y_{14}y_{15}+y_4^3y_{11}^2y_{15}^2\\
&\quad +y_4^3y_{11}y_{12}y_{14}y_{15}+y_4^3y_{12}^2y_{13}y_{15}\\
&\quad +y_4^3y_{12}y_{13}^2y_{14}+y_4^2y_6y_{10}y_{12}y_{13}y_{15}\\
&\quad +y_4^2y_6y_{10}y_{13}^2y_{14}+y_4^2y_6y_{11}^2y_{13}y_{15}+y_4^2y_6y_{11}y_{13}^3+y_4^2y_7^2y_{12}y_{15}^2\\
&\quad +y_4^2y_7y_8y_{13}^2y_{15}+y_4^2y_7y_{11}^2y_{12}y_{15}+y_4^2y_7y_{11}y_{12}y_{13}^2+y_4^2y_8^2y_{13}^2y_{14}\\
&\quad +y_4^2y_8y_{10}^2y_{14}^2+y_4^2y_8y_{10}y_{11}y_{12}y_{15}+y_4^2y_8y_{11}^2y_{12}y_{14}+y_4^2y_8y_{12}^4\\
&\quad +y_4^2y_{10}^3y_{11}y_{15}+y_4^2y_{10}^3y_{13}^2+y_4^2y_{10}^2y_{11}^2y_{14}\\
&\quad +y_4^2y_{10}y_{11}^3y_{13}+y_4y_6^3y_{12}y_{15}^2\\
&\quad +y_4y_6^3y_{13}y_{14}y_{15}+y_4y_6^2y_7y_{12}y_{14}y_{15}+y_4y_6^2y_8y_{10}y_{15}^2\\
&\quad +y_4y_6^2y_8y_{11}y_{14}y_{15}+y_4y_6^2y_8y_{12}y_{13}y_{15}+y_4y_6^2y_8y_{13}^2y_{14}\\
&\quad +y_4y_6^2y_{10}^2y_{13}y_{15}+y_4y_6^2y_{10}^2y_{14}^2+y_4y_6^2y_{11}^3y_{15}\\
&\quad +y_4y_6^2y_{12}^4+y_4y_6y_7^2y_{10}y_{15}^2\\
&\quad +y_4y_6y_7y_8y_{13}^3+y_4y_6y_8^2y_{10}y_{13}y_{15}+y_4y_6y_8y_{10}^2y_{11}y_{15}\\
&\quad +y_4y_6y_8y_{10}y_{11}^2y_{14}+y_4y_6y_8y_{11}^3y_{13}+y_4y_7^3y_{12}^2y_{15}+y_4y_7^2y_8y_{12}y_{13}^2
\end{aligned}
$$

$$
\begin{aligned}
&+y_4y_7^2y_{10}^2y_{13}^2+y_4y_7^2y_{11}^2y_{12}^2+y_4y_7y_8^2y_{11}^2y_{15}+y_4y_7y_8^2y_{11}y_{13}^2\\
&+y_4y_7y_8y_{11}^3y_{12}+y_4y_8^4y_{14}^2+y_4y_8^2y_{11}^4+y_4y_{10}^6+y_6^2y_7^2y_8y_{15}^2\\
&+y_6^2y_7^2y_{12}y_{13}^2+y_6^2y_8y_{11}^4+y_6^2y_{10}^4y_{12}\\
&+y_6^2y_{10}^3y_{11}^2+y_6y_7^2y_{11}^4+y_6y_8^4y_{13}^2\\
&+y_7^4y_8y_{14}^2+y_7^4y_{10}y_{13}^2+y_8^5y_{12}^2+y_8^4y_{10}^2y_{12}+y_8^4y_{10}y_{11}^2+y_8^3y_{10}^4\\
&+y_4^4y_7y_{11}y_{15}^2+y_4^4y_7y_{13}^2y_{15}+y_4^4y_7y_{13}y_{14}^2\\
&+y_4^4y_{11}y_{12}^2y_{13}+y_4^3y_6y_8^2y_{15}^2\\
&+y_4^3y_6y_{11}^3y_{13}+y_4^3y_7y_8^2y_{14}y_{15}+y_4^3y_7y_8y_{11}^2y_{15}+y_4^3y_7y_{10}^3y_{15}\\
&+y_4^3y_7y_{10}^2y_{11}y_{14}+y_4^3y_7y_{11}^3y_{12}+y_4^2y_6^2y_7^2y_{15}^2+y_4^2y_6^2y_8^2y_{14}^2\\
&+y_4^2y_6^2y_{10}^2y_{11}y_{13}+y_4^2y_6^2y_{10}^2y_{12}^2\\
&+y_4^2y_6y_7y_8^2y_{12}y_{15}+y_4^2y_6y_7y_8^2y_{13}y_{14}\\
&+y_4^2y_6y_7y_8y_{10}^2y_{15}+y_4^2y_6y_7y_8y_{11}^2y_{13}+y_4^2y_7^4y_{13}y_{15}+y_4^2y_7^4y_{14}^2\\
&+y_4^2y_7^3y_{11}^2y_{13}+y_4^2y_7^2y_8^2y_{11}y_{15}+y_4^2y_7^2y_8^2y_{12}y_{14}+y_4^2y_7^2y_8^2y_{13}^2\\
&+y_4^2y_7^2y_8y_{10}^2y_{14}+y_4^2y_7^2y_8y_{11}^2y_{12}+y_4^2y_8^4y_{11}y_{13}+y_4y_6^5y_{15}^2\\
&+y_4y_6^4y_7y_{14}y_{15}+y_4y_6^4y_{10}y_{13}^2+y_4y_6^2y_7y_8^2y_{10}y_{15}+y_4y_6^2y_7y_8^2y_{11}y_{14}\\
&+y_4y_6y_7^4y_{11}y_{15}+y_4y_6y_7^2y_8^2y_{11}y_{13}+y_4y_7^5y_{11}y_{14}+y_4y_7^3y_8^3y_{15}\\
&+y_4y_7^3y_8^2y_{11}y_{12}+y_6^6y_{14}^2+y_6^5y_7y_{12}y_{15}+y_6^5y_7y_{13}y_{14}+y_6^5y_{10}y_{11}y_{13}\\
&+y_6^4y_7^2y_{11}y_{15}+y_6^4y_7^2y_{12}y_{14}+y_6^4y_7^2y_{13}^2+y_6^4y_7y_{11}^3+y_6^4y_8^2y_{11}y_{13}\\
&+y_6^4y_8^2y_{12}^2+y_6^3y_7y_8^3y_{15}+y_6^2y_7^2y_8^3y_{14}+y_6^2y_7y_8^4y_{13}+y_6^2y_8^4y_{10}^2\\
&+y_6y_7^5y_8y_{15}+y_6y_7^3y_8^3y_{13}+y_7^7y_{15}+y_7^6y_8y_{14}+y_7^5y_8^2y_{13}+y_7^4y_8^3y_{12}\\
&+y_7^3y_8^4y_{11}+y_4^6y_{13}^2y_{14}+y_4^5y_7y_{11}^2y_{15}+y_4^5y_7y_{11}y_{13}^2+y_4^4y_6^2y_{11}^2y_{14}\\
&+y_4^4y_6y_7y_{10}^2y_{15}+y_4^4y_6y_7y_{11}^2y_{13}+y_4^4y_7^2y_8y_{13}^2+y_4^4y_7^2y_{11}^2y_{12}\\
&+y_4^3y_6^2y_7^2y_{13}^2+y_4^3y_7^4y_{11}y_{13}+y_4^2y_6^5y_{13}^2\\
&+y_4^2y_6^4y_7y_{10}y_{15}+y_4^2y_6^4y_7y_{11}y_{14}\\
&+y_4^2y_6^4y_7y_{12}y_{13}+y_4^2y_6^4y_8y_{11}y_{13}+y_4y_6^6y_{11}y_{13}+y_4y_6^4y_7^3y_{15}\\
&+y_4y_6^2y_7^5y_{13}+y_4y_7^7y_{11}+y_4y_7^4y_8^4+y_6^7y_7y_{15}+y_6^6y_7^2y_{14}+y_6^6y_8y_{10}^2\\
&+y_6^5y_7^3y_{13}+y_6^4y_7^4y_{12}+y_6^4y_8^5+y_7^8y_8+y_4^7y_6y_{15}^2+y_4^7y_7y_{14}y_{15}\\
&+y_4^6y_6^2y_{14}^2+y_4^6y_6y_7y_{12}y_{15}+y_4^6y_6y_7y_{13}y_{14}
\end{aligned}
$$

$$
\begin{aligned}
&+y_4^6y_7^2y_{11}y_{15}+y_4^6y_7^2y_{12}y_{14}\\
&+y_4^6y_{10}^4+y_4^5y_6^2y_7y_{10}y_{15}+y_4^5y_6^2y_7y_{11}y_{14}\\
&+y_4^5y_6y_7^2y_{11}y_{13}+y_4^5y_7^3y_8y_{15}\\
&+y_4^5y_7^3y_{11}y_{12}+y_4^4y_6^4y_{12}^2+y_4^4y_6^3y_7y_8y_{15}\\
&+y_4^4y_6^2y_7^2y_8y_{14}+y_4^4y_6y_7^3y_8y_{13}\\
&+y_4^4y_7^5y_{13}+y_4^4y_7^4y_8y_{12}+y_4^4y_8^6+y_4^2y_6^6y_7y_{13}+y_4^2y_6^6y_{10}^2+y_4^2y_6^4y_7^3y_{11}\\
&+y_4^2y_6^4y_8^4+y_4^2y_7^8+y_6^8y_8^2+y_4^9y_{14}^2+y_4^8y_7y_{10}y_{15}+y_4^8y_7y_{11}y_{14}\\
&+y_4^8y_7y_{12}y_{13}+y_4^8y_8y_{11}y_{13}+y_4^8y_8y_{12}^2+y_4^8y_{10}^2y_{12}+y_4^8y_{10}y_{11}^2\\
&+y_4^7y_7^3y_{15}+y_4^7y_7^2y_{11}^2+y_4^6y_6^3y_7y_{15}+y_4^6y_6^2y_7^2y_{14}+y_4^6y_6^2y_8y_{10}^2\\
&+y_4^6y_6y_7^3y_{13}+y_4^6y_7^4y_{12}+y_4^6y_8^5+y_4^5y_6^4y_7y_{13}+y_4^5y_6^4y_{10}^2+y_4^5y_6^2y_7^3y_{11}\\
&+y_4^5y_6^2y_8^4+y_4^2y_6^8y_8+y_4y_6^{10}.
\end{aligned}
$$

Remark 3.2 The total Stiefel–Whitney classes of the induced representation

$$\lambda_n^2\colon \mathrm{Spin}(n) \to O\big(\tbinom{n}{2}\big)$$

for $3 \le n \le 9$ are given as follows:

$$
\begin{aligned}
&w(\lambda_3^2) = 1,\\
&w(\lambda_4^2) = 1,\\
&w(\lambda_5^2) = 1 + y_4,\\
&w(\lambda_6^2) = (1 + y_4 + y_6)^2,\\
&w(\lambda_7^2) = (1 + y_4 + y_6 + y_7)^3,\\
&w(\lambda_8^2) = (1 + y_4 + y_6 + y_7)^4,\\
&w(\lambda_9^2) = (1 + y_4 + y_6 + y_7 + y_8)(1 + y_4 + y_6 + y_7)^4.
\end{aligned}
$$

4 The Stiefel–Whitney classes of the spin representation

In this section, by making use of Theorem 2.4, we calculate the Stiefel–Whitney classes of the spin representations $(\Delta_n)_{\mathbb{R}}$ and $(\Delta_n^{\pm})_{\mathbb{R}}$ for $n \le 15$. Firstly, one can obtain them for $n \le 7$ as follows:

$$
\begin{aligned}
&w((\Delta_3)_{\mathbb{R}}) = p(\Delta_3) = 1 + u_4,\\
&w((\Delta_4^+)_{\mathbb{R}}) = p(\Delta_4^+) = 1 + u_4,
\end{aligned}
$$

$$\begin{aligned}
w((\Delta_4^-)_{\mathbb{R}}) &= p(\Delta_4^-) = 1 + y_4 + u_4, \\
w((\Delta_5)_{\mathbb{R}}) &= p(\Delta_5) = 1 + y_4 + u_8, \\
w((\Delta_6)_{\mathbb{R}}) &= c(\Delta_6) = 1 + y_4 + y_6 + u_8, \\
w(\Delta_7) &= 1 + y_4 + y_6 + y_7 + u_8.
\end{aligned}$$

