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Evens norm, transfers and characteristic classes for extraspecial p–groups

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Let P be the extraspecial p–group of order p^{2n+1}, of p–rank $n+1$, and of exponent p if $p>2$. Let Z be the center of P and let $\kappa_{n,r}$ be the characteristic classes of degree 2^n-2^r (resp. $2(p^n-p^r)$) for $p=2$ (resp. $p>2$), $0\le r\le n-1$, of a degree p^n faithful irreducible representation of P. It is known that, modulo nilradical, the ιth powers of the $\kappa_{n,r}$'s belong to $\mathcal{T}=\operatorname{Im}\big(H^*(P/Z,\mathbf{F}_p)/\sqrt{0}\xrightarrow{\operatorname{Inf}}H^*(P,\mathbf{F}_p)/\sqrt{0}\big)$, with $\iota=1$ if $p=2$, $\iota=p$ if $p>2$. We obtain formulae in $H^*(P,\mathbf{F}_p)/\sqrt{0}$ relating the $\kappa_{n,r}^{\iota}$ terms to the ones of fewer variables. For $p>2$ and for a given sequence $r_0,\dots,r_{n-1}$ of non-negative integers, we also prove that, modulo-nilradical, the element $\prod_{r_i}\kappa_{n,i}^{r_i}$ belongs to $\mathcal{T}$ if and only if either $r_0\ge 2$, or all the r_i are multiple of p. This gives the determination of the subring of invariants of the symplectic group $Sp_{2n}(\mathbf{F}_p)$ in $\mathcal{T}$.

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1 Introduction

Let p be a prime number. For a given group P , denote by $H^*(P)$ the mod–p cohomology algebra of P. We are interested in the case where $P=P_n$, the extraspecial group p–group of order p^{2n+1}, of p–rank $n+1$, and of exponent p if $p>2$. It is known (see the work of Green–Leary [4] or Quillen[12]) that, for $p>2$ (resp. $p=2$), there are exactly n Chern (resp. Stiefel–Wihitney) classes $\kappa_{n,r}$ of degree $2(p^n-p^r)$ (resp. 2^n-2^r), $0\le r\le n-1$, of a degree p^n faithful irreducible representation of P; these classes restrict to maximal elementary abelian subgroups of P as Dickson invariants.

Set $E=E_n=P/Z$, with Z the center P, E is then a vector space of dimension $2n$ over $\mathbf{F}_p$. Set $h^*(P)=H^*(P)/\sqrt{0}$ (so $h^*(P)=H^*(P)$ for $p=2$, by [12] and denote by $\mathcal{T}=\mathcal{T}_n$ the subring of $h^*(P)$ equal to the image of the inflation Inf_P^E, modulo nilradical. For $p=2$, it follows from [12] that all the $\kappa_{n,r}$ terms belong to $\mathcal{T}$. For $p>2$, this fact does not hold, as shown by Green and Minh [3; 5]; however, in [5], it is also proved that all pth powers of the $\kappa_{n,r}$ terms, are in turn, belonging to $\mathcal{T}$.

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For convenience, set $\iota = 1$ for $p = 2$, and $\iota = p$ for $p > 2$. It follows that, for $0 \le r \le n-1$, there exists $f_{n,r} \in H^*(E)$ such that $\mathrm{Inf}(f_{n,r}) \doteqdot \kappa_{n,r}^{\iota}$. Here and in what follows $a \doteqdot b$ means $a = b$ modulo $\sqrt{0}$. Via the inflation map, as elements of $h^*(P)$, $f_{n,r}$ can be identified with $\kappa_{n,r}^{\iota}$, $0 \le r \le n-1$. The first aim of this paper is to get an alternating formula expressing $f_{n,r}$ by means of $f_{n-1,r-1}$ and $f_{n-1,r}$. This work is motivated from the elegant formula, for $p > 2$ (resp. $p = 2$), expressing Chern (resp. Stiefel–Whitney) classes of the regular representation r_A of the elementary abelian p–group A via such classes of fewer variables. It is known that, if A is of rank m and $p > 2$ (resp. $p = 2$), then the $2(p^m - p^r)$th Chern (resp. $(2^m - 2^r)$th Stiefel–Whitney) class of r_A is the Dickson invariant $Q_{m,r}$ of the same degree with variables in a basis $x_1, \dots, x_m$ of $\beta H^1(A)$ (resp. $H^1(A)$), with the Bockstein homomorphism. These invariants are related by

$$Q_{m,r} = Q_{m-1,r} V_m^{p-1} + Q_{m-1,r-1}^p,$$

where

$$V_m = \prod_{\lambda_i \in \mathbf{F}_p} (\lambda_1 x_1 + \cdots + \lambda_{m-1} x_{m-1} + x_m) = (-1)^{m-1} \sum_{s=0}^{m-1} (-1)^s Q_{m-1,s} x_m^{p^s},$$

is the Mùi invariant.

In so doing, we need to use the Evens norm and transfers from maximal subgroups of P. Some basic properties of the Evens norm, in the relation with modular invariants, are recalled in Section 2. In Section 3, we show how to obtain characteristic classes of P by means of the Evens norm (Theorem 3.7). Theorem 3.8 describes the image of such classes via the Evens norm. From this, we obtain formulae relating characteristic classes with such classes of fewer variables (Corollary 3.9).

Let $r_0, \dots, r_{n-1}$ be a sequence of non-negative integers. In Section 4, we prove that, for $p > 2$, modulo nilradical, the product $\prod_{i \ge 0} \kappa_{n,i}^{r_i}$ belongs to $\mathcal{T}$ if and only if either $r_0 \ge 2$, or all the r_i are multiple of p Theorem 4.1. This generalizes a result, given by Green and Leary [3; 4], proving that $\kappa_{n,0}^s$ belongs to $\mathcal{T}$ provided either $s \ge 2^n$, or $s \ge 2$ and $n \le 2$. As a consequence, we obtain in the last section the determination of the subring of invariants of the symplectic group in $\mathcal{T}$ Theorem 5.1.

For convenience, given a subgroup K of a group G, any element of $H^*(G)$ is also considered as an element of $H^*(K)$ via the restriction map Res_K^G. Also, if K is normal in G, then any element of $H^*(G/K)$ can be considered as an element of $H^*(G)$ via the inflation map $\mathrm{Inf}_K^{G/K}$.

2 Evens norm and Mùi invariants

Given a polynomial algebra $F = \mathbf{F}_p[t_1, \dots, t_k]$ and $1 \le m \le k-1$, define the Mùi invariant [10]

$$V_{m+1} = V_{m+1}(t_1, \dots, t_{m+1}) = \prod_{\lambda_i \in \mathbf{F}_p} (\lambda_1 t_1 + \dots + \lambda_m t_m + t_{m+1}). \tag{1}$$

It follows from the work of L E Dickson [1] that

$$V_{m+1} = (-1)^m \sum_{s=0}^{m} (-1)^s Q_{m,s} t_{m+1}^{p^s}$$

with $Q_{m,s} = Q_{m,s}(t_1, \dots, t_m)$ the Dickson invariants defined inductively as follows (we shall omit the variables, if no confusion can arise).

$$\begin{aligned} Q_{m,m} &= 1 \\ Q_{m,0} &= \prod_{\substack{\lambda_j \in \mathbf{F}_p \\ \lambda_j \text{ not all equal } 0}} (\lambda_1 t_1 + \dots + \lambda_m t_m) \\ Q_{m,s} &= Q_{m-1,s} V_m^{p-1} + Q_{m-1,s-1}^p. \end{aligned}$$

By (1) the $Q_{m,s}$ are independent of the choice of generators $t_1, \dots, t_m$ of $\mathbf{F}_p[t_1, \dots, t_m]$. Hence, if $(t_1, \dots, t_m)$ is a basis of $H^1(W)$ (resp. $\beta H^1(W)$) with W an elementary abelian 2–group (resp. p–group with $p > 2$) of rank m, we may write

$$\begin{aligned} Q_{m,s}(t_1, \dots, t_m) &= Q_s(W) \\ V_{m+1}(t_1, \dots, t_k, X) &= V(W, X). \end{aligned}$$

The Mùi invariants can be obtained by means of Evens norm map $\mathcal{N}_{U \to W}$ with U a subgroup of W (see Corollary 2.2 below). Let us recall that, for every maximal subgroup K of a p–group G, and for $\xi \in H^r(K)$, we may define the Evens norm map

$$\mathcal{N}_{K \to G}(\xi) \in H^{pr}(G).$$

Here are some properties of $\mathcal{N}_{K \to G}$. For details of the proof, the reader can refer to the work of Evens [2], Minh [9] or Mùi [10].

