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The Lambda algebra and Sq^0

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The action of Sq on the cohomology of the Steenrod algebra is induced by an endomorphism θ of the Lambda algebra. This paper studies the behavior of θ in order to understand the action of Sq; the main result is that Sq is injective in filtrations less than 4, and its kernel on the 4–line is computed.

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1 Introduction

The Lambda algebra Λ was constructed in the "six author paper" [2], and numerous authors have studied it over the past 40 years. In [12], Wang studied an algebra endomorphism θ of Λ, and this endomorphism is the main focus of this paper.

Λ is the bigraded differential $\mathbb{F}_2$–algebra with generators $\lambda_n \in \Lambda^{1,n+1}$ for $n \geq 0$, with relations and differential given as follows:

$$\sum_{i+j=n} \binom{i+j}{i} \lambda_{i-1+m}\lambda_{j-1+2m} \qquad \text{for } m \geq 1, n \geq 0,$$

$$d(\lambda_{n-1}) = \sum_{i+j=n} \binom{i+j}{j} \lambda_{i-1}\lambda_{j-1}.$$

Let A be the mod 2 Steenrod algebra. The following result and its unstable analogue are the main reasons for studying the Lambda algebra.

Theorem 1.1 (Bousfield et al. [2]) $H^{s,t}(\Lambda) \cong \mathrm{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$. *Indeed, one can view Λ as the E_1–term of the classical Adams spectral sequence converging to the 2–component of the stable homotopy groups of spheres.*

Given the free algebra $F = \mathrm{alg}_{\mathbb{F}_2}(\lambda_0, \lambda_1, \dots)$ of which Λ is a quotient, define a map $\widetilde{\theta}\colon F \to F$ by $\widetilde{\theta}(\lambda_n) = \lambda_{2n+1}$.

Proposition 1.2 (Wang [12]) *$\widetilde{\theta}$ induces a map $\theta\colon \Lambda \to \Lambda$ of differential graded algebras, and this map is one-to-one.*

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Corollary 1.3 *The subalgebra of* Λ *generated by* $\{\lambda_{2n+1} : n \geq 0\}$ *is a sub-differential graded algebra, which is isomorphic to* Λ*.*

Note that the isomorphism $\theta\colon \Lambda \xrightarrow{\cong} \operatorname{im}(\theta)$ doubles internal degrees. There is an endomorphism, called Sq^0, of $H^*\Lambda = \mathrm{Ext}_A^*(\mathbb{F}_2, \mathbb{F}_2)$ which also doubles internal degrees. One way to describe this map is as follows: the graded dual of A is a commutative Hopf algebra over $\mathbb{F}_2$, and the Frobenius map is a Hopf algebra map. Sq^0 is the induced map on $\mathrm{Ext}_A^*(\mathbb{F}_2, \mathbb{F}_2)$. See May [6, Proposition 11.10] for this result, and see Palmieri [7] for more discussion of Sq^0 acting on Ext.

Proposition 1.4 $\theta\colon \Lambda \to \Lambda$ *induces the map*

$$\mathrm{Sq}^0\colon \mathrm{Ext}_A^*(\mathbb{F}_2, \mathbb{F}_2) \to \mathrm{Ext}_A^*(\mathbb{F}_2, \mathbb{F}_2).$$

Bob Bruner told me how to prove this.

Proof According to Priddy [8], λ_n is represented in the cobar construction by $[\xi_1^{n+1}]$. According to May [6, Proposition 11.10], Sq^0 is induced by the dual to the Frobenius, so Sq^0 takes $[\xi_1^{n+1}]$ to $[\xi_1^{2n+2}]$; thus it agrees with the map induced by θ, which sends λ_n to λ_{2n+1}. □

The goal of this paper is to use the DGA endomorphism $\theta\colon \Lambda \to \Lambda$ to study the action of Sq^0 on Ext over the Steenrod algebra. The main result is the following; the result when $s \leq 3$ was first proved by Wang.

Theorem 1.5 *Consider* $\mathrm{Sq}^0\colon \mathrm{Ext}_A^{s,*}(\mathbb{F}_2, \mathbb{F}_2) \to \mathrm{Ext}_A^{s,*}(\mathbb{F}_2, \mathbb{F}_2)$. *When* $s \leq 3$, *this map is injective. When* $s = 4$, *the kernel is one-dimensional:*

$$\ker(\mathrm{Sq}^0 \,|\, \mathrm{Ext}_A^4) = \mathrm{Span}(h_0^4).$$

This theorem is proved in Section 3 using a Bockstein spectral sequence argument. The Bockstein spectral sequence is also of interest in its own right. We also invert θ to construct the "cocomplete Lambda algebra" in Section 4.

This paper studies some of the same issues as [7], but uses a different approach. As with that paper, this work raises more questions than it answers; here are some.

Question (a) Is $(\mathrm{Sq}^0)^{-1}\,\mathrm{Ext}_A^*(\mathbb{F}_2, \mathbb{F}_2)$ finite dimensional in each bidegree?

(b) Even better, for fixed s, is there a uniform bound on

$$\dim (\mathrm{Sq}^0)^{-1}\,\mathrm{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2)?$$

(c) Hư̛ng has asked the following: for fixed s, is there a bound on the "Sq^0–nilpotence height" of elements in $\mathrm{Ext}_A^{s,*}(\mathbb{F}_2,\mathbb{F}_2)$? That is, is there a number $N = N(s)$ so that, for $z \in \mathrm{Ext}_A^{s,*}(\mathbb{F}_2,\mathbb{F}_2)$, either $(\mathrm{Sq}^0)^N(z) = 0$ or $(Sq^0)^i(z) \neq 0$ for all i? See Hu'ng [4] for more details.

The Bockstein spectral sequence of Section 3 may be helpful in answering the last question, and perhaps the cocomplete Lambda algebra (Section 4) could be helpful for the first two.

