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The odd-primary Kudo–Araki–May algebra of algebraic Steenrod operations and invariant theory

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We describe bialgebras of lower-indexed algebraic Steenrod operations over the field with p elements, p an odd prime. These go beyond the operations that can act nontrivially in topology, and their duals are closely related to algebras of polynomial invariants under subgroups of the general linear groups that contain the unipotent upper triangular groups. There are significant differences between these algebras and the analogous one for $p=2$, in particular in the nature and consequences of the defining Adem relations.

16W22; 16W30, 16W50, 55S10, 55S12, 55S99, 57T05

1 Introduction and statement of results

Mod p "lower-indexed" operations D_i $(i \geq 0)$ arising via $\mathbb{F}_p$–equivariance, for odd primes p, were constructed by Steenrod [19] for the cohomology of topological spaces and by Dyer and Lashof [3] for the homology of iterated loop spaces. The operations were described in a more general algebraic context by May [12], who computed implicit relations among the D_i. Until recently, however, all investigators immediately dropped the D_i for i not congruent to 0 or -1 $\bmod(p-1)$, because such operations act trivially in the cohomology and homology of topological spaces [19, page 104]. Furthermore, for reasons of grading under composition, those D_i that can act nontrivially in topology were converted to upper-indexed "Steenrod" operations, P^j and βP^j in cohomology, generating the odd-primary Steenrod algebra with "Adem" relations, and Q^k and βQ^k in homology, generating the Dyer–Lashof algebra. For instance, we recall that on even degrees in cohomology, $D_i\colon H^{2q}(X;\mathbb{F}_p) \to H^{2qp-i}(X;\mathbb{F}_p)$, and for $u \in H^{2q}(X;\mathbb{F}_p)$, Steenrod defined $P^j\colon H^{2q}(X;\mathbb{F}_p) \to H^{2q+2(p-1)j}(X;\mathbb{F}_p)$ by $P^j u = (-1)^{q-j} D_{2(p-1)(q-j)} u$. This displays the discarding of most of the operations and also shows that the composition algebras of operations generated by the D's versus the P's will be dramatically different in structure, especially since the degree of the

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underlying class, which varies during composition, is involved in converting betwixt them.

The structure generated by all the D_i subject to their universal "Adem" relations has a richness going beyond topology and is finding application in the study of algebras of polynomial invariants, of which we shall give a new example here. We have also used this structure in work to appear [16] to give a minimal presentation for the mod p cohomology of $\mathbb{C}P(\infty)$ as a module over the Steenrod algebra, which in turn allows us to give a minimal presentation of the cohomology of the classifying space BU (ie the algebra of symmetric invariants) as an algebra over the Steenrod algebra. Corresponding results at the prime 2, some joint with Peterson, appear in [15; 17; 18]. In [17] we analyzed the analogous algebra of operations D_i at the prime 2 and named it the Kudo–Araki–May algebra $\mathcal{K}$. Where results in this paper are completely analogous to those in [17], we shall omit their proofs; proofs that are not analogous or immediate will be given in subsequent sections.

A major contrast with odd primes is that at the prime 2 all operations D_i can act nontrivially for spaces, and they all convert to Steenrod or Dyer–Lashof operations, unlike the operations we are pursuing here. In work in progress, of which we give an example in this paper, we extend the results of Campbell [2] and Kechagias [10; 11] to describe certain algebras of odd primary polynomial invariants as subalgebras of the dual of the bialgebra generated by the D_{2i}, along with their structure as algebras over the Steenrod algebra. This will involve the broader set of operations going beyond those that can act nontrivially for spaces.

We shall also see a surprising algebraic difference in the larger algebra of operations generated by the D_{2i}: the Adem relations between these operations are no longer entirely determined just by those generated by "inadmissibles", as happened at the prime 2 and in the odd-primary Steenrod algebra. Nonetheless our study of this bialgebra and some of its subalgebras and quotient algebras, relying on analysis of Adem relations via formal power series, will produce bases consisting of certain (but not necessarily all) admissibles (see Definition 1.11), analogous to the Steenrod algebra, along with other features such as a generalized Nishida action. Our Nishida action provides structure over the Steenrod algebra that can be compared with the action independently enjoyed by algebras of invariants.

In our applications we deal with polynomial invariants or cohomology of spaces concentrated in even degrees, so we shall simplify by eliminating Bocksteins and thus deal only with the even D's, which we shall denote by $e_i = D_{2i}$, for $i \geq 0$. We begin our analysis of the algebraic structure generated by the e_i, and adopt the notational convention henceforth that $e_i = 0$ unless i is a nonnegative integer.

To prepare for applications to polynomial invariants, and because we must study the implementation of the Adem relations very carefully, we begin formally with just the free algebra on the e_i, before the imposition of any Adem relations.

Definition 1.1 Let $\widehat{\mathcal{U}}$ be the free noncommutative $\mathbb{F}_p$–algebra generated by elements e_i, for $i \geq 0$, of *topological* degree $|e_i| = 2i$. The topological degree of a product xy in $\widehat{\mathcal{U}}$ is given by $|xy| = |x| + p\,|y|$ (cf [17]). Note that $\widehat{\mathcal{U}}$ is bigraded, by topological degree t and by length n. We write $\widehat{\mathcal{U}}_{n,t}$ for the component in this bidegree. We caution that the reader should not assume that e_0 is the identity. It is not: $e_0 \in \widehat{\mathcal{U}}_{1,0}$, whereas $1 \in \widehat{\mathcal{U}}_{0,0}$. Note too that the nature of the formula for topological degree on products, in fact the principal purpose of its skewed nature, will be to make all relations homogeneous in topological degree in addition to length; this will be apparent from any of the formulations of the relations below.

If we restrict attention to May's algebraic Steenrod operations e_i applied only to even-dimensional classes, we see that his universal formulas [12, page 180] involving the algebraic Steenrod operations are induced by imposing on $\widehat{\mathcal{U}}$ the relations

$$\sum_k (-1)^{k+s}\binom{s-(p-1)k}{k} e_{r+(pk-s)(p-1)}e_{s-k(p-1)} \sim \sum_l (-1)^{l+r}\binom{r-(p-1)l}{l} e_{s+(pl-r)(p-1)}e_{r-l(p-1)} \qquad \text{for each fixed } r,s \geq 0,$$

which we shall refer to as May's relations.

We note that since $e_i = 0$ for i negative, these sums are finite for each pair r,s. But these relations are extremely difficult to use in practice; for instance, a particular monomial may appear in multiple relations on either side. Happily, the effect of these elaborate relations can be unraveled into equivalent relations that are more tractable and of more familiar form, first by interpreting them in terms of formal power series by defining

$$e(u) = \sum_{i=0}^{\infty} e_i u^i.$$

A short calculation followed by a change of variables gives the following theorem.

Theorem 1.2 *May's relations above for the e_i can be encoded as a formal power series identity expressing a certain symmetry:*

$$e(u)e(v(v^{p-1}-u^{p-1})) \sim e(v)e(u(u^{p-1}-v^{p-1})),$$

in which the coefficients of the monomials $u^r v^s$ are the corresponding individual relations above.

To obtain equivalent relations that have many of the useful features of the familiar Adem relations for the Steenrod and Dyer–Lashof algebras, we can use the residue method of Bullett and Macdonald [1] and Steiner [20], as we did in [17, page 1461], to obtain the next theorem.

Theorem 1.3 (proved in Section 2) *May's relations are equivalent to the relations*

$$e_i e_j \sim \sum_k (-1)^{\frac{pk-i}{p-1}} \binom{k-j-1}{\frac{pk-i}{p-1}-j} e_{i+pj-pk} e_k \qquad \text{for all integers } i, j \in \mathbb{Z},$$

where the numerator in a binomial coefficient may be any integer, and a term is present only if the fraction shown is an integer.

The very complicated relations of May can thus be replaced by these equivalent relations, which are at least somewhat like traditional Adem relations, in the sense that each two-fold monomial is now related only to a single sum of two-fold monomials. However, notice that these replacement relations can be nontrivial for arbitrary integer indices i and j, since even though the left side is automatically zero if i or j is negative, the right side may not be, for instance, $e_{-(p-1)} e_p \sim -2 e_1 e_{p-1}$. Together we shall call these the *full relations*, since they are bi-indexed by all $i, j \in \mathbb{Z}$, whereas we will call the proper subset that is bi-indexed only by $i, j \geq 0$ the *Adem relations*.

Since the monomials $e_i e_j$ for $i, j \geq 0$ form a basis for the two-fold operations before relations are imposed, we wish to do our linear algebra with only the Adem relations, rather than the full relations. Thus we will initially develop our resulting quotient algebra(s) formed by imposing only the Adem relations. Once we understand various important aspects of these, we will be in a position to prove, in Theorem 1.14, that the additional relations, ie for negative i or j, are actually redundant.

We shall also be interested in two sub/quotient algebras of $\widehat{\mathcal{U}}$:

Definition 1.4 Let $\widetilde{\mathcal{U}}$ be the subalgebra of $\widehat{\mathcal{U}}$ generated by the e_i for which i is even, and let $\mathcal{U}$ be the subalgebra of $\widehat{\mathcal{U}}$ generated by the e_i for which i is divisible by $p-1$. In dealing with $\mathcal{U}$ we shall use the notation $d_i = e_{i(p-1)}$ for its algebra generators, all brought together in the formal power series notation $d(t) = \sum_{i=0}^{\infty} d_i t^i$. (On spaces concentrated in even degrees, only the operations d_i can act nontrivially [19, page 104]. It was the d_i that were originally used by Steenrod to construct the reduced power operations P^j of the Steenrod algebra $\mathcal{A}$.) Note that $\widetilde{\mathcal{U}}$ and $\mathcal{U}$ can also be regarded as quotient algebras, namely as $\widehat{\mathcal{U}}$ modulo the two-sided ideals generated by the e_i for which i is odd, and the e_i for which i is not divisible by $p-1$, respectively. In fact this is how they will be considered henceforth.