Proposition 2.1 *Let G, G' be p–groups and let K be a subgroup of G.*

(i) *If N is a subgroup of K, then*

$$\mathcal{N}_{N \to G} = \mathcal{N}_{K \to G} \circ \mathcal{N}_{N \to K}.$$

(ii) *If H is a subgroup of G and $G=\coprod_{x\in D} KxH$, then, for $\xi\in H^r(G)$,*

$$\mathrm{Res}^G_H \mathcal{N}_{K\to G}(\xi)=\prod_{x\in D}\mathcal{N}_{H\cap {}^xK\to H}\mathrm{Res}^{{}^xK}_{H\cap {}^xK}({}^x\xi).$$

(iii) *If K is a subgroup of G' and $f\colon G'\to G$ is a homomorphism such that $f(K')\subset K$ and f induces a bijection $G'/K'\cong G/K$ of coset spaces, then, for $\xi\in H^r(K)$,*

$$\mathcal{N}_{K'\to G'}(f_{|K'})^*(\xi)=f^*\left(\mathcal{N}_{K\to G}(\xi)\right).$$

In particular, if N is a normal subgroup of G and $N\subset K$, then, for $\xi\in H^r(K/N)$,

$$\mathcal{N}_{K\to G}\mathrm{Inf}^{K/N}_K(\xi)=\mathrm{Inf}^{G/N}_G\mathcal{N}_{K/N\to G/N}(\xi).$$

(iv) *If $\xi,\xi'\in H^n(K)$, then*

$$\mathcal{N}_{K\to G}(\xi+\xi')=\mathcal{N}_{K\to G}(\xi)+\mathcal{N}_{K\to G}(\xi')$$

modulo a sum of transfers from proper subgroups of G containing the intersection of the conjugates of K. Hence, the norm map is in general non-additive, although $\mathcal{N}_{K\to G}\circ\mathrm{Res}^G_K$ is.

(v) *If $\xi\in H^r(K)$, $\xi'\in H^s(K)$, and $[G:K]=n$, then*

$$\mathcal{N}_{K\to G}(\xi.\xi')=(-1)^{\frac{n(n-1)}{2}rs}\mathcal{N}_{K\to G}(\xi)\mathcal{N}_{K\to G}(\xi').$$

(vi) *Assume that $G=K\times E$, with $E=(\mathbf{F}_p)^m$. Consider E as the group of all translations on a vector space S of dimension m over $\mathbf{F}_p$ and let $W(m)$ be an E–free acyclic complex with augmentation $\epsilon\colon W(m)\to\mathbf{F}_p$. Let C be a cochain complex of which the cohomology is $H^*(K)$ and set $C^S=\otimes_{c\in S}C_c$, $C_c=C$. Then*

$$\mathcal{N}_{K\to K\times E}=d^*_m P_m,$$

where $P_m\colon H^r(C)\to H_E^{p^m r}\left(W(m)\otimes C^S\right)$ is the Steenrod power map, and $d^*_m\colon H_E^{p^m r}\left(W(m)\otimes C^S\right)\to H^*(E)\otimes H^*(C)$ is induced by the diagonal $C\to C^S$ and the Künneth formula.

In the rest of this section, suppose that W is an elementary abelian p–group of rank $n+1$ and U a subgroup of W of index p^m.

By Proposition 2.1(vi), $\mathcal{N}_{U\to W}=d^*_m P_m$. The first part of the following corollary is then originally due to Mùi [10] and reproved by Okuyama and Sasaki [11]; the second one was given by Hưng and Minh [6, Proof of Theorem B].

Corollary 2.2 *For $p = 2$ (resp. $p > 2$) and for every $x \in H^1(W)$ (resp. $x \in \beta H^1(W)$),*

(i) $\mathcal{N}_{U\to W}\big(\mathrm{Res}^W_U(x)\big) = V(W/U, x)$

(ii) *with T the maximal subgroup of W satisfying $\mathrm{Res}^W_T(x) = 0$ then*

$$(-1)^r \mathcal{N}_{T\to W}(Q_r(T)) = \sum_{i=r}^{n} (-1)^i Q_i^p(T) x^{p^{i+1}-p^{r+1}} - \Big[\sum_{i=r+1}^{n} (-1)^i Q_i(T) x^{p^i - p^{r+1}}\Big]\Big[\sum_{i=0}^{n} (-1)^i Q_i(T) x^{p^i}\Big]^{p-1}.$$

In the following corollary, G is supposed to be a p–group given by a central extension and

$$\{0\} \longrightarrow \mathbb{Z}/p \longrightarrow G \stackrel{j}{\longrightarrow} W \longrightarrow \{0\}$$

and $K = j^{-1}(U)$. Set

$$H^{ev}(U) = \begin{cases} H^*(U) & p = 2, \\ \sum_{n\ge 0} H^{2n}(U) & p > 2. \end{cases}$$

The following is straightforward from Proposition 2.1.

Corollary 2.3 *The composition map*

$$H^{ev}(U) \stackrel{\mathrm{Inf}^U_K}{\longrightarrow} H^*(K) \stackrel{\mathcal{N}_{K\to G}}{\longrightarrow} H^*(G)$$

is a ring homomorphism.

In [9] we proved the following proposition.

Proposition 2.4 *Let $\xi \in H^q(G)$. Set $\mu(q) = (-1)^{hq} h!$ with $h = (p-1)/2$ for $p > 2$. If $K = \ker(u)$ with $u \in H^1(G)$, $u \neq 0$, then, by setting $v = \beta(u)$, we have*

$$\mathcal{N}_{K\to G}(\mathrm{Res}^G_K(\xi)) = \begin{cases} \sum_i Sq^i(\xi) u^{q-i} & \text{p=2} \\ \mu(q) \displaystyle\sum_{\substack{\epsilon=0,1 \\ 0\le 2i\le q-\epsilon}} (-1)^{\epsilon+i} \beta^\epsilon \mathcal{P}^i(\xi) v^{(q-2i)h-\epsilon} u^\epsilon & \text{p>2.} \end{cases}$$

where Sq^i (resp. $\mathcal{P}^i$) denotes the Steenrod operation for $p = 2$ (resp. $p > 2$).

3 Characteristic classes for extraspecial p–groups

Let $E = E_n$, $n \geq 1$, be the elementary abelian p–group of rank $2n$. Let $x_1, \ldots, x_{2n}$ be a basis of $H^1(E) = \mathrm{Hom}(E, \mathbf{F}_p)$ and define

$$y_i = \begin{cases} x_i & p = 2 \\ \beta(x_i) & p > 2. \end{cases}$$

$1 \leq i \leq 2n$, with β the Bockstein homomorphism. We have

$$H^*(E) = \begin{cases} \mathbf{F}_p[y_1, \ldots, y_{2n}] & p = 2 \\ \Lambda[x_1, \ldots, x_{2n}] \otimes \mathbf{F}_p[y_1, \ldots, y_{2n}] & p > 2, \end{cases}$$

with $\Lambda[s, t, \ldots]$ (resp. $\mathbf{F}_p[s, t, \ldots]$) the exterior (resp. polynomial) algebra with generators $s, t, \ldots$ over $\mathbf{F}_p$. Let $P = P_n$ be the extraspecial p–group given by the central extension

$$\{1\} \longrightarrow Z/p \xrightarrow{i} P_n \longrightarrow E \longrightarrow \{1\}$$

classified by the cohomology class $x_1x_2 + \cdots + x_{2n-1}x_{2n} \in H^2(E)$. The following notation will be used. Set $Z = i(\mathbb{Z}/p)$, the center of P. For every elementary subgroup A of P containing Z, write $A/Z = Z'$, so $A = A' \times Z$, and A' is of rank n if A maximal elementary abelian in P. Fix a generator γ of $H^1(Z)$ (resp. $\beta H^1(Z)$) for $p = 2$ (resp. $p > 2$). This element, and also every element of $H^*(A')$, are then considered as elements of $H^*(A)$ via the inflation maps.

