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Unstable modules over the Steenrod algebra revisited

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A new and natural description of the category of unstable modules over the Steenrod algebra as a category of comodules over a bialgebra is given; the theory extends and unifies the work of Carlsson, Kuhn, Lannes, Miller, Schwartz, Zarati and others. Related categories of comodules are studied, which shed light upon the structure of the category of unstable modules at odd primes. In particular, a category of bigraded unstable modules is introduced; this is related to the study of modules over the motivic Steenrod algebra.

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1 Introduction

The Steenrod algebra $\mathcal{A}$ over a prime field $\mathbb{F}$ of characteristic p is a fundamental mathematical object; it is defined in algebraic topology to be the algebra of stable cohomology operations for singular cohomology with coefficients in $\mathbb{F}$. The algebra $\mathcal{A}$ is graded and acts on the cohomology ring of a space; the underlying graded $\mathcal{A}$–module is *unstable*, a condition which is usually defined in terms of the Steenrod reduced power operations and the Bockstein operator.

This paper shows that the well-known description of the dual of the Steenrod algebra, which is due to Milnor, has an extension which leads to an entirely algebraic description of the category $\mathcal{U}$ of unstable modules over the Steenrod algebra as a category of comodules (defined with respect to a completed tensor product) over a bialgebra, without imposing an external instability condition. The existence of such a description is implied by the general theory of tensor abelian categories at the prime two; in the odd prime case, the method requires the super-algebra setting, namely using $\mathbb{Z}/2$–gradings to introduce the necessary sign conventions for commutativity.

Theorem 1 *The category of unstable modules over the Steenrod algebra at an odd prime is equivalent to the category of right comodules (with respect to a completed tensor product) over a $\mathbb{Z}/2$–graded bialgebra $\widetilde{\mathcal{A}}^*$.*

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At an odd prime, the theory introduces an auxiliary bialgebra $\mathcal{B}$, which can be defined as the bialgebra of endomorphisms of the additive group (in the super-algebra setting). The category $\mathcal{U}(\mathcal{B})$ of graded comodules over this coalgebra is equipped with an exact forgetful functor $\mathcal{U}(\mathcal{B}) \to \mathcal{U}$ to the category of unstable modules, which is induced by a surjective morphism of bialgebras $\mathcal{B} \twoheadrightarrow \widetilde{\mathcal{A}}^*$. Moreover, $\mathcal{U}(\mathcal{B})$ is equivalent to a representation category which generalizes the description given by Kuhn of the category of unstable modules over the field $\mathbb{F}_2$. This part of the theory extends and unifies existing approaches to the category of unstable modules.

The surjection $\mathcal{B} \twoheadrightarrow \widetilde{\mathcal{A}}^*$ factorizes across a bialgebra $\overline{\mathcal{B}}$; the category $\mathcal{U}^{\mathrm{bi.gr}}$ of comodules over this bialgebra is a bigraded analogue of the category of unstable modules. This category has not hitherto been studied. The structure of $\mathcal{U}^{\mathrm{bi.gr}}$ is of interest since it sheds light on and provides new approaches to the structure of the classical category of unstable modules at odd primes and also in connection with unstable modules defined over the motivic Steenrod algebra (at all primes). At an odd prime, these tensor abelian categories are related by exact functors which are induced by corestriction

$$\mathcal{U}(\mathcal{B}) \xrightarrow{\Psi} \mathcal{U}^{\mathrm{bi.gr}} \xrightarrow{\Theta} \mathcal{U}.$$

The functorial point of view on the category of unstable modules, developed by Kuhn [6] from the work of Henn, Lannes and Schwartz [5] arises naturally in the theory via the free commutative algebra functor; forgetting the algebra structure, this can be considered as an object of the category $\mathcal{F}$ of functors from finite-dimensional $\mathbb{F}$–vector spaces to $\mathbb{F}$–vector spaces. Key examples of objects of $\mathcal{F}$ are given by the exterior power functors Λ^a and the divided power functors Γ^b.

The one-sided Morita equivalence theory of [6] gives rise to functors $r_{\mathcal{B}}\colon \mathcal{F} \to \mathcal{U}(\mathcal{B})$ and $r'\colon \mathcal{F} \to \mathcal{U}'$, where $\mathcal{U}' \subset \mathcal{U}$ denotes the full subcategory of unstable modules concentrated in even degree. These give a very natural description of the projective generators of the category $\mathcal{U}^{\mathrm{bi.gr}}$.