2 Recollections on the Lambda algebra

In this section, we recall some facts about the Lambda algebra. Λ was first constructed in the "six author paper" [2]. Wang [12] gave alternate forms for the relations and the differential, described the endomorphism θ, and used Lambda algebra calculations to prove the Hopf invariant one problem. See also Ravenel's green book [10, Section 3.3], which summarizes the basic facts about Λ, and also describes the Curtis algorithm; Priddy's Koszul resolutions paper [8] gives a purely algebraic construction of Λ; and Richter has given elementary proofs of some of the properties of Λ in [11].

Λ is a bigraded differential $\mathbb{F}_2$–algebra; it is defined to be

$$\Lambda = \mathrm{alg}_{\mathbb{F}_2}(\lambda_0, \lambda_1, \lambda_2, \dots)/(\text{relations}),$$

graded by putting λ_n in bidegree $(1, n+1)$. There are two forms for the relations and the differential, the "symmetric" form

$$\sum_{i+j=n} \binom{i+j}{i} \lambda_{i-1+m}\lambda_{j-1+2m} \qquad \text{for} \quad m \geq 1, n \geq 0, \tag{1}$$

$$d(\lambda_{n-1}) = \sum_{i+j=n} \binom{i+j}{j} \lambda_{i-1}\lambda_{j-1}, \tag{2}$$

and the "admissible" form

$$\lambda_i\lambda_{2i+1+n} = \sum_{j\geq 0} \binom{n-j-1}{j} \lambda_{i+n-j}\lambda_{2i+1+j} \qquad \text{for} \quad i, n \geq 0, \tag{3}$$

$$d(\lambda_n) = \sum_{j\geq 1} \binom{n-j}{j} \lambda_{n-j}\lambda_{j-1}. \tag{4}$$

Wang [12] showed that these two sets of relations generate the same ideal in the free algebra on the λ_n's. One can show that d behaves formally like taking the commutator with λ_{-1}; Bruner noted this in [3].

We refer to the first grading in Λ as the *homological* grading, and the second as the *internal* grading. Note that some authors define λ_n to have degree $(1,n)$ rather than $(1,n+1)$, in which case the second grading is the *stem* grading, the difference between the internal and homological gradings.

Definition 2.1 A monomial $\lambda_{i_1}\cdots\lambda_{i_s}$ is *admissible* if $2i_r \geq i_{r+1}$ for $1 \leq r \leq s-1$.

Proposition 2.2 (Bousfield et al. [2]) *The admissible monomials form a basis for* Λ.

Note that, while it is easy to show from the relations above that the admissible monomials span Λ, it is not trivial to show that they are linearly independent. Similarly, it is not obvious that the differential d respects the relations. See [11] for elementary proofs.

As remarked in Theorem 1.1, the cohomology $H^*(\Lambda, d)$ is Λ's raison d'être.

3 A Bockstein spectral sequence

Let

$$\Lambda' = \Lambda/\theta\Lambda,$$

and consider the short exact sequence of chain complexes

$$0 \to \Lambda \xrightarrow{\theta} \Lambda \to \Lambda' \to 0. \tag{5}$$

After taking cohomology, this gives a Bockstein spectral sequence with

$$E_1^{s,t,u} = \begin{cases} H^{s+t,u}(\Lambda'), & \text{when } s \geq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

$$d_r \colon E_r^{s,t,u} \to E_r^{s+r,t-r+1,u/2^r}, \tag{7}$$

converging to $H^{s+t,u}(\Lambda) = \operatorname{Ext}_A^{s+t,u}(\mathbb{F}_2, \mathbb{F}_2)$. The phrase "Bockstein spectral sequence" is perhaps ambiguous, so some details may be helpful. One way to construct it is to filter Λ by setting $F^s\Lambda = \operatorname{im}\theta^s$. Because θ is injective, $F^s\Lambda$ is isomorphic to Λ; more precisely, $F^s\Lambda$ in bidegree $(i, 2^s j)$ is isomorphic to Λ in bidegree (i, j). Up to a similar doubling of internal degrees, $F^s\Lambda/F^{s+1}\Lambda$ is isomorphic to Λ'. The Bockstein spectral sequence is the one which arises from taking cohomology of this filtered chain complex. Thus for fixed $s \geq 0$ and t we have $E_1^{s,t} \cong E_1^{s+i,t-i}$ for all i with $i \geq 0$. Given a class $x \in E_1^{s,0}$, we write $\overline{\theta}^{-i}x$ for the image of x under this isomorphism. The differentials are "$\overline{\theta}$–periodic" and determined by their effect on $E_r^{*,0}$. For any $x \in E_r^{s,0}$, there is a differential

$$d_r \colon x \longmapsto \overline{\theta}^r y,$$

if and only if for any $i \geq 0$, there is a differential

$$d_r \colon \overline{\theta}^i x \longmapsto \overline{\theta}^{i+r} y.$$

Furthermore, as the notation suggests, the differentials determine extensions in that they reflect the action of the map induced by θ, namely Sq^0. That is, if $x \in H^n(\Lambda')$ is an infinite cycle and not a boundary in the spectral sequence, then it survives to a "Sq^0–periodic" element $\overline{x}$ in $H^n(\Lambda)$. More precisely, in this situation $\overline{\theta}^i x$ survives to $E_\infty^{n+i,-i}$ for each $i \geq 0$, and this infinite family corresponds in the abutment to the family

$$\{(\mathrm{Sq}^0)^i \overline{x} \in H^{2^i n}(\Lambda) : i \geq 0\}.$$

The presence of a differential $d_r \colon x \longmapsto y$ from a class $x \in H^{n,u}(\Lambda')$ to a class $y \in H^{n+1,u/2^r}(\Lambda')$ means that r "copies" of y survive to E_∞: the elements $\overline{\theta}^i y$ survive for $0 \leq i \leq r-1$. These elements correspond in the abutment to a finite Sq^0–family

$$\{\overline{y}, \mathrm{Sq}^0\, \overline{y}, (\mathrm{Sq}^0)^2 \overline{y}, \ldots, (\mathrm{Sq}^0)^{r-1} \overline{y}\},$$

where $\overline{y} \in H^{n+1,u/2^r}(\Lambda)$. Thus differentials coming from the n–line in $H^*(\Lambda')$ give information about the Sq^0–action on $H^{n+1}(\Lambda)$.