We give $\widehat{\mathcal{U}}$ the structure of a bialgebra by defining a component coalgebra structure with diagonal map given by

$$\Delta(e_i) = \sum_a e_a \otimes e_{i-a}.$$

In formal power series, this becomes

$$\Delta e(u) = e(u) \otimes e(u),$$

ie $e(u)$ is *grouplike*. With this definition, we see that $\widetilde{\mathcal{U}}$ and $\mathcal{U}$ become quotient bialgebras.

To prepare for our linear algebra on $\widehat{\mathcal{U}}$, $\widetilde{\mathcal{U}}$, and $\mathcal{U}$, note that since there is exactly one Adem relation for each basis element of $\widehat{\mathcal{U}}_{2,*}$, we can encode them in an endomorphism.

Definition 1.5 Let $\theta\colon \widehat{\mathcal{U}}_{2,*} \to \widehat{\mathcal{U}}_{2,*}$ be defined by the formula

$$\theta(e_i e_j) = \sum_k (-1)^{\frac{pk-i}{p-1}} \binom{k-j-1}{\frac{pk-i}{p-1} - j} e_{i+pj-pk} e_k \qquad \text{for all } i, j \geq 0,$$

in other words, assign the right side of an Adem relation to its left side for each basis element $e_i e_j \in \widehat{\mathcal{U}}_{2,*}$. NB: Here it is critical that $i, j \geq 0$, since we are defining θ using a basis of $\widehat{\mathcal{U}}_{2,*}$.

Since we will sometimes also need to use the right side of a full relation even if i, j do not satisfy the nonnegativity requirement of the definition of θ on basis elements of $\widehat{\mathcal{U}}_{2,*}$, we extend the notation to all $i, j \in \mathbb{Z}$.

Definition 1.6 Let the notation $\theta(i, j)$ be defined by

$$\theta(i, j) = \sum_k (-1)^{\frac{pk-i}{p-1}} \binom{k-j-1}{\frac{pk-i}{p-1} - j} e_{i+pj-pk} e_k \qquad \text{for all } i, j \in \mathbb{Z},$$

recognizing that this is merely a function on pairs $i, j \in \mathbb{Z}$, since $e_i e_j$ is a basis element of $\widehat{\mathcal{U}}_{2,*}$ only when $i, j \geq 0$. In fact $e_i e_j = 0$ by definition when i or j is negative, while $\theta(i, j)$ as defined may not be.

Regarding relations induced on $\widetilde{\mathcal{U}}$ and $\mathcal{U}$, careful examination of the full relations on $\widehat{\mathcal{U}}$ in Theorem 1.3 shows that, on the quotients (or subalgebras) $\widetilde{\mathcal{U}}$ and $\mathcal{U}$, they induce full relations, and also a corresponding endomorphism θ using the Adem relations. We also have notation $\theta(i,j)$ for all appropriate $i,j \in \mathbb{Z}$, merely by restricting all subscripts to those of the respective generators of each quotient. In particular the full relations on $\mathcal{U}$ can be written more succinctly.

Theorem 1.7 *In $\mathcal{U}$ the full relations are*

$$d_i d_j \sim \sum_l (-1)^{pl-i} \binom{(p-1)(l-j)-1}{pl-i-(p-1)j} d_{i+pj-pl} d_l \qquad \text{for all } i,j \in \mathbb{Z}.$$

For expressing a formal power series identity in $\mathcal{U}$ equivalent to these, and envisaging the analogy to what we began with for May's relations in $\widehat{\mathcal{U}}$, we first define two helpful functions.

Definition 1.8 (1) $\varphi(a,b) = a(a^{p-1} - b^{p-1})$, and

(2) $\psi(a,b) = a(a-b)^{p-1}$.

Note that the formal power series identity for May's relations in $\widehat{\mathcal{U}}$ can then be written just as

$$e(u)e(\varphi(v,u)) \sim e(v)e(\varphi(u,v)).$$

The full relations on the quotient $\mathcal{U}$ are then equivalent to the following formal power series identity.

Theorem 1.9 (proved in Section 2) *The relations in $\mathcal{U}$ induced by May's relations in $\widehat{\mathcal{U}}$ are*

$$d(u)d(\psi(v,u)) \sim d(v)d(\psi(u,v)), \text{ ie}$$
$$d(u)d(v(v-u)^{p-1}) \sim d(v)d(u(u-v)^{p-1}).$$

Now we are ready to consider the quotients by Adem relations.

Definition 1.10 We shall denote the algebra quotients of $\widehat{\mathcal{U}}$, $\widetilde{\mathcal{U}}$, and $\mathcal{U}$ by their Adem relations by $\widehat{\mathcal{K}}$, $\widetilde{\mathcal{K}}$ and $\mathcal{K}$, respectively. (Recall that later we will show that the full relations with negative i or j are redundant.) Note that since, as remarked above, the relations are homogeneous in both bidegrees, $\widehat{\mathcal{K}}$, $\widetilde{\mathcal{K}}$, and $\mathcal{K}$ will inherit these bidegrees as well. By analogy with the prime 2 [17], and because it consists of the algebra of operations acting in topology on even degrees, we shall call $\mathcal{K}$ the even topological Kudo–Araki–May algebra. Obviously $\widehat{\mathcal{K}}$ and $\widetilde{\mathcal{K}}$ provide larger, purely algebraic, versions.

Relationships between $\mathcal{K}$ and the Steenrod and Dyer–Lashof algebras are inherent in the consequences of the conversion formulas [12, pp 161–2, 182]

$$P^j u = (-1)^{q-j}\, d_{q-j} u, \text{ where } u \text{ is a cohomology class of degree } 2q,$$

and $$Q^j u = (-1)^{j-q}\, d_{j-q} u, \text{ where } u \text{ is a homology class of degree } 2q.$$

Under composition using these conversions, the Adem relations in $\mathcal{K}$ produce, respectively, the traditional Adem relations in the Steenrod and Dyer–Lashof algebras. However, the reader should not imagine that this conversion provides anything as simple as an algebra map between $\mathcal{K}$ and either of the other two algebras. For instance, in either case the operation $d_0 = e_0$ always converts to the unstable p–th power operation on any class. Since the degree of a class is involved in the conversion of any operation on that class, and this degree changes during composition, the relationship that arises is that of a "sheared algebra map" as described in [17].

We are now ready to describe bases for $\widehat{\mathcal{K}}$, $\widetilde{\mathcal{K}}$, and $\mathcal{K}$.

Definition 1.11 We shall call a monomial $e_{i_1} \cdots e_{i_n}$ or $d_{i_1} \cdots d_{i_n}$ in any of our algebras *admissible* if $i_1 \le \cdots \le i_n$, otherwise *inadmissible*. And if $e_i e_j$ is admissible, we call the nonnegative number $j - i$ its *excess*.

For $p = 2$ the inadmissible Adem relations (ie for $i > j$) completely determine the admissible relations (ie for $i \le j$), making the latter redundant [5; 17], and thus the admissible monomials form a basis. We shall see that this is also the case at odd primes for the Adem relations in $\mathcal{U}$ (with generators $d_i = e_{i(p-1)}$) that create $\mathcal{K}$, and thus the admissible monomials form a basis for $\mathcal{K}$ (and correspondingly for the Steenrod algebra) at odd primes. However, the situation is very different in $\widehat{\mathcal{U}}$ and in $\widetilde{\mathcal{U}}$, where the subscripts on generators are not necessarily divisible by $p-1$. The following proposition illustrates some of the serious consequences of the nonredundancy of admissible Adem relations outside $\mathcal{U}$ and $\mathcal{K}$.

Proposition 1.12 (proved in Section 3) (a) *In the quotient $\widehat{\mathcal{K}}$, if $i \not\equiv j \mod (p-1)$ then $e_i e_j = 0$.*

(b) *In the quotient $\widehat{\mathcal{K}}$, if either i or j is odd, then $e_i e_j = 0$.*

Since part (b) tells us that any products not occurring entirely inside $\widetilde{\mathcal{K}}$ are zero, we see that $\widehat{\mathcal{K}}$ and $\widetilde{\mathcal{K}}$ are the same in lengths greater than one. And even within $\widetilde{\mathcal{K}}$, part (a) tells us that admissibles not having mutually congruent indices will be zero, forcing a sparseness to any possible basis. We are able to show that these two phenomena are the total extent of the collapsing effects of the admissible Adem relations, leaving the rest intact.

Theorem 1.13 (proved in Section 3) (a) *A vector space basis for $\widetilde{\mathcal{K}}$ (and for $\widehat{\mathcal{K}}$ in lengths exceeding one) is given by the monomials $e_{i_1} \cdots e_{i_n}$ for which all i_k are even, $0 \le i_1 \le \cdots \le i_n$, and $i_k \equiv i_{k-1} \mod (p-1)$ for $2 \le k \le n$.*

(b) *A vector space basis for $\mathcal{K}$ is given by the monomials $d_{i_1} \cdots d_{i_n}$ for which $0 \le i_1 \le \cdots \le i_n$.*

At this point we introduce important endomorphisms, which are essential for proving that the basis theorem above ensures that the negative relations are redundant, as alluded to earlier, and which then produce endomorphisms of the quotients by the Adem relations.