Denote by $\mathcal{A}$ the set of maximal elementary abelian subgroups of P. Set $h^*(P) = H^*(P)/\sqrt{0}$. By the work of Quillen [12], the map induced by the restrictions

$$h^*(P) \xrightarrow{\mathrm{Res}} \prod_{A \in \mathcal{A}} h^*(A)$$

is injective. Therefore the maps

$$h^*(E) \xrightarrow{\mathrm{Inf}_P^E} h^*(P) \qquad \text{and} \qquad h^*(E) \xrightarrow{\mathrm{Res}} \prod_{A \in \mathcal{A}} h^*(A')$$

have the same kernel. Let $\mathcal{T} = \mathcal{T}_n$ be the subring of $h^*(P)$ equal to the image of the inflation Inf_P^E. For elements ξ, η of $h^*(E)$, it follows that $\mathrm{Inf}_P^E(\xi) = \mathrm{Inf}_P^E(\eta)$ if and only if $\mathrm{Res}_{A'}^E(\xi) = \mathrm{Res}_{A'}^E P(\eta)$, for every $A \in \mathcal{A}$.

We are now interested in Chern (resp. Stiefel–Whitney) classes, for $p > 2$ (resp $p = 2$), of a degree p^n faithful irreducible representation of P. Fix a nontrivial linear character χ of Z. We have then an irreducible character $\hat{\chi}$ of P given by

$$\hat{\chi}(g) = \begin{cases} p^n \chi(g) & g \in Z \\ 0 & \text{otherwise.} \end{cases}$$

Let ρ be a representation affording the character $\hat{\chi}$ and set

$$\zeta = \zeta_n = \begin{cases} c_{p^n}(\rho) & p > 2 \\ w_{2^n}(\rho) & p = 2, \end{cases}$$

$$\kappa_{n,r} = \begin{cases} (-1)^{n-r} c_{p^n - p^r}(\rho) & p > 2 \\ w_{2^n - 2^r}(\rho) & p = 2, \end{cases}$$

$0 \le r \le n$. We have the following theorem.

Theorem 3.1 (Green–Leary [4], Quillen [12])

(i) *In $h^*(P)$,*

$$1 - \kappa_{n,n-1} + \cdots + (-1)^n \kappa_{n,0} + \zeta_n = \begin{cases} c(\rho) & p>2 \\ w(\rho) & p=2, \end{cases}$$

the subring of $h^*(P)$ generated by non-nilpotent Chern classes is generated by

$$\kappa_{n,0}, \ldots, \kappa_{n,n-1}, y_1, \ldots, y_{2n}, \zeta_n.$$

(ii) *For every $0 \le i \le n-1$ and for every $A \in \mathcal{A}$*

$$\mathrm{Res}^P_A(\zeta) = V(A', \gamma)$$
$$\mathrm{Res}^P_A(\kappa_{n,i}) = Q_i(A').$$

In the article [5] by Green and Minh, Chern classes of P are also obtained by means of the inflation Inf^E_P and transfer maps tr^K_P with K maximal in P. Similar results for the case $p = 2$ can also be obtained by using the same argument. The result can be stated as follows. Let x be a non-zero element of $H^1(P)$ and set $H_x = \ker(x)$. Pick a rank one subgroup $U \neq Z$ of the center of H_x. So $H_x = P_{n-1} \times U$. By the Künneth formula, we can consider any element of $H^*(P_{n-1})$ (and of $H^*(U)$) as an element of $H^*(H_x)$. For $0 \le r \le n-1$, set

$$\chi_{r,x} = \begin{cases} \mathrm{tr}^{H_x}_P(\kappa_{n-1,r}\zeta^{p-1}_{n-1}) & n \ge 2 \\ \mathrm{tr}^{H_x}_P(\zeta^{p-1}_{n-1}) & n = 1. \end{cases}$$

For $r \geq 0$, define

$$z_n^{(r)} = \begin{cases} \Sigma_{i=1}^{i=n} y_{2i} y_{2i-1} & p = 2, r = 0 \\ \Sigma_{i=1}^{i=n} (y_{2i-1}^{\iota p^r} y_{2i} - y_{2i-1} y_{2i}^{\iota p^r}) & \text{otherwise,} \end{cases}$$

with $\iota = 1$ for $p = 2$, and $\iota = p$ for $p > 2$. Let A_0 be an element of $\mathcal{A}$. We have the following theorem.

Theorem 3.2 (Green–Minh [5])

(i) *In $H^*(P)$ for $0 \leq r \leq n-1$,*

$$\kappa_{n,r} = Q_r(P/A_0) - \sum_{x \in \mathbb{P}H^1(P/A_0)} \chi_{r,x}.$$

(ii) *There exist $f_{n,0}, \dots, f_{n,n-1} \in H^*(E)$, viewed as elements of $H^*(P)$ via the inflation map, such that*

$$z_n^{(n)} + \sum_{i=0}^{n-1} (-1)^{n-i} z_n^{(i)} f_{n,i} = 0,$$

and, for every $A \in \mathcal{A}$, $\mathrm{Res}_A^P(\kappa_{n,r}^\iota) = \mathrm{Res}_A^P(f_{n,r})$, $0 \leq r \leq n-1$.

(iii) *There exist h_i, $0 \leq i \leq n-1$, and a unique η of $H^*(E)$ such that*

$$z_n^{(n-1)} = y_{2n}\eta + \sum_{i=0}^{n-2} h_i z_n^{(i)},$$

and in $h^*(P)$, $\chi_{n-1,x_{2n}}^\iota = -\mathrm{Inf}(\eta^{p-1})$. Furthermore, for all $0 \leq r \leq n-1$ and all $\phi \in \mathbb{P}H^1(E)$, $\chi_{r,\phi}^\iota \in \mathrm{Im}(\mathrm{Inf}_P^E)$, as elements of $h^*(P)$.

By Quillen [12] it is known that, for $p = 2$, all the $\kappa_{n,r}$ and $\chi_{r,\phi}$ belong to $\mathcal{T}$. For $p > 2$, it follows that the above theorem that all p^{th}–powers of the $\kappa_{n,r}$ and $\chi_{r,\phi}$ belong to $\mathcal{T}$. In fact, by setting

$$\varphi = \begin{cases} y_{2n-1}^p - y_{2n-1} y_{2n}^{p-1} & p > 2 \\ y_{2n-1} & p = 2, \end{cases}$$

we have the following corollary.

Corollary 3.3 *In $h^*(P)$,*

(i) $\kappa_{n,r}^\iota = f_{n,r}$, $0 \leq r \leq n-1$

(ii)

$$\eta = (-1)^{n-1}\left[\varphi f_{n-1,0} + \sum_{i=1}^{n-1}(-1)^i\left(y_{2n-1}^{\iota p^i} - y_{2n-1}y_{2n}^{\iota p^i-1}\right)f_{n-1,i}\right]$$

(iii) *with* $K = \ker(x_{2n})$,

$$\eta = \begin{cases}\mathrm{tr}_P^K(\zeta_{n-1}) & p=2\\ (-1)^{n-1}\mathrm{tr}_P^K\left[\zeta_{n-1}^{p-1}\sum_{i=0}^{n-1}(-1)^i y_{2n-1}^{p^i}\kappa_{n-1,i}\right] & p>2;\end{cases}$$

(iv) *for* $0 \le r \le n-1$,

$$\chi^{\iota}_{r,x_{2n}} = -f_{n-1,r}\left[\varphi f_{n-1,0} + \sum_{i=1}^{n-1}(-1)^i\left(y_{2n-1}^{\iota p^i} - y_{2n-1}y_{2n}^{\iota p^i-1}\right)f_{n-1,i}\right]^{p-1}.$$

Proof Part (i) follows from Theorem 3.2 (ii), by noting that the restriction map from $h^*(P)$ to $\prod_{A\in\mathcal{A}} H^*A$ is injective.