To compute the rth differential on an element $x \in E_r$, lift x to a class in $H^*(\Lambda')$, represent it by a class in Λ, and take the coboundary. Since x has survived to the E_r–term, this coboundary is in the image of θ^r, so apply $(\theta^r)^{-1}$. Project back to a class in $H^*(\Lambda')$. For example, $\lambda_2\lambda_2\lambda_1$ represents a class at the E_1–term. Lift it back to the class with the same name in Λ and take its coboundary. The result is λ_1^4. This is in the image of θ, and applying θ^{-1} gives λ_0^4. The class λ_0^4 projects to a nonzero class in $H^*(\Lambda')$. Thus there is a differential $d_1(\lambda_2\lambda_2\lambda_1) = \lambda_0^4$. More generally, there is a differential

$$d_1(\lambda_2\lambda_2\lambda_1^n) = \lambda_0^{n+3} \tag{8}$$

for $n \geq 1$, reflecting the fact that λ_0^{n+3} represents a nonzero class in $H^*(\Lambda)$, while $\lambda_1^{n+3} = \theta(\lambda_0^{n+3})$ is a boundary in Λ, and thus $\mathrm{Sq}^0[\lambda_0^{n+3}] = 0$ for all $n \geq 1$.

This may not be a good way to compute the cohomology of the Lambda algebra, but it is a way to study Sq^0 acting on it. A little analysis leads one to believe that most of the cohomology of Λ is "Sq^0–periodic". The word "most" in the previous sentence doesn't have any meaning really, but consider this: given a monomial $M = \lambda_{n_1} \cdots \lambda_{n_k}$

in Λ which maps to a nonzero class in Λ', then at least one n_i is even. Recall that the differential is a derivation, and note that $d(\lambda_{2m})$ is a sum of terms $\lambda_i\lambda_j$ with one of i and j even, the other odd. Thus applying d to M (in Λ) yields a sum of monomials which, before applying the Adem relations, have the same number of even λs as M does.

Furthermore, the Adem relations preserve the parity of the number of even λs. So there is really only one way to get cohomology classes in Λ' which are not images of cohomology classes in Λ: all terms in the differential must contain an even number of even λs, and the Adem relations on those terms must convert them into terms with no even λs. The resulting boundary is zero in the quotient Λ', and hence one gets a cocycle in $H(\Lambda')$ which supports a boundary in the spectral sequence. Thus such cocycles are precisely those classes in $H(\Lambda')$ which are cocycles but which are not in the image of any cocycle in $H(\Lambda)$.

Let's look for such things. First, here is a summary of the previous paragraphs.

Lemma 3.1 *If a class $x \in H^*(\Lambda')$ supports a nonzero differential in the Bockstein spectral sequence (6), then x lifts to a sum of monomials in Λ' which each contain a positive and even number of even λs.*

Here are two other useful observations in our search.

Lemma 3.2 (Wang [12, Proposition 1.9]) *Given $x \in \Lambda'$ in positive stem degree, if $d(x) = 0$ and x is not a boundary, then x is homologous to an element y with "odd ending integers" – that is, every term in the admissible expression for y ends in an odd lambda.*

(Wang's proof works Λ'.)

Lemma 3.3 (Wang [12, Proposition 1.8.3]) *Let $\lambda_{n_1}\cdots\lambda_{n_r} \in \Lambda$ be an admissible sequence. Then $(d\lambda_{n_1})\lambda_{n_2}\cdots\lambda_{n_r}$ is a sum of admissible terms with leading integers at most $n_1 - 1$.*

Note 3.4 There is an obvious vector space splitting of the short exact sequence

$$0 \to \Lambda \xrightarrow{\theta} \Lambda \to \Lambda' \to 0,$$

in which one views Λ' as being spanned by the admissible monomials containing at least one even lambda. We will often take this point of view.

The following proposition and theorem are the main results in this section.

Proposition 3.5 *Consider the Bockstein spectral sequence (6).*

(a) *There are no nonzero differentials coming from* $H^1(\Lambda')$, *and thus* Sq^0 *is injective on* $H^2\Lambda = \mathrm{Ext}^2_A(\mathbb{F}_2, \mathbb{F}_2)$.

(b) *There are no nonzero differentials coming from* $H^2(\Lambda')$, *and thus* Sq^0 *is injective on* $H^3\Lambda = \mathrm{Ext}^2_A(\mathbb{F}_2, \mathbb{F}_2)$.

Wang proved this result in [12, Proposition 2.3].

Proof (a) Lemma 3.1 implies that any class which supports a nonzero differential must be in homological degree at least 2.

(b) By Lemma 3.1, if a class supports a differential, then it is a sum of terms of the form $\lambda_{2m}\lambda_{2n}$, and hence is in even stem degree. Also, Lemma 3.2 says that any cohomology class is cohomologous to a sum of odd-ending monomials. Therefore parity considerations imply that any class in an even stem on the 2–line is cohomologous to a sum of terms of the form $\lambda_{2i+1}\lambda_{2j+1}$, and hence is zero in Λ'. □

Theorem 3.6 *In the Bockstein spectral sequence (6), the only differential emanating from* $H^3(\Lambda')$ *is* $d_1\colon \lambda_2\lambda_2\lambda_1 \longmapsto \lambda_0^4$. *Hence the kernel of* Sq^0 *on* Ext^4 *is spanned by* h_0^4.