The results on the projective generators of $\mathcal{U}^{\mathrm{bi.gr}}$ lead, at odd primes, to a new and natural analysis of the free unstable modules $F(n)$. The following result provides an odd-primary analogue of the well-known analysis of the structure of the free unstable modules at the prime two, where $\mathcal{O}$ denotes the forgetful functor $\mathcal{U}' \to \mathcal{U}$.

Proposition 2 *For p an odd prime and n a positive integer, $F(n)$ has a finite filtration with associated graded*

$$\bigoplus_{\substack{a+2b=n \\ a\geq 0}} \mathcal{O}r'(\Lambda^a \otimes \Gamma^b) \oplus \bigoplus_{\substack{a+2b=n \\ a\geq 1}} \Sigma\mathcal{O}r'(\Lambda^{a-1} \otimes \Gamma^b).$$

There is a corresponding analysis of the injective cogenerators of $\mathcal{U}$, which is presented in Section 7.4. These results are applied in Section 8 to give new proofs of the fundamental results on nil-localization in the odd characteristic case, using the results on $\mathcal{U}(\mathcal{B})$ provided by the representation-theoretic framework.

The final section of the paper indicates a modification of the theory at the prime two which introduces an abelian category $\mathcal{U}_{\mathcal{M}}$ which is related to the study of unstable modules over the motivic Steenrod algebra. This allows the comparison of the category of unstable modules and a suitable category of bigraded unstable modules at the prime two.

1.1 Related results

There are related results which occur in the literature; the results on unstable modules at the prime two are implicit in the work of Kuhn [6]. The bialgebra which is used to define the category of unstable modules occurs in Bisson–Joyal [1, Section 4], where it is termed the extended Milnor–Hopf algebra. The addendum [1, page 260] indicates the fact that the category of unstable modules corresponds to the category of comodules over this bialgebra. A second reference for related material in a similar context is Smirnov [12, page 116, Chapter 5]. This reference also provides a related statement for the odd prime case.

1.2 Organization of the paper

The first part of the paper is devoted to the introduction of the categories of (generalized) unstable modules which are of interest here. Section 2 provides a survey of the theory of tensor abelian categories which motivates the constructions of the paper. Section 3 constructs the bialgebra $\mathcal{B}$, by considering the endomorphisms of the additive group; this is a generalization of Milnor's approach to calculating the dual Steenrod algebra. The other bialgebras used in the paper are constructed as quotients of $\mathcal{B}$. Section 4 defines the categories of generalized unstable modules as categories of comodules; the simple objects of these categories are considered briefly, together with associated suspension functors.

The second part of the paper establishes the connection with the functorial point of view. Section 5 reviews the category $\mathcal{F}$ of functors, the notion of an exponential functor and then establishes the relation with the bialgebra $\mathcal{B}$ defined in the first part. Section 6 reviews and extends the results of Kuhn on representation categories, motivated by the considerations of this paper.

The third part is devoted to an analysis of the projective and injective objects of the category $\mathcal{U}^{\mathrm{bi.gr}}$ at an odd prime. Section 7 considers the standard projective generators and the standard injective cogenerators of $\mathcal{U}^{\mathrm{bi,gr}}$; the results are new and are applied to obtain a new analysis of the structure of the projective and injective objects of the category $\mathcal{U}$ of unstable modules. Section 8 uses the theory to provide a new proof of the injectivity of $H^*(BV)$ in the odd prime case, more in the spirit of [6].

The fourth part of the paper corresponds to Section 9; this indicates how bigraded unstable modules appear at the prime two. This theory is related to the study of modules over the motivic Steenrod algebra. This material will be returned to in greater detail elsewhere.

The appendix reviews certain results on comodules which are required in the paper.

Part I Basic structure

2 Tensor abelian categories

The category of unstable modules over the Steenrod algebra is a tensor abelian category to which Tannakian theory can be applied. The relevant theory of tensor abelian categories is reviewed in this section; for further details, the reader is referred to Deligne–Milne [3].