Define the three algebra maps $\widehat{\alpha}\colon \widehat{\mathcal{U}} \to \widehat{\mathcal{U}}$, $\widetilde{\alpha}\colon \widetilde{\mathcal{U}} \to \widetilde{\mathcal{U}}$, and $\alpha\colon \mathcal{U} \to \mathcal{U}$ by the formulas $\widehat{\alpha}(e_i) = e_{i-1}$, $\widetilde{\alpha}(e_i) = e_{i-2}$, $\alpha(e_i) = e_{i-(p-1)}$ on their respective algebra generators.

Theorem 1.14 (proved in Section 3) *The basis elements of $\widehat{\mathcal{K}}$, $\widetilde{\mathcal{K}}$ and $\mathcal{K}$ given by Theorem 1.13 remain linearly independent if we impose the additional (negative) relations*

$$e_i e_j \sim \sum_k (-1)^{\frac{pk-i}{p-1}} \binom{k-j-1}{\frac{pk-i}{p-1} - j} e_{i+pj-pk} e_k \qquad \text{for } i < 0 \text{ or } j < 0.$$

Thus imposing either the full relations or just the Adem relations produces the same quotients.

Now that we know that the full relations and their proper subset the Adem relations are equivalent impositions, we may use them interchangeably in studying the effects of imposing them. In particular, it is easy to see from comparing their formal power series formulations in Theorem 1.2 and Theorem 1.9 with the power series formulations $\widehat{\alpha}\,(e(u)) = ue(u)$, $\widetilde{\alpha}\,(e(u)) = u^2 e(u)$, and $\alpha\,(d(u)) = ud(u)$, that $\widetilde{\alpha}$ and α commute with their respective relations (which can also be verified by direct calculation). Thus they induce algebra endomorphisms on $\widetilde{\mathcal{K}}$ and $\mathcal{K}$, respectively. Note, however, that $\widehat{\alpha}$ fails to commute with its associated relations by a sign.

NB: Since $\widehat{\mathcal{K}}$ degenerates to $\widetilde{\mathcal{K}}$ in lengths greater than two, we focus most of our attention henceforth on $\widetilde{\mathcal{K}}$ and $\mathcal{K}$.

The next theorem follows immediately from the formal power series formulas for the relations and for Δ.

Theorem 1.15 *The diagonal maps Δ in $\widetilde{\mathcal{U}}$ and $\mathcal{U}$ respect the relations, and hence $\widetilde{\mathcal{K}}$ and $\mathcal{K}$ inherit the structure of bialgebras.*

In preparation for connections to algebras of invariants via dualization, we are interested in the coalgebra primitives in these bialgebras. The components of each of these bialgebras in fixed length degree n are coalgebras. Their primitive elements are given in the following theorem. Parts 1, 2, and 3 are immediate, while parts 4 and 5 are proved analogously to Theorem A of [17].

Theorem 1.16 (1) *A basis for the coalgebra primitives in $\widehat{\mathcal{U}}_{n,*}$ consists of the elements e_0^n and $e_0^a e_1 e_0^{n-a-1}$, for $0 \le a \le n-1$.*

(2) *For those in $\widetilde{\mathcal{U}}_{n,*}$: e_0^n and $e_0^a e_2 e_0^{n-a-1}$, for $0 \le a \le n-1$.*

(3) *For those in $\mathcal{U}_{n,*}$: $d_0^n = e_0^n$ and $d_0^a d_1 d_0^{n-a-1} = e_0^a e_{p-1} e_0^{n-a-1}$, for $0 \le a \le n-1$.*

(4) *For those in $\widetilde{\mathcal{K}}_{n,*}$: e_2^n and $e_0^a e_{p-1}^{n-a}$, for $1 \le a \le n$.*

(5) *For those in $\mathcal{K}_{n,*}$: $d_0^a d_1^{n-a} = e_0^a e_{p-1}^{n-a}$, for $0 \le a \le n$.*

Now we discuss applications via dualization connecting these coalgebras to algebras of polynomial invariants over the Steenrod algebra $\mathcal{A}$. There are already some known results of interest. For instance, it was proved by Kechagias in [10, Theorem 2.23] that in length degree n, the dual algebra to the coalgebra $\widehat{\mathcal{U}}_{n,*}$ is isomorphic to the invariants of the polynomial ring $S = \mathbb{F}_p[t_1, \dots, t_n]$ in n variables of degree 2, under the action of $\widehat{T}_n$, the upper triangular group with 1's down the principal diagonal. We shall discuss more examples below, but first we wish to introduce additional structure on our coalgebras for comparison with the natural action of the Steenrod algebra on algebras of invariants.

We shall create a downward action of $\mathcal{K}^{\mathrm{op}}$ (and hence of $\mathcal{A}^{\mathrm{op}}$) on $\widehat{\mathcal{U}}$,

$$\mathcal{K}^{\mathrm{op}}_{m,i} \otimes \widehat{\mathcal{U}}_{n,j} \to \widehat{\mathcal{U}}_{n,(i+j)/p^m},$$

whose contragradient then automatically produces an unstable action of $\mathcal{A}$ on the dual of $\widehat{\mathcal{U}}$. In Theorem 1.20 below, and in work in progress, we will relate this to the action of $\mathcal{A}$ on certain algebras of invariants.

Analogously to the prime 2 [17], we denote this action by the symbol $*$, refer to it as the *Nishida* action, since it generalizes the interaction discovered by Nishida between the actions of the Steenrod and Dyer–Lashof algebras on the homology of infinite loop spaces, and define it inductively.

Definition 1.17 Define first an action of $\mathcal{U}^{\mathrm{op}}$ by

$$d_i * 1 = \begin{cases} 1, & \text{if } i = 0 \\ 0, & \text{otherwise} \end{cases}$$

and $$d_i * e_j e_L = \sum_k (-1)^{i-k} \binom{i+\frac{j-i}{p}}{i-k} e_{i+\frac{j-i}{p}-(p-1)k}(d_k * e_L),$$

where e_L is any monomial in the e's.

The reader may calculate that our Nishida action is encoded in the formal power series identity
$$d(u^{p-1}) * [e(v) \cdot \_\_] = e(\varphi(v,u))[d(\varphi(u,v)^{p-1}) * \_\_],$$
where $\varphi(a,b)$ is the function defined above. One can also check straightforwardly that both $\widetilde{\mathcal{U}}$ and $\mathcal{U}$ inherit the action of $\mathcal{U}^{\mathrm{op}}$, considered either as subalgebras or quotient algebras of $\widehat{\mathcal{U}}$.

Theorem 1.18 (proved in Section 2) *The formula for $*$ respects the Adem relations in $\mathcal{U}^{\mathrm{op}}$ and hence defines a genuine action of $\mathcal{K}^{\mathrm{op}}$ on $\widehat{\mathcal{U}}$, $\widetilde{\mathcal{U}}$ and $\mathcal{U}$. Furthermore, it also respects the Adem relations in $\widehat{\mathcal{U}}$, $\widetilde{\mathcal{U}}$ and $\mathcal{U}$, and hence induces an action of $\mathcal{K}^{\mathrm{op}}$ on $\widehat{\mathcal{K}}$, $\widetilde{\mathcal{K}}$ and $\mathcal{K}$.*

Note that our Nishida action is itself a map of coalgebras, as may easily be verified by induction on length in $\widehat{\mathcal{U}}$ using the formal power series formulation above of the Nishida action, since $e(t)$ (as noted above) and $d(t)$ are both grouplike. This means that the induced contragradient $\mathcal{A}$–action mentioned above turns the duals of $\widehat{\mathcal{U}}$, $\widetilde{\mathcal{U}}$, $\mathcal{U}$, $\widehat{\mathcal{K}}$, $\widetilde{\mathcal{K}}$, and $\mathcal{K}$ into unstable algebras over the Steenrod algebra. Clearly from the theorem the natural maps between these are maps over the Steenrod algebra.

We comment at this point on the relationship between the action we have defined above, the traditional Nishida relations in topology, and the natural Steenrod algebra action in invariant theory; the gist is that they are all in agreement. Formulas compatible with those of our definition above, valid in the homology of infinite loop spaces, can be derived from Theorem 9.4 (ii) of May [12], so our formulas will agree with those of the traditional Nishida relations when restricted to the Dyer–Lashof algebra, ie the homology of QS^0. It is also not hard to check (as we did in the 2–primary case in [17]) that $\mathcal{K}$ is isomorphic as a coalgebra to the mod p Dyer–Lashof algebra modulo Bocksteins [10]. Hence the dual to $\mathcal{K}_{n,*}$ is isomorphic to the n–th Dickson algebra of GL_n invariants [21], and in fact this is an isomorphism of algebras over the Steenrod algebra [21; 13, page 33f]. So we see that our Nishida action will induce the same $\mathcal{A}$–action as that from invariant theory. This could also be verified by direct calculations similar to those in our proof of Theorem 1.20 below.

More broadly, we can now enlarge any inquiry comparing the duals of $\widehat{\mathcal{U}}$, $\widetilde{\mathcal{U}}$, $\mathcal{U}$, $\widetilde{\mathcal{K}}$, and $\mathcal{K}$ to algebras of invariants: Beyond just comparing the dualization of their coalgebra structures to certain algebra structures, we can actually compare unstable algebras over the Steenrod algebra, since both they and algebras of invariants now have independent defined structure as algebras over the Steenrod algebra. Despite suggestions in the literature that such isomorphisms automatically respect the $\mathcal{A}$–action, it seems to us that this still requires proof, and is one of the most interesting features to ponder.