Proof Suppose that $y \in H^3(\Lambda')$ supports a differential. By Lemma 3.2, we may assume that y is a sum of admissible monomials each of which ends in an odd lambda, and by Lemma 3.1, each term in y has two even lambdas, and thus y is a sum of terms of the form $\lambda_{\mathrm{even}}\lambda_{\mathrm{even}}\lambda_{\mathrm{odd}}$.

Following Wang, we write y in the form

$$y = \lambda_{2n}y_1 + y',$$

where y' is a sum of admissible terms with leading term less than λ_{2n} (and hence leading term no larger than λ_{2n-2}), and y_1 is a polynomial in homological degree 2; note that the stem degree of y_1 is odd. We also assume that y is chosen from its cohomology class so that its leading term, in the lexicographic ordering, is as small as possible.

View y as being an element of Λ, via the splitting mentioned in Note 3.4. Since y is a cocycle in Λ', the boundary of y in Λ must consist of all odd terms. So by Lemma 3.3, y_1 must be a cocycle in Λ. If y_1 were a coboundary, then y would be cohomologous to a class with smaller leading term, so this can't happen by the minimality assumption. Now we appeal to Wang's computation of $H^2(\Lambda)$. By [12, Proposition 2.4], the only odd ending cocycles in odd stems are $\lambda_0\lambda_{2^m-1}$, which in admissible form is

$$\sum_{j=1}^{m-1} \lambda_{2^m-2^j}\lambda_{2^j-1} = \lambda_{2^m-2}\lambda_1 + \lambda_{2^m-4}\lambda_3 + \text{other terms.}$$

Thus y is of the form

$$y = (2n, 2^m-2, 1) + (2n, 2^m-4, 3) + (\text{smaller terms}),$$

where "smaller" is with respect to the lexicographic ordering. The coboundary of y is

$$d(y) = (2n-1, 2^m-3, 1, 1) + (2n-1, 2^m-4, 2, 1) + (\text{smaller terms}).$$

The first term is all odd, so is zero in Λ', but the second term cannot be canceled by any of the smaller terms. Thus it must not be present, which means that m must be 2: y is of the form

$$y = (2n, 2, 1) + (\text{smaller terms}).$$

The leading term of each "smaller term" is at most λ_{2n-2}. We will show that if n is bigger than 1, then the coboundary of y cannot be zero in Λ'. To do this, we will examine the coboundary of $\lambda_{2n}\lambda_2\lambda_1$, and find terms in it which cannot be canceled. There are three cases, depending on the congruence class of n mod 4.

Case 1 ($n \equiv 0 \mod 4$) Modulo terms with leading term less than λ_{2n-4}, here are the coboundaries of all monomials in the appropriate bidegree:

$$\begin{aligned}
d(\lambda_{2n}\lambda_2\lambda_1) &= \lambda_{2n-1}\lambda_1\lambda_1\lambda_1 + \lambda_{2n-2}\lambda_1\lambda_2\lambda_1 + \lambda_{2n-4}\lambda_3\lambda_2\lambda_1,\\
d(\lambda_{2n-2}\lambda_2\lambda_3) &= \lambda_{2n-2}\lambda_1\lambda_2\lambda_1 + \lambda_{2n-5}\lambda_2\lambda_2\lambda_3,\\
d(\lambda_{2n-2}\lambda_4\lambda_1) &= \lambda_{2n-2}\lambda_2\lambda_1\lambda_1 + \lambda_{2n-3}\lambda_3\lambda_1\lambda_1 + \lambda_{2n-3}\lambda_2\lambda_2\lambda_1 + \lambda_{2n-5}\lambda_2\lambda_4\lambda_1,\\
d(\lambda_{2n-4}\lambda_6\lambda_1) &= \lambda_{2n-4}\lambda_3\lambda_2\lambda_1 + \lambda_{2n-5}\lambda_5\lambda_1\lambda_1 + \lambda_{2n-5}\lambda_4\lambda_2\lambda_1 + \lambda_{2n-5}\lambda_3\lambda_3\lambda_1\\
&\qquad + \lambda_{2n-6}\lambda_4\lambda_3\lambda_1 + \lambda_{2n-6}\lambda_3\lambda_4\lambda_1,\\
d(\lambda_{2n-4}\lambda_4\lambda_3) &= \lambda_{2n-5}\lambda_2\lambda_2\lambda_3 + \lambda_{2n-6}\lambda_2\lambda_3\lambda_3.
\end{aligned}$$

Any terms which are all odd go to zero in Λ', so we may ignore them. Given the monomial $\lambda_{2n}\lambda_2\lambda_1$, the only way to cancel $\lambda_{2n-4}\lambda_3\lambda_2\lambda_1$ in its coboundary is to add $\lambda_{2n-4}\lambda_6\lambda_1$, but then there is no way to cancel the term $\lambda_{2n-5}\lambda_4\lambda_2\lambda_1$.

Case 2 ($n \equiv 2 \mod 4$) If $n = 2$, then the leading term of y is $\lambda_4\lambda_2\lambda_1$. This leads to a permanent cycle in the spectral sequence: the element

$$\lambda_5\lambda_1\lambda_1 + \lambda_4\lambda_2\lambda_1 + \lambda_2\lambda_2\lambda_3$$

is a cocycle in Λ, and maps to $\lambda_4\lambda_2\lambda_1 + \lambda_2\lambda_2\lambda_3$ in Λ'.