2.1 General theory

A tensor category is a category $\mathcal{C}$ which is equipped with a symmetric monoidal structure $(\mathcal{C}, \otimes, \mathbf{1})$, where $\mathbf{1}$ denotes the unit. For a field $\mathbb{F}$, let $\mathcal{V}_{\mathbb{F}}$ denote the category of $\mathbb{F}$–vector spaces, equipped with the usual abelian tensor structure.

Definition 2.1.1

(1) An abelian tensor category is a tensor category $(\mathcal{C}, \otimes, \mathbf{1})$ for which the category $\mathcal{C}$ is abelian and the functor $\otimes$ is biadditive.

(2) For $(\mathcal{C}, \otimes, \mathbf{1})$ an abelian tensor category such that $\mathrm{End}(\mathbf{1}) = \mathbb{F}$ is a field, a fibre functor is a faithful, exact, $\mathbb{F}$–linear tensor functor $\mathcal{C} \to \mathcal{V}_{\mathbb{F}}$.

An affine monoid scheme over $\mathbb{F}$ is a scheme of the form $M := \mathrm{Spec}(B)$, where B is a bialgebra over $\mathbb{F}$ (not necessarily equipped with a conjugation), for which the underlying $\mathbb{F}$–algebra is commutative[1]. An affine group scheme is an affine monoid scheme of the form $G := \mathrm{Spec}(H)$, where H is a Hopf algebra over $\mathbb{F}$ (a bialgebra with conjugation).

[1] This usage of the term bialgebra conflicts with the terminology of [3].

Notation 2.1.2 For $M = \mathrm{Spec}(B)$ an affine monoid scheme over $\mathbb{F}$, let $\mathrm{Rep}_{\mathbb{F}}(M)$ (the category of representations of M) denote the category of right B–comodules.

Proposition 2.1.3 *For M an affine monoid scheme, the category $\mathrm{Rep}_{\mathbb{F}}(M)$ is a tensor abelian category, equipped with a canonical fibre functor.*

Example 2.1.4 Let $\mathbb{F}$ be a field.

(1) The category of comodules over the multiplicative group $\mathbb{G}_m \cong \mathrm{Spec}(\mathbb{F}[x^{\pm 1}])$ is equivalent to the category of graded vector spaces.

(2) The category of comodules over $G := \mathrm{Spec}(\mathbb{F}[t]/(t^2-1))$ is equivalent to the category of $\mathbb{Z}/2$–graded vector spaces.

(3) [3, Example 1.25] Let $\mathcal{V}_{\mathbb{F}}^{\mathbb{Z}/2}$ denote the category of $\mathbb{Z}/2$–graded vector spaces, equipped with the symmetric monoidal tensor structure involving the Koszul sign convention. Then $\mathcal{V}_{\mathbb{F}}^{\mathbb{Z}/2}$ is a tensor abelian category but it is not of the form $\mathrm{Rep}_{\mathbb{F}}(M)$, for any affine monoid scheme M.

All of the above examples are *rigid* tensor categories, which means that duality behaves well. In the first two cases, this follows from the fact that the categories are representations of affine group schemes rather than just affine monoid schemes.

The following is the part of the theory of neutral Tannakian categories which is relevant to this paper.

Theorem 2.1.5 (Deligne and Milne [3, Proposition 2.14]) *Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a $\mathbb{F}$–linear abelian tensor category such that $\mathrm{End}(\mathbf{1}) = \mathbb{F}$ and let $\omega\colon \mathcal{C} \to \mathcal{V}_{\mathbb{F}}^{\mathrm{fd}}$ be a fibre functor to the category of finite-dimensional vector spaces. There exists an affine monoid scheme $M = \mathrm{Spec}(B)$ over $\mathbb{F}$ such that $(\mathcal{C}, \otimes, \mathbf{1})$ is equivalent to the category $\mathrm{Rep}_{\mathbb{F}}(M)$ equipped with the canonical abelian tensor category structure and fibre functor.*

Remark 2.1.6 The above theory can be generalized to the $\mathbb{Z}/2$–graded situation with the Koszul sign convention. In this context, the fibre functor is a tensor functor $\omega_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{V}_{\mathbb{F}}^{\mathbb{Z}/2}$.

This generalization is essential for the topological considerations, where the algebras to be considered are naturally $\mathbb{Z}/2$–graded and are commutative with respect to the Koszul sign convention.