Secondly, it also follows from Theorem 2.4 that the total Stiefel–Whitney class of the spin representation Δ_8^+ is given by

$$w(\Delta_8^+) = 1 + y_4 + y_6 + y_7 + u_8.$$

According to Adams [2], the outer automorphism group Out(Spin(8)) of Spin(8) is isomorphic to the symmetric group Σ_3 of degree 3 which acts on the set of the representations λ^1, Δ_8^+ and Δ_8^-, where λ^1 is the natural projection $\mathrm{Spin}(8) \to SO(8)$. Then there is an automorphism $\sigma\colon \mathrm{Spin}(8) \to \mathrm{Spin}(8)$ such that $\sigma^*(\lambda^1) = \Delta_8^-$, $\sigma^*(\Delta_8^+) = \lambda^1$ and $\sigma^*(\Delta_8^-) = \Delta_8^+$. Since $Bf_7^* w(\Delta_8^-) = w(\Delta_7)$, we can write as follows:

$$w(\Delta_8^-) = 1 + y_4 + y_6 + y_7 + u_8 + a_1 y_8,$$

where $a_1 \in \mathbb{Z}/2$. Then we have

$$\begin{aligned}
B\sigma^* y_8 &= B\sigma^* w_8(\lambda_8^1) = w_8(\Delta_8^-) = u_8 + a_1 y_8, \\
B\sigma^* u_8 &= B\sigma^* w_8(\Delta_8^+) = w_8(\lambda_8^1) = y_8,
\end{aligned}$$

and $$u_8 = w_8(\Delta_8^+) = B\sigma^* w_8(\Delta_8^-) = y_8 + a_1(u_8 + a_1 y_8).$$

Thus we obtain $a_1 = 1$, and the total Stiefel–Whitney class is given by

$$w(\Delta_8^-) = 1 + y_4 + y_6 + y_7 + (u_8 + y_8).$$

Since the induced representation $f_8^*\Delta_9$ is isomorphic to $\Delta_8^+ \oplus \Delta_8^-$ by (2–2), the total Stiefel–Whitney class of $f_8^*\Delta_9$ is given by

$$\begin{aligned}
w(f_8^*\Delta_9) &= w(\Delta_8^+) w(\Delta_8^-) \\
&= 1 + (y_8 + y_4^2) + (y_4 y_8 + y_6^2) + (y_6 y_8 + y_7^2) + y_7 y_8 + (u_8^2 + y_8 u_8).
\end{aligned}$$

Since $Bf_8^*\colon H^*(B\,\mathrm{Spin}(9);\mathbb{Z}/2) \to H^*(B\,\mathrm{Spin}(8);\mathbb{Z}/2)$ is a monomorphism, the total Stiefel–Whitney class of the spin representation Δ_9 is given by

$$w(\Delta_9) = 1 + (y_8 + y_4^2) + (y_4 y_8 + y_6^2) + (y_6 y_8 + y_7^2) + y_7 y_8 + u_{16}. \tag{4–1}$$

Recall from (2–3) that we have $\widetilde{r}_5^*\Delta_{10}^+ = 1 \oplus \lambda^2 \oplus \lambda^4$, where λ^i is the i–th exterior representation $SU(5) \to SU(\binom{5}{i})$. In a similar way to the Stiefel–Whitney classes of i–the exterior representations, we can calculate the mod 2 Chern classes of the i–th exterior representations λ^i using the Borel–Hirzebruch method; the mod 2 total Chern classes are given by

$$\begin{aligned} c(\lambda^2) &= 1 + c_2 + c_3 + (c_4 + c_2^2) + c_5 + (c_3^2 + c_2^3) + c_2^2 c_3 + (c_3 c_5 + c_2^2 c_4 + c_2 c_3^2) \\ &\qquad + (c_3^3 + c_2^2 c_5) + (c_5^2 + c_2 c_3 c_5 + c_3^2 c_4), \\ c(\lambda^4) &= 1 + c_2 + c_3 + c_4 + c_5, \end{aligned}$$

where $c_i \in H^*(BSU(5);\mathbb{Z}/2)$ is the mod 2 i–th Chern class. By (2–1), we obtain

$$w_{16}(\widetilde{r}_5^*(\Delta_{10}^+)_{\mathbb{R}}) = c_8(\widetilde{r}_5^*\Delta_{10}^+) = c_3 c_5 + c_4^2 + c_2^4.$$

Since $B\widetilde{r}_5^*\colon H^{16}(B\,\mathrm{Spin}(10);\mathbb{Z}/2) \to H^{16}(BSU(5);\mathbb{Z}/2)$ is an isomorphism, the above equality gives

$$w_{16}((\Delta_{10}^+)_{\mathbb{R}}) = y_6 y_{10} + y_8^2 + y_4^4.$$

Using the Wu formula, we obtain the total Stiefel–Whitney class

$$\begin{aligned} w((\Delta_{10}^+)_{\mathbb{R}}) &= c(\Delta_{10}^+) \\ &= 1 + (y_6 y_{10} + y_8^2 + y_4^4) \\ &\quad + (y_4 y_{10}^2 + y_6 y_8 y_{10} + y_4^2 y_6 y_{10} + y_4^2 y_8^2 + y_6^4) \\ &\quad + (y_8 y_{10}^2 + y_4^2 y_{10}^2 + y_4 y_6 y_8 y_{10} + y_6^3 y_{10} + y_6^2 y_8^2 + y_7^4) \\ &\quad + (y_{10}^3 + y_4 y_6 y_{10}^2 + y_6^2 y_8 y_{10} + y_7^2 y_8^2) + u_{32}. \end{aligned} \tag{4–2}$$

Note that $c(\Delta_{10}^+) = c(\Delta_{10}^-)$, since Δ_{10}^+ and Δ_{10}^- are conjugate to each other. Using it, we can obtain the total Chern class $c(\rho_6)$ for the representation $\rho_6\colon E_6 \to SU(27)$ mentioned in the introduction, since we have by Corollary 8.3 of [2] that the induced representation

$$\mathrm{Spin}(10) \longrightarrow E_6 \longrightarrow SU(27)$$

is a direct sum of the one dimensional trivial representation, Δ_{10}^+ and the composition map

$$\mathrm{Spin}(10) \longrightarrow SO(10) \longrightarrow SU(10).$$

Recall from (2–2) that $f_{10}^*((\Delta_{11})_{\mathbb{C}}) = \Delta_{10}^+ \oplus \Delta_{10}^-$, which gives $w(f_{10}^*(\Delta_{11})_{\mathbb{R}}) = w((\Delta_{10}^+)_{\mathbb{R}}) w((\Delta_{10}^-)_{\mathbb{R}})$. We consider the homomorphism

$$Bf_{10}^*\colon H^*(B\,\mathrm{Spin}(11);\mathbb{Z}/2) \longrightarrow H^*(B\,\mathrm{Spin}(10);\mathbb{Z}/2),$$

where we see that a basis of $\operatorname{Ker} Bf_{10}^*$ of degree 32 is given by

$$\{y_{10} y_{11}^2,\ y_4 y_6 y_{11}^2,\ y_6 y_7 y_8 y_{11},\ y_7^3 y_{11},\ y_4^2 y_6 y_7 y_{11}\}.$$

So we can write as follows:

$$\begin{aligned}w_{32}((\Delta_{11})_{\mathbb{R}}) = a_1 y_{10} y_{11}^2 + a_2 y_4 y_6 y_{11}^2 + a_3 y_6 y_7 y_8 y_{11} + a_4 y_7^3 y_{11} \\ + a_5 y_4^2 y_6 y_7 y_{11} + y_6^2 y_{10}^2 + y_8^4 + y_4^8,\end{aligned}$$

where $a_i \in \mathbb{Z}/2$. By Theorem 2.4, we have

$$0 = \mathrm{Sq}^1\, w_{32}((\Delta_{11})_{\mathbb{R}}) = a_1 y_{11}^3 + a_2 y_4 y_7 y_{11}^2 + a_3 y_7^2 y_8 y_{11} + a_5 y_4^2 y_7^2 y_{11},$$

which implies $a_1 = a_2 = a_3$ and $a_5 = 0$. Again by Theorem 2.4, we have

$$0 = \mathrm{Sq}^4\, w_{32}((\Delta_{11})_{\mathbb{R}}) = a_4 y_7^2 y_{11}^2 + y_7^2 y_{11}^2,$$

which implies $a_4 = 1$. Further by Theorem 2.4, we have

$$0 = \mathrm{Sq}^{30}\, w_{32}((\Delta_{11})_{\mathbb{R}}) = (a_1 + 1)(y_4^2 y_6^2 y_{10}^2 y_{11}^2 + y_6^4 y_8^2 y_{11}^2 + y_{10}^4 y_{11}^2),$$

which implies $a_1 = 1$. Thus we obtain the total Stiefel–Whitney class

$$\begin{aligned}
w((\Delta_{11})_{\mathbb{R}}) &= c((\Delta_{11})_{\mathbb{C}}) = p(\Delta_{11}) \\
&= 1 + (y_{10} y_{11}^2 + y_4 y_6 y_{11}^2 + y_6 y_7 y_8 y_{11} + y_7^3 y_{11} + y_6^2 y_{10}^2 + y_8^4 + y_4^8) \\
&\quad + (y_8^2 y_{10} y_{11}^2 + y_4 y_6 y_8^2 y_{11}^2 + y_4^4 y_{10} y_{11}^2 + y_4^5 y_6 y_{11}^2 + y_4 y_6^2 y_{10} y_{11}^2 \\
&\qquad + y_6^3 y_8 y_{11}^2 + y_6 y_7 y_8^3 y_{11} + y_6 y_{10}^2 y_{11}^2 + y_4^4 y_6 y_7 y_8 y_{11} \\
&\qquad + y_4^2 y_7^3 y_8 y_{11} + y_4^3 y_7^2 y_{11}^2 + y_4^4 y_7^3 y_{11} + y_7^3 y_8^2 y_{11} + y_6^2 y_8^2 y_{10}^2 \\
&\qquad + y_4^2 y_{10}^4 + y_4^4 y_6^2 y_{10}^2 + y_4^4 y_8^4 + y_6^8) \\
&\quad + (y_6^3 y_8^2 y_{11}^2 + y_4^2 y_8^2 y_{10} y_{11}^2 + y_4^3 y_6 y_8^2 y_{11}^2 + y_4 y_6^5 y_{11}^2 + y_6^4 y_{10} y_{11}^2 \\
&\qquad + y_4 y_{10}^3 y_{11}^2 + y_4^3 y_6^2 y_{10} y_{11}^2 + y_4^2 y_6^3 y_8 y_{11}^2 + y_6 y_8 y_{10}^2 y_{11}^2 \\
&\qquad + y_4^2 y_6 y_7 y_8^3 y_{11} + y_6^5 y_7 y_8 y_{11} + y_6^4 y_7^3 y_{11} + y_4^2 y_6^2 y_8^2 y_{10}^2 + y_6^6 y_{10}^2 \\
&\qquad + y_8^2 y_{10}^4 + y_4^4 y_{10}^4 + y_6^4 y_8^4 + y_7^8) \\
&\quad + (y_6^3 y_{10}^2 y_{11}^2 + y_4^2 y_{10}^3 y_{11}^2 + y_4^3 y_6 y_{10}^2 y_{11}^2 + y_4 y_6^4 y_{10} y_{11}^2 + y_6^5 y_8 y_{11}^2 \\
&\qquad + y_8 y_{10}^3 y_{11}^2 + y_6^3 y_7 y_8^3 y_{11} + y_6^4 y_7^2 y_{11}^2 + y_7^7 y_{11} + y_4^2 y_7^2 y_8^2 y_{11}^2 \\
&\qquad + y_4^2 y_6^2 y_{10}^4 + y_6^4 y_8^2 y_{10}^2 + y_8^2 y_{11}^4 + y_{10}^6 + y_4^4 y_{11}^4 + y_7^4 y_8^4) \\
&\quad + u_{64}.
\end{aligned}$$

Recall from (2–3) that $\widetilde{r}_6^*(\Delta_{12}^+)_{\mathbb{C}} = 2 \oplus \lambda^2 \oplus \lambda^4$, where λ^i is the i–th exterior representation $SU(6) \to SU(\binom{6}{i})$. In a similar way to the case of the i–th exterior representation $SU(5) \to SU(\binom{5}{i})$, we obtain the mod 2 total Chern classes

$$\begin{aligned} c(\lambda^2) = c(\lambda^4) = 1 &+ (c_3c_5 + c_4^2 + c_2^4) \\ &+ (c_2c_5^2 + c_3^2c_6 + c_3c_4c_5 + c_2^2c_3c_5 + c_2^2c_4^2 + c_3^4 + c_6^2) \\ &+ (c_3c_5c_6 + c_4c_5^2 + c_2^2c_5^2 + c_2c_3^2c_6 + c_2c_3c_4c_5 + c_3^3c_5 + c_3^2c_4^2) \\ &+ (c_5^3 + c_2c_3c_5^2 + c_3^3c_6 + c_3^2c_4c_5), \end{aligned}$$

where $c_i \in H^*(BSU(6); \mathbb{Z}/2)$ is the mod 2 i–th Chern class. Then by (2–1),

$$w_{32}(\widetilde{r}_6^*(\Delta_{12}^+)_{\mathbb{R}}) = c_{16}(\widetilde{r}_6(\Delta_{12}^+)_{\mathbb{C}}) = c_3^2c_5^2 + c_4^4 + c_2^8.$$