Now assume that $n \geq 6$. Modulo terms with leading term less than λ_{2n-5}, here are the coboundaries of all monomials in the appropriate bidegree:

$$\begin{aligned}
d(\lambda_{2n}\lambda_2\lambda_1) &= \lambda_{2n-1}\lambda_1\lambda_1\lambda_1 + \lambda_{2n-2}\lambda_1\lambda_2\lambda_1 + \lambda_{2n-5}\lambda_4\lambda_2\lambda_1,\\
d(\lambda_{2n-2}\lambda_2\lambda_3) &= \lambda_{2n-2}\lambda_1\lambda_2\lambda_1 + \lambda_{2n-5}\lambda_2\lambda_2\lambda_3,\\
d(\lambda_{2n-2}\lambda_4\lambda_1) &= \lambda_{2n-2}\lambda_2\lambda_1\lambda_1 + \lambda_{2n-3}\lambda_3\lambda_1\lambda_1 + \lambda_{2n-3}\lambda_2\lambda_2\lambda_1 + \lambda_{2n-5}\lambda_2\lambda_4\lambda_1,\\
d(\lambda_{2n-4}\lambda_6\lambda_1) &= \lambda_{2n-4}\lambda_3\lambda_2\lambda_1 + \lambda_{2n-5}\lambda_5\lambda_1\lambda_1 + \lambda_{2n-5}\lambda_4\lambda_2\lambda_1 + \lambda_{2n-5}\lambda_3\lambda_3\lambda_1,\\
d(\lambda_{2n-4}\lambda_4\lambda_3) &= \lambda_{2n-5}\lambda_2\lambda_2\lambda_3.
\end{aligned}$$

The only way to cancel the term $\lambda_{2n-5}\lambda_4\lambda_2\lambda_1$ in the coboundary of $\lambda_{2n}\lambda_2\lambda_1$ is to add $\lambda_{2n-4}\lambda_6\lambda_1$ to it, but the coboundary of the resulting sum has a term $\lambda_{2n-4}\lambda_3\lambda_2\lambda_1$, which cannot be canceled.

Case 3 (n odd) Assume that $2n > 2$. Then the first "other term" in the coboundary of y is $\lambda_{2n-3}\lambda_2\lambda_2\lambda_1$. To cancel this, the only appropriate smaller terms are of the form $\lambda_{2n-2}\lambda_{2j}\lambda_{2k+1}$, where $2j + 2k + 1 = 5$. So the terms are $\lambda_{2n-2}\lambda_2\lambda_3$ and $\lambda_{2n-2}\lambda_4\lambda_1$. Modulo terms with leading term less than λ_{2n-3}, their coboundaries are as follows: λ_{2n-3}:

$$\begin{aligned}
d(\lambda_{2n}\lambda_2\lambda_1) &= \lambda_{2n-1}\lambda_1\lambda_1\lambda_1 + \lambda_{2n-3}\lambda_2\lambda_2\lambda_1,\\
d(\lambda_{2n-2}\lambda_2\lambda_3) &= \lambda_{2n-2}\lambda_1\lambda_2\lambda_1,\\
d(\lambda_{2n-2}\lambda_4\lambda_1) &= \lambda_{2n-2}\lambda_2\lambda_1\lambda_1 + \lambda_{2n-3}\lambda_3\lambda_1\lambda_1 + \lambda_{2n-3}\lambda_2\lambda_2\lambda_1.
\end{aligned}$$

While we can cancel the term $\lambda_{2n-3}\lambda_2\lambda_2\lambda_1$, we cannot cancel $\lambda_{2n-2}\lambda_2\lambda_1\lambda_1$.

So if y supports a differential in the Bockstein spectral sequence, then y has leading term $\lambda_2\lambda_2\lambda_1$. There are no smaller terms in the same bidegree, so the potential cocycle equals its leading term: $y = \lambda_2\lambda_2\lambda_1$. □

This proof is not ideal. The technical aspects are cumbersome, and the proof will not generalize well to higher dimensions.

3.1 Comments on Sq^0 on the 5–line

What about the next degree? All of the classes in the ideal generated by λ_0^4 must be hit by differentials

$$d_1\colon \lambda_{2^n-1}\lambda_2\lambda_2\lambda_1 \longmapsto \lambda_{2^{n-1}-1}\lambda_0^4, \quad \text{for } n \geq 5.$$

Why $n \geq 5$? Note that the class $\lambda_{2^{n-1}-1}\lambda_0^4$ is a coboundary in Λ' when $n = 2$, since $\lambda_1\lambda_0$ is the coboundary of λ_2. Thus $\lambda_1\lambda_0^4$ cannot be the target of a differential in the spectral sequence. It is a coboundary when $n = 3$, since $\lambda_3\lambda_0\lambda_0$ is cohomologous to λ_1^3, and $\lambda_1^3\lambda_0$ is the coboundary of $\lambda_1\lambda_1\lambda_2$. It is a coboundary when $n = 4$: a calculation shows that it is the coboundary of

$$\begin{gathered}\lambda_8\lambda_0\lambda_0\lambda_0 + \lambda_6\lambda_2\lambda_0\lambda_0 + \lambda_5\lambda_1\lambda_2\lambda_0 + \lambda_4\lambda_4\lambda_0\lambda_0 + \lambda_4\lambda_2\lambda_2\lambda_0 + \lambda_4\lambda_1\lambda_1\lambda_2 \\ +\lambda_3\lambda_3\lambda_2\lambda_0 + \lambda_2\lambda_4\lambda_1\lambda_1 + \lambda_2\lambda_2\lambda_4\lambda_0 + \lambda_2\lambda_2\lambda_2\lambda_2 + \lambda_1\lambda_2\lambda_4\lambda_1 + \lambda_1\lambda_1\lambda_2\lambda_4.\end{gathered}$$

As above, suppose that $y = \lambda_n y_1 + y'$ is a cycle in Λ' with odd-ending integers, in which the leading integers in the polynomial y_1 are at most $2n$, and the leading integers in y' are at most $n-1$. We may also assume that among classes in Λ' with odd-ending integers which are homologous to y, y has the smallest leading integer.

(We will also view y as an element of Λ, using the obvious splitting of the quotient map Note 3.4.)