Example 2.1.7 Let $\mathbb{F}$ be the prime field of characteristic p and let $\mathcal{U}$ denote the category of unstable modules over the $\mathbb{F}$–Steenrod algebra.

(1) For $p = 2$, the category $\mathcal{U}$ is an $\mathbb{F}_2$–linear abelian tensor category, equipped with a fibre functor to the category of $\mathbb{F}_2$–vector spaces.

(2) For $p > 2$, the category $\mathcal{U}$ is an $\mathbb{F}_p$–linear abelian tensor category, equipped with a fibre functor to the category $\mathcal{V}_{\mathbb{F}}^{\mathbb{Z}/2}$ of $\mathbb{Z}/2$–graded $\mathbb{F}_p$–vector spaces, equipped with the tensor structure with the Koszul sign convention.

The theory of Tannakian categories is developed in terms of finite-dimensional representations. For this reason, Tannakian theory does not apply directly in considering unstable modules, but requires modification using completed tensor products in the comodule structures. Modulo this addendum, Tannakian theory implies that the category of unstable modules (for $p = 2$ and for p odd) has a description as a category of comodules.

3 Endomorphisms of the additive group

This section constructs the bialgebra $\mathcal{B}$, which is the fundamental mathematical object of this paper, as the endomorphism bialgebra of the additive group in the category of $\mathbb{Z}/2$–graded algebras. Throughout the section, $\mathbb{F}$ denotes a prime field of odd characteristic.

The $\mathbb{F}$–algebra structures considered in this paper are graded commutative, with respect to the Koszul sign convention. The foundations are developed in the super-algebra context, namely using $\mathbb{Z}/2$–gradings, to avoid imposing an *a priori* $\mathbb{Z}$–grading. The fact that the foundations of algebraic geometry can be generalized to the super-algebra context is well known (see Deligne [2, Section 0.3]).

3.1 Super algebras

Let $\mathcal{V}_{\mathbb{F}}^{\mathbb{Z}/2}$ denote the category of $\mathbb{Z}/2$–vector spaces, equipped with the symmetric monoidal structure provided by the graded tensor product with the Koszul sign convention [3, Example 1.25]. Let $Alg_{\mathbb{F}}^{\mathbb{Z}/2}$ denote the category of unital commutative monoids in $\mathcal{V}_{\mathbb{F}}^{\mathbb{Z}/2}$; this is the category of unital $\mathbb{Z}/2$–graded $\mathbb{F}$–algebras which are graded commutative, with respect to the Koszul sign convention. The tensor product of $\mathcal{V}_{\mathbb{F}}^{\mathbb{Z}/2}$ induces the coproduct in the category $Alg_{\mathbb{F}}^{\mathbb{Z}/2}$; in particular, the tensor product of two objects of $Alg_{\mathbb{F}}^{\mathbb{Z}/2}$ is in $Alg_{\mathbb{F}}^{\mathbb{Z}/2}$.

The category $Bialg_{\mathbb{F}}^{\mathbb{Z}/2}$ of $\mathbb{Z}/2$–graded bialgebras is the category of comonoid objects in $Alg_{\mathbb{F}}^{\mathbb{Z}/2}$. Namely an object $B \in Bialg_{\mathbb{F}}^{\mathbb{Z}/2}$ is a $\mathbb{Z}/2$–graded algebra $B \in Alg_{\mathbb{F}}^{\mathbb{Z}/2}$ which is equipped with a morphism[2] $B \to B \otimes B$ in $Alg_{\mathbb{F}}^{\mathbb{Z}/2}$, which is coassociative and counital with respect to the counit morphism $B \to \mathbb{F}$ in $Alg_{\mathbb{F}}^{\mathbb{Z}/2}$.

3.2 The bialgebra $\mathcal{B}$

Notation 3.2.1 Let H denote the free $\mathbb{Z}/2$–graded algebra on the $\mathbb{Z}/2$–graded vector space $\langle x, y\rangle$, where y has degree 1 and x has degree 0.

The algebra H has the structure of a $\mathbb{Z}/2$–graded Hopf algebra, with underlying algebra $\Lambda(y) \otimes \mathbb{F}[x]$, which is primitively-generated. The Hopf algebra H represents the additive group in the context of $\mathbb{Z}/2$–graded algebras.