In the case of the projection of $\mathcal{U}$ onto $\mathcal{K}$, this has already been explored by Kechagias [11, Theorem 4.11], who proved that the natural inclusion of the Dickson algebra of general linear group invariants into the algebra of invariants under T_n, the upper triangular group with arbitrary units on the diagonal, is dual to the coalgebra surjection of $\mathcal{U}_{n,*}$ onto $\mathcal{K}_{n,*}$, as indicated in the commutative diagram below. While the corresponding vertical maps are over the Steenrod algebra by naturality, it is not yet clear to us whether the isomorphism between the dual of $\mathcal{U}_{n,*}$ and the triangular invariants respects their independent $\mathcal{A}$–actions.

$$\begin{tikzcd} S^{T_n} & \mathcal{U}_n^* \arrow[l, "\cong"'] \\ S^{GL_n} \arrow[u] & \mathcal{K}_n^* \arrow[l, "\cong"'] \arrow[u] \end{tikzcd}$$

As an example of our broader aims, we intend to enlarge the situation above to a second commutative diagram:

$$\begin{tikzcd} S^{\widetilde{T}_n} & \widetilde{\mathcal{U}}_n^* \arrow[l, "\cong"', "\omega"] \\ S^{\widetilde{SL}_n} \arrow[u, "\tau"] & \widetilde{\mathcal{K}}_n^* \arrow[l, "\cong"'] \arrow[u, "\sigma"'] \end{tikzcd}$$

Note that the entire first diagram maps by inclusions into the second, forming a cube, and the reader may check that the faces of the cube joining the first diagram to the second will also commute. Here $\widetilde{T}_n$ consists of the upper triangular matrices whose diagonal elements are ± 1, and $\widetilde{SL}_n$ is the group of matrices whose determinants are ± 1. In this diagram, the horizontal isomorphisms are those provided by the following two theorems. All the maps will arise from matching natural generators for the algebras in question. Theorem 1.21 below treats the commutativity of this diagram.

Theorem 1.19 (proved in Section 4) *The dual of $\widetilde{\mathcal{U}}_{n,*}$ is isomorphic to the ring $\mathbb{F}_p[t_1, \ldots, t_n]^{\widetilde{T}_n}$ of invariants under the action of the group $\widetilde{T}_n$.*

Theorem 1.20 (proved in Section 4) *The dual of $\widetilde{\mathcal{K}}_{n,*}$ is isomorphic as an algebra over the Steenrod algebra to $\mathbb{F}_p[t_1, \ldots, t_n]^{\widetilde{SL}_n}$, the ring of invariants under the action of the group $\widetilde{SL}_n$.*

There are two main points to this theorem. The first is that we are dealing with operations that lie below the radar of the classical lower-indexed operations associated with the Steenrod and Dyer–Lashof algebras. The second is that the Steenrod algebra structure is entirely determined by our Nishida formula in $\widetilde{\mathcal{U}}$. It is also the case that the Steenrod algebra structure on the Dickson algebra $\mathbb{F}_p[t_1,\dots,t_n]^{GL_n}$ is determined by the Nishida formula in $\mathcal{U}$. In work now in progress, we shall compute the relationships of other quotient bi-algebras of $\widehat{\mathcal{U}}$ to algebras of invariants of other subgroups of GL_n that contain the upper triangular group $\widehat{T}_n$ defined above.

We now turn to the commutativity of the diagram above. We begin by defining the maps in the diagram. Identify $\widetilde{\mathcal{K}}_{n,*}{}^* = \mathbb{F}_p[t_1,\dots,t_n]^{\widetilde{SL}_n}$ via the isomorphism described in the proof of Theorem 1.20. Define maps

$$\begin{aligned}
\sigma &\colon \widetilde{\mathcal{K}}_{n,*}{}^* \to \widetilde{\mathcal{U}}^*_{n,*},\\
\tau &\colon \widetilde{\mathcal{K}}_{n,*}{}^* = \mathbb{F}_p[t_1,\dots,t_n]^{\widetilde{SL}_n} \to \mathbb{F}_p[t_1,\dots,t_n]^{\widetilde{T}_n}, \text{ and}\\
\omega &\colon \widetilde{\mathcal{U}}^*_{n,*} \to \mathbb{F}_p[t_1,\dots,t_n]^{\widetilde{T}_n}, \quad \text{as follows.}
\end{aligned}$$

The map σ is the dual of the map that imposes Adem relations, the map τ is the map induced by the inclusion $\widetilde{T}_n \subseteq \widetilde{SL}_n$, and the map ω is the isomorphism described in the proof of Theorem 1.19.

Theorem 1.21 (proved in Section 4) *We have $\omega \circ \sigma = \tau$.*

We conclude this section by listing two important self-maps and concomitant properties of these algebras.

(1) Let κ denote multiplication by e_0. Since the element $e_0 = d_0$ satisfies $\Delta(e_0) = e_0 \otimes e_0$, its dual, κ^*, is an algebra endomorphism on the duals of the various algebras we have defined.

(2) The p–th power map on the dual algebras is known as the Frobenius map. Its dual V, known as the Verschiebung, is given on generators by $V(e_i) = e_{i/p}$, where we recall the convention that $e_a = 0$ if a is not a nonnegative integer. We extend the map multiplicatively to products. Since $V(e(u)) = e(u^p)$, it is easy to check, using the power series formulation of the Adem relations, that V is well-defined on $\widetilde{\mathcal{K}}$ and $\mathcal{K}$.

Remark 1.22 Relations between these maps and the Nishida action include:

(A) $d_i * \kappa(e_J) = \kappa(V(d_i) * e_J)$, and

(B) $d_0 * e_J = V(e_J)$.

(Here $J = (j_1,\dots,j_n)$ is a multi-index and $e_J = e_{j_1} \cdots e_{j_n}$.)

2 Proofs of Theorem 1.3, Theorem 1.9 and Theorem 1.18

Proof of Theorem 1.3 We will use the residue method of Jacobi [8] (see [7; 6; 4, Section 1.1, 1.2]), and leave straightforward calculations to the reader. If $f(x) = \sum_k a_k x^k$ is a formal Laurent series (ie $k \in \mathbb{Z}$ is bounded below) with coefficients in a ring with unity, define its "residue" $\mathrm{res}_x \sum_k a_k x^k$ to be a_{-1}. Then Jacobi's change of variables formula implies [4, Section 1.1, 1.2] that if $y = g(x)$ is a formal power series with coefficients in the same coefficient ring, and if $g(0) = 0$ and $g'(0)$ is invertible in the coefficient ring, then $\mathrm{res}_y f(y) = \mathrm{res}_x f(g(x))\, g'(x)$, where g' is the formal derivative.

First we will show that the full relations follow from the formal power series identity form of May's relations.

In preparation, our setting is the ring $\big(\widehat{\mathcal{U}}((u))\big)\big((v)\big)$ of formal Laurent series in v with coefficients in the ring of formal Laurent series in u (with coefficients in $\widehat{\mathcal{U}}$). Note that $w = g(v) = v(v^{p-1} - u^{p-1})$ satisfies the hypotheses for a Jacobi change of variables. Note too for use below that since g is a very simple multiplicatively invertible polynomial in v, one can compute its powers in our ring, both positive and negative, by writing $g(v)^m = \big(-u^{p-1}v(1 + (-(\frac{v}{u})^{p-1}))\big)^m$ and using the geometric/binomial power series expansion for $(1+x)^m$, for any $m \in \mathbb{Z}$.

Now using a Jacobi change of variables and the formal power series identity for May's relations, we have, for any $i, j \in \mathbb{Z}$,

$$
\begin{aligned}
e_i e_j &= \operatorname*{res}_u \left(\operatorname*{res}_w \frac{e(u)\, e(w)}{u^{i+1} w^{j+1}} \right) \\
&= \operatorname*{res}_u \left(\operatorname*{res}_v \frac{e(u)}{u^{i+1}} \frac{e\big(v(v^{p-1} - u^{p-1})\big)}{\big(v(v^{p-1} - u^{p-1})\big)^{j+1}} \frac{d}{dv} \Big(v(v^{p-1} - u^{p-1}) \Big) \right) \\
&\sim \operatorname*{res}_u \left(\operatorname*{res}_v \frac{e(v) e(u(u^{p-1} - v^{p-1}))\, \big(-u^{p-1}\big)}{u^{i+1} \big(v(v^{p-1} - u^{p-1})\big)^{j+1}} \right).
\end{aligned}
$$

From here a straightforward calculation of the latter using expanded formal power series yields

$$
\sum_k (-1)^{\frac{pk-i}{p-1}} \binom{k-j-1}{\frac{pk-i}{p-1} - j} e_{i+pj-pk} e_k \,,
$$

so the full relations follow from May's relations.