If n is odd, then y_1 must be a cycle in Λ'. In this case the boundary of y is

$$dy = \lambda_n(dy_1) + (d\lambda_n)y_1 + dy'.$$

The first term goes to zero in Λ'. We may assume that y_1 is nonzero in Λ', and so is not all odd. We may also assume that y_1 is not a boundary. So either y_1 is a cycle in Λ, or is a non-cycle but has an all-odd boundary. Suppose that y_1 is a cycle in Λ. By Wang's results, we know all of the cycles in Λ^3, and all of the even-dimensional ones are homologous to an all-odd monomial, and hence to something zero in Λ'. Therefore we can ignore this case, and we may assume that y_1 is a cycle in Λ' which is not the image of a cycle in Λ. By our previous calculation, this means that $y_1 = \lambda_2\lambda_2\lambda_1$:

$$y = \lambda_{2m+1}\lambda_2\lambda_2\lambda_1 + y'.$$

By an analysis similar to the proof of Theorem 3.6, we can show that if $2m+1 \equiv 1 \pmod 4$, then y cannot be the leading term of a cocycle. If $2m+1 \equiv 3 \pmod 4$, then $\lambda_{2m+1}\lambda_2\lambda_2\lambda_1$ is the leading term in the coboundary of $\lambda_{2n+4}\lambda_2\lambda_1$, and thus

$$\lambda_{2m+1}\lambda_2\lambda_2\lambda_1 + y' + d(\lambda_{2n+4}\lambda_2\lambda_1)$$

is homologous to y but has smaller leading term.

As a consequence, we may assume that the leading integer n is even. In this case, then y_1 must be a cycle in Λ, and as in the proof of Theorem 3.6, the minimality assumption on the leading integer of y means that y_1 cannot be a boundary. A result of Wang [12, Proposition 2.10] lists representatives of all of the cohomology classes of length 3 in an even stem: they are

$$\lambda_2\lambda_3\lambda_3, \ \lambda_0\lambda_{2^i-1}\lambda_{2^j-1}.$$

Thus we could write down all of the possibilities for the term $\lambda_{2m}y_1$. This is somewhat complicated, though, because these are not the admissible forms for the degree 3 cycles, and even if we had them, left multiplication by λ_{2m} could make some of the resulting terms inadmissible.

Computer calculations in internal degrees up to 82 have found two phenomena: in this range, there is only one differential other than those coming from the ideal generated by $\lambda_2\lambda_2\lambda_1$:

$$d_1(\lambda_{14}\lambda_6\lambda_5\lambda_3 + (\text{smaller terms})) = \lambda_6\lambda_2\lambda_1\lambda_1\lambda_1 + (\text{smaller terms}).$$

(This reflects the fact that $Ph_2 \in \mathrm{Ext}_A^{5,16}(\mathbb{F}_2, \mathbb{F}_2)$ is in the kernel of Sq^0.) Also, the source of every differential in this range except for $\lambda_2\lambda_2\lambda_1\lambda_1$ is cohomologous to a class of the form

$$\lambda_{2m}\lambda_6\lambda_5\lambda_3 + (\text{smaller terms}).$$

This is a relatively small range of dimensions, but one wonders if these patterns hold for the rest of the 4–line. Since we don't have enough evidence to make a good conjecture, we pose questions.

Question (a) Is the kernel of Sq^0 on Ext^5 is spanned by the "obvious" classes $h_ih_0^4$ with $i \geq 5$, plus the class Ph_2 in the 11–stem?

(b) Also, consider an element of the form $y = \lambda_{2k}y_1 + y' \in \Lambda'$ which supports a differential in the spectral sequence. Assume that y is not cohomologous to $\lambda_2\lambda_2\lambda_1\lambda_1$. Does the lexicographically smallest element in the cohomology class of y have leading term $\lambda_{2m}\lambda_6\lambda_5\lambda_3$?

These may be interesting questions, but the methods used in proving Theorem 1.5 and Theorem 3.6 are only going to get harder in higher filtrations, so some other ideas are needed.

4 The cocomplete Lambda algebra

Let $\theta^{-1}\Lambda$ be the direct limit of the diagram

$$\Lambda \xrightarrow{\theta} \Lambda \xrightarrow{\theta} \Lambda \xrightarrow{\theta} \cdots .$$

Call $\theta^{-1}\Lambda$ the *cocomplete Lambda algebra*. For any real number r, write $\mathbb{Z}[\frac{1}{2}]_{>r}$ for the set of elements of $\mathbb{Z}[\frac{1}{2}]$ which are greater than r, and similarly for $\mathbb{Z}[\frac{1}{2}]_{\geq r}$.

Proposition 4.1 *In this proposition, all indices i, j, m, n, and i_r are assumed to be in $\mathbb{Z}[\frac{1}{2}]$.*

(a) *$\theta^{-1}\Lambda$ is a $\mathbb{Z}\times\mathbb{Z}[\frac{1}{2}]_{>0}$–graded $\mathbb{F}_2$–algebra with generators λ_n in bidegree $(1, n+1)$ for all $n > -1$.*

(b) *The relations in $\theta^{-1}\Lambda$ are generated by*

$$\sum_{i+j=n} \binom{i+j}{i} \lambda_{i-1+m}\lambda_{j-1+2m} \qquad \text{for } m > 0,\ n \geq 0.$$

This is the symmetric form of the relations.

(c) *Alternatively, the relations are generated by*

$$\lambda_i \lambda_{2i+1+n} = \sum_{j \geq 0} \binom{n-j-2^{-N(j,n)}}{j} \lambda_{i+n-j}\lambda_{2i+1+j} \qquad \text{for } i > -1,\ n \geq 0,$$

where (after [5, 1.7]) $N = N(j,n)$ is the least integer so that $2^N j$ and $2^N n$ are integers. This is the admissible form of the relations.