Remark 3.2.2 The algebra H has a Hausdorff filtration $(I^n H)$ given by powers of the augmentation ideal, hence it is possible to form half-completed tensor products $H \widehat{\otimes} V := \lim_{\leftarrow}(H/I^n H) \otimes V$, for any vector space V.

The half-completed tensor product is sufficient to be able to define a general notion of completed comodule structure over a bialgebra B, where the structure morphism is given by a morphism $\psi\colon H \to H \widehat{\otimes} B$.

In the applications, all objects will have a $\mathbb{Z}$–grading and all completed tensor products can be understood in the usual graded context. For this reason, the details concerning the usage of the half-completed tensor product are left to be supplied by the interested reader.

All comodule structures in this section are understood to be defined with respect to a half completed tensor product.

Definition 3.2.3 For $B \in Bialg_{\mathbb{F}}^{\mathbb{Z}/2}$ a $\mathbb{Z}/2$–graded bialgebra, a multiplicative right B–comodule algebra structure on H is a morphism of super-algebras $\psi\colon H \to H \widehat{\otimes} B$ which induces a comodule structure on H.

Proposition 3.2.4

(1) *There exists a $\mathbb{Z}/2$–graded bialgebra $\mathcal{B}$ with underlying $\mathbb{F}$–algebra the free super-algebra*

$$\mathcal{B} \cong \Lambda(w, \tau_i | i \geq 0) \otimes \mathbb{F}[u, \xi_j \,|\, j \geq 0]$$

[2] Here, as elsewhere, the graded tensor product is denoted simply $\otimes$.

and with coproduct

$$\begin{aligned} u &\mapsto u \otimes u + w \otimes \tau_0 \\ w &\mapsto u \otimes w + w \otimes \xi_0 \\ \tau_i &\mapsto \sum_{s=0}^{i} \xi_{i-s}^{p^s} \otimes \tau_s + \tau_i \otimes u \\ \xi_j &\mapsto \sum_{s=0}^{j} \xi_{j-s}^{p^s} \otimes \xi_s + \tau_j \otimes w. \end{aligned}$$

(2) *The underlying super–algebra H admits a $\mathcal{B}$–comodule structure $\psi\colon H \to H \hat{\otimes} \mathcal{B}$ such that*

 (a) *ψ is a morphism of super-algebras, determined by*

$$y \mapsto y \otimes u + \sum_{i \geq 0} x^{p^i} \otimes \tau_i \quad \text{and} \quad x \mapsto y \otimes w + \sum_{j \geq 0} x^{p^j} \otimes \xi_j ;$$

 (b) *the coproduct $H \to H \otimes H$ is a morphism of $\mathcal{B}$–comodules, where $H \otimes H$ is given the tensor product $\mathcal{B}$–comodule structure.*

Proof (Indications) The construction of the bialgebra, its coproduct and the comodule structure is a straightforward generalization of Milnor's method [7] for calculating the dual of the Steenrod algebra. □

Remark 3.2.5

(1) The bialgebra $\mathcal{B}$ can be interpreted as the endomorphism bialgebra of the additive algebraic group; this implies that it satisfies a universal property, the formulation of which is left to the reader.

(2) The case of the prime field of characteristic two is similar, but more elementary, since the algebras considered are commutative in the ungraded sense and the respective Hopf algebra H is the polynomial algebra on a single generator. The universal bialgebra $\mathcal{B}_2$ is $\mathbb{F}_2[\xi_j \,|\, j \geq 0]$, equipped with the coproduct

$$\Delta \xi_j = \sum_{s=0}^{j} \xi_{j-s}^{2^s} \otimes \xi_s.$$

This is the extended Milnor–Hopf algebra of Bisson–Joyal [1, Section 4].

(3) It is well-known that the Steenrod algebra, for $p = 2$, can be regarded as automorphisms of the additive formal group and it is folklore that this can be extended, for p odd, by considering the super-algebra setting. The above

generalizes this point of view, by considering the full endomorphism ring in the $\mathbb{Z}/2$–graded setting.

3.3 Bialgebras derived from the universal bialgebra $\mathcal{B}$

There are quotient bialgebras of $\mathcal{B}$ which are of importance in considering the category of unstable modules over the Steenrod algebra. The ideals $\langle w \rangle$ and $\langle w, \xi_0 - u^2 \rangle$ are Hopf ideals in $\mathcal{B}$, which allows the following definition.