We have, by Theorem 3.2,

$$\begin{aligned}
z_n^{(n-1)} &= z_{n-1}^{(n-1)} + \left(y_{2n-1}^{\iota p^{n-1}}y_{2n} - y_{2n-1}y_{2n}^{\iota p^{n-1}}\right)\\
&= \left(y_{2n-1}^{\iota p^{n-1}}y_{2n} - y_{2n-1}y_{2n}^{\iota p^{n-1}}\right) + (-1)^n\sum_{i=0}^{n-2}(-1)^i z_{n-1}^{(i)} f_{n-1,i}\\
&= \left(y_{2n-1}^{\iota p^{n-1}}y_{2n} - y_{2n-1}y_{2n}^{\iota p^{n-1}}\right)\\
&\qquad + (-1)^n\Bigg[\sum_{i=0}^{n-2}(-1)^i z_n^{(i)} f_{n-1,i} - y_{2n}\varphi f_{n-1,0}\\
&\qquad\qquad - \sum_{i=1}^{n-2}(-1)^i\left(y_{2n-1}^{\iota p^i}y_{2n} - y_{2n-1}y_{2n}^{\iota p^i}\right)f_{n-1,i}\Bigg]\\
&= (-1)^n\left[\sum_{i=0}^{n-2}(-1)^i z_n^{(i)} f_{n-1,i} - y_{2n}X\right],
\end{aligned}$$

with

$$X = \varphi f_{n-1,0} + \sum_{i=1}^{n-1}(-1)^i(y_{2n-1}^{\iota p^i} - y_{2n-1}y_{2n}^{\iota p^i-1})f_{n-1,i}.$$

So $\eta = (-1)^{n+1}X$; (ii) and (iv) are proved.

Pick an element $A \in \mathcal{A}$. By [5, Lemma 7.1] and its proof, we have

$$\mathrm{Res}^P_A \eta = \begin{cases} -V(B', y_{2n-1})^\iota & A \subseteq K \\ 0 & \text{otherwise.} \end{cases}$$

Set

$$Y = \begin{cases} \mathrm{tr}^K_P(\zeta_{n-1}) & p = 2 \\ (-1)^{n-1}\mathrm{tr}^K_P[\zeta^{p-1}_{n-1}\sum_{i=0}^{n-1} y^{p^i}_{2n-1}\kappa_{n-1,i}] & p > 2. \end{cases}$$

If $A \not\subset K$, then $\mathrm{Res}^P_A(Y) = 0$, by the Mackey formula. Suppose $A \subset K$. Set $B = A \cap P_{n-1}$. For $p > 2$, we have

$$\begin{aligned} \mathrm{Res}^P_A(Y) &= (-1)^{n-1} \sum_{g \in P/K} {}^g \sum_{i=0}^{n-1} \mathrm{Res}^K_A((-1)^i y^{p^i}_{2n-1}\kappa_{n-1,i}\zeta^{p-1}_{n-1}) \\ &= (-1)^{n-1} \sum_{g \in P/K} {}^g \sum_{i=0}^{n-1} (-1)^i y^{p^i}_{2n-1} Q_i(B')\zeta^{p-1}_{n-1} \\ &= (-1)^{n-1} \sum_{i=0}^{n-1} (-1)^i y^{p^i}_{2n-1} Q_i(B') \sum_{g \in P/K} {}^g\zeta^{p-1}_{n-1} \end{aligned}$$

since the y_{2n-1} and the $Q_i(B')$ are invariant under the action of P/K. Thus

$$\mathrm{Res}^P_A(Y) = \begin{cases} \mathrm{Res}^P_A(\mathrm{tr}^K_P(\zeta_{n-1})) & p = 2 \\ V(B', y_{2n-1})\mathrm{Res}^P_K(\mathrm{tr}^K_P(\zeta^{p-1}_{n-1})) & p > 2. \end{cases}$$

Following [5, Proposition 4.4] we have

$$\mathrm{Res}^P_A(\mathrm{tr}^K_P(\zeta^{p-1}_{n-1})) = -V(B', y_{2n-1})^{p-1}.$$

So $\mathrm{Res}^P_A(Y) = -V(B', y_{2n-1})^\iota$.

Since $\eta - Y$ restricts trivially to every element of $\mathcal{A}$, it follows that $\eta \doteqdot Y$. □

Proposition 3.4 *For* $0 \le r \le n-1$,

$$\kappa_{n,r} \doteqdot - \sum_{x \in \mathbb{P}H^1(P)} \chi_{r,x}.$$

Proof Let A be an element of $\mathcal{A}$. There exist exactly $\frac{p^n-1}{p-1}$ elements of $\mathbb{P}H^1(P)$ of which the kernel contains A. The subset of those elements is nothing but $\mathbb{P}H^1(P/A)$.

Let x be an element of $\mathbb{P}H^1(P)$. It is clear that $\mathrm{Res}^P_A(\chi_{r,x}) = 0$ if $x \notin \mathbb{P}H^1(P/A)$.

Hence

$$\mathrm{Res}^P_A\Big(\sum_{x\in\mathbb{P}H^1(P)}\chi_{r,x}\Big)=\mathrm{Res}^P_A\Big(\sum_{x\in\mathbb{P}H^1(P/A)}\chi_{r,x}\Big).$$

Therefore, by [5, Theorem 5.2]

$$\begin{aligned}\mathrm{Res}^P_A\Big(\sum_{x\in\mathbb{P}H^1(P)}\chi_{r,x}\Big)&=\mathrm{Res}^P_A\Big(Q_r(P/A)-\kappa_{n,r}\Big)\\&=-Q_r(A'),\end{aligned}$$

since $\mathrm{Res}^P_A(Q_r(P/A))=0$ and $\mathrm{Res}^P_A(\kappa_{n,r})=Q_r(A')$. The proposition follows. □

We are now going to obtain characteristic classes of P by using the Evens norm map. We first need the following lemma.

Lemma 3.5 *Fix a generator e of Z. Let $H=A\cap B$ with $A,B\in\mathcal{A}$ and let $(h_1,\dots,h_k,e)$ be a basis of H. Then there exist elements $g_1,\dots,g_k$ of P satisfying*

(i) $[g,g_i]=1$, $[g_i,h_j]=\begin{cases}1 & i\neq j\\ e & i=j\end{cases}$ and $1\le i,j\le k$.

(ii) $P=\coprod_{g\in G}AgB$ *is a double coset decomposition of P with $G=\langle g_1,\dots,g_k\rangle$.*

Proof The existence of the g_j satisfying (i) follows from [8]. Assume that $agb=a'g'b'$ with $a,a'\in A$, $b,b'\in B$, $g,g'\in G$. It follows that $[g,h_i]=[g',h_i]$ and $1\le i\le k$, hence $g=g'$. As $|G|=p^k$, (ii) is obtained. □

The following notation will be used. Let C be the cyclic group of order p and fix a generator u of $H^1(C)$ (resp. $H^2(C)$) for $p=2$ (resp. $p>2$). Set $\Gamma=P\times C$. If H is a subgroup of P, every element of $H^*(H)$ (resp. $H^*(C)$) can be considered as an element of $H^*(H\times C)$. We have the following lemma.