Recall from (2–2) that $f_{11}^*\Delta_{12}^+ = \Delta_{11}$, which gives $f_{11}^*w((\Delta_{12}^+)_{\mathbb{R}}) = w((\Delta_{11})_{\mathbb{R}})$. We consider the homomorphism

$$Bf_{11}^* \oplus B\widetilde{r}_6^*\colon H^*(B\,\mathrm{Spin}(12); \mathbb{Z}/2) \to H^*(B\,\mathrm{Spin}(11); \mathbb{Z}/2) \oplus H^*(BSU(6); \mathbb{Z}/2),$$

where we see that a basis of $\mathrm{Ker}\; Bf_{11}^* \oplus B\widetilde{r}_6^*$ of degree 32 is given by

$$\{y_6y_7^2y_{12}\}.$$

So we can write as follows:

$$\begin{aligned} w_{32}((\Delta_{12}^+)_{\mathbb{R}}) = a_1y_6y_7^2y_{12} + y_{10}y_{11}^2 + y_4y_6y_{11}^2 + y_6y_7y_8y_{11} \\ + y_7^3y_{11} + y_6^2y_{10}^2 + y_8^4 + y_4^8, \end{aligned}$$

where $a_1 \in \mathbb{Z}/2$. By Theorem 2.4 we have

$$0 = \mathrm{Sq}^1\, w_{32}((\Delta_{12}^+)_{\mathbb{R}}) = a_1y_7^3y_{12} + y_{11}^3 + y_4y_7y_{11}^2 + y_7^2y_8y_{11},$$

which implies $a_1 = 1$. Thus we have

$$w_{32}((\Delta_{12}^+)_{\mathbb{R}}) = y_6y_7^2y_{12} + y_{10}y_{11}^2 + y_4y_6y_{11}^2 + y_6y_7y_8y_{11} + y_7^3y_{11} + y_6^2y_{10}^2 + y_8^4 + y_4^8,$$

and the total Stiefel–Whitney class is given by

$$\begin{aligned} w((\Delta_{12}^+)_{\mathbb{R}}) &= c((\Delta_{12}^+)_{\mathbb{C}}) = p(\Delta_{12}^+) \\ &= 1 + w_{32}((\Delta_{12}^+)_{\mathbb{R}}) + \mathrm{Sq}^{16}\, w_{32}((\Delta_{12}^+)_{\mathbb{R}}) + \mathrm{Sq}^{24}\, w_{32}((\Delta_{12}^+)_{\mathbb{R}}) \\ &\quad + \mathrm{Sq}^{28}\, w_{32}((\Delta_{12}^+)_{\mathbb{R}}) + u_{64}. \end{aligned} \tag{4–3}$$

In a similar way, we can obtain $w_{32}((\Delta_{12}^-)_{\mathbb{R}})$ which is equal to $w_{32}((\Delta_{12}^+)_{\mathbb{R}})$. In order to determine the total Stiefel–Whitney class of $(\Delta_{12}^-)_{\mathbb{R}}$, we need to calculate $w_{64}((\Delta_{12}^-)_{\mathbb{R}})$. From (2–4) we have $\widetilde{r}_6^*((\Delta_{12}^-)_{\mathbb{C}}) = \lambda^1 \oplus \lambda^3 \oplus \lambda^5$, for λ^i the i–th exterior representation $SU(6) \to SU(\binom{6}{i})$. Similarly to the case of the i–th exterior representation $SU(5) \to SU(\binom{5}{i})$, we obtain the mod 2 total Chern class

$$\begin{aligned} c(\lambda^3) &= 1 + c_2^2 + c_3^2 + (c_4^2 + c_2^4) + c_5^2 + (c_6^2 + c_3^4 + c_2^6) + c_2^4 c_3^2 \\ &\quad + (c_3^2 c_5^2 + c_2^4 c_4^2 + c_2^2 c_3^4) + (c_2^4 c_5^2 + c_3^6) + (c_5^4 + c_2^4 c_6^2 + c_2^2 c_3^2 c_5^2 + c_3^4 c_4^2), \\ c(\lambda^1) &= c(\lambda^5) = 1 + c_2 + c_3 + c_4 + c_5 + c_6. \end{aligned}$$

Then by (2–1) we obtain

$$w_{64}(\tilde{r}_6^*(\Delta_{12}^-)_{\mathbb{R}}) = c_{32}(\tilde{r}_6^*(\Delta_{12}^-)_{\mathbb{C}}) = c_6^2(c_5^4 + c_2^4 c_6^2 + c_2^2 c_3^2 c_5^2 + c_3^4 c_4^2).$$

Recall from (2–2) that $f_{11}^* \Delta_{12}^- = \Delta_{11}$, which gives $w(f_{11}^*(\Delta_{12}^-)_{\mathbb{R}}) = w((\Delta_{11})_{\mathbb{R}})$, and hence we obtain

$$w_{64}((\Delta_{12}^-)_{\mathbb{R}}) \equiv u_{64} + y_{10}^4 y_{12}^2 + y_4^4 y_{12}^4 + y_4^2 y_6^2 y_{10}^2 y_{12}^2 + y_6^4 y_8^2 y_{12}^2 \quad \text{mod Ker } Bf_{11}^* \oplus B\tilde{r}_6^*,$$

where
$$\begin{aligned} Bf_{11}^*&\colon H^*(B\,\mathrm{Spin}(12);\mathbb{Z}/2) \longrightarrow H^*(B\,\mathrm{Spin}(11);\mathbb{Z}/2), \\ B\tilde{r}_6^*&\colon H^*(B\,\mathrm{Spin}(12);\mathbb{Z}/2) \longrightarrow H^*(BSU(6);\mathbb{Z}/2). \end{aligned}$$

In order to determine $w_{64}((\Delta_{12}^-)_{\mathbb{R}})$ exactly, we use the method of indeterminate coefficients; using the equations $\mathrm{Sq}^n\, w_{64}((\Delta_{12}^-)_{\mathbb{R}} = 0$ for $n = 1, 2, 4, 62$, we can determine all the indeterminate coefficients, and hence the 64–th Stiefel–Whitney class is given as follows:

$$\begin{aligned} w_{64}((\Delta_{12}^-)_{\mathbb{R}}) &= u_{64} + y_{10}^4 y_{12}^2 + y_{10}^3 y_{11}^2 y_{12} + y_7^4 y_{12}^3 + y_4^2 y_{10} y_{11}^2 y_{12}^2 \\ &\quad + y_4 y_7^2 y_{11}^2 y_{12}^2 + y_6^3 y_{11}^2 y_{12}^2 + y_7^3 y_8 y_{11} y_{12}^2 + y_7^2 y_8^2 y_{11}^2 y_{12} \\ &\quad + y_4^4 y_{12}^4 + y_4^2 y_6 y_7^2 y_{12}^3 + y_4^3 y_6 y_{11}^2 y_{12}^2 + y_4^2 y_6 y_7 y_8 y_{11} y_{12}^2 \\ &\quad + y_4^2 y_7^3 y_{11} y_{12}^2 + y_4 y_6^3 y_7 y_{11} y_{12}^2 + y_4^2 y_6^2 y_{10}^2 y_{12}^2 + y_6^4 y_8^2 y_{12}^2 \\ &\quad + y_6^3 y_7^2 y_8 y_{12}^2 + y_4^2 y_6^2 y_{10} y_{11}^2 y_{12} + y_4 y_7^3 y_8^2 y_{11} y_{12} \\ &\quad + y_6^3 y_7 y_8^2 y_{11} y_{12} + y_7^4 y_8^3 y_{12}. \end{aligned}$$

Thus we obtain the total Stiefel–Whitney class

$$\begin{aligned} w((\Delta_{12}^-)_{\mathbb{R}}) &= c((\Delta_{12}^-)_{\mathbb{C}}) = p(\Delta_{12}^-) \\ &= 1 + w_{32}((\Delta_{12}^+)_{\mathbb{R}}) + \mathrm{Sq}^{16}\, w_{32}((\Delta_{12}^+)_{\mathbb{R}}) + \mathrm{Sq}^{24}\, w_{32}((\Delta_{12}^+)_{\mathbb{R}}) \\ &\quad + \mathrm{Sq}^{28}\, w_{32}((\Delta_{12}^+)_{\mathbb{R}}) + w_{64}((\Delta_{12}^-)_{\mathbb{R}}). \end{aligned} \tag{4–4}$$

Using it, we can obtain the total Pontrjagin class $p(\rho_7)$ for $\rho_7\colon E_7 \to Sp(28)$, the representation mentioned in the introduction, since we have by Corollary 8.2 of [2] that the induced representation

$$\mathrm{Spin}(12) \longrightarrow E_7 \longrightarrow Sp(28)$$

is a direct sum of

$$\Delta_{12}^-\colon \mathrm{Spin}(12) \longrightarrow Sp(16)$$

and the composition map

$$\mathrm{Spin}(12) \longrightarrow SO(12) \longrightarrow SU(12) \longrightarrow Sp(12).$$

Recall from (2–2) that $f_{12}^*\Delta_{13} = \Delta_{12}^+ \oplus \Delta_{12}^-$. Then we have

$$w(f_{12}^*(\Delta_{13})_{\mathbb{R}} = w((\Delta_{12}^+)_{\mathbb{R}})w((\Delta_{12}^-)_{\mathbb{R}}).$$

In order to determine $w_{64}((\Delta_{13})_{\mathbb{R}})$, we use the method of indeterminate coefficients; using the equations $\mathrm{Sq}^n\, w_{64}((\Delta_{13})_{\mathbb{R}}) = 0$ for $n = 1, 2, 4, 8, 62$, we can determine all the indeterminate coefficients, and hence the 64–th Stiefel–Whitney class is given as follows:

$$\begin{aligned}
w_{64}((\Delta_{13})_{\mathbb{R}}) = {} & y_{12}y_{13}^4 + y_6^2y_{13}^4 + y_4y_{10}y_{11}y_{13}^3 + y_7^2y_{11}y_{13}^3 + y_4y_{11}^2y_{12}y_{13}^2 \\
& + y_7y_8y_{11}y_{12}y_{13}^2 + y_6y_{10}^2y_{12}y_{13}^2 + y_6y_{10}y_{11}^2y_{13}^2 + y_8y_{10}^2y_{11}y_{12}y_{13} \\
& + y_{10}^4y_{11}y_{13} + y_4^3y_{13}^4 + y_4^2y_6y_{11}y_{13}^3 + y_4y_7^3y_{13}^3 + y_6^3y_8y_{12}y_{13}^2 \\
& + y_6^2y_7^2y_{12}y_{13}^2 + y_6^2y_7y_8y_{11}y_{13}^2 + y_6y_7^3y_{11}y_{13}^2 + y_4^2y_8y_{11}y_{12}^2y_{13} \\
& + y_4y_6y_7y_{11}^2y_{12}y_{13} + y_4^2y_{10}^2y_{11}y_{12}y_{13} + y_6^3y_{10}y_{11}y_{12}y_{13} \\
& + y_6^2y_8^2y_{11}y_{12}y_{13} + y_6y_7^2y_8y_{11}y_{12}y_{13} + y_7^4y_{11}y_{12}y_{13} + y_7^3y_8y_{11}^2y_{13} \\
& + y_4^3y_6y_7y_{13}^3 + y_4^3y_7^2y_{12}y_{13}^2 + y_4^2y_6^3y_{12}y_{13}^2 + y_4^2y_6^2y_7y_{11}y_{13}^2 \\
& + y_4^2y_7^2y_8^2y_{13}^2 + y_4y_6^2y_7^2y_8y_{13}^2 + y_4y_6y_7^4y_{13}^2 + y_4^3y_6y_{10}y_{11}y_{12}y_{13} \\
& + y_4^2y_6y_7^2y_{11}y_{12}y_{13} + y_4y_6^4y_{11}y_{12}y_{13} + y_6^4y_7y_8y_{12}y_{13} + y_4y_7^5y_{12}y_{13} \\
& + y_4^2y_6^2y_{10}^2y_{11}y_{13} + y_6^4y_8^2y_{11}y_{13} + y_4y_7^4y_8y_{11}y_{13} + y_7^5y_8^2y_{13} + y_{10}^4y_{12}^2 \\
& + y_{10}^3y_{11}^2y_{12} + y_7^4y_{12}^3 + y_4^2y_{10}y_{11}^2y_{12}^2 + y_4y_7^2y_{11}^2y_{12}^2 + y_6^3y_{11}^2y_{12}^2 \\
& + y_7^3y_8y_{11}y_{12}^2 + y_7^2y_8^2y_{11}^2y_{12} + y_4^4y_{12}^4 + y_4^2y_6y_7^2y_{12}^3 + y_4^3y_6y_{11}^2y_{12}^2 \\
& + y_4^2y_6y_7y_8y_{11}y_{12}^2 + y_4^2y_7^3y_{11}y_{12}^2 + y_4y_6^3y_7y_{11}y_{12}^2 + y_4^2y_6^2y_{10}^2y_{12}^2 \\
& + y_6^4y_8^2y_{12}^2 + y_6^3y_7^2y_8y_{12}^2 + y_4^2y_6^2y_{10}y_{11}^2y_{12} + y_4y_7^3y_8^2y_{11}y_{12} \\
& + y_6^3y_7y_8^2y_{11}y_{12} + y_7^4y_8^3y_{12} + y_6^2y_7^4y_{12}^2 + y_{10}^2y_{11}^4 + y_4^2y_6^2y_{11}^4 \\
& + y_6^2y_7^2y_8^2y_{11}^2 + y_7^6y_{11}^2 + y_6^4y_{10}^4 + y_8^8 + y_4^{16}.
\end{aligned}$$