Definition 3.3.1

(1) Let $\overline{\mathcal{B}}$ denote the quotient $\mathcal{B}/(w)$ in $Bialg_{\mathbb{F}}^{\mathbb{Z}/2}$.

(2) Let $\widetilde{\mathcal{A}}^*$ denote the quotient a $\mathcal{B}/(w, \xi_0 - u^2)$ in $Bialg_{\mathbb{F}}^{\mathbb{Z}/2}$.[3]

(3) Let $\mathcal{B}'' \in Bialg_{\mathbb{F}}^{\mathbb{Z}/2}$ denote the sub-bialgebra of $\widetilde{\mathcal{A}}^*$ which is generated by the elements ξ_j together with u.

Remark 3.3.2

(1) The bialgebras $\mathcal{B}, \overline{\mathcal{B}}, \widetilde{\mathcal{A}}^* \in Bialg_{\mathbb{F}}^{\mathbb{Z}/2}$ do not have $\mathbb{Z}/2$–graded Hopf algebra structures. For example, $\mathbb{F}[\xi_0]$ is a sub-bialgebra of $\overline{\mathcal{B}}$ which does not have the structure of a Hopf algebra, since ξ_0 is grouplike and not invertible.

(2) The dual of the Steenrod algebra, $\mathcal{A}^*$, is obtained as the quotient of the bialgebra $\widetilde{\mathcal{A}}^*$ by the Hopf ideal generated by $u - 1$.[4]

(3) The bialgebra $\widetilde{\mathcal{A}}^*$ is related to the bigraded algebra J_*^* which was constructed by Miller from the Brown–Gitler modules (cf Schwartz [10, Theorem 2.4.8]).

Lemma 3.3.3 *There is a commutative diagram of $\mathbb{Z}/2$–graded bialgebras:*

$$\begin{tikzcd}
\mathcal{B} \arrow[r] & \overline{\mathcal{B}} \arrow[r] \arrow[d] & \widetilde{\mathcal{A}}^* \arrow[d] \\
& \mathbb{F}[\xi_0, u] \arrow[r, "\xi_0 \mapsto u^2"'] & \mathbb{F}[u]
\end{tikzcd}$$

where the bialgebras $\mathbb{F}[\xi_0, u], \mathbb{F}[u]$ are generated by group-like elements. Moreover, the vertical morphisms of the diagram are split surjections of bialgebras.

3.4 Introducing $\mathbb{Z}$–gradings

The category of non-negatively graded $\mathbb{F}$–vector spaces is equivalent to the category of right comodules over $\mathbb{F}[u]$. The following results show that there are natural gradings on the bialgebras under consideration, which induce the $\mathbb{Z}/2$–gradings.

Lemma 3.4.1 *For B one of the bialgebras $\mathcal{B}, \overline{\mathcal{B}}, \mathcal{B}'', \widetilde{\mathcal{A}}^*$,*

(1) *B admits a morphism of bialgebras $B \to \mathbb{F}[u]$, which factors across $\widetilde{\mathcal{A}}^* \to \mathbb{F}[u]$;*

[3] The notation reflects the relation with the dual of the Steenrod algebra.

[4] It is the desire for compatibility with the usual notation for the dual of the Steenrod algebra which imposes the cumbersome notation for $\widetilde{\mathcal{A}}^*$.

(2) *B is naturally bigraded with respect to the corestricted left and right $\mathbb{F}[u]$–comodule structures;*

(3) *B is of finite type with respect to the grading induced by the left comodule structure.*

Proof The morphism $B \to \mathbb{F}[u]$ is provided by Lemma 3.3.3; the remainder of the Lemma is straightforward. □

Definition 3.4.2 Let B be a bialgebra as above, equipped with the bigrading induced by the $\mathbb{F}[u]$–comodule structures; the total degree of a bihomogeneous element of bidegree (m,n) is the integer $m-n$.

The following result implies that all the $\mathbb{Z}/2$–gradings which are considered are the reduction of a natural $\mathbb{Z}$–grading on the algebra $\mathcal{B}$. There are similar results for the other bialgebras.