Lemma 3.6 *Let A, B be elements of $\mathcal{A}$ and let $v\in H^1(A\times C)$ (resp. $H^2(A\times C)$) for $p=2$ (resp. $p>2$). Assume that $\mathrm{Res}^{A\times C}_{Z\times C}(v)=\mu\gamma+\lambda u$ with $\mu,\lambda\in\mathbf{F}_p$, then*

$$\mathrm{Res}^{\Gamma}_{B\times C}\mathcal{N}_{A\times C\to\Gamma}(v)=\mu V(B',\gamma)+\lambda V(B',u).$$

Proof Let $P=\cup_{g\in G}AgB$ be the double coset decomposition of P given in Lemma 3.5. Set $H=B\cap A$. We have

$$\begin{aligned}
\mathrm{Res}^{\Gamma}_{B\times C}\mathcal{N}_{A\times C\to\Gamma}(v) &= \prod_{g\in G}\mathcal{N}_{(B\cap A^g)\times C\to B\times C}\mathrm{Res}^{A^g\times C}_{(B\cap A^g)\times C}({}^g v)\\
&= \prod_{g\in G}\mathcal{N}_{H\times C\to B\times C}\circ\mathrm{Res}^{A\times C}_{H\times C}({}^g v) && (A \text{ is normal})\\
&= \mathcal{N}_{H\times C\to B\times C}\Big(\prod_{g\in G}\mathrm{Res}^{A\times C}_{H\times C}({}^g v)\Big) && (\text{Corollary 2.3})\\
&= \mathcal{N}_{H\times C\longrightarrow B\times C}\Big(\prod_{g\in G}(\mu^g\gamma+\lambda u)\Big)\\
&= \mathcal{N}_{H\times C\longrightarrow B\times C}\big(V(H',\mu\gamma+\lambda u)\big) && (\text{Lemma 3.5})\\
&= \mathcal{N}_{Z\times C\longrightarrow B\times C}(\mu\chi+\lambda u) && (\text{Corollary 2.2})\\
&= \mu V(B',\gamma)+\lambda V(B',u) && (\text{Corollary 2.3})
\end{aligned}$$

as required. □

The following shows that characteristic classes of P can be obtained by means of the Evens norm map.

Theorem 3.7 *Let A be an element of $\mathcal{A}$ and let v be an. element of $H^1(A)$ (resp. $\beta H^1(A)$) for $p=2$ (resp. $p>2$) satisfying $\mathrm{Res}^A_Z(v)=\gamma$. Set*

$$\zeta_{A,v} = \mathcal{N}_{A\times C\longrightarrow\Gamma}(v+u)-\mathcal{N}_{A\times C\longrightarrow\Gamma}(v).$$

As elements of $h^*(P)$ then

$$\zeta_{A,v} = (-1)^n\sum_{s=0}^{n}(-1)^s\kappa_{n,s}u^{p^s}.$$

Proof For every $B\in\mathcal{A}$, by Lemma 3.6 we have

$$\begin{aligned}
\mathrm{Res}^{\Gamma}_{B\times Z}(\zeta_{A,v}) &= \mathrm{Res}^{\Gamma}_{B\times Z}\mathcal{N}_{A\times Z\longrightarrow\Gamma}(v+u)-\mathrm{Res}_{B\times Z}\mathcal{N}_{A\times Z\longrightarrow\Gamma}(v)\\
&= V(B',\gamma+u)-V(B',\gamma)\\
&= V(B',u),
\end{aligned}$$

since $V(B',X)$, as a function on X, is additive. By Theorem 3.1 (ii),

$$\mathrm{Res}^{\Gamma}_{B\times Z}(\zeta_{A,v}) = \mathrm{Res}^{\Gamma}_{B\times Z}[(-1)^n\sum_{s=0}^{n}(-1)^s\kappa_{n,s}u^{p^s}].$$

So $\zeta_{A,v}\doteqdot(-1)^n\sum_{s=0}^{n}(-1)^s\kappa_{n,s}u^{p^s}$. □

Remark Write $\mathcal{N}_{A\times C\longrightarrow\Gamma}=\mathcal{N}$. It follows from the above theorem and from Corollary 2.2 that

$$\mathcal{N}(v+u)-\mathcal{N}(v)-\mathcal{N}(u)\doteqdot(-1)^n\sum_{s=0}^{n}(-1)^s[\kappa_{n,s}-Q_s(P/A)]u^{p^s}.$$

According to Proposition 2.1(iv), the $\kappa_{n,s}$ can be expressed as sums of transfers from maximal subgroups of $P\times C$. Such formulae are the ones given in Theorem 3.2.

Let $a_1,a_2,\dots,a_{2n-1},a_{2n}$ be elements of P satisfying $x_i(a_j)=\delta_{ij}$ with δ_{ij} the Kronecker symbol, $1\le i,j\le 2n$. Suppose that K is a maximal subgroup of P given by $K=\ker(x_{2n})$. So $K\cong P_{n-1}\times\langle a_{2n-1}\rangle\cong P_{n-1}\times\mathbb{Z}/p$. Write $y_{2n}=y$, $\mathcal{N}_{K\times C\longrightarrow\Gamma}=\mathcal{N}$, and, for $0\le r\le n-1$, $\chi_{r,x_{2n}}=\chi_r$. Define

$$\theta_{n-1,r}=\mathrm{Res}^P_K(\chi_r)\in H^*(K),$$

and

$$\theta_{n-1}=u^{p^n}+\sum_{r=0}^{n-1}\Big[(-1)^{n-r}u^{p^r}(-\theta_{n-1,r}+\kappa^p_{n-1,r-1})\Big]\in H^*(K\times C)$$

with the convention that $\kappa_{n-1,-1}=0$.

Theorem 3.8 *As elements of $h^*(P)$,*

$$(-1)^n\mathcal{N}(\theta_{n-1})=\sum_{s=0}^{n}(-1)^s\kappa^p_{n,s}u^{p^{s+1}}$$

$$(-1)^r\mathcal{N}(\kappa^\iota_{n-1,r})=\sum_{i=r}^{n-1}(-1)^i\kappa^{\iota p}_{n-1,i}y^{\iota(p^{i+1}-p^{r+1})}$$
$$-\Bigg(\sum_{i=r+1}^{n-1}(-1)^i\kappa^\iota_{n-1,i}y^{\iota(p^i-p^{r+1})}\Bigg)\Bigg[\sum_{i=0}^{n-1}(-1)^i\kappa^\iota_{n-1,i}y^{\iota p^i}\Bigg]^{p-1},$$

for $0\le r\le n-2$.

Proof For convenience, write $\kappa_{n-1,r}=\kappa_r$ for $0\le r\le n-1$. Let A be an element of $\mathcal{A}$ and set $X=\mathcal{N}(\theta_{n-1})$ and $Y_r=\mathcal{N}(\kappa^\iota_{n-1,r})$. Let

$$Z_r=\sum_{i=r}^{n-1}(-1)^i\kappa_i^{\iota p}y^{\iota(p^{i+1}-p^{r+1})}-\Bigg(\sum_{i=r+1}^{n-1}(-1)^i\kappa^\iota_i y^{\iota(p^i-p^{r+1})}\Bigg)\Bigg[\sum_{i=0}^{n-1}(-1)^i\kappa^\iota_i y^{\iota p^i}\Bigg]^{p-1}$$

for $0\le r\le n-2$. Consider the following cases:

Case 1 $A \subset K$ By setting $B = A \cap P_{n-1}$, we have $A = B \times \langle a_{2n-1} \rangle$. So

$$\mathrm{Res}^{\Gamma}_{K\times C}(X) = \prod_{x\in\langle a_{2n}\rangle} {}^{x}\theta_{n-1}.$$

As the θ_{n-1} belong to $\mathrm{Im}(\mathrm{Res}^{P}_{K})$, they are invariant under the action of a_{2n}. Hence

$$\begin{aligned}
\mathrm{Res}^{\Gamma}_{A\times C}(X) &= \prod_{x\in\langle a_{2n}\rangle} \mathrm{Res}^{K\times C}_{A\times C}({}^{x}\theta_{n-1}) \\
&= \mathrm{Res}^{K\times C}_{A\times C}(\theta^{p}_{n-1}) \\
&= u^{p^{n+1}} + \Bigg[\sum_{r=0}^{n-1}(-1)^{n-r} u^{p^{r}} \left(Q_r(B')V(B', y_{2n-1})^{p-1} + Q^{p}_{r-1}(B') \right)\Bigg]^{p} \\
&= V(A', u)^{p} \\
&= \mathrm{Res}^{\Gamma}_{A\times C}\left((-1)^{n} \sum_{s=0}^{n} (-1)^{s} \kappa^{p}_{n,s} u^{p^{s+1}} \right).
\end{aligned}$$

Also, for $0 \le r \le n-2$,

$$\begin{aligned}
\mathrm{Res}^{\Gamma}_{A\times C}(Y_r) &= \textstyle\prod_{x\in\langle a_{2n}\rangle} \mathrm{Res}^{K\times C}_{A\times C}\left({}^{x}\kappa^{\iota}_{n-1,r}\right) \\
&= \left[\mathrm{Res}^{K\times C}_{A\times C}(\kappa_{n-1,r})^{\iota}\right]^{p} = Q^{\iota p}_{r}(B') = (-1)^{r}\mathrm{Res}^{\Gamma}_{A\times C}(Z_r).
\end{aligned}$$