Now we turn to the converse, to prove the formal power series identity relations under the hypothesis of the full relations. This time we prepare for a Jacobi change of variables using the power series

$$\rho(w) = -u \sum_{i\geq 0} \left(\frac{w}{u^p}\right)^{p^i},$$

again with coefficients in $\widehat{\mathcal{U}}((u))$, and satisfying the Jacobi requirements since $\rho(0)=0$ and $\rho'(w) = -u^{-(p-1)}$ (our characteristic is p), which is invertible in $\widehat{\mathcal{U}}((u))$. Note also that

$$\left(\frac{\rho(w)}{u}\right)^p = \frac{\rho(w)}{u} + \frac{w}{u^p},$$

and thus

$$w = \rho(w)^p - u^{p-1}\rho(w).$$

In other words, $g(\rho(w)) = w$, so ρ is the composition inverse of g, and thus $\rho(g(v)) = v$ holds too [4, Section 1.1], which can easily be verified by direct calculation and will also be needed below.

Now we assume the full relations as given, and will derive May's relations. From the full relations and the calculations mentioned above, we have, for every $i, j \in \mathbb{Z}$,

$$\begin{aligned} e_i e_j &\sim \sum_k (-1)^{\frac{pk-i}{p-1}} \binom{k-j-1}{\frac{pk-i}{p-1} - j} e_{i+pj-pk} e_k \\ &= \operatorname*{res}_u \left(\operatorname*{res}_t \frac{e(t) e(u(u^{p-1} - t^{p-1}))\left(-u^{p-1}\right)}{u^{i+1}\left(t(t^{p-1} - u^{p-1})\right)^{j+1}} \right). \end{aligned}$$

(Here we use t to avoid confounding at this stage with the v in our desired final identity.) Next we make our change of variables $t = \rho(w)$, and use the polynomial equation above satisfied by $\rho(w)$, simplifying to

$$e_i e_j \sim \operatorname*{res}_u \left(\operatorname*{res}_w \frac{1}{u^{i+1} w^{j+1}} e(\rho(w)) e(u(u^{p-1} - \rho(w)^{p-1})) \right),$$

so

$$e(u)e(w) \sim e(\rho(w)) e\big(u(u^{p-1} - \rho(w)^{p-1})\big).$$

Now we substitute $g(v) = v(v^{p-1} - u^{p-1})$ for w in this equality, and use the fact that $\rho(g(v)) = v$, producing

$$e(u) e\big(v(v^{p-1} - u^{p-1})\big) \sim e(v) e\big(u(u^{p-1} - v^{p-1})\big),$$

as desired. □

Proof of Theorem 1.9 We begin with May's relations in $\widehat{\mathcal{U}}$ in the form

$$e(u)e(v(v^{p-1} - u^{p-1})) \sim e(v)e(u(u^{p-1} - v^{p-1})).$$

Now the map to the quotient $\mathcal{U}$ sends $e_{i(p-1)}$ to d_i, and all other e's to 0, so the relations become

$$d(u^{p-1})d\big(v^{p-1}(v^{p-1}-u^{p-1}\big)^{p-1}) \sim d(v^{p-1})d\big(u^{p-1}(u^{p-1}-v^{p-1})^{p-1}\big),$$

and substituting u for u^{p-1} and v for v^{p-1} produces

$$d(u)d(v(v-u)^{p-1}) \sim d(v)d(u(u-v)^{p-1}),$$

as desired. $\square$

In preparation for the proof of Theorem 1.18, note the identity

$$\varphi\left(\varphi\left(a,b\right),\varphi\left(c,b\right)\right) = \varphi\left(\varphi\left(a,c\right),\varphi\left(b,c\right)\right)$$

for the function φ defined in the introduction.

Proof of Theorem 1.18 Let $\cdot$ denote the multiplication in $\widehat{\mathcal{U}}$ and $\bullet$ be the multiplication in $\mathcal{U}^{\mathrm{op}}$. For the first part of the theorem, we must show that

$$\begin{aligned}&\big(d(\psi(v^{p-1},u^{p-1}))\bullet d(u^{p-1})\big)*\big(e(w)\cdot\_\_\big)\\ &\sim\big(d(\psi(u^{p-1},v^{p-1}))\bullet d(v^{p-1})\big)*\big(e(w)\cdot\_\_\big).\end{aligned}$$

We compute, assuming the result true inductively for lower length in $\widehat{\mathcal{U}}$:

$$\begin{aligned}&\big(d(\psi(v^{p-1},u^{p-1}))\bullet d(u^{p-1})\big)*\big(e(w)\cdot__\big)\\
&=d(\varphi(v,u)^{p-1})*[d(u^{p-1})*\big(e(w)\cdot__\big)]\\
&=d(\varphi(v,u)^{p-1})*\big[e(\varphi(w,u))\cdot\big(d\big(\varphi(u,w)^{p-1}\big)*__\big)\big]\\
&=e\big(\varphi(\varphi(w,u),\varphi(v,u))\big)\cdot\big[d\big(\varphi(\varphi(v,u),\varphi(w,u))^{p-1}\big)*\big(d\big(\varphi(u,w)^{p-1}\big)*__\big)\big]\\
&=e\big(\varphi(\varphi(w,u),\varphi(v,u))\big)\cdot\big[d\big(\varphi(\varphi(v,w),\varphi(u,w))^{p-1}\big)*\big(d\big(\varphi(u,w)^{p-1}\big)*__\big)\big]\\
&=e\big(\varphi(\varphi(w,u),\varphi(v,u))\big)\cdot\Big[\Big(d\big(\varphi(\varphi(v,w),\varphi(u,w))^{p-1}\big)\bullet d\big(\varphi(u,w)^{p-1}\big)\Big)*__\Big]\\
&=e\big(\varphi(\varphi(w,u),\varphi(v,u))\big)\cdot\Big[\Big(d\big(\psi(\varphi(v,w)^{p-1},\varphi(u,w)^{p-1})\big)\bullet d\big(\varphi(u,w)^{p-1}\big)\Big)*__\Big]\\
&\sim e\big(\varphi(\varphi(w,v),\varphi(u,v))\big)\cdot\Big[\Big(d\big(\psi(\varphi(u,w)^{p-1},\varphi(v,w)^{p-1})\big)\bullet d\big(\varphi(v,w)^{p-1}\big)\Big)*__\Big].\end{aligned}$$

We may now reverse these steps to see that this last term is equal to the desired

$$\big(d(\psi(u^{p-1},v^{p-1}))\bullet d(v^{p-1})\big)*\big(e(w)\cdot\_\_\big).$$

For the second part of Theorem 1.18, we must show that

$$d(u^{p-1})*\big[e(v)\cdot e(\varphi(w,v))\cdot\_\_\big]\sim d(u^{p-1})*\big[e(w)\cdot e(\varphi(v,w))\cdot\_\_\big].$$

We compute

$$\begin{aligned}
&d(u^{p-1}) * \big[e(v)\cdot e(\varphi(w,v))\cdot __\big] \\
&= e(\varphi(v,u))\cdot\big[d\big(\varphi(u,v)^{p-1}\big) * \big(e\big(\varphi(w,v)\big)\cdot __\big)\big] \\
&= e(\varphi(v,u))\cdot\Big[e\big(\varphi\big(\varphi(w,v),\varphi(u,v)\big)\cdot\Big(d\big(\varphi\big(\varphi(u,v),\varphi(w,v)\big)\big)^{p-1} * __\Big)\Big] \\
&= \big[e(\varphi(v,u))\cdot e\big(\varphi(\varphi(w,v),\varphi(u,v)\big)\big]\cdot\Big(d\big(\varphi\big(\varphi(u,v),\varphi(w,v)\big)\big)^{p-1} * __\Big) \\
&= \big[e(\varphi(v,u))\cdot e\big(\varphi(\varphi(w,u),\varphi(v,u)\big)\big]\cdot\Big(d\big(\varphi\big(\varphi(u,w),\varphi(v,w)\big)\big)^{p-1} * __\Big) \\
&\sim \big[e(\varphi(w,u))\cdot e\big(\varphi(\varphi(v,u),\varphi(w,u)\big)\big]\cdot\Big(d\big(\varphi\big(\varphi(u,w),\varphi(v,w)\big)\big)^{p-1} * __\Big).
\end{aligned}$$

By reversing these steps, this last term is equal to $d(u^{p-1}) * \big[e(w)\cdot e(\varphi(v,w))\cdot \_\_\big]$, as desired. □

3 Proofs of Proposition 1.12, Theorem 1.13 and Theorem 1.14

To begin this section, we collect some facts about the full relations in $\widehat{\mathcal{U}}_{2,*}$. We use the notation $\theta(i,j)$ from the introduction for the right side of any full relation.

Lemma 3.1 *For any $i,j\in\mathbb{Z}$, if e_le_k appears with a nonzero coefficient in $\theta(i,j)$, then $k\equiv i \mod (p-1)$ and $l\equiv j \mod (p-1)$.*

(So the second index in each term that appears on the right-hand side of any full relation is congruent mod $(p-1)$ to the first index on the left-hand side, and similarly for the other pair.)

Proof The fraction that occurs as the exponent of -1 must be an integer, so $k\equiv i \mod (p-1)$. The second congruence then follows from $i+pj=l+pk$. □

Thus we have the following lemma.

Lemma 3.2 *For any $i,j\in\mathbb{Z}$, if e_le_k appears with a nonzero coefficient in $\theta(i,j)$, then $l-k\equiv-(i-j) \mod (p-1)$.*

(So the difference of indices in each term that appears on the right-hand side of any full relation is congruent mod $(p-1)$ to the negation of the corresponding index difference on the left. Thus $\widehat{\mathcal{U}}_{2,*}$ splits into two subspaces, generated respectively by those e_ie_j for which either $i\equiv j$ or $i\not\equiv j \mod (p-1)$, and every full relation involves terms that lie in only one of these subspaces.)