Thus we obtain the total Stiefel–Whitney class

$$\begin{aligned} w((\Delta_{13})_{\mathbb{R}}) = c((\Delta_{13})_{\mathbb{C}}) &= p(\Delta_{13}) \\ &= 1 + w_{64}((\Delta_{13})_{\mathbb{R}}) + \mathrm{Sq}^{32}\, w_{64}((\Delta_{13})_{\mathbb{R}}) + \mathrm{Sq}^{48}\, w_{64}((\Delta_{13})_{\mathbb{R}}) \\ &\quad + \mathrm{Sq}^{56}\, w_{64}((\Delta_{13})_{\mathbb{R}}) + \mathrm{Sq}^{60}\, w_{64}((\Delta_{13})_{\mathbb{R}}) + u_{128}. \end{aligned}$$

Recall from (2–3) that if λ^i is the i–th exterior representation $SU(7) \to SU(\binom{7}{i})$ then $\widetilde{r}_7^*((\Delta_{14}^+)_{\mathbb{C}}) = 1 \oplus \lambda^2 \oplus \lambda^4 \oplus \lambda^6$. In a similar way to the case of the i–th exterior representation $SU(5) \to SU(\binom{5}{i})$, we obtain the mod 2 total Chern classes

$$\begin{aligned} c(\lambda^2) = {} & 1+c_2+c_3+c_4+c_5+c_6+c_7+(c_3c_5+c_4^2+c_2^4)+(c_2c_3c_5+c_2c_4^2+c_2^5) \\ & +(c_3^2c_5+c_3c_4^2+c_2^4c_3) \\ & +(c_6^2+c_2c_5^2+c_3^2c_6+c_4^3+c_2^2c_3c_5+c_2^2c_4^2+c_3^4+c_2^4c_4+c_2c_3c_7) \\ & +(c_3c_5^2+c_4^2c_5+c_2^4c_5) \\ & +(c_2c_6^2+c_4^2c_6+c_4c_5^2+c_3^3c_5+c_3^2c_4^2+c_2^4c_6+c_2^3c_3c_5+c_2^3c_4^2+c_2c_3^4 \\ & \quad +c_2c_5c_7+c_7^2) \\ & +(c_3c_6^2+c_5^3+c_2^2c_3^2c_5+c_2^2c_3c_4^2+c_3^5+c_3c_5c_7+c_4^2c_7+c_2^4c_7) \\ & +(c_4c_6^2+c_2c_3c_5c_6+c_3^2c_4c_6+c_3c_4^2c_5+c_2^3c_5^2+c_2^2c_3^2c_6+c_2^2c_4^3+c_2c_3^3c_5 \\ & \quad +c_2c_3^2c_4^2+c_3^4c_4+c_2c_3c_4c_7+c_3^3c_7+c_2^3c_3c_7+c_2c_7^2) \\ & +(c_5c_6^2+c_2^2c_3c_5^2+c_2^2c_4^2c_5+c_3^3c_4^2+c_3c_7^2) \\ & +(c_6^3+c_2c_5^2c_6+c_3^2c_6^2+c_3c_5^3+c_4^2c_5^2+c_2^2c_3c_5c_6+c_2^2c_4^2c_6+c_2^2c_4c_5^2 \\ & \quad +c_2c_3^2c_4c_6+c_2c_3^2c_5^2+c_2c_3c_4^2c_5+c_3^2c_4^3+c_2c_3c_6c_7+c_2c_4c_5c_7+c_2^3c_5c_7 \\ & \quad +c_2^2c_3c_4c_7+c_4c_7^2) \\ & +(c_3c_5^2c_6+c_2^2c_5^3+c_2c_3^2c_5c_6+c_3^3c_4c_6+c_3^3c_5^2+c_6^2c_7+c_3^2c_6c_7+c_3c_4c_5c_7 \\ & \quad +c_2^2c_3c_5c_7+c_2^2c_4^2c_7+c_2c_3^2c_4c_7+c_5c_7^2+c_2c_3c_7^2) \\ & +(c_3c_5c_6^2+c_4c_5^2c_6+c_5^4+c_2^2c_5^2c_6+c_2c_3^2c_6^2+c_2c_3c_4c_5c_6+c_2c_3c_5^3+c_3^2c_4^2c_6 \\ & \quad +c_3^2c_4c_5^2+c_2c_5c_6c_7+c_3c_5^2c_7+c_2^2c_4c_5c_7+c_6c_7^2+c_2^3c_7^2) \\ & +(c_5^3c_6+c_2c_3c_5^2c_6+c_3^3c_6^2+c_3^2c_4c_5c_6+c_3c_5c_6c_7+c_4c_5^2c_7+c_2c_3c_4c_5c_7 \\ & \quad +c_3^2c_4^2c_7+c_2c_5c_7^2+c_2^2c_3c_7^2+c_7^3), \\ c(\lambda^4) = {} & 1+c_2^2+c_3^2+c_3c_5+c_5^2+(c_2c_5^2+c_3^2c_6+c_3c_4c_5+c_2c_3c_7) \\ & +(c_3c_5c_6+c_4c_5^2+c_2^2c_5^2+c_2c_3^2c_6+c_2c_3c_4c_5+c_2c_5c_7+c_2^2c_3c_7) \end{aligned}$$

$$
\begin{aligned}
&+(c_5^3+c_2c_3c_5^2+c_3^3c_6+c_3^2c_4c_5+c_2c_3^2c_7)\\
&+(c_2^2c_6^2+c_3^2c_5^2+c_3c_4^2c_5+c_4^4+c_2^3c_5^2+c_2^2c_3^2c_6+c_2^2c_3c_4c_5+c_2^8+c_2^2c_5c_7\\
&\quad+c_3^3c_7+c_2^3c_3c_7)\\
&+(c_3^2c_6^2+c_3c_5^3+c_4^2c_5^2+c_2^2c_3c_5c_6+c_2^2c_4c_5^2+c_2c_3^2c_5^2+c_3^4c_6+c_3^3c_4c_5\\
&\quad+c_2^4c_5^2+c_2^3c_3^2c_6+c_2^3c_3c_4c_5+c_2^3c_5c_7+c_2c_3^3c_7+c_2^4c_3c_7+c_2^2c_7^2)\\
&+(c_2^2c_5^3+c_2^3c_3c_5^2+c_2^2c_3^3c_6+c_2^2c_3^2c_4c_5+c_2^3c_3^2c_7)\\
&+(c_3c_5c_6^2+c_5^4+c_2c_4^2c_5^2+c_3^3c_5c_6+c_3^2c_4^2c_6+c_3^2c_4c_5^2+c_3c_4^3c_5+c_2^4c_6^2\\
&\quad+c_2^2c_3c_4^2c_5+c_2^2c_4^4+c_2c_3^4c_6+c_2c_3^3c_4c_5+c_2^5c_5^2+c_2^4c_3^2c_6+c_2^4c_3c_4c_5+c_2^{10}\\
&\quad+c_3c_5^2c_7+c_2^2c_3c_6c_7+c_2c_3c_4^2c_7+c_3^3c_4c_7+c_2^4c_5c_7+c_2^5c_3c_7+c_3^2c_7^2\\
&\quad+c_2^3c_7^2)\\
&+(c_3^2c_5^3+c_2c_3^3c_5^2+c_3^5c_6+c_3^4c_4c_5+c_2c_3^4c_7)\\
&+(c_5^2c_6^2+c_2c_5^4+c_3^2c_5^2c_6+c_3c_4^2c_5c_6+c_3c_4c_5^3+c_4^3c_5^2+c_2^2c_3c_5^3+c_2c_3^2c_4^2c_6\\
&\quad+c_2c_3c_4^3c_5+c_3^4c_5^2+c_3^3c_4^2c_5+c_3^2c_4^4+c_2^4c_3c_5c_6+c_2^4c_4c_5^2+c_2^6c_5^2+c_2^5c_3^2c_6\\
&\quad+c_2^5c_3c_4c_5+c_2^8c_3^2+c_5^3c_7+c_2^2c_5c_6c_7+c_2c_3c_5^2c_7+c_2c_4^2c_5c_7+c_3^2c_4c_5c_7\\
&\quad+c_2^3c_3c_6c_7+c_2^2c_3c_4^2c_7+c_2c_3^3c_4c_7+c_3^5c_7+c_2^5c_5c_7+c_2^6c_3c_7+c_3c_5c_7^2\\
&\quad+c_2c_3^2c_7^2)\\
&+(c_4^2c_5^3+c_2c_3c_4^2c_5^2+c_3^3c_4^2c_6+c_3^2c_4^3c_5+c_2^4c_5^3+c_2^5c_3c_5^2+c_2^4c_3^3c_6+c_2^4c_3^2c_4c_5\\
&\quad+c_3^2c_5^2c_7+c_2^2c_3^2c_6c_7+c_2c_3^3c_5c_7+c_2c_3^2c_4^2c_7+c_3^4c_4c_7+c_2^5c_3^2c_7+c_2^2c_5c_7^2\\
&\quad+c_3^3c_7^2+c_2^3c_3c_7^2)\\
&+(c_6^4+c_2c_5^2c_6^2+c_3^2c_6^3+c_3c_4c_5c_6^2+c_3c_5^3c_6+c_4c_5^4+c_2^2c_3c_5c_6^2+c_2^2c_4^2c_6^2\\
&\quad+c_2^2c_5^4+c_2c_3^2c_5^2c_6+c_2c_3c_4c_5^3+c_2c_3^4c_5^2+c_3^6c_6+c_3^5c_4c_5+c_2^6c_6^2+c_2^4c_3^2c_5^2\\
&\quad+c_2^4c_3c_4^2c_5+c_2^4c_4^4+c_3^8+c_2^7c_5^2+c_2^6c_3^2c_6+c_2^6c_3c_4c_5+c_2^8c_3c_5+c_2c_3c_6^2c_7\\
&\quad+c_3c_4c_5^2c_7+c_2^3c_5c_6c_7+c_2^2c_3c_4c_6c_7+c_2^2c_4^2c_5c_7+c_2c_3^5c_7+c_2^6c_5c_7\\
&\quad+c_2^4c_3^3c_7+c_2^7c_3c_7+c_5^2c_7^2+c_2^3c_4c_7^2)\\
&+(c_5^5+c_2c_3c_5^4+c_3^3c_5^2c_6+c_3^2c_4c_5^3+c_2c_3^2c_5^2c_7)\\
&+(c_3c_5c_6^3+c_4c_5^2c_6^2+c_2^2c_5^2c_6^2+c_2c_3^2c_6^3+c_2c_3c_4c_5c_6^2+c_3^3c_5c_6^2+c_3^2c_4^2c_6^2\\
&\quad+c_3^5c_5c_6+c_3^4c_4c_5^2+c_2^4c_3^2c_6^2+c_2^4c_3c_5^3+c_2^4c_4^2c_5^2+c_2^2c_3^4c_5^2+c_2c_3^6c_6\\
&\quad+c_2c_3^5c_4c_5+c_2^6c_3c_5c_6+c_2^6c_4c_5^2+c_2^5c_3^2c_5^2+c_2^4c_3^4c_6+c_2^4c_3^3c_4c_5+c_2^7c_3^2c_6
\end{aligned}
$$