Case 2 $A \not\subset K$ By setting $H = K \cap A$, we have

$$\begin{aligned}
\mathrm{Res}^{\Gamma}_{A\times C}(X) &= \mathcal{N}_{H\times C\to A\times C}\mathrm{Res}^{K\times C}_{H\times C}(X) \\
&= \mathcal{N}_{H\times C\to A\times C}(V(H', u)^{p}) \\
&= V(A', u)^{p} \\
&= \mathrm{Res}^{\Gamma}_{A\times C}\left((-1)^{n} \sum_{s=0}^{n} (-1)^{s} \kappa^{p}_{n,s} u^{p^{s+1}} \right).
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Res}^{\Gamma}_{A\times C}(Y_r) &= \mathcal{N}_{H\times C\to A\times C}\mathrm{Res}^{K\times C}_{H\times C}(Y_r) \\
&= \mathcal{N}_{H\times C\to A\times C}(Q^{\iota}_{r}(H/Z)) \\
&= (-1)^{r}\mathrm{Res}^{\Gamma}_{A\times C}(Z_r).
\end{aligned}$$

This completes the proof. □

Formulae relating the $\kappa^{\iota}_{n,r}$ to such classes of fewer variables are given by the following corollary.

Corollary 3.9 *For $0 \le r \le n-1$, as elements of $h^*(P)$,*

$$\kappa^{\iota}_{n,r} = \kappa^{\iota p}_{n-1,r-1} + \kappa^{\iota}_{n-1,r}\left[\sum_{i=0}^{n-1}(-1)^i \kappa^{\iota}_{n-1,i}\, y^{\iota p^i}\right]^{p-1} + \left[\kappa^{\iota}_{n-1,0}\varphi + \sum_{i=1}^{n-1}(-1)^i \kappa^{\iota}_{n-1,i}\,(y^{\iota p^i}_{2n-1} - y_{2n-1}\, y^{\iota p^i - 1})\right]^{p-1}.$$

Proof By Corollary 2.3 we have

$$\begin{aligned}
\mathcal{N}(\theta_{n-1}) &= \mathcal{N}(u^{p^n}) + \sum_{r=0}^{n-1} \mathcal{N}\left[\left(-1\right)^{n-r} u^{p^r}(-\theta_{n-1,r} + \kappa^{p}_{n-1,r-1})\right] \\
&= V(y,u)^{p^n} + \sum_{r=0}^{n-1}(-1)^{n-r} V(y,u)^{p^r}\left[\mathcal{N}(-\theta_{n-1,r}) + \mathcal{N}(\kappa^{p}_{n-1,r-1})\right] \\
&= (u^p - uy^{p-1})^{p^n} \\
&\quad + \sum_{r=0}^{n-1}(-1)^{n-r}(u^p - uy^{p-1})^{p^r}\left[\mathcal{N}(-\theta_{n-1,r}) + \mathcal{N}(\kappa^{p}_{n-1,r-1})\right] \\
&= u^{p^{n+1}} + \sum_{r=0}^{n-1}(-1)^{n-r} u^{p^{r+1}}\Big[\mathcal{N}(-\theta_{n-1,r}) + \mathcal{N}(\kappa^{p}_{n-1,r-1}) \\
&\qquad + y^{(p-1)p^{r+1}}\left(\mathcal{N}(-\theta_{n-1,r+1}) + \mathcal{N}(\kappa^{p}_{n-1,r})\right)\Big] \\
&\quad + (-1)^n u y^{p-1} \mathcal{N}(-\theta_{n-1,0}).
\end{aligned}$$

By the Frobenius formula, the cup-product of χ_r with each of x_{2n}, y_{2n} vanishes. As the transfer commutes with Steenrod operations, we have, by Proposition 2.4 and Theorem 3.8,

$$\begin{aligned}
u^{p^{n+1}} &+ \sum_{r=0}^{n-1}(-1)^{n-r}\kappa^{p}_{n,r} u^{p^{r+1}} = \mathcal{N}(\theta_{n-1}) \\
&= u^{p^{n+1}} + \sum_{r=0}^{n-1}(-1)^{n-r} u^{p^{r+1}}\Big[-\chi^p_r + \mathcal{N}(\kappa^{p}_{n-1,r-1}) \\
&\qquad + y^{(p-1)p^{r+1}}\left(-\chi^p_{r+1} + \mathcal{N}(\kappa^{p}_{n-1,r})\right)\Big] \\
&= u^{p^{n+1}} + \sum_{r=0}^{n-1}(-1)^{n-r} u^{p^{r+1}}\left[-\chi^p_r + \mathcal{N}(\kappa^{p}_{n-1,r-1}) + y^{(p-1)p^{r+1}}\mathcal{N}(\kappa^{p}_{n-1,r})\right].
\end{aligned}$$

Therefore

$$\sum_{r=0}^{n-1}(-1)^{n-r}\kappa_{n,r}^{\iota}u^{\iota p^r}=\sum_{r=0}^{n-1}(-1)^{n-r}u^{\iota p^r}\Big[-\chi_r^{\iota}+\mathcal{N}(\kappa_{n-1,r-1}^{\iota})+y^{\iota(p-1)p^r}\mathcal{N}(\kappa_{n-1,r}^{\iota})\Big].$$

Hence

$$\kappa_{n,r}^{\iota}=-\chi_r^{\iota}+\mathcal{N}(\kappa_{n-1,r-1}^{\iota})+y^{(p-1)\iota p^r}\mathcal{N}(\kappa_{n-1,r}^{\iota}).$$

Since

$$\mathcal{N}(\kappa_{n-1,r-1}^{\iota})+y^{(p-1)\iota p^r}\mathcal{N}(\kappa_{n-1,r}^{\iota})\doteqdot\kappa_{n-1,r-1}^{\iota p}+\kappa_{n-1,r}^{\iota}\left[\sum_{i=0}^{n-1}(-1)^i\kappa_{n-1,i}^{\iota}y^{\iota p^i}\right]^{p-1},$$

by Theorem 3.8, we obtain

$$\kappa_{n,r}^{\iota}\doteqdot-\chi_r^{\iota}+\kappa_{n-1,r-1}^{\iota p}+\kappa_{n-1,r}^{\iota}\left[\sum_{i=0}^{n-1}(-1)^i\kappa_{n-1,i}^{\iota}y^{\iota p^i}\right]^{p-1}.$$

The corollary follows from Corollary 3.3. □

4 The subring $\mathbf{F}_p[\kappa_{n,0},\dots,\kappa_{n,n-1}]\cap\mathcal{T}$

In this section, p is supposed to be an odd prime. It was proved by Green and Leary [3; 4] that $\kappa_{n,0}^s\in\mathcal{T}$, provided that $s\geq 2^n$, or $s\geq 2$ and $n\leq 2$. This result can be sharpened as follows. Let $\mathbb{R}_n$ be the set consisting of sequences $R=(r_0,r_1,\dots,r_{n-1})$ of non-negative integers. For $R=(r_0,\dots,r_{n-1})\in\mathbb{R}_n$ and for $m>0$, set

$$s_R=\sum_{i\geq 0}r_i,$$

$$\kappa_m^R=\begin{cases}\prod_{i=0}^{i=m-1}\kappa_{m,i}^{r_i} & m\leq n\\ \prod_{i=0}^{i=n-1}\kappa_{m,i}^{r_i} & m>n.\end{cases}$$

The main purpose of this section is to prove the following theorem.

Theorem 4.1 *Let $R=(r_0,\dots,r_{n-1})$ be an element of $\mathbb{R}_n$. As an element of $h^*(P)$, κ_n^R belongs to $\mathcal{T}$ if and only if one of the following conditions is satisfied:*
(R_1) *$r_0\geq 2$;*
(R_2) *$r_0=0$ and all the r_i terms with $i>0$, are multiples of p.*

The rest of the section is devoted to the proof of the theorem.