(d) *The differential in $\theta^{-1}\Lambda$ is given by*

$$d(\lambda_{n-1}) = \sum_{\substack{i+j=n \\ i,j>0}} \binom{i+j}{j} \lambda_{i-1}\lambda_{j-1} \qquad \text{(symmetric form).}$$

(e) *Alternatively, the differential is given by*

$$d(\lambda_n) = \sum_{j>0} \binom{n-j}{j} \lambda_{n-j}\lambda_{j-1} \qquad \text{(admissible form).}$$

(f) *The admissible monomials form a basis for $\theta^{-1}\Lambda$. (As in Λ, a monomial $\lambda_{i_1}\cdots\lambda_{i_s}$ in $\theta^{-1}\Lambda$ is* admissible *if $2i_r \geq i_{r+1}$ for $1 \leq r \leq s-1$.)*

(g) *The cohomology of $\theta^{-1}\Lambda$ is equal to $(\mathrm{Sq}^0)^{-1}\operatorname{Ext}^*_A(\mathbb{F}_2, \mathbb{F}_2)$, which in turn is equal to $\operatorname{Ext}^*_{\widehat{A}}(\mathbb{F}_2, \mathbb{F}_2)$, where $\widehat{A}$ is the "complete Steenrod algebra," as studied in [1] and [5].*

Note that for integers a and b, $\binom{a}{b} \equiv \binom{2a}{2b} \pmod 2$, and this allows one to define mod 2 binomial coefficients for elements of $\mathbb{Z}[\frac{1}{2}]$. As a consequence, all of the sums here are finite. This is essentially because $\theta^{-1}\Lambda$ is constructed as a direct limit; in contrast, the complete Steenrod algebra $\widehat{A}$ is constructed as an inverse limit, and the Adem relations there are infinite sums – see [5, 1.7]. Similarly, the admissible monomials in $\widehat{A}$ do not span, while the admissible monomials in $\theta^{-1}\Lambda$ do. The relation with $\widehat{A}$ also explains the terminology "cocomplete Lambda algebra."

Proof In general, all of this follows from the colimit description of $\theta^{-1}\Lambda$. In more detail: for part (b), applying θ to the symmetric Adem relation (1) indexed by m and n yields

$$\begin{aligned}
(\text{symmetric Adem relation})_{m,n} &= \sum_{i+j=n} \tbinom{i+j}{i} \lambda_{i-1+m}\lambda_{j-1+2m} \\
&\overset{\theta}{\longmapsto} \sum_{i+j=n} \tbinom{i+j}{i} \lambda_{2i-1+2m}\lambda_{2j-1+4m} \\
&= \sum_{2i+2j=2n} \tbinom{2i+2j}{2i} \lambda_{2i-1+2m}\lambda_{2j-1+4m} \\
&= \sum_{i+j=2n} \tbinom{i+j}{i} \lambda_{i-1+2m}\lambda_{j-1+4m} \\
&= (\text{symmetric Adem relation})_{2m,2n}.
\end{aligned}$$

In Λ, one has symmetric Adem relations for all integers $m \geq 1$ and $n \geq 0$; thus after inverting θ, one needs relations for all $m, n \in \mathbb{Z}[\frac{1}{2}]$ with $m > 0$ and $n \geq 0$.

Similarly, for part (c), applying θ to the admissible Adem relation (3) indexed by i and n yields

$$\begin{aligned}
(\text{admissible Adem relation})_{i,n} &= \lambda_i\lambda_{2i+1+n} + \sum_{j\geq 0} \tbinom{n-j-1}{j} \lambda_{i+n-j}\lambda_{2i+1+j} \\
&\overset{\theta}{\longmapsto} \lambda_{2i+1}\lambda_{4i+3+2n} + \sum_{j\geq 0} \tbinom{n-j-1}{j} \lambda_{2i+1+2n-2j}\lambda_{4i+3+2j} \\
&= \lambda_{2i+1}\lambda_{4i+3+2n} + \sum_{j\geq 0} \tbinom{2n-2j-2}{2j} \lambda_{2i+1+2n-2j}\lambda_{4i+3+2j} \\
&= \lambda_{2i+1}\lambda_{4i+3+2n} + \sum_{j\geq 0} \tbinom{2n-j-2}{j} \lambda_{2i+1+2n-j}\lambda_{4i+3+j} \\
&= (\text{admissible Adem relation})_{2i+1,2n}
\end{aligned}$$

In Λ, one has admissible Adem relations for all non-negative integers i and n, so in $\theta^{-1}\Lambda$, one gets admissible Adem relations for all $i \in \mathbb{Z}[\frac{1}{2}]_{>-1}$ and $n \in \mathbb{Z}[\frac{1}{2}]_{\geq 0}$. The change of the binomial coefficient from $\binom{n-j-1}{j}$ to $\binom{2n-j-2}{j}$ explains the presence of the integer $N(j,n)$ in the formula.

The two forms of the differentials in $\theta^{-1}\Lambda$ are derived similarly.

Since the admissible monomials form a basis for Λ, and since θ is injective on basis elements, part (f) follows.

Since homology commutes with colimits, part (g) follows. See also [7, 5.3] for the isomorphism between $(\mathrm{Sq}^0)^{-1}\operatorname{Ext}^*_A(\mathbb{F}_2,\mathbb{F}_2)$ and $\operatorname{Ext}^*_{\widehat{A}}(\mathbb{F}_2,\mathbb{F}_2)$. □

The action of θ on $\theta^{-1}\Lambda$ yields an action of an infinite cyclic group; this action is free in each positive degree in $\theta^{-1}\Lambda$, and trivial in degree zero. Under this action, the λ_n's are partitioned into orbits, and each orbit contains a unique λ_n with n an even integer. Among these, λ_0 is the only cocycle. Hence $H^1(\theta^{-1}\Lambda)$ is spanned by $\{\theta^k(\lambda_0) : k \in \mathbb{Z}\}$.

One would hope that computing $H^*(\theta^{-1}\Lambda)$ in higher dimensions would be simpler than computing $H^*(\Lambda)$, because of this symmetry. We have been unable to take advantage of this so far, unfortunately.