$$
\begin{aligned}
&+c_2^7c_3c_4c_5+c_2c_5c_6^2c_7+c_3c_5^2c_6c_7+c_4c_5^3c_7+c_2^2c_4c_5c_6c_7+c_2^2c_5^3c_7\\
&+c_2c_3^2c_5c_6c_7+c_2c_3c_4c_5^2c_7+c_3^3c_4c_6c_7+c_3^3c_5^2c_7+c_3^2c_4^2c_5c_7+c_2c_3^4c_5c_7\\
&+c_2^2c_3^5c_7+c_2^7c_5c_7+c_2^5c_3^3c_7+c_2^8c_3c_7+c_2c_5^2c_7^2+c_3^2c_6c_7^2+c_3c_4c_5c_7^2\\
&+c_2^3c_6c_7^2+c_2^2c_4^2c_7^2+c_2c_3^2c_4c_7^2+c_2^6c_7^2+c_2c_3c_7^3)\\
+&(c_5^3c_6^2+c_2c_3c_5^2c_6^2+c_3^3c_6^3+c_3^2c_4c_5c_6^2+c_3^4c_5^3+c_2c_3^5c_5^2+c_3^7c_6+c_3^6c_4c_5\\
&+c_2^6c_5^3+c_2^7c_3c_5^2+c_2^6c_3^3c_6+c_2^6c_3^2c_4c_5+c_5^4c_7+c_2^2c_5^2c_6c_7+c_2c_3^2c_6^2c_7\\
&+c_2c_3c_5^3c_7+c_3^2c_4c_5^2c_7+c_2c_3^6c_7+c_2^7c_3^2c_7+c_3c_5^2c_7^2+c_2^2c_3c_6c_7^2\\
&+c_2^2c_4c_5c_7^2+c_2^3c_7^3)\\
+&(c_3^2c_5^2c_6^2+c_3c_5^5+c_4^2c_5^4+c_2c_3^2c_5^4+c_3^4c_5^2c_6+c_3^3c_4c_5^3+c_2^4c_3c_5c_6^2+c_2^4c_5^4\\
&+c_2^2c_3^4c_6^2+c_3^6c_5^2+c_3^5c_4^2c_5+c_3^4c_4^4+c_2^5c_4^2c_5^2+c_2^4c_3^3c_5c_6+c_2^4c_3^2c_4^2c_6\\
&+c_2^4c_3^2c_4c_5^2+c_2^4c_3c_4^3c_5+c_2^3c_3^4c_5^2+c_2^2c_3^6c_6+c_2^2c_3^5c_4c_5+c_2^8c_6^2+c_2^6c_3c_4^2c_5\\
&+c_2^6c_4^4+c_2^5c_3^4c_6+c_2^5c_3^3c_4c_5+c_2^2c_3^8+c_5^3c_6c_7+c_2c_3c_5^2c_6c_7+c_3^3c_6^2c_7\\
&+c_3^2c_4c_5c_6c_7+c_2c_3^3c_5^2c_7+c_2^4c_3c_5^2c_7+c_2^2c_3^4c_5c_7+c_3^7c_7+c_2^6c_3c_6c_7\\
&+c_2^5c_3c_4^2c_7+c_2^4c_3^3c_4c_7+c_2^3c_3^5c_7+c_2^8c_5c_7+c_3c_5c_6c_7^2+c_4c_5^2c_7^2+c_2^2c_5^2c_7^2\\
&+c_2c_3c_4c_5c_7^2+c_3^2c_4^2c_7^2+c_2^4c_3^2c_7^2+c_2^7c_7^2+c_2c_5c_7^3+c_2^2c_3c_7^3+c_7^4)\\
+&(c_2^4c_3^2c_5^3+c_2^5c_3^3c_5^2+c_2^4c_3^5c_6+c_2^4c_3^4c_4c_5+c_2^5c_3^4c_7)\\
+&(c_3^3c_5^3c_6+c_3^2c_4c_5^4+c_2^4c_5^2c_6^2+c_2^2c_3^2c_5^4+c_2c_3^4c_5^2c_6+c_2c_3^3c_4c_5^3+c_3^6c_6^2+c_3^5c_5^3\\
&+c_3^4c_4^2c_5^2+c_2^5c_5^4+c_2^4c_3^2c_5^2c_6+c_2^4c_3c_4^2c_5c_6+c_2^4c_3c_4c_5^3+c_2^4c_4^3c_5^2\\
&+c_2^2c_3^5c_5c_6+c_2^2c_3^4c_4c_5^2+c_2c_3^6c_5^2+c_3^8c_6+c_3^7c_4c_5+c_2^6c_3c_5^3+c_2^5c_3^2c_4^2c_6\\
&+c_2^5c_3c_4^3c_5+c_2^4c_3^3c_4^2c_5+c_2^4c_3^2c_4^4+c_2^3c_3^6c_6+c_2^3c_3^5c_4c_5+c_3^{10}+c_2c_3^2c_5^3c_7\\
&+c_2^4c_5^3c_7+c_2^2c_3^3c_5^2c_7+c_2^6c_5c_6c_7+c_2^5c_3c_5^2c_7+c_2^5c_4^2c_5c_7+c_2^4c_3^2c_4c_5c_7\\
&+c_2^3c_3^4c_5c_7+c_2c_3^7c_7+c_2^7c_3c_6c_7+c_2^6c_3c_4^2c_7+c_2^5c_3^3c_4c_7+c_2^4c_3c_5c_7^2\\
&+c_2^2c_3^4c_7^2+c_2^5c_3^2c_7^2)\\
+&(c_3^2c_5^5+c_2c_3^3c_5^4+c_3^5c_5^2c_6+c_3^4c_4c_5^3+c_2^4c_4^2c_5^3+c_2^2c_3^4c_5^3+c_2^5c_3c_4^2c_5^2\\
&+c_2^4c_3^3c_4^2c_6+c_2^4c_3^2c_4^3c_5+c_2^3c_3^5c_5^2+c_2^2c_3^7c_6+c_2^2c_3^6c_4c_5+c_2c_3^4c_5^2c_7\\
&+c_2^4c_3^2c_5^2c_7+c_2^6c_3^2c_6c_7+c_2^5c_3^3c_5c_7+c_2^5c_3^2c_4^2c_7+c_2^4c_3^4c_4c_7+c_2^3c_3^6c_7\\
&+c_2^6c_5c_7^2+c_2^4c_3^3c_7^2+c_2^7c_3c_7^2)\\
+&(c_5^4c_6^2+c_2c_5^6+c_3^2c_5^4c_6+c_3c_4c_5^5+c_2^4c_6^4+c_2^2c_3^2c_5^2c_6^2+c_2^2c_3c_5^5+c_2^2c_4^2c_5^4
\end{aligned}
$$

$$
\begin{aligned}
&\quad +c_3^4c_4^2c_6^2+c_3^4c_5^4+c_2^5c_5^2c_6^2+c_2^4c_3^2c_6^3+c_2^4c_3c_4c_5c_6^2+c_2^4c_3c_5^3c_6+c_2^4c_4c_5^4\\
&\quad +c_2^3c_3^2c_5^4+c_2^2c_3^4c_5^2c_6+c_2^2c_3^3c_4c_5^3+c_2c_3^4c_4^2c_5^2+c_3^7c_5c_6+c_3^6c_4^2c_6+c_3^6c_4c_5^2\\
&\quad +c_3^5c_4^3c_5+c_2^6c_3c_5c_6^2+c_2^6c_4^2c_6^2+c_2^6c_5^4+c_2^5c_3^2c_5^2c_6+c_2^5c_3c_4c_5^3+c_2^4c_3^4c_6^2\\
&\quad +c_2^2c_3^5c_4^2c_5+c_2^2c_3^4c_4^4+c_2c_3^8c_6+c_2c_3^7c_4c_5+c_3^9c_5+c_5^5c_7+c_2^2c_5^3c_6c_7\\
&\quad +c_2c_3c_5^4c_7+c_3^3c_5^2c_6c_7+c_2^3c_3c_5^2c_6c_7+c_2^2c_3^3c_6^2c_7+c_2^2c_3^2c_4c_5c_6c_7\\
&\quad +c_2c_3^4c_5c_6c_7+c_3^5c_4c_6c_7+c_3^4c_4^2c_5c_7+c_2^5c_3c_6^2c_7+c_2^4c_3c_4c_5^2c_7+c_2^3c_3^3c_5^2c_7\\
&\quad +c_2^2c_3^5c_6c_7+c_2c_3^5c_4^2c_7+c_3^7c_4c_7+c_2^7c_5c_6c_7+c_2^6c_3c_4c_6c_7+c_2^6c_4^2c_5c_7\\
&\quad +c_2^4c_3^4c_5c_7+c_3c_5^3c_7^2+c_2^2c_3c_5c_6c_7^2+c_2^2c_4c_5^2c_7^2+c_3^4c_6c_7^2+c_3^3c_4c_5c_7^2\\
&\quad +c_2^3c_3c_4c_5c_7^2+c_2^2c_3^2c_4^2c_7^2+c_2c_3^4c_4c_7^2+c_3^6c_7^2+c_2^3c_3^4c_7^2+c_2^7c_4c_7^2+c_2^3c_5c_7^3\\
&\quad +c_2c_3^3c_7^3+c_2^4c_3c_7^3)\\
&+(c_2^4c_5^5+c_3^6c_5^3+c_2^5c_3c_5^4+c_2^4c_3^3c_5^2c_6+c_2^4c_3^2c_4c_5^3+c_2c_3^7c_5^2+c_3^9c_6+c_3^8c_4c_5\\
&\quad +c_2^5c_3^2c_5^2c_7+c_2c_3^8c_7)\\
&+(c_3c_5^5c_6+c_4c_5^6+c_2^2c_5^6+c_2c_3^2c_5^4c_6+c_2c_3c_4c_5^5+c_3^3c_5^5+c_3^2c_4^2c_5^4+c_2^4c_3c_5c_6^3\\
&\quad +c_2^4c_4c_5^2c_6^2+c_2^2c_3^3c_5^3c_6+c_2^2c_3^2c_4c_5^4+c_3^5c_4^2c_5c_6+c_3^4c_4^3c_5^2+c_2^6c_5^2c_6^2\\
&\quad +c_2^5c_3^2c_6^3+c_2^5c_3c_4c_5c_6^2+c_2^4c_3^3c_5c_6^2+c_2^4c_3^2c_4^2c_6^2+c_2^4c_3^2c_5^4+c_2^3c_3^4c_5^2c_6\\
&\quad +c_2^3c_3^3c_4c_5^3+c_2^2c_3^5c_5^3+c_2c_3^6c_4^2c_6+c_2c_3^5c_4^3c_5+c_3^7c_4^2c_5+c_3^6c_4^4+c_2c_5^5c_7\\
&\quad +c_2^2c_3c_5^4c_7+c_3^4c_5^3c_7+c_2^5c_5c_6^2c_7+c_2^4c_3c_5^2c_6c_7+c_2^4c_4c_5^3c_7+c_2^3c_3^2c_5^3c_7\\
&\quad +c_2^2c_3^4c_5c_6c_7+c_2c_3^4c_4^2c_5c_7+c_3^6c_4c_5c_7+c_2^6c_4c_5c_6c_7+c_2^6c_5^3c_7\\
&\quad +c_2^5c_3^2c_5c_6c_7+c_2^5c_3c_4c_5^2c_7+c_2^4c_3^3c_4c_6c_7+c_2^4c_3^2c_4^2c_5c_7+c_2^3c_3^5c_6c_7\\
&\quad +c_2^2c_3^5c_4^2c_7+c_2c_3^7c_4c_7+c_3^9c_7+c_2^2c_3^2c_5^2c_7^2+c_3^5c_5c_7^2+c_2^5c_5^2c_7^2+c_2^4c_3^2c_6c_7^2\\
&\quad +c_2^4c_3c_4c_5c_7^2+c_2c_3^6c_7^2+c_2^7c_6c_7^2+c_2^6c_4^2c_7^2+c_2^5c_3^2c_4c_7^2+c_2^5c_3c_7^3)\\
&+(c_5^7+c_2c_3c_5^6+c_3^3c_5^4c_6+c_3^2c_4c_5^5+c_2^4c_5^3c_6^2+c_2^2c_3^2c_5^5+c_3^4c_4^2c_5^3+c_2^5c_3c_5^2c_6^2\\
&\quad +c_2^4c_3^3c_6^3+c_2^4c_3^2c_4c_5c_6^2+c_2^3c_3^3c_5^4+c_2^2c_3^5c_5^2c_6+c_2^2c_3^4c_4c_5^3+c_2c_3^5c_4^2c_5^2\\
&\quad +c_3^7c_4^2c_6+c_3^6c_4^3c_5+c_2c_3^2c_5^4c_7+c_2^4c_5^4c_7+c_3^6c_5^2c_7+c_2^6c_5^2c_6c_7+c_2^5c_3^2c_6^2c_7\\
&\quad +c_2^5c_3c_5^3c_7+c_2^4c_3^2c_4c_5^2c_7+c_2^3c_3^4c_5^2c_7+c_2^2c_3^6c_6c_7+c_2c_3^7c_5c_7+c_2c_3^6c_4^2c_7\\
&\quad +c_3^8c_4c_7+c_2^4c_3c_5^2c_7^2+c_2^2c_3^4c_5c_7^2+c_3^7c_7^2+c_2^6c_3c_6c_7^2+c_2^6c_4c_5c_7^2+c_2^3c_3^5c_7^2\\
&\quad +c_2^7c_7^3),
\end{aligned}
$$

$$c(\lambda^6) = 1+c_2+c_3+c_4+c_5+c_6+c_7.$$

By (2–1), we obtain

$$\begin{aligned}
c_{32}(1\oplus\lambda^2\oplus\lambda^4\oplus\lambda^6) = {}& c_3c_5^3c_7^2+c_5^5c_7+c_5^4c_6^2+c_2^3c_5c_7^3+c_2^2c_3c_5c_6c_7^2+c_2^2c_4c_5^2c_7^2\\
&+c_2^2c_5^3c_6c_7+c_2c_3^3c_7^3+c_3^4c_6c_7^2+c_3^3c_4c_5c_7^2+c_3^3c_5^2c_6c_7\\
&+c_2^4c_3c_7^3+c_2^4c_5^2c_7^2+c_2^4c_6^4+c_2^3c_3c_4c_5c_7^2+c_2^3c_3c_5^2c_6c_7\\
&+c_2^2c_3^3c_5c_7^2+c_2^2c_3^3c_6^2c_7+c_2^2c_3^2c_4^2c_7^2+c_2^2c_3^2c_4c_5c_6c_7\\
&+c_2^2c_3^2c_5^3c_7+c_2^2c_3^2c_5^2c_6^2+c_2c_3^4c_4c_7^2+c_2c_3^4c_5c_6c_7+c_3^6c_7^2\\
&+c_3^5c_4c_6c_7+c_3^4c_4^2c_5c_7+c_3^4c_4^2c_6^2+c_3^4c_5^4+c_4^8+c_2^{16}.
\end{aligned}$$