Proof By Corollary 3.3, $\kappa_n^R \in \mathcal{T}$ if R satisfies (R_2). We shall prove the following proposition.

Proposition 4.2 *If $R \in \mathbb{R}_n$ satisfies (R_1), then $\kappa_n^R \in \mathcal{T}$.*

By [4; 7], the proposition holds for $n = 1$. Suppose inductively that it holds for $n-1$. Set $K = \ker(x_{2n}) = P_{n-1} \times \mathbb{Z}/p$ and $\mathcal{T}' = \mathrm{Im}(\mathrm{Inf}_K^{K/Z}) + \sqrt{0}$. Write $w = \mathrm{Res}_K^P(y_{2n-1})$ and $\psi = (-1)^{n-1} \sum_{j=0}^{n-1} (-1)^j \kappa_{n-1,j} w^{p^j}$. We have

$$\mathrm{Res}_K^P(\kappa_{n,j}) \doteqdot \kappa_{n-1,j-1}^p + \kappa_{n-1,j} \psi^{p-1}, \quad 0 \le j \le n.$$

So, for every element $R \in \mathbb{R}_n$, as elements of $h^*(K)$,

$$\begin{aligned}
\mathrm{Res}_K^P(\kappa_n^R) &= \prod_{j=0}^{n-1} \left[\kappa_{n-1,j-1}^p + \kappa_{n-1,j} \psi^{p-1} \right]^{r_j} \\
&= \kappa_{n-1,0}^{r_0} \psi^{r_0(p-1)} \prod_{j=1}^{n-1} \left[\kappa_{n-1,j-1}^p + \kappa_{n-1,j} \psi^{p-1} \right]^{r_j} \\
&= \kappa_{n-1}^R \psi^{(p-1)s_R} + \sum_{r_0 \le t < s_R} \rho_t \psi^{(p-1)t} \qquad (2)
\end{aligned}$$

with $\rho_t \in h^*(P_{n-1})$.

Lemma 4.3 *Let $S = (s_0, \ldots, s_{n-1})$ be an element of $\mathbb{R}_n$ with $s_0 \ge 1$, and let x be a non-zero element of $H^1(P)$. Then*

$$\kappa_n^S \chi_{0,x} \in \mathcal{T}.$$

Proof Without loss of generality, we may assume that $x = x_{2n}$. So $K = \ker(x)$. Since $s_0 \ge 1$, by (2), we have

$$\mathrm{Res}_K^P(\kappa_n^S) \doteqdot \sum_{U \in \mathcal{U}} \kappa_{n-1}^U w^{t_U} \psi \qquad (3)$$

with $\mathcal{U}$ a subset of

$$\{R = (r_0, \ldots, r_{n-2}) \in \mathbb{R}_{n-1} | r_0 \ge 1\}.$$

Let $U = (u_0, \ldots, u_{n-2})$ be an element of $\mathcal{U}$. Since

$$\kappa_{n-1}^U \kappa_{n-1,0} = \kappa_{n-1,0}^{u_0+1} \prod_{i=1}^{n-2} \kappa_{n-1,i}^{u_i}$$

and $u_0+1\geq 2$, it follows from the inductive hypothesis that $\kappa_{n-1}^U\kappa_{n-1,0}$, and hence $\kappa_{n-1}^U\kappa_{n-1,0}w^{t_U}$ belong to $\mathcal{T}'$. So, via the inflation map, $\kappa_{n-1}^U\kappa_{n-1,0}w^{t_U}$ belongs to $\mathcal{T}$.

We then have as elements of $h^*(P)$,

$$\begin{aligned}\kappa_n^S\chi_{0,x} &= \kappa_n^S \mathrm{tr}_P^K(\kappa_{n-1,0}\zeta_{n-1}^{p-1})\\ &= \mathrm{tr}_P^K(\mathrm{Res}_K^P(\kappa_K^S).\kappa_{n-1,0}\zeta_{n-1}^{p-1}) && \text{by Frobenius formula}\\ &= \sum_{U\in\mathcal{U}} \mathrm{tr}_P^K(\kappa_{n-1}^U\kappa_{n-1,0}w^{t_U}\psi\zeta_{n-1}^{p-1}) && \text{by (3)}\\ &= \sum_{U\in\mathcal{U}} \kappa_{n-1}^U\kappa_{n-1,0}w^{t_U}.\mathrm{tr}_P^K(\psi\zeta_{n-1}^{p-1}),\end{aligned}$$

which implies $\kappa_n^S\chi_{0,x}\in\mathcal{T}$, by Corollary 3.3 (iii). □

Proof of Proposition 4.2 Let $R=(r_0,\dots,r_{n-1})$ be an element of $\mathbb{R}_n$. By Corollary 3.3(i), $\kappa_n^R\in\mathcal{T}$ if R satisfies (R_2). Suppose that $r_0\geq 2$. Set $S=(r_0-1,r_1,\dots,r_{n-1})$. We then have

$$\kappa_n^R=\kappa_n^S\kappa_{n,0}=-\sum_{x\in\mathbb{P}H^1(P)}\kappa_n^S\chi_{0,x},$$

by Proposition 3.4. Since $r_0-1\geq 1$ by Lemma 4.4 $\kappa_n^S\chi_{0,x}\in\mathcal{T}$, for every $x\in\mathbb{P}H^1(P)$; so $\kappa_n^R\in\mathcal{T}$. The proposition is proved. □

Consider ψ, and also the right hand side of (2), as polynomials with variable w and with coefficients in $h^*(P_{n-1})$. We have the following lemma.

Lemma 4.4 *Let $R=(r_0,\dots,r_{n-1})$ be an element of $\mathbb{R}_n$ with $s_R\neq 0$ mod p. Then for $0\leq i\leq n-2$,*

(i) $\mathrm{Res}_K^P(\kappa_n^R)\doteq s_R(-1)^{i+n}\kappa_{n-1}^R\kappa_{n-1,i}w^{p^{n-1}[(p-1)s_R-1]+p^i}+$ *other terms;*

(ii) $\kappa_{n-1}^R\kappa_{n-1,i}\in\mathcal{T}$ *if* $\kappa_n^R\in\mathcal{T}$.

Proof For $t<s_R$, $\deg(\psi^{(p-1)t})\leq p^{n-1}(p-1)s_R-p^n+p^{n-1}$; hence

$$\deg(\psi^{(p-1)t})<\min(p^{n-1}(p-1)s_R-1,\ p^{n-1}[(p-1)s_R-2]+p^i).$$

So (i) follows from (2) and the fact that

$$\psi^{(p-1)s_R} = [\sum_{i=0}^{n-1}(-1)^i \kappa_{n-1,i} w^{p^i}]^{(p-1)s_R}$$
$$= -s_R \sum_{i=0}^{n-2}(-1)^{i+n-1}\kappa_{n-1,i} w^{p^{n-1}[(p-1)s_R-1]+p^i} + \text{other terms.}$$

Write

$$\mathrm{Res}^P_K(\kappa_n^R) = \sum_{i\geq 0}\rho_i w^i$$

with $\rho_i \in h^*(P_{n-1})$. If $\kappa_n^R \in \mathcal{T}$, then $\mathrm{Res}^P_K(\kappa_n^R)$ belongs to $\mathcal{T}'$, so all the ρ_i lie in $\mathcal{T}'$; (ii) is then a direct consequence of (i). □

The proof of the theorem is completed by Proposition 4.2 and the following.

Lemma 4.5 *If $\kappa_n^R \in \mathcal{T}$ with $R = (r_0, \dots, r_{n-1}) \in \mathbb{R}_n$, then R satisfies (R_1) or (R_2).*

Proof By Leary [7], the lemma holds for $n = 1$. Assume that it holds for $n-1$.