Lemma 3.3 *For any $i, j \in \mathbb{Z}$, if $i > j$ and $e_l e_k$ appears with a nonzero coefficient in $\theta(i, j)$, then $l \leq k$.*

(That is, any full relation for an inadmissible monomial rewrites the monomial in terms of admissibles.)

Proof For the denominator of the binomial coefficient to be nonnegative, we must have $pk \geq i+(p-1)j$, so $i+pj-pk \leq j$, whence $k \geq i+pj-pk+(k-j) = l+(k-j)$. Since $i > j$, the first inequality also yields $pk > pj$, whence $k > j$. Combining these facts, we see that $l \leq k$. □

Lemma 3.4 *For any $i, j \in \mathbb{Z}$, if $i \leq j$ and $e_l e_k$ appears with a nonzero coefficient in $\theta(i, j)$, then the numerator of the binomial coefficient is negative. Conversely, if $i > j$ and $e_l e_k$ appears with a nonzero coefficient, then the numerator of the binomial coefficient is nonnegative.*

(That is, any full relation for an admissible monomial produces exclusively terms on the right with negative numerators in their binomial coefficients, and for an inadmissible monomial produces exclusively terms with nonnegative numerators.)

Proof Consider first $i \leq j$. If the numerator were nonnegative, $k \geq j+1$, and the binomial coefficient is nonzero, then $k-j-1 \geq \frac{pk-i}{p-1} - j$, from which we obtain $i-(p-1) \geq k$. Then since $i \leq j$, we have $j-(p-1) \geq k \geq j+1$, a contradiction. Hence $k < j+1$, ie the numerator is negative.

Next consider $i > j$, and the binomial coefficient nonzero, so the denominator is nonnegative, ie $pk-i \geq pj-j$. Then $k-j \geq (i-j)/p > 0$, so $k-j-1 \geq 0$, ie the numerator is nonnegative. □

Lemma 3.5 *Let $1 \leq b \leq p-1$. Then $\theta(i, i-b) = \theta(e_i e_{i-b}) = 0$ for any $i \geq b$.*

(This expresses an important "edge effect", in which Adem relations for "nearly admissible" inadmissibles are zero on the right side.)

Proof We compute

$$e_i e_{i-b} \sim \sum_k (-1)^{\frac{pk-i}{p-1}} \binom{k-i+b-1}{\frac{pk-i}{p-1}-i+b} e_{i+p(i-b)-pk} e_k.$$

Since $b \geq 1$, the left side is inadmissible, so by Lemma 3.4 the numerator of the binomial coefficient is nonnegative. Thus for it to be nonzero, we must have $k-$

$i+b-1 \geq \frac{pk-i}{p-1} - i + b$, whence $i-(p-1) \geq k$. But also $k \geq i-b+1$. Since $i-(p-1)-(i-b+1) = b-p$, there are no values of k that satisfy both inequalities if $1 \leq b \leq p-1$. $\square$

In preparation for the proof of Theorem 1.13, we follow Le Minh Ha [5], who proved algebraically that the Adem relations for admissibles in the Steenrod algebra are redundant, using the self-map of the Steenrod algebra dual to multiplication by ξ_1 in the dual Steenrod algebra. We imitate his self-map in our setting, defining

$$\eta\colon \widehat{\mathcal{U}}_{2,*} \longrightarrow \widehat{\mathcal{U}}_{2,*+p-1}$$

by
$$\eta(e_i e_j) = e_{i+p-1} e_j + e_{i-(p-1)^2} e_{j+p-1} \text{ for } i,j \geq 0.$$

NB: Like the formula for $\theta(e_i e_j)$, this formula only applies when $i,j \geq 0$. We need to be extremely careful to pay attention to this, and will say "η is defined by formula" for emphasis when this is the case.

Lemma 3.6 *Consider $e_i e_j$ for $i \geq (p-1)^2$ and $j \geq 0$.*

(a) *If $0 \leq i \leq j$ (ie $e_i e_j$ is admissible), then $\theta(\eta(e_i e_j)) = \eta(\theta(e_i e_j))$.*

(b) *If $0 \leq j < i$ (ie $e_i e_j$ is inadmissible), and $i - j \leq \frac{i}{p} + p - 1$, then $\theta(\eta(e_i e_j)) = \eta(\theta(e_i e_j))$.*

Proof First note, regarding the left side of the equality, that since $i \geq (p-1)^2$, both terms in the formula above for $\eta(e_i e_j)$ have entirely nonnegative indices, and thus θ is defined on them by the formula for Adem relations in its definition.

Second, regarding the right side of the equality, in both cases (a) and (b) we shall check that if $e_l e_k$ is a term in $\theta(e_i e_j)$ with nonzero binomial coefficient, then l and k are nonnegative, and hence η is defined by formula on $e_l e_k$. Indeed, in case (a), by Lemma 3.4, $k \leq j$. So $l = i + pj - pk \geq i \geq 0$. And $\frac{pk-i}{p-1} - j \geq 0$, whence $pk \geq i + (p-1)j$, so $k \geq 0$. In case (b), by Lemma 3.4, $k \geq j+1 \geq 0$. So also $\frac{pk-i}{p-1} - j \leq k - j - 1$, thus $pk - i \leq (p-1)k - (p-1)$, and so $k \leq i - (p-1)$. Then

$$\begin{aligned} l = i + pj - pk &= (p+1)i - p(i-j) - pk \\ &\geq (p+1)i - p(i-j) - pi + p(p-1) \\ &= i - p(i-j) + p(p-1) \\ &\geq i - (i + (p-1)p) + p(p-1) = 0, \end{aligned}$$

using the additional hypothesis for the last inequality.

Now, since we have shown that all terms on both sides of the claimed equality

$$\theta(\eta(e_i e_j)) = \eta(\theta(e_i e_j))$$

will be calculated using the defining formulas for θ and η on basis elements, it remains to show that these match up. We leave the details to the reader, noting that it boils down first to combining four terms to three via Pascal's ordinary binomial coefficient identity, followed by the mod p identity $\binom{M}{N} \equiv \binom{M-p}{N} + \binom{M-p}{N-p}$ also used by Le Minh Ha [5]. □

Lemma 3.7 *For $i, j \geq 0$, if $i \leq j$, then in the quotient $\widehat{\mathcal{K}}$, the admissible $e_i e_j$ can be rewritten by application of Adem relations as*

$$e_i e_j \ = \gamma e_i e_j + \text{a sum of admissibles of lesser excess,}$$

with coefficient $\gamma = 0$ if $i \not\equiv j \mod (p-1)$, and $\gamma = (-1)^i = (-1)^j$ if $i \equiv j \mod (p-1)$.

Proof In the quotient $\widehat{\mathcal{K}}$ by the Adem relations, we have

$$e_i e_j = \sum_k (-1)^{\frac{pk-i}{p-1}} \binom{k-j-1}{\frac{pk-i}{p-1} - j} e_l e_k, \quad \text{where } l = i + pj - pk.$$

Since $i \leq j$, by Lemma 3.4, for nonzero terms we have $k \leq j$. By Lemma 3.1, there is no term with $k = j$ unless $j = k \equiv i \mod (p-1)$), in which case the coefficient is $(-1)^{-j}$, the sign claimed for γ.

Now the remaining terms in the sum are either admissible of lesser excess, since $k < j$ and $i + pj = l + pk$, or inadmissible. If inadmissible, ie $l > k$, we proceed as follows for any nonzero term.

First, since we must have

$$\frac{pk-i}{p-1} - j \geq 0,$$

we get
$$pk \geq i + (p-1)j.$$

Then
$$\begin{aligned} l &= i + pj - pk \\ &\leq i + pj - (i + (p-1)j) = j. \end{aligned}$$

Now applying another Adem relation, from Lemma 3.3 we have

$$e_l e_k = \sum_m (-1)^{\frac{pm-l}{p-1}} \binom{m-k-1}{\frac{pm-l}{p-1} - k} e_{i+pj-pm} e_m,$$

where the nonzero terms on the right are all admissible. By Lemma 3.4, $m > k$. Also, $m-k-1 \geq \frac{pm-l}{p-1} - k$, whence $l-(p-1) \geq m$, so $j-(p-1) \geq m$ from above, and thus $j > m$. Write $n = i + pj - pm$, so we now have $n > i$. Thus $m - n < j - i$, so $e_n e_m$ is an admissible of lesser excess than $e_i e_j$. □

Proof of Proposition 1.12 Clearly we may assume $i, j \geq 0$. We begin with part (a). First, consider $i > j$. Lemma 3.2 and Lemma 3.3, along with the hypothesis $i \not\equiv j \mod (p-1)$, ensure that in the quotient $\widehat{\mathcal{K}}$ by the Adem relations, $e_i e_j$ is a linear combination of terms $e_l e_k$ with $l \not\equiv k \mod (p-1)$ and $l \leq k$. This reduces the proof to considering terms for which $i \leq j$. So let $0 \leq i \leq j$ and $i \not\equiv j \mod (p-1)$. Then from Lemma 3.2 and Lemma 3.7, $e_i e_j$ can be written as a sum of admissibles of lower excess satisfying the same noncongruence condition. By Fermat's method of descent the proof is complete.