Recall that $f_{13}^*\Delta_{14}^+=(\Delta_{13})_{\mathbb{C}}$, which gives $w(f_{13}^*(\Delta_{14}^+)_{\mathbb{R}})=w((\Delta_{13})_{\mathbb{R}})$. In order to determine $w_{64}((\Delta_{14}^+)_{\mathbb{R}})$, we use the method of indeterminate coefficients; using the equations $\mathrm{Sq}^n\, w_{64}((\Delta_{14}^+)_{\mathbb{R}})=0$ for $n=1,2,4$, we can determine all the coefficients, and hence the 64–th Stiefel–Whitney class is given as follows:

$$\begin{aligned}
w_{64}((\Delta_{14}^+)_{\mathbb{R}}) = {}& y_{11}y_{13}^3y_{14}+y_4y_6y_{13}^2y_{14}^2+y_6^2y_{11}y_{13}y_{14}^2+y_4y_{10}y_{11}^2y_{14}^2+y_6y_{10}^3y_{14}^2\\
&+y_4y_7y_{13}^3y_{14}+y_4y_{10}y_{11}y_{12}y_{13}y_{14}+y_7^2y_{11}y_{12}y_{13}y_{14}\\
&+y_7y_8y_{11}^2y_{13}y_{14}+y_6y_{10}y_{11}^2y_{12}y_{14}+y_8y_{10}^2y_{11}^2y_{14}+y_{10}^5y_{14}\\
&+y_4^3y_{10}y_{14}^3+y_4y_6^3y_{14}^3+y_4^3y_{11}y_{13}y_{14}^2+y_4^2y_6y_{10}y_{12}y_{14}^2\\
&+y_4y_6y_7^2y_{12}y_{14}^2+y_6^4y_{12}y_{14}^2+y_4y_6y_7y_8y_{11}y_{14}^2+y_6^3y_7y_{11}y_{14}^2\\
&+y_4^2y_8y_{10}^2y_{14}^2+y_6^3y_8y_{10}y_{14}^2+y_4^2y_8^2y_{13}^2y_{14}+y_4^2y_6y_{11}y_{12}y_{13}y_{14}\\
&+y_4y_7^3y_{12}y_{13}y_{14}+y_4^2y_7y_{11}^2y_{13}y_{14}+y_4^2y_8y_{10}y_{11}y_{13}y_{14}\\
&+y_4y_7^2y_8y_{11}y_{13}y_{14}+y_7^3y_8^2y_{13}y_{14}+y_4^2y_7y_{11}y_{12}^2y_{14}+y_4y_6^2y_{11}^2y_{12}y_{14}\\
&+y_6y_7^3y_{11}y_{12}y_{14}+y_4^2y_{10}^3y_{12}y_{14}+y_6^3y_{10}^2y_{12}y_{14}+y_6y_7^2y_8y_{11}^2y_{14}\\
&+y_4^4y_6y_{14}^3+y_4^4y_7y_{13}y_{14}^2+y_4^3y_6y_7y_{11}y_{14}^2+y_4^4y_{10}^2y_{14}^2\\
&+y_4^3y_6y_8y_{10}y_{14}^2+y_4^2y_6^3y_{10}y_{14}^2+y_4^2y_6^2y_8^2y_{14}^2+y_4y_6^4y_8y_{14}^2\\
&+y_4y_6^3y_7^2y_{14}^2+y_6^6y_{14}^2+y_4^3y_6y_8y_{11}y_{13}y_{14}+y_4^3y_7^2y_{11}y_{13}y_{14}\\
&+y_4^2y_6y_7y_8^2y_{13}y_{14}+y_4^2y_6^3y_{12}^2y_{14}+y_4^3y_6y_{10}^2y_{12}y_{14}\\
&+y_4^2y_6^2y_8y_{10}y_{12}y_{14}+y_4y_6^4y_{10}y_{12}y_{14}+y_4^2y_7^2y_8^2y_{12}y_{14}\\
&+y_4y_6^2y_7^2y_8y_{12}y_{14}+y_6^5y_8y_{12}y_{14}+y_4y_6y_7^4y_{12}y_{14}+y_4^2y_6^2y_8y_{11}^2y_{14}\\
&+y_4y_6y_7^3y_8y_{11}y_{14}+y_4^2y_6^2y_{10}^3y_{14}+y_6^4y_8^2y_{10}y_{14}+y_6^2y_7^2y_8^3y_{14}\\
&+y_6y_7^4y_8^2y_{14}+y_{12}y_{13}^4+y_6^2y_{13}^4+y_4y_{10}y_{11}y_{13}^3+y_7^2y_{11}y_{13}^3
\end{aligned}$$

$$
\begin{aligned}
&+y_4y_{11}^2y_{12}y_{13}^2+y_7y_8y_{11}y_{12}y_{13}^2+y_6y_{10}^2y_{12}y_{13}^2+y_6y_{10}y_{11}^2y_{13}^2\\
&+y_8y_{10}^2y_{11}y_{12}y_{13}+y_{10}^4y_{11}y_{13}+y_4^3y_{13}^4+y_4^2y_6y_{11}y_{13}^3+y_4y_7^3y_{13}^3\\
&+y_6^3y_8y_{12}y_{13}^2+y_6^2y_7^2y_{12}y_{13}^2+y_6^2y_7y_8y_{11}y_{13}^2+y_6y_7^3y_{11}y_{13}^2\\
&+y_4^2y_8y_{11}y_{12}^2y_{13}+y_4y_6y_7y_{11}^2y_{12}y_{13}+y_4^2y_{10}^2y_{11}y_{12}y_{13}\\
&+y_6^3y_{10}y_{11}y_{12}y_{13}+y_6^2y_8^2y_{11}y_{12}y_{13}+y_6y_7^2y_8y_{11}y_{12}y_{13}\\
&+y_7^4y_{11}y_{12}y_{13}+y_7^3y_8y_{11}^2y_{13}+y_4^3y_6y_7y_{13}^3+y_4^3y_7^2y_{12}y_{13}^2\\
&+y_4^2y_6^3y_{12}y_{13}^2+y_4^2y_6^2y_7y_{11}y_{13}^2+y_4^2y_7^2y_8^2y_{13}^2+y_4y_6^2y_7^2y_8y_{13}^2\\
&+y_4y_6y_7^4y_{13}^2+y_4^3y_6y_{10}y_{11}y_{12}y_{13}+y_4^2y_6y_7^2y_{11}y_{12}y_{13}\\
&+y_4y_6^4y_{11}y_{12}y_{13}+y_6^4y_7y_8y_{12}y_{13}+y_4y_7^5y_{12}y_{13}+y_4^2y_6^2y_{10}^2y_{11}y_{13}\\
&+y_6^4y_8^2y_{11}y_{13}+y_4y_7^4y_8y_{11}y_{13}+y_7^5y_8^2y_{13}+y_{10}^4y_{12}^2+y_{10}^3y_{11}^2y_{12}\\
&+y_7^4y_{12}^3+y_4^2y_{10}y_{11}^2y_{12}^2+y_4y_7^2y_{11}^2y_{12}^2+y_6^3y_{11}^2y_{12}^2+y_7^3y_8y_{11}y_{12}^2\\
&+y_7^2y_8^2y_{11}^2y_{12}+y_4^4y_{12}^4+y_4^2y_6y_7^2y_{12}^3+y_4^3y_6y_{11}^2y_{12}^2+y_4^2y_6y_7y_8y_{11}y_{12}^2\\
&+y_4^2y_7^3y_{11}y_{12}^2+y_4y_6^3y_7y_{11}y_{12}^2+y_4^2y_6^2y_{10}^2y_{12}^2+y_6^4y_8^2y_{12}^2\\
&+y_6^3y_7^2y_8y_{12}^2+y_4^2y_6^2y_{10}y_{11}^2y_{12}+y_4y_7^3y_8^2y_{11}y_{12}+y_6^3y_7y_8^2y_{11}y_{12}\\
&+y_7^4y_8^3y_{12}+y_6^2y_7^4y_{12}^2+y_{10}^2y_{11}^4+y_4^2y_6^2y_{11}^4+y_6^2y_7^2y_8^2y_{11}^2+y_7^6y_{11}^2\\
&+y_6^4y_{10}^4+y_8^8+y_4^{16}.
\end{aligned}
$$

Thus we obtain the total Stiefel–Whitney class

$$
\begin{aligned}
w((\Delta_{14}^+)_{\mathbb{R}}) &= c(\Delta_{14}^+)\\
&= 1+w_{64}((\Delta_{14}^+)_{\mathbb{R}})+\mathrm{Sq}^{32}\,w_{64}((\Delta_{14}^+)_{\mathbb{R}})+\mathrm{Sq}^{48}\,w_{64}((\Delta_{14}^+)_{\mathbb{R}})\\
&\quad+\mathrm{Sq}^{56}\,w_{64}((\Delta_{14}^+)_{\mathbb{R}})+\mathrm{Sq}^{60}\,w_{64}((\Delta_{14}^+)_{\mathbb{R}})+\mathrm{Sq}^{62}\,w_{64}((\Delta_{14}^+)_{\mathbb{R}})\\
&\quad+u_{128}.
\end{aligned}
$$

Note that $c(\Delta_{14}^+)=c(\Delta_{14}^-)$, since Δ_{14}^+ and Δ_{14}^- are conjugate to each other.

Recall from (2–2) that $f_{14}^*\Delta_{15}=(\Delta_{14}^+)_{\mathbb{R}}$, which gives $w(f_{14}^*\Delta_{15})=w((\Delta_{14}^+)_{\mathbb{R}})$. In order to determine $w_{64}(\Delta_{15})$, we use the method of indeterminate coefficients; using the equations $\mathrm{Sq}^n\,w_{64}(\Delta_{15})=0$ for $n=1,2,4,8$, we can determine all the indeterminate coefficients, and hence the 64–th Stiefel–Whitney class is given as follows:

$$
\begin{aligned}
w_{64}(\Delta_{15}) &= y_{10}y_{13}^3y_{15}+y_4^2y_{13}^2y_{15}^2+y_4y_6y_{11}y_{13}y_{15}^2+y_7^3y_{13}y_{15}^2+y_4y_{10}^3y_{15}^2\\
&\quad+y_4y_7y_{11}y_{13}y_{14}y_{15}+y_4y_{10}^2y_{11}y_{14}y_{15}+y_4y_8y_{11}y_{13}^2y_{15}
\end{aligned}
$$