Suppose that $\kappa_n^R \in \mathcal{T}$ with $R = (r_0, \dots, r_{n-1})$ and $r_0 < 2$. It follows that $\zeta = \mathrm{Res}^P_K(\kappa_n^R) \in \mathcal{T}'$. Consider ζ as a polynomial with variable w and with coefficients in $h^*(P_{n-1})$. By (2), we have

$$\zeta = \kappa_{n-1}^R \psi^{(p-1)s_R} + \sum_{0\leq t<s_R} \rho_t \psi^{(p-1)t}$$
$$= \kappa_{n-1}^R w^{p^{n-1}(p-1)s_R} + \text{other terms,} \qquad \text{by (3)}$$

which implies $\kappa_{n-1}^R \in \mathcal{T}'$. By the induction hypothesis, $r_0 = 0$ and $r_1, \dots, r_{n-2}$ are multiples of p. So $s_R = r_{n-1} \bmod p$. If $s_R \neq 0 \bmod p$, it follows from Lemma 4.5 that

$$\kappa_{n-1,1}^{r_1} \dots \kappa_{n-1,n-3}^{r_{n-3}} \kappa_{n-1,n-2}^{r_{n-2}+1} = \kappa_{n-1,n-2}\kappa_{n-1}^R \in \mathcal{T}'$$

which contradicts the induction hypothesis, since $r_{n-2} = 0 \bmod p$ implies $r_{n-2}+1 \neq 0 \bmod p$. So $s_R = 0 \bmod p$, hence $r_{n-1} = 0 \bmod p$. The lemma follows. □

This completes the proof of Theorem 4.1. □

Let x be a non-zero element of $H^1(P)$. By Theorem 3.2(iii), there exists a unique $\eta_x \in H^*(E)$ such that, as elements of $H^*(E)/(z_n^{(1)}, \dots, z_n^{(n-2)})$,

$$z_n^{(n-1)} = \begin{cases} \eta_x \beta(x) & p \text{ odd,} \\ \eta_x x & p = 2. \end{cases} \tag{4}$$

Note that $H_x = \ker(x)$ can be identified with $P_{n-1} \times \mathbb{Z}/p$. Pick a non-zero element u_x of $H^1(P)$ satisfying

$$0 \neq \mathrm{Res}^P_{H_x}(u_x) \in \ker(\mathrm{Res}^{H_x}_{P_{n-1}}) \quad \text{set } v_x = \begin{cases} u_x & p = 2 \\ \beta(u_x) & p \text{ odd,} \end{cases}$$

and define $\psi_x = (-1)^{n-1} \sum_{j=0}^{n-1} (-1)^j \kappa_{n-1,j} v_x^{p^j}$.

Let R be an element of $\mathbb{R}_n$ satisfying (R_1) or (R_2). By Theorem 4.1, κ_n^R and κ_{n-1}^R both belong to $\mathcal{T}$. It is then interesting to find out a formulae relating κ_n^R and κ_{n-1}^R. If R satisfies (R_2), the formula can be derived from Corollary 3.9. In the case where R satisfies (R_1) the formula follows from the next corollary.

Corollary 4.6 *Let $R = (r_0, \dots, r_{n-1})$ be an element of $\mathbb{R}_n$ with $r_0 \geq 2$. Then, as elements of $\mathcal{T}$,*

$$\kappa_n^R = -\kappa_{n-1}^{r_0} \sum_{x \in \mathbb{P}H^1(P)} \eta_x \psi_x^{(r_0-1)(p-1)-1} \prod_{j=1}^{n-1} [\kappa_{n-1,j-1}^p + \kappa_{n-1,j} \psi_x^{p-1}]^{r_j}.$$

Proof Set $S = (r_0 - 1, r_2, \dots, r_{n-1})$ and $U = (r_0 - 2, r_1, \dots, r_{n-1})$. It follows from the proof of Proposition 4.2 that

$$\begin{aligned}
\kappa_n^R &= -\sum_{x \in \mathbb{P}H^1(P)} \kappa_n^S \chi_{0,x} \\
&= -\sum_{x \in \mathbb{P}H^1(P)} \kappa_n^S \mathrm{tr}_P^{H_x}(\kappa_{n-1,0} \zeta_{n-1}^{p-1}) \\
&= -\sum_{x \in \mathbb{P}H^1(P)} \mathrm{tr}_P^{H_x} \left[\mathrm{Res}^P_{H_x}(\kappa_n^S) \kappa_{n-1,0} \zeta_{n-1}^{p-1} \right] \\
&= -\sum_{x \in \mathbb{P}H^1(P)} \mathrm{tr}_P^{H_x} \left[\mathrm{Res}^P_{H_x}(\kappa_n^U) \kappa_{n-1,0}^2 \zeta_{n-1}^{p-1} \right] \\
&= -\sum_{x \in \mathbb{P}H^1(P)} \mathrm{tr}_P^{H_x} \left(\psi \zeta_{n-1}^{p-1} \kappa_{n-1,0}^{r_0} \psi_x^{(r_0-1)(p-1)-1} \prod_{j=1}^{n-1} [\kappa_{n-1,j-1}^p + \kappa_{n-1,j} \psi_x^{p-1}]^{r_j} \right).
\end{aligned}$$

Since $r_0 \geq 2$, it follows from Theorem 4.1 that

$$\rho_x = \kappa_{n-1,0}^{r_0} \psi_x^{(r_0-1)(p-1)-1} \prod_{j=1}^{n-1} \left[\kappa_{n-1,j-1}^p + \kappa_{n-1,j} \psi_x^{p-1} \right]^{r_j}$$

belongs to $\mathcal{T}$, for any $x \in \mathbb{P}H^1(P)$. Hence

$$\begin{aligned}\kappa_n^R &= -\sum_{x\in\mathbb{P}H^1(P)} \rho_x \mathrm{tr}_P^{H_x}(\psi_x \zeta_{n-1}^{p-1}) \\ &= -\sum_{x\in\mathbb{P}H^1(P)} \rho_x \eta_x \qquad \text{by (4).}\end{aligned}$$

This completes the proof. □

5 Symplectic invariants

Recall that the symplectic group $Sp_{2n} = Sp_{2n}(\mathbf{F}_p)$ is the group consisting of E which preserve the nondegenerate symplectic form $x_1x_2+\ldots+x_{2n-1}x_{2n}$ of $H^2(E)$. Clearly $z_n^{(0)}, \ldots, z_n^{(n-1)}$ belong to the subring of invariants of Sp_{2n} in $\mathbf{F}_p[y_1, \ldots, y_{2n}]$. According to a result of Quillen [12] for $p=2$, and of Tezuka–Yagita [13] for $p>2$,

$$\mathcal{T} = \mathbf{F}_p[y_1, \ldots, y_{2n}]/(z_n^{(0)}, \ldots, z_n^{(n-1)}).$$

There is then an induced action of Sp_{2n} on $\mathcal{T}$. Set

$$\mathbb{R}' = \{R \in \mathbb{R} | R \text{ satisfies } (R_1) \text{ or } (R_2)\},$$

and let $\mathbb{R}''$ be the subset of $\mathbb{R}_n$ consisting of elements $R = (r_0, r_1, \ldots, r_{n-1})$ of $\mathbb{R}_n$ satisfying the following two conditions:

- $0 \le r_i \le p-1$ for $i > 0$
- $r_0 = 3$, or $r_0 = 2$ and $r_1, \ldots, r_{n-1}$ are not all equal to 0.

Let $\mathcal{T}^{Sp_{2n}}$ be the ring of invariants of Sp_{2n} in $\mathcal{T}$. The following is then straightforward from Theorem 4.1 and [3, Proposition 21].

Theorem 5.1 *$\mathcal{T}^{Sp_{2n}}$ is the subring of $\mathbf{F}_p[\kappa_{n,1}, \ldots, \kappa_{n,n-1}]$ given by:*

(i) *for $p=2$, $\mathcal{T}^{Sp_{2n}} = \mathbf{F}_p[\kappa_{n,0}, \ldots, \kappa_{n,n-1}]$;*

(ii) *for $p > s2$,*

  (a) *as a vector space over $\mathbf{F}_p$, $\mathcal{T}^{Sp_{2n}}$ has a basis $\{\kappa_n^R | R \in \mathbb{R}'\}$;*

  (b) *as a module over polynomial algebra $\mathbf{F}_p[\kappa_{n,0}^2, \kappa_{n,1}^p, \ldots, \kappa_{n,n-1}^p]$, $\mathcal{T}^{Sp_{2n}}$ is freely generated by $\{1, \kappa_n^R | R \in \mathbb{R}''\}$.*

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