For part (b), we first note that thanks to part (a) we need only consider the case where $i \equiv j \mod (p-1)$. Then Lemma 3.1, Lemma 3.2, and Lemma 3.3 ensure that in the quotient $\widehat{\mathcal{K}}$ by the Adem relations, $e_i e_j$ is a linear combination of terms $e_l e_k$ with $l \equiv k \mod (p-1)$, $0 \leq l \leq k$, and l, k odd. Now Lemma 3.7 writes $2e_l e_k$ as a sum of terms of lesser excess, that by our Lemmas also satisfy the same conditions of index congruence, admissibility, and oddness. Again by descent we are finished. □

Proof of Theorem 1.13 Since we are working to analyze $\widetilde{\mathcal{K}}$, a quotient of $\widetilde{\mathcal{U}}$, henceforth all subscripts on e's shall be even. According to Lemma 3.3 and Proposition 1.12, we need only prove that the elements $e_{i_1} \cdots e_{i_n}$ for which all i_k are even, $0 \leq i_1 \leq \cdots \leq i_n$, and $i_k \equiv i_{k-1} \mod (p-1)$ for $2 \leq k \leq n$, are linearly independent in $\widetilde{\mathcal{K}}$. This proof will use the method of Le Minh Ha [5]. From a little linear algebra and Lemma 3.2 and Lemma 3.7, it is enough to show that in $\widetilde{\mathcal{U}}$, the Adem relations for $e_i e_j$, for $0 \leq i \leq j$ and $i - j \equiv 0 \mod (p-1)$, are consequences of the relations for which $i > j$. Let $\Theta(i,j) = e_i e_j - \theta(e_i e_j)$ for $i, j \geq 0$. Our goal is thus to prove that in $\widetilde{\mathcal{U}}$ every $\Theta(i,j)$ for which $j - i \equiv 0 \mod (p-1)$ and $i \leq j$ (i, j even) is a linear combination of $\Theta(l,k)$'s for which $l > k$. Thus we consider $\Theta(i, i+s)$ for $i, s \geq 0$, and proceed by induction on s. Since $s \equiv 0 \mod (p-1)$, we may write $s = c(p-1)$ and induct on $c \geq 0$.

Case One (base step) Consider $s = c(p-1)$, where $0 \leq c \leq p$. We have

$$\theta(e_i e_j) = \theta(e_i e_{i+s}) = \sum_k (-1)^{\frac{pk-i}{p-1}} \binom{k-i-s-1}{\frac{pk-i}{p-1} - i - s} e_{(p+1)i+ps-pk} e_k .$$

For the binomial coefficient to be nonzero, we must have (using Lemma 3.4 to know that the numerator is negative)

$$\begin{aligned} i+s \geq k &\geq i+s-\frac{s}{p} \\ &\geq i+c(p-1)-\frac{c(p-1)}{p} \\ &= i+c(p-1)-c+\frac{c}{p}. \end{aligned}$$

So
$$i+c(p-1) \geq k \geq i+c(p-1)-c+\frac{c}{p}.$$

By Lemma 3.1, there is one possible value for k in this range, namely $k=i+c(p-1)$, unless $c=p$, in which case $k=i+(c-1)(p-1)$ is also a possibility. For $k=i+c(p-1)$, we get the term

$$\begin{aligned} &(-1)^{\frac{pk-i}{p-1}}\binom{k-i-s-1}{\frac{pk-i}{p-1}-i-s}e_{(p+1)i+ps-pk}e_k \\ &= (-1)^{i+pc}\binom{-1}{c}e_ie_{i+c(p-1)} \\ &= (-1)^i e_ie_{i+s}. \end{aligned}$$

Thus since i is even, we have shown that for $0 \leq c < p$, the Adem relation for $e_ie_{i+c(p-1)}$ reduces to

$$\Theta\,(i,i+s) = e_ie_{i+s} - e_ie_{i+s} = 0,$$

while for $c=p$ there is one possible remaining surviving term, with $k=i+(p-1)^2$, a multiple of $e_{i+p^2-p}e_{i+(p-1)^2}$. But $\theta(e_{i+p^2-p}e_{i+(p-1)^2}) = 0$ by Lemma 3.5, so $e_{i+p^2-p}e_{i+(p-1)^2} = \Theta(i+p^2-p, i+(p-1)^2)$, expressing the dependence of $\Theta\,(i,i+s)$ on an inadmissible relation, as desired.

Case Two (inductive step) Let $c \geq p+1$. Inductively, we assume that for $0 \leq q < c$ we have

$$\Theta(a,a+q(p-1)) = \sum_{m \in M_q} \gamma_m \Theta(a+mp(p-1), a+(q-m)(p-1)),$$

where γ_m is a scalar and

$$M_q = \{m \mid mp \leq q \leq m(p+1)-1\}.$$

We note that the relations appearing on the right hand side of this formula are inadmissible, and that the Case One results do satisfy this assumption.

By Lemma 3.6(a), we have

$$\Theta(i, i + c(p-1)) = \eta\Theta(i + (p-1)^2, i + (c-1)(p-1)) - \Theta(i + p(p-1), i + (c-1)(p-1)).$$

We focus first on the second term on the right side. Since $c - 1 \geq p$, by the inductive hypothesis we have

$$\Theta(i + p(p-1), i + (c-1)(p-1)) = \sum_{m' \in M'_{q'}} \gamma_{m'} \Theta(a' + m' p(p-1), a' + (q' - m')(p-1)),$$

where $a' = i + p(p-1)$, $q' = c - 1 - p$, and

$$M'_{q'} = \{m' | m' p \leq q' \leq m'(p+1) - 1\}.$$

Here $\Theta(a' + m' p(p-1), a' + (q' - m')(p-1))$
$$= \Theta(i + (m'+1)p(p-1), i + (c - 1 - m')(p-1)).$$

And $m' p \leq c - 1 - p \leq m'(p+1) - 1$, so $(m'+1)p + 1 \leq c \leq (m'+1)(p+1) - 1$.

Now focusing on the first term on the right side, we can again use the inductive hypothesis to obtain

$$\eta\Theta(i + (p-1)^2, i + (c-1)(p-1)) = \eta \sum_{m'' \in M''_{q''}} \gamma_{m''} \Theta(a'' + m'' p(p-1), a'' + (q'' - m'')(p-1)),$$

where $a'' = i + (p-1)^2$, $q'' = c - p$, and

$$M''_{q''} = \{m'' \,|\, m'' p \leq q'' \leq m''(p+1) - 1\}.$$

One checks that $(a'' + m'' p(p-1), a'' + (q'' - m'')(p-1))$ satisfies the hypotheses of Lemma 3.6(b). Hence

$$\begin{aligned} \eta\Theta(a'' + m'' p(p-1), a'' + (q'' - m'')(p-1)) &= \Theta(i + (m''+1)p(p-1), i + (c - 1 - m'')(p-1)) \\ &\quad + \Theta(i + m'' p(p-1), i + (c - m'')(p-1)). \end{aligned}$$

Considering the first term on the right here, we have $m'' p \leq c - p \leq m''(p+1) - 1$, so $(m''+1)p \leq c \leq (m''+1)(p+1) - 2$, and therefore the term is inadmissible and meets the inductive requirement. For the second term, $m''p < (m''+1)p \leq c$; and either $c \leq m''(p+1)-1$ or $m''(p+1) \leq c \leq (m''+1)(p+1)-2$. In the first instance, the term is inadmissible and the inductive assumption is met. In the latter case, we have an admissible for which $0 \leq c - m''(p+1) \leq p-1$, so by Case 1, $\Theta(i+m''p(p-1), i+(c-m'')(p-1)) = 0$.

We conclude that the elements of the form $e_a e_{a+q}$ for a even, $q \geq 0$, and q divisible by $p-1$, are linearly independent in $\widetilde{\mathcal{K}}_{2,*}$. Both parts of the theorem follow. $\square$

Proof of Theorem 1.14 We will use the notation $\Theta(i,j)$ of the previous proof for the Adem relations, but extended to the full relations for $i,j \in \mathbb{Z}$ via $\Theta(i,j) = e_i e_j - \theta(i,j)$, where $\theta(i,j)$ is our notation from the introduction for the right side of any full relation (and differs in general, we recall, from $\theta\left(e_i e_j\right)$, which is zero when either i or j is negative). Now for any $i,j \in \mathbb{Z}$, we have $\widetilde{\alpha}^n\left(\Theta(i,j)\right) = \Theta\left(i-2n, j-2n\right)$. This is essentially contained in the fact that the formal power series formulation of the Adem relations clearly commutes with $\widetilde{\alpha}$ since $\widetilde{\alpha}\left(e(u)\right) = u^2 e(u)$, but it may be verified directly by comparing the explicit formulas.

We will first prove our claim for the negative Adem relations on $\widetilde{\mathcal{U}}$, ie we will show that for any relation $\Theta(i,j)$ on $\widetilde{\mathcal{U}}$ (ie i,j even) for which $i<0$ or $j<0$, the relation is already trivial once the Adem relations on $\widetilde{\mathcal{U}}$ are imposed, ie it is trivial in $\widetilde{\mathcal{K}}$. First, since any nonzero term in the formula for $\theta(i,j)$ must satisfy $i+(p-1)j \leq pk \leq i+pj$, we see that if $j<0$, there are none, so $\Theta(i,j)$ is trivial in $\widetilde{\mathcal{K}}$. Furthermore, if $i \not\equiv j \mod (p-1)$, then all terms in $\theta(i,j)$ also have their indices noncongruent mod $(p-1)$ from Lemma 3.2, so by Proposition 1.12 all such terms are also zero in $\widetilde{\mathcal{K}}$. Thus it remains only to consider the case when $i<0$, $j \geq 0$. and $i \equiv j \mod (p-1)$.