We remark that Λ is a Koszul algebra. Let A_{Lie} be the "Steenrod algebra for simplicial Lie algebras," which is generated by elements Sq^n and satisfies the usual Adem relations, but has $\mathrm{Sq}^0 = 0$. Priddy provided a criterion in [8, 5.3] to check whether an algebra is a Koszul algebra, and he showed in [9, 8.3–4] that both A_{Lie} and Λ satisfy this condition. Furthermore, he showed in [9, 9.1] that $\operatorname{Ext}^*_{A_{\mathsf{Lie}}}(\mathbb{F}_2,\mathbb{F}_2)$ is isomorphic to the opposite algebra to Λ, and pointed out in [8, 9.4] that

$$\operatorname{Ext}^*_{\operatorname{Ext}^*_{A_{\mathsf{Lie}}}(\mathbb{F}_2,\mathbb{F}_2)}(\mathbb{F}_2,\mathbb{F}_2) \cong A_{\mathsf{Lie}}.$$

In other words, Λ is the "Koszul dual" of A_{Lie}.

As a consequence, $\theta^{-1}\Lambda$ is a Koszul algebra, in a slightly unconventional sense (since it is $\mathbb{Z}\times\mathbb{Z}[\frac{1}{2}]$–graded, rather than $\mathbb{Z}\times\mathbb{Z}$–graded). So one should be able to compute $\operatorname{Ext}^*_{\theta^{-1}\Lambda}(\mathbb{F}_2,\mathbb{F}_2)$ pretty easily; the result should be the "complete Steenrod algebra for simplicial Lie algebras" (which is built from A_{Lie} using an inverse limit, just as $\widehat{A}$ is built from A). The details are left for the interested reader.

Appendix A The cohomology of $\Lambda/\theta\Lambda$ through the 14–stem

Figure 1 contains a table showing the result of hand and computer calculations of $H^{s,t}(\Lambda/\theta\Lambda)$ up to the 14–stem.

Columns are indexed by stem degree $t-s$, and rows are indexed by filtration degree s. Each cohomology class is represented by the leading term of a polynomial representing it, and that leading term is listed just by giving the subscripts on the lambdas involved; for example, the entry "61" in the 7–stem represents $\lambda_6\lambda_1 + \lambda_4\lambda_3$. The entry "$0^n$" stands for λ_0^n, while "221^n" stands for $\lambda_2\lambda_2\lambda_1^n$.

This table is complete in the range $t-s \leq 14$ except that it is missing the higher powers of λ_0 in the 0–stem. Underlined classes support differentials in the Bockstein spectral sequence; in this range, the only differentials are the d_1's sending $\lambda_2\lambda_2\lambda_1^n$ to λ_0^{n+3}.

By comparing the well-known computations of $H^{s,t}(\Lambda)$ in this range with what the Bockstein spectral sequence gives using the visible differentials, one can conclude that there are no differentials entering this picture from higher stems. Therefore the classes $[\lambda_0^i]$ with $i \geq 4$ are in the kernel of Sq^0, and for all other classes x in this range, either $\mathrm{Sq}^0(x)$ is nonzero or $\mathrm{Sq}^0(x)$ is out of this range.

filtration degree

12	0^{12}														$\underline{221^{10}}$
11	0^{11}													$\underline{221^{9}}$	
10	0^{10}												$\underline{221^{8}}$		
9	0^{9}											$\underline{221^{7}}$			
8	0^{8}										$\underline{221^{6}}$				
7	0^{7}									$\underline{221^{5}}$		1124111			
6	0^{6}								$\underline{221111}$		124111	224111			344111
5	0^{5}							$\underline{22111}$		24111		44111			51233
4	0^{4}						$\underline{2211}$	4111		1233					6611
3	0^{3}					$\underline{221}$		421	431						671
2	0^{2}			21				61							
1	0														
0															
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

stem degree

Figure 1: Calculations of $H^{s,t}(\Lambda')$ up to the 14–stem

References

[1] **D Arnon**, *Generalized Dickson invariants*, Israel J. Math. 118 (2000) 183–205 MR1776082

[2] **A K Bousfield**, **E B Curtis**, **D M Kan**, **D G Quillen**, **D L Rector**, **J W Schlesinger**, *The mod–p lower central series and the Adams spectral sequence*, Topology 5 (1966) 331–342 MR0199862

[3] **R Bruner**, *An example in the cohomology of augmented algebras*, J. Pure Appl. Algebra 55 (1988) 81–84 MR968570

[4] **N H V Hưng**, *The cohomology of the Steenrod algebra and representations of the general linear groups*, Trans. Amer. Math. Soc. 357 (2005) 4065–4089 MR2159700

[5] **I Llerena**, **N H V Hưng**, *The complete Steenrod algebra and the generalized Dickson algebra*, from: "Algebraic topology: new trends in localization and periodicity (Sant Feliu de Guíxols, 1994)", Progr. Math. 136, Birkhäuser, Basel (1996) 271–284

[6] **J P May**, *A general algebraic approach to Steenrod operations*, from: "The Steenrod Algebra and its Applications (Proc. Conf. to Celebrate N. E. Steenrod's Sixtieth Birthday, Battelle Memorial Inst., Columbus, Ohio, 1970)", Lecture Notes in Mathematics, Vol. 168, Springer, Berlin (1970) 153–231 MR0281196

[7] **J H Palmieri**, *Some quotient Hopf algebras of the dual Steenrod algebra*, Trans. Amer. Math. Soc. 358 (2006) 671–685 MR2177035

[8] **S B Priddy**, *Koszul resolutions*, Trans. Amer. Math. Soc. 152 (1970) 39–60 MR0265437

[9] **S B Priddy**, *Primary cohomology operations for simplicial Lie algebras*, Illinois J. Math. 14 (1970) 585–612 MR0270364

[10] **D C Ravenel**, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics 121, Academic Press, Orlando, FL (1986) MR860042

[11] **W Richter**, *Lambda algebra unstable composition products and the Lambda EHP sequence* Preprint

[12] **J S P Wang**, *On the cohomology of the mod–2 Steenrod algebra and the non-existence of elements of Hopf invariant one*, Illinois J. Math. 11 (1967) 480–490 MR0214065

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