$$
\begin{aligned}
&+y_6y_7y_{11}y_{12}y_{13}y_{15}+y_4y_{10}y_{11}^2y_{13}y_{15}+y_6y_8y_{11}^2y_{13}y_{15}+y_7^2y_{11}^2y_{13}y_{15}\\
&+y_6y_{10}^3y_{13}y_{15}+y_6y_{10}^2y_{11}y_{12}y_{15}+y_8y_{10}^3y_{11}y_{15}+y_4^3y_7y_{15}^3\\
&+y_4^2y_7^2y_{12}y_{15}^2+y_4^3y_{11}^2y_{15}^2+y_4^2y_7y_8y_{11}y_{15}^2+y_4y_6^2y_7y_{11}y_{15}^2+y_6y_7^4y_{15}^2\\
&+y_4^2y_6y_7y_{14}^2y_{15}+y_4^3y_{10}y_{13}y_{14}y_{15}+y_4y_6^3y_{13}y_{14}y_{15}\\
&+y_4^2y_6y_{10}y_{11}y_{14}y_{15}+y_4y_6y_7^2y_{11}y_{14}y_{15}+y_4^2y_6y_{11}^2y_{13}y_{15}\\
&+y_4y_6y_7y_8y_{11}y_{13}y_{15}+y_4y_6^2y_{10}^2y_{13}y_{15}+y_4y_6y_8^2y_{10}y_{13}y_{15}\\
&+y_6^3y_8y_{10}y_{13}y_{15}+y_6y_7^2y_8^2y_{13}y_{15}+y_7^4y_8y_{13}y_{15}+y_4^2y_6y_{11}y_{12}^2y_{15}\\
&+y_4y_7^3y_{12}^2y_{15}+y_6^2y_7^2y_{11}y_{12}y_{15}+y_4y_7y_8^2y_{11}^2y_{15}+y_6^2y_7y_8y_{11}^2y_{15}\\
&+y_6y_7^3y_{11}^2y_{15}+y_4^2y_{10}^3y_{11}y_{15}+y_6^3y_{10}^2y_{11}y_{15}+y_4^5y_{14}y_{15}^2+y_4^4y_8y_{10}y_{15}^2\\
&+y_4^3y_6y_8^2y_{15}^2+y_4^2y_6^3y_8y_{15}^2+y_4^2y_6^2y_7^2y_{15}^2+y_4y_6^5y_{15}^2+y_4^4y_6y_{13}y_{14}y_{15}\\
&+y_4^4y_7y_{12}y_{14}y_{15}+y_4^3y_7y_8^2y_{14}y_{15}+y_4^2y_6^2y_7y_8y_{14}y_{15}+y_4^4y_7y_{13}^2y_{15}\\
&+y_4^4y_8y_{12}y_{13}y_{15}+y_4^3y_6^2y_{12}y_{13}y_{15}+y_4^2y_6^2y_8^2y_{13}y_{15}+y_6^6y_{13}y_{15}\\
&+y_4^3y_6y_7y_{12}^2y_{15}+y_4^4y_{10}y_{11}y_{12}y_{15}+y_4^2y_6y_7y_8^2y_{12}y_{15}+y_4y_6^2y_7^3y_{12}y_{15}\\
&+y_4^3y_6y_{10}^2y_{11}y_{15}+y_4y_6^4y_{10}y_{11}y_{15}+y_4^2y_7^2y_8^2y_{11}y_{15}+y_4y_6^2y_7^2y_8y_{11}y_{15}\\
&+y_6^5y_8y_{11}y_{15}+y_4y_6y_7^4y_{11}y_{15}+y_6^4y_7^2y_{11}y_{15}+y_4y_7^3y_8^3y_{15}+y_6^3y_7y_8^3y_{15}\\
&+y_6^2y_7^3y_8^2y_{15}+y_6y_7^5y_8y_{15}+y_7^7y_{15}+y_4^6y_{10}y_{15}^2+y_4^5y_7^2y_{15}^2+y_4^4y_6^3y_{15}^2\\
&+y_4^6y_{11}y_{14}y_{15}+y_4^4y_6^2y_7y_{14}y_{15}+y_4^6y_{12}y_{13}y_{15}+y_4^5y_6y_{10}y_{13}y_{15}\\
&+y_4^4y_6^2y_8y_{13}y_{15}+y_4^4y_6y_7^2y_{13}y_{15}+y_4^4y_7^3y_{12}y_{15}+y_4^5y_7y_{11}^2y_{15}\\
&+y_4^4y_7^2y_8y_{11}y_{15}+y_{11}y_{13}^3y_{14}+y_4y_6y_{13}^2y_{14}^2+y_6^2y_{11}y_{13}y_{14}^2\\
&+y_4y_{10}y_{11}^2y_{14}^2+y_6y_{10}^3y_{14}^2+y_4y_7y_{13}^3y_{14}+y_4y_{10}y_{11}y_{12}y_{13}y_{14}\\
&+y_7^2y_{11}y_{12}y_{13}y_{14}+y_7y_8y_{11}^2y_{13}y_{14}+y_6y_{10}y_{11}^2y_{12}y_{14}+y_8y_{10}^2y_{11}^2y_{14}\\
&+y_{10}^5y_{14}+y_4^3y_{10}y_{14}^3+y_4y_6^3y_{14}^3+y_4^3y_{11}y_{13}y_{14}^2+y_4^2y_6y_{10}y_{12}y_{14}^2\\
&+y_4y_6y_7^2y_{12}y_{14}^2+y_6^4y_{12}y_{14}^2+y_4y_6y_7y_8y_{11}y_{14}^2+y_6^3y_7y_{11}y_{14}^2\\
&+y_4^2y_8y_{10}^2y_{14}^2+y_6^3y_8y_{10}y_{14}^2+y_4^2y_8^2y_{13}^2y_{14}+y_4^2y_6y_{11}y_{12}y_{13}y_{14}\\
&+y_4y_7^3y_{12}y_{13}y_{14}+y_4^2y_7y_{11}^2y_{13}y_{14}+y_4^2y_8y_{10}y_{11}y_{13}y_{14}\\
&+y_4y_7^2y_8y_{11}y_{13}y_{14}+y_7^3y_8^2y_{13}y_{14}+y_4^2y_7y_{11}y_{12}^2y_{14}+y_4y_6^2y_{11}^2y_{12}y_{14}\\
&+y_6y_7^3y_{11}y_{12}y_{14}+y_4^2y_{10}^3y_{12}y_{14}+y_6^3y_{10}^2y_{12}y_{14}+y_6y_7^2y_8y_{11}^2y_{14}\\
&+y_4^4y_6y_{14}^3+y_4^4y_7y_{13}y_{14}^2+y_4^3y_6y_7y_{11}y_{14}^2+y_4^4y_{10}^2y_{14}^2+y_4^3y_6y_8y_{10}y_{14}^2
\end{aligned}
$$

$$
\begin{aligned}
&+y_4^2y_6^3y_{10}y_{14}^2+y_4^2y_6^2y_8^2y_{14}^2+y_4y_6^4y_8y_{14}^2+y_4y_6^3y_7^2y_{14}^2+y_6^6y_{14}^2\\
&+y_4^3y_6y_8y_{11}y_{13}y_{14}+y_4^3y_7^2y_{11}y_{13}y_{14}+y_4^2y_6y_7y_8^2y_{13}y_{14}+y_4^2y_6^3y_{12}^2y_{14}\\
&+y_4^3y_6y_{10}^2y_{12}y_{14}+y_4^2y_6^2y_8y_{10}y_{12}y_{14}+y_4y_6^4y_{10}y_{12}y_{14}+y_4^2y_7^2y_8^2y_{12}y_{14}\\
&+y_4y_6^2y_7^2y_8y_{12}y_{14}+y_6^5y_8y_{12}y_{14}+y_4y_6y_7^4y_{12}y_{14}+y_4^2y_6^2y_8y_{11}^2y_{14}\\
&+y_4y_6y_7^3y_8y_{11}y_{14}+y_4^2y_6^2y_{10}^3y_{14}+y_6^4y_8^2y_{10}y_{14}+y_6^2y_7^2y_8^3y_{14}\\
&+y_6y_7^4y_8^2y_{14}+y_{12}y_{13}^4+y_6^2y_{13}^4+y_4y_{10}y_{11}y_{13}^3+y_7^2y_{11}y_{13}^3\\
&+y_4y_{11}^2y_{12}y_{13}^2+y_7y_8y_{11}y_{12}y_{13}^2+y_6y_{10}^2y_{12}y_{13}^2+y_6y_{10}y_{11}^2y_{13}^2\\
&+y_8y_{10}^2y_{11}y_{12}y_{13}+y_{10}^4y_{11}y_{13}+y_4^3y_{13}^4+y_4^2y_6y_{11}y_{13}^3+y_4y_7^3y_{13}^3\\
&+y_6^3y_8y_{12}y_{13}^2+y_6^2y_7^2y_{12}y_{13}^2+y_6^2y_7y_8y_{11}y_{13}^2+y_6y_7^3y_{11}y_{13}^2\\
&+y_4^2y_8y_{11}y_{12}^2y_{13}+y_4y_6y_7y_{11}^2y_{12}y_{13}+y_4^2y_{10}^2y_{11}y_{12}y_{13}\\
&+y_6^3y_{10}y_{11}y_{12}y_{13}+y_6^2y_8^2y_{11}y_{12}y_{13}+y_6y_7^2y_8y_{11}y_{12}y_{13}+y_7^4y_{11}y_{12}y_{13}\\
&+y_7^3y_8y_{11}^2y_{13}+y_4^3y_6y_7y_{13}^3+y_4^3y_7^2y_{12}y_{13}^2+y_4^2y_6^3y_{12}y_{13}^2+y_4^2y_6^2y_7y_{11}y_{13}^2\\
&+y_4^2y_7^2y_8^2y_{13}^2+y_4y_6^2y_7^2y_8y_{13}^2+y_4y_6y_7^4y_{13}^2+y_4^3y_6y_{10}y_{11}y_{12}y_{13}\\
&+y_4^2y_6y_7^2y_{11}y_{12}y_{13}+y_4y_6^4y_{11}y_{12}y_{13}+y_6^4y_7y_8y_{12}y_{13}+y_4y_7^5y_{12}y_{13}\\
&+y_4^2y_6^2y_{10}^2y_{11}y_{13}+y_6^4y_8^2y_{11}y_{13}+y_4y_7^4y_8y_{11}y_{13}+y_7^5y_8^2y_{13}+y_{10}^4y_{12}^2\\
&+y_{10}^3y_{11}^2y_{12}+y_7^4y_{12}^3+y_4^2y_{10}y_{11}^2y_{12}^2+y_4y_7^2y_{11}^2y_{12}^2+y_6^3y_{11}^2y_{12}^2\\
&+y_7^3y_8y_{11}y_{12}^2+y_7^2y_8^2y_{11}^2y_{12}+y_4^4y_{12}^4+y_4^2y_6y_7^2y_{12}^3+y_4^3y_6y_{11}^2y_{12}^2\\
&+y_4^2y_6y_7y_8y_{11}y_{12}^2+y_4^2y_7^3y_{11}y_{12}^2+y_4y_6^3y_7y_{11}y_{12}^2+y_4^2y_6^2y_{10}^2y_{12}^2\\
&+y_6^4y_8^2y_{12}^2+y_6^3y_7^2y_8y_{12}^2+y_4^2y_6^2y_{10}y_{11}^2y_{12}+y_4y_7^3y_8^2y_{11}y_{12}\\
&+y_6^3y_7y_8^2y_{11}y_{12}+y_7^4y_8^3y_{12}+y_6^2y_7^4y_{12}^2+y_{10}^2y_{11}^4+y_4^2y_6^2y_{11}^4+y_6^2y_7^2y_8^2y_{11}^2\\
&+y_7^6y_{11}^2+y_6^4y_{10}^4+y_8^8+y_4^{16}.
\end{aligned}
$$

Using the above result on $w_{64}(\Delta_{15})$, we have the following theorem.

Theorem 4.1 *The total Stiefel–Whitney class of the representation* $\Delta_{15}\colon \mathrm{Spin}(15) \to O(128)$ *is given by*

$$
\begin{aligned}
w(\Delta_{15}) = 1 &+ w_{64}(\Delta_{15}) + \mathrm{Sq}^{32}\, w_{64}(\Delta_{15}) + \mathrm{Sq}^{48}\, w_{64}(\Delta_{15}) + \mathrm{Sq}^{56}\, w_{64}(\Delta_{15})\\
&+ \mathrm{Sq}^{60}\, w_{64}(\Delta_{15}) + \mathrm{Sq}^{62}\, w_{64}(\Delta_{15}) + \mathrm{Sq}^{63}\, w_{64}(\Delta_{15}) + u_{128},
\end{aligned}
$$

where $u_{128} = w_{128}(\Delta_{15})$.

5 The Stiefel–Whitney classes of the induced representation from the adjoint representation of E_8

Summing up all the calculations in the previous section, we have the following:

Theorem 5.1 *For the representation* $\Delta_{15} \oplus \lambda^1_{15} \oplus \lambda^2_{15}$: $\mathrm{Spin}(15) \to SO(248)$, *the Stiefel–Whitney classes of degree* 2^i *are given as follows:*

$$
\begin{aligned}
w_1(\Delta_{15}\oplus\lambda^1_{15} \oplus \lambda^2_{15}) &= w_1(\lambda^1_{15} \oplus \lambda^2_{15}) = 0,\\
w_2(\Delta_{15}\oplus\lambda^1_{15} \oplus \lambda^2_{15}) &= w_2(\lambda^1_{15} \oplus \lambda^2_{15}) = 0,\\
w_4(\Delta_{15}\oplus\lambda^1_{15} \oplus \lambda^2_{15}) &= w_4(\lambda^1_{15} \oplus \lambda^2_{15}) = 0,\\
w_8(\Delta_{15}\oplus\lambda^1_{15} \oplus \lambda^2_{15}) &= w_8(\lambda^1_{15} \oplus \lambda^2_{15}) = 0,\\
w_{16}(\Delta_{15}\oplus\lambda^1_{15} \oplus \lambda^2_{15}) &= w_{16}(\lambda^1_{15} \oplus \lambda^2_{15}) = y_4^4,\\
w_{32}(\Delta_{15}\oplus\lambda^1_{15} \oplus \lambda^2_{15}) &= w_{32}(\lambda^1_{15} \oplus \lambda^2_{15})\\
&= y_4^2 y_{11} y_{13} + y_6^2 y_7 y_{13} + y_6^2 y_{10}^2 + y_7^3 y_{11} + y_4^8,\\
w_{64}(\Delta_{15}\oplus\lambda^1_{15} \oplus \lambda^2_{15}) &= w_{64}(\Delta_{15}) + w_{64}(\lambda^1_{15} \oplus \lambda^2_{15})\\
&= y_{10} y_{13}^3 y_{15} + y_{11} y_{13}^3 y_{14} + y_{12} y_{13}^4 + y_4^2 y_{13}^2 y_{15}^2\\
&\quad + y_4 y_6 y_{11} y_{13} y_{15}^2 + y_4 y_6 y_{13}^2 y_{14}^2 + y_4 y_7 y_{11} y_{13} y_{14} y_{15}\\
&\quad + y_4 y_7 y_{13}^3 y_{14} + y_4 y_8 y_{11} y_{13}^2 y_{15} + y_4 y_{10}^3 y_{15}^2\\
&\quad + y_4 y_{10}^2 y_{11} y_{14} y_{15} + y_4 y_{10} y_{11}^2 y_{13} y_{15} + y_4 y_{10} y_{11}^2 y_{14}^2\\
&\quad + y_4 y_{10} y_{11} y_{12} y_{13} y_{14} + y_4 y_{11}^2 y_{12} y_{13}^2 + y_6^2 y_{11} y_{13} y_{14}^2\\
&\quad + y_6 y_7 y_{11} y_{12} y_{13} y_{15} + y_6 y_8 y_{11}^2 y_{13} y_{15} + y_6 y_{10}^3 y_{13} y_{15}\\
&\quad + y_6 y_{10}^3 y_{14}^2 + y_6 y_{10}^2 y_{11} y_{12} y_{15} + y_6 y_{10}^2 y_{12} y_{13}^2\\
&\quad + y_6 y_{10} y_{11}^2 y_{12} y_{14} + y_7^3 y_{13} y_{15}^2 + y_7^2 y_{11}^2 y_{13} y_{15}\\
&\quad + y_7^2 y_{11} y_{12} y_{13} y_{14} + y_7 y_8 y_{11}^2 y_{13} y_{14} + y_7 y_8 y_{11} y_{12} y_{13}^2\\
&\quad + y_8 y_{10}^3 y_{11} y_{15} + y_8 y_{10}^2 y_{11}^2 y_{14} + y_8 y_{10}^2 y_{11} y_{12} y_{13} + y_{10}^5 y_{14}\\
&\quad + y_{10}^4 y_{12}^2 + y_{10}^3 y_{11}^2 y_{12} + y_4^3 y_7 y_{15}^3 + y_4^3 y_{10} y_{13} y_{14} y_{15}\\
&\quad + y_4^3 y_{10} y_{14}^3 + y_4^3 y_{11}^2 y_{15}^2 + y_4^3 y_{11} y_{13}^2 y_{15} + y_4^3 y_{11} y_{13} y_{14}^2\\
&\quad + y_4^2 y_6 y_7 y_{14}^2 y_{15} + y_4^2 y_6 y_{10} y_{11} y_{14} y_{15} + y_4^2 y_6 y_{10} y_{12} y_{14}^2\\
&\quad + y_4^2 y_6 y_{11} y_{12}^2 y_{15} + y_4^2 y_6 y_{11} y_{12} y_{13} y_{14} + y_4^2 y_7^2 y_{12} y_{15}^2
\end{aligned}
$$