In this case, we first examine $\Theta(-i, j-2i)$,which is also a relation on $\widetilde{\mathcal{U}}$ satisfying the same congruence condition on its entries, but now with both entries positive. Thus from the proof of Theorem 1.13, if it is not already inadmissible it can be written as a linear combination of inadmissibles, ie in $\widetilde{\mathcal{U}}$,

$$\Theta(-i, j-2i) = \sum_{k>l} \gamma_{k,l} \Theta\left(k,l\right).$$

Now we apply $\widetilde{\alpha}^{-i}$ (recall that $-i>0$), which from our observation above, about how $\widetilde{\alpha}$ passes through Θ, yields

$$\Theta(i,j) = \sum_{k>l} \gamma_{k,l} \Theta\left(k+2i, l+2i\right)$$

in $\widetilde{\mathcal{U}}$. Now if $l+2i<0$, we know from above that $\Theta\left(k+2i, l+2i\right) = 0$ in $\widetilde{\mathcal{U}}$, and if $l+2i \geq 0$, then since $k>l$, we see that $\Theta\left(k+2i, l+2i\right)$ is one of the Adem relations already imposed in the definition of $\widetilde{\mathcal{K}}$.

To extend our claim to cover the negative relations on $\widehat{\mathcal{U}}$, notice that if either i or j is odd, then from Lemma 3.1 and Lemma 3.2, any nonzero term in the formula for $\theta(i,j)$ must have one of its indices odd as well, so from Proposition 1.12 all such terms are zero once the Adem relations are imposed; thus the relation $\Theta(i,j)$ is trivial after imposition of the Adem relations on $\widehat{\mathcal{U}}$. □

4 Proofs of Theorem 1.19, Theorem 1.20 and Theorem 1.21

Proof of Theorem 1.19 We have seen that the coalgebra primitives in $\widetilde{\mathcal{U}}_{n,*}$ are the monomial basis elements e_0^n and $e_0^a e_2 e_0^{n-a-1}$, for $0 \le a \le n-1$. Let $\widetilde{v}_{n,a} = (e_0^a e_2 e_0^{n-a-1})^*$, the dual element to $e_0^a e_2 e_0^{n-a-1}$. Note that the topological degree of $\widetilde{v}_{n,a}$ is $4p^a$. Also note that since $\Delta(e_0^n) = e_0^n \otimes e_0^n$, its dual is 1, the algebra identity in $\widetilde{\mathcal{U}}^*_{n,*}$. The multiplication in $\widetilde{\mathcal{U}}^*_{n,*}$ is commutative and obeys the usual degree convention for products. On basis elements of $\widetilde{\mathcal{U}}_{n,*}$, the correspondence

$$e_{2i_1} \cdots e_{2i_n} \mapsto \widetilde{v}_{n,0}^{i_1} \cdots \widetilde{v}_{n,n-1}^{i_n}$$

provides a bijection of graded vector spaces from $\widetilde{\mathcal{U}}_{n,*}$ to $\mathbb{F}_p[\widetilde{v}_{n,0}, \dots, \widetilde{v}_{n,n-1}]$. It follows from calculations of Mùi [14] and Kechagias [9; 10, Corollary 4.22] that the equality $\mathbb{F}_p[t_1, \dots, t_n]^{\widetilde{T}_n} = \mathbb{F}_p[\widetilde{V}_1, \dots, \widetilde{V}_n]$ holds, where the degree of $\widetilde{V}_i$ (the square of Mùi's invariant V_i) is $4p^{i-1}$, so we obtain our desired result by mapping one set of generators to the other. □

Proof of Theorem 1.20 First we calculate the structure of the dual of $\widetilde{\mathcal{K}}_{n,*}$ as an algebra over the Steenrod algebra. The coalgebra primitives in $\widetilde{\mathcal{K}}_{n,*}$ are e_2^n, and $e_0^a e_{p-1}^{n-a}$ for $1 \le a \le n$, and they are elements of the basis of Theorem 1.13. Denote their dual elements in $\widetilde{\mathcal{K}}_{n,*}{}^*$ by $\widetilde{s}_{n,0}$ and $c_{n,a}$, respectively (note that $c_{n,n}$ is the unit in $\widetilde{\mathcal{K}}^*_{n,*}$). The topological degree of e_2^n is $4(1+p+\cdots+p^{n-1})$ and that of $e_0^a e_{p-1}^{n-a}$ is $2(p^n - p^a)$. It is easy to see, as in the preceding proposition, that $\widetilde{\mathcal{K}}_{n,*}{}^*$ is a polynomial algebra on the elements $\widetilde{s}_{n,0}$ and $c_{n,a}$, $1 \le a \le n-1$.

We shall determine the resulting action of the Steenrod algebra. Calculating with the Nishida formulas in $\widetilde{\mathcal{K}}_{n,*}$, we obtain, for $n \ge 1$ and $1 \le i \le n-1$,

$$\begin{aligned} d_{p^{n-1}+2p^{n-2}+\cdots+2p+2} * e_2^{n-1} e_{p+1} &= -2e_2^n, \\ d_{p^n - p^i - p^{i-1}} * e_0^{i-1} e_{p-1}^{n-i+1} &= -e_0^i e_{p-1}^{n-i}, \\ d_{p^n - p^{n-1} - p^i} * e_0^i e_{p-1}^{n-i-1} e_{2p-2} &= e_0^i e_{p-1}^{n-i}. \end{aligned}$$

Converting these to $\widetilde{\mathcal{K}}_{n,*}{}^*$ and moving from the action of $\mathcal{K}$ to that of $\mathcal{A}$, by freely using earlier lemmas and theorems about $\widetilde{\mathcal{K}}_{n,*}$, and the sparseness of $\widetilde{\mathcal{K}}_{n,*}{}^*$ in low degrees, we get the following formulas, for $n \geq 1$ and $1 \leq i \leq n-1$:

$$P^{p^{n-1}}\widetilde{s}_{n,0} = 2\widetilde{s}_{n,0}c_{n,n-1},$$
$$P^{p^{i-1}}c_{n,i} = c_{n,i-1}$$
$$P^{p^{n-1}}c_{n,i} = -c_{n,i}c_{n,n-1}.$$

(Note that the occurrence in these formulas of $c_{n,0}$ really denotes $(e^n_{p-1})^* = \widetilde{s}^{(p-1)/2}_{n,0}$.) All other operations P^{p^i} are zero on the generators since their images lie in degrees in which there are no nonzero elements.

These calculated values from our Nishida action on the coalgebra $\widetilde{\mathcal{K}}_{n,*}$ coincide with the structure over the Steenrod algebra of $\mathbb{F}_p[t_1,\dots,t_n]^{\widetilde{SL}_n}$ as a ring of invariants, as can easily be deduced from the known results in Kech [9, page 945; 10, page 280] and Wilkerson [21], so they are isomorphic. Under the isomorphism, $c_{n,a} \in \widetilde{\mathcal{K}}_{n,*}$ maps to the Dickson invariant of the same name in $\mathbb{F}_p[t_1,\dots,t_n]^{\widetilde{SL}_n}$, and $\widetilde{s}_{n,0}$ maps to the remaining polynomial generator of the ring of invariants, which has the formula $\prod_{i=1}^n \widetilde{V}_i$ in terms of the elements in the proof above for the $\widetilde{T}_n$ invariants [9, page 945; 10, page 280], see also Mùi [14] and Wilkerson [21]. □

Proof of Theorem 1.21 We will use notation from the two proofs above. Observe, using our various results, that

(1) In $\widetilde{\mathcal{K}}_{n,*}$, one has $e_ie_j = e_2^2$ if and only if $i = j = 2$, and $e_ie_j = e_0e_{p^r(p-1)}$ if and only if $(i,j) = (0, p^r(p-1))$ or $(i,j) = (p^{r+1}(p-1), 0)$ (use the Verschiebung map V, described at the end of the first section of the paper),

(2) In $\widetilde{\mathcal{U}}^*_{n,*}$, we have
$$(e_{i_1}\cdots e_{i_n})^* \cdot (e_{j_1}\cdots e_{j_n})^* = (e_{i_1+j_1}\cdots e_{i_n+j_n})^*,$$
where all i's and j's are even.

Now, from the first observation we can calculate that

$$\sigma(c_{n,i}) = \sum_{\substack{0\leq j_1<\cdots<j_{n-i}\leq n\\ j_0=0}} \left\{ \left(\prod_{s=1}^{n-i} e_0^{j_s-j_{s-1}-1} e_{(p-1)p^{i+s-j_s}} \right) e_0^{n-j_{n-i}} \right\}^*.$$

And from the second, we find that

$$\left\{ \left(\prod_{s=1}^{n-i} e_0^{j_s-j_{s-1}-1} e_{(p-1)p^{i+s-j_s}} \right) e_0^{n-j_{n-i}} \right\}^* = \prod_{s=1}^{n-i} \left(\widetilde{v}_{n,j_s-1}\right)^{\frac{p-1}{2}p^{i+s-j_s}}.$$

But by the known formulas for the Dickson invariants [10, page 280; 11, page 224; 14],

$$\tau(c_{n,i}) = \sum_{1\le j_1<\cdots<j_{n-i}\le n} \left\{ \prod_{s=1}^{n-i} \widetilde{V}_{j_s}^{\frac{p-1}{2}p^{i+s-j_s}} \right\}.$$

A similar calculation shows that $\omega(\sigma(\widetilde{s}_{n,0})) = \tau(\widetilde{s}_{n,0})$, and the proposition follows. □

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