$$
\begin{aligned}
&+ y_4^2 y_7 y_8 y_{11} y_{15}^2 + y_4^2 y_7 y_{11} y_{12}^2 y_{14} + y_4^2 y_7 y_{11} y_{12} y_{13}^2 \\
&+ y_4^2 y_8^2 y_{13}^2 y_{14} + y_4^2 y_8 y_{10}^2 y_{14}^2 + y_4^2 y_8 y_{10} y_{11} y_{13} y_{14} \\
&+ y_4^2 y_8 y_{11} y_{12}^2 y_{13} + y_4^2 y_{10}^3 y_{11} y_{15} + y_4^2 y_{10}^3 y_{12} y_{14} \\
&+ y_4^2 y_{10}^2 y_{11} y_{12} y_{13} + y_4^2 y_{10} y_{11}^2 y_{12}^2 + y_4 y_6^3 y_{13} y_{14} y_{15} \\
&+ y_4 y_6^3 y_{14}^3 + y_4 y_6^2 y_7 y_{11} y_{15}^2 + y_4 y_6^2 y_{10}^2 y_{13} y_{15} \\
&+ y_4 y_6^2 y_{11}^2 y_{12} y_{14} + y_4 y_6 y_7^2 y_{11} y_{14} y_{15} + y_4 y_6 y_7^2 y_{12} y_{14}^2 \\
&+ y_4 y_6 y_7 y_8 y_{11} y_{13} y_{15} + y_4 y_6 y_7 y_8 y_{11} y_{14}^2 \\
&+ y_4 y_6 y_7 y_{11}^2 y_{12} y_{13} + y_4 y_6 y_8^2 y_{10} y_{13} y_{15} + y_4 y_7^3 y_{11} y_{13} y_{15} \\
&+ y_4 y_7^3 y_{12}^2 y_{15} + y_4 y_7^3 y_{12} y_{13} y_{14} + y_4 y_7^2 y_8 y_{11} y_{13} y_{14} \\
&+ y_4 y_7^2 y_{11}^2 y_{12}^2 + y_4 y_7 y_8^2 y_{11}^2 y_{15} + y_6^4 y_{12} y_{14}^2 \\
&+ y_6^3 y_7 y_{11} y_{13} y_{15} + y_6^3 y_7 y_{11} y_{14}^2 + y_6^3 y_8 y_{10} y_{13} y_{15} \\
&+ y_6^3 y_8 y_{10} y_{14}^2 + y_6^3 y_8 y_{12} y_{13}^2 + y_6^3 y_{10}^2 y_{11} y_{15} \\
&+ y_6^3 y_{10}^2 y_{12} y_{14} + y_6^3 y_{10} y_{11} y_{12} y_{13} + y_6^3 y_{11}^2 y_{12}^2 \\
&+ y_6^2 y_7^2 y_{11} y_{12} y_{15} + y_6^2 y_7^2 y_{11} y_{13} y_{14} + y_6^2 y_7^2 y_{12} y_{13}^2 \\
&+ y_6^2 y_7 y_8 y_{11}^2 y_{15} + y_6^2 y_8^2 y_{11} y_{12} y_{13} + y_6 y_7^4 y_{15}^2 \\
&+ y_6 y_7^3 y_{11}^2 y_{15} + y_6 y_7^3 y_{11} y_{12} y_{14} + y_6 y_7^2 y_8^2 y_{13} y_{15} \\
&+ y_6 y_7^2 y_8 y_{11}^2 y_{14} + y_6 y_7^2 y_8 y_{11} y_{12} y_{13} + y_7^4 y_8 y_{13} y_{15} \\
&+ y_7^4 y_{12}^3 + y_7^3 y_8^2 y_{13} y_{14} + y_7^3 y_8 y_{11} y_{12}^2 + y_7^2 y_8^2 y_{11}^2 y_{12} \\
&+ y_4^5 y_{14} y_{15}^2 + y_4^4 y_6 y_{13} y_{14} y_{15} + y_4^4 y_6 y_{14}^3 + y_4^4 y_7 y_{11} y_{15}^2 \\
&+ y_4^4 y_7 y_{12} y_{14} y_{15} + y_4^4 y_8 y_{10} y_{15}^2 + y_4^4 y_8 y_{12} y_{13} y_{15} \\
&+ y_4^4 y_{10} y_{11} y_{12} y_{15} + y_4^4 y_{11} y_{12}^2 y_{13} + y_4^3 y_6^2 y_{12} y_{13} y_{15} \\
&+ y_4^3 y_6 y_7 y_{11} y_{13} y_{15} + y_4^3 y_6 y_7 y_{11} y_{14}^2 + y_4^3 y_6 y_7 y_{12}^2 y_{15} \\
&+ y_4^3 y_6 y_8^2 y_{15}^2 + y_4^3 y_6 y_8 y_{10} y_{14}^2 + y_4^3 y_6 y_8 y_{11} y_{13} y_{14} \\
&+ y_4^3 y_6 y_{10}^2 y_{11} y_{15} + y_4^3 y_6 y_{10}^2 y_{12} y_{14} + y_4^3 y_6 y_{10} y_{11} y_{12} y_{13} \\
&+ y_4^3 y_6 y_{11}^2 y_{12}^2 + y_4^3 y_7 y_8^2 y_{14} y_{15} + y_4^2 y_6^3 y_8 y_{15}^2 + y_4^2 y_6^3 y_{10} y_{14}^2 \\
&+ y_4^2 y_6^3 y_{12}^2 y_{14} + y_4^2 y_6^3 y_{12} y_{13}^2 + y_4^2 y_6^2 y_7 y_8 y_{14} y_{15} \\
&+ y_4^2 y_6^2 y_8^2 y_{13} y_{15} + y_4^2 y_6^2 y_8^2 y_{14}^2 + y_4^2 y_6^2 y_8 y_{10} y_{12} y_{14} \\
&+ y_4^2 y_6^2 y_8 y_{11}^2 y_{14} + y_4^2 y_6^2 y_{10}^3 y_{14} + y_4^2 y_6^2 y_{10}^2 y_{12}^2
\end{aligned}
$$

$$
\begin{aligned}
&+ y_4^2y_6^2y_{10}y_{11}^2y_{12} + y_4^2y_6y_7^2y_{11}y_{12}y_{13} + y_4^2y_6y_7^2y_{12}^3 \\
&+ y_4^2y_6y_7y_8^2y_{12}y_{15} + y_4^2y_6y_7y_8^2y_{13}y_{14} + y_4^2y_6y_7y_8y_{11}y_{12}^2 \\
&+ y_4^2y_7^3y_{11}y_{12}^2 + y_4^2y_7^2y_8^2y_{11}y_{15} + y_4^2y_7^2y_8^2y_{12}y_{14} \\
&+ y_4^2y_8^4y_{11}y_{13} + y_4y_6^5y_{15}^2 + y_4y_6^4y_8y_{14}^2 + y_4y_6^4y_{10}y_{11}y_{15} \\
&+ y_4y_6^4y_{10}y_{12}y_{14} + y_4y_6^4y_{11}y_{12}y_{13} + y_4y_6^3y_7^2y_{13}y_{15} \\
&+ y_4y_6^3y_7^2y_{14}^2 + y_4y_6^3y_7y_{11}y_{12}^2 + y_4y_6^2y_7^3y_{12}y_{15} \\
&+ y_4y_6^2y_7^3y_{13}y_{14} + y_4y_6^2y_7^2y_8y_{11}y_{15} + y_4y_6^2y_7^2y_8y_{12}y_{14} \\
&+ y_4y_6y_7^4y_{11}y_{15} + y_4y_6y_7^4y_{12}y_{14} + y_4y_6y_7^3y_8y_{11}y_{14} \\
&+ y_4y_7^3y_8^3y_{15} + y_4y_7^3y_8^2y_{11}y_{12} + y_6^5y_8y_{11}y_{15} + y_6^5y_8y_{12}y_{14} \\
&+ y_6^4y_7^2y_{13}^2 + y_6^4y_7y_8y_{12}y_{13} + y_6^4y_8^2y_{10}y_{14} + y_6^4y_8^2y_{12}^2 \\
&+ y_6^4y_{10}^4 + y_6^3y_7^2y_8y_{12}^2 + y_6^3y_7y_8^3y_{15} + y_6^3y_7y_8^2y_{11}y_{12} \\
&+ y_6^2y_7^3y_8^2y_{15} + y_6^2y_7^2y_8^3y_{14} + y_6^2y_7y_8^4y_{13} + y_6^2y_8^4y_{10}^2 \\
&+ y_6y_7^5y_8y_{15} + y_6y_7^4y_8^2y_{14} + y_7^6y_{11}^2 + y_7^4y_8^3y_{12} + y_7^3y_8^4y_{11} \\
&+ y_4^5y_7y_{11}^2y_{15} + y_4^5y_{10}^2y_{11}y_{13} + y_4^5y_{11}^4 + y_4^4y_6y_7y_{10}^2y_{15} \\
&+ y_4^4y_6y_7y_{11}^2y_{13} + y_4^4y_7^2y_{10}^2y_{14} + y_4^4y_7^2y_{11}^2y_{12} \\
&+ y_4^4y_7y_8y_{10}^2y_{13} + y_4^4y_7y_8y_{11}^3 + y_4^7y_6y_{15}^2 + y_4^7y_7y_{14}y_{15} \\
&+ y_4^7y_{10}y_{11}y_{15} + y_4^7y_{10}y_{13}^2 + y_4^7y_{11}^2y_{14} + y_4^7y_{11}y_{12}y_{13} \\
&+ y_4^6y_6^2y_{13}y_{15} + y_4^6y_6^2y_{14}^2 + y_4^6y_6y_7y_{12}y_{15} + y_4^6y_6y_7y_{13}y_{14} \\
&+ y_4^6y_6y_8y_{13}^2 + y_4^6y_7^2y_{12}y_{14} + y_4^6y_7^2y_{13}^2 + y_4^6y_7y_8y_{10}y_{15} \\
&+ y_4^6y_7y_8y_{11}y_{14} + y_4^6y_7y_8y_{12}y_{13} + y_4^6y_8^2y_{11}y_{13} + y_4^6y_{10}^4 \\
&+ y_4^4y_6^4y_{11}y_{13} + y_4^4y_6^4y_{12}^2 + y_4^4y_6^2y_8^2y_{10}^2 + y_4^4y_7^5y_{13} \\
&+ y_4^{10}y_{11}y_{13} + y_4^8y_6^2y_7y_{13} + y_4^8y_6^2y_{10}^2 + y_4^8y_7^3y_{11} + y_4^4y_6^8 \\
&+ y_4^{16},
\end{aligned}
$$

$$
\begin{aligned}
w_{128}(\Delta_{15} \oplus \lambda_{15}^1 \oplus \lambda_{15}^2) &= w_{128}(\Delta_{15}) + w_{112}(\Delta_{15})w_{16}(\lambda_{15}^1 \oplus \lambda_{15}^2) \\
&\quad + w_{96}(\Delta_{15})w_{32}(\lambda_{15}^1 \oplus \lambda_{15}^2) + w_{64}(\Delta_{15})w_{64}(\lambda_{15}^1 \oplus \lambda_{15}^2) \\
&\equiv u_{128} \quad \text{mod decomposables.}
\end{aligned}
$$

Since the element $w_{128}(\Delta_{15} \oplus \lambda^1_{15} \oplus \lambda^2_{15})$ is a member of a system of generators of $H^*(B\,\mathrm{Spin}(15); \mathbb{Z}/2)$ as an algebra over the Steenrod algebra, we obtain the following corollary.

Corollary 5.2 *The Stiefel–Whitney class $w_{128}(\mathrm{Ad}_{E_8})$ of the adjoint representation $\mathrm{Ad}_{E_8}\colon E_8 \to SO(248)$ can be chosen as a member of a system of generators of $H^*(BE_8; \mathbb{Z}/2)$ as an algebra over the Steenrod algebra.*

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