Proposition 3.4.3 *The $\mathbb{Z}/2$–grading of $\mathcal{B}$ is the mod 2 reduction of the $\mathbb{Z}$–grading of $\mathcal{B}$ given by the total degree.*

Remark 3.4.4 The usual grading of the dual Steenrod algebra $\mathcal{A}^*$ can be recovered from the total degree, using the total degree defined on $\widetilde{\mathcal{A}}^*$ together with the observation that the total degree of u is zero.

4 Categories of comodules related to unstable modules

This section defines the categories of comodules which are of interest in this paper, in relation to the category of unstable modules over the Steenrod algebra.

Throughout this section, let $\mathbb{F}$ be the prime field of characteristic p, where $p>2$; the underlying category of vector spaces is taken to be the category of $\mathbb{Z}/2$–graded vector spaces, equipped with the Koszul–sign tensor structure.

The results presented in this section have analogues for the case $p=2$.

4.1 Categories of graded comodules

Definition 4.1.1 Let $B \in Bialg_{\mathbb{F}}^{\mathbb{Z}/2}$ be a bialgebra which is equipped with a morphism of bialgebras $B \to \mathbb{F}[u]$ and hence has a left and right grading. Suppose that B is of finite type with respect to the left grading.

Let $\mathit{Comod}_B^{\mathrm{gr}}$ be the category of graded, B–comodules, in the following sense. An object of $\mathit{Comod}_B^{\mathrm{gr}}$ is a non-negatively graded vector space M, with grading defined by the comodule structure $M \to M \otimes \mathbb{F}[u]$, together with the comodule structure morphism $M \to M \widehat{\otimes} B$ which satisfies the condition that the diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d] & M \widehat{\otimes} B \arrow[d] \\ M \otimes \mathbb{F}[u] \arrow[r] & M \widehat{\otimes} \mathbb{F}[u] \end{tikzcd}$$

commutes. In particular, the morphism $M \to M \widehat{\otimes} B$ is a morphism of graded vector spaces, where the grading on the right is induced by the right grading of B.

Proposition 4.1.2 *Let B be a bialgebra which is equipped with a morphism of bialgebras $B \to \mathbb{F}[u]$. Suppose that B is of finite type with respect to the grading induced by the associated left $\mathbb{F}[u]$–comodule structure, then the category $\mathit{Comod}_B^{\mathrm{gr}}$ is an abelian tensor category.*

The bialgebras $\mathcal{B}, \overline{\mathcal{B}}, \mathcal{B}'', \widetilde{\mathcal{A}}^*$ satisfy the hypotheses of Definition 4.1.1, by Lemma 3.4.1; hence the above definition can be applied.

Definition 4.1.3 Define the following tensor abelian categories:

$$\mathcal{U}(\mathcal{B}) := \mathit{Comod}_{\mathcal{B}}^{\mathrm{gr}} \qquad \mathcal{U}^{\mathrm{bi.gr}} := \mathit{Comod}_{\overline{\mathcal{B}}}^{\mathrm{gr}}$$
$$\mathcal{U}'' := \mathit{Comod}_{\mathcal{B}''}^{\mathrm{gr}} \qquad \mathcal{U} := \mathit{Comod}_{\widetilde{\mathcal{A}}^*}^{\mathrm{gr}}$$
$$\mathcal{U}' := \mathit{Comod}_{\mathbb{F}[\xi_j \,|\, j \geq 0]}^{\mathrm{gr}}$$

The notation $\mathcal{U}$ does not conflict with the usual usage, by the following result.

Theorem 4.1.4 *The category $\mathit{Comod}_{\widetilde{\mathcal{A}}^*}^{\mathrm{gr}}$ is equivalent to the category of unstable modules over the mod–p Steenrod algebra $\mathcal{A}$.*

Proof The category of unstable modules over the Steenrod algebra is usually defined as the category of graded modules over the Steenrod algebra, $\mathcal{A}$, subject to an instability condition in terms of the operation of elements derived from the dual basis to the basis of monomials in the elements ξ_j, τ_i. It is elementary to show that this condition is equivalent to a condition on the adjoint coaction involving the terms ξ_1^t. It is a straightforward exercise to show that this implies that a graded module over the Steenrod algebra is unstable if and only if the adjoint coaction extends to a graded right $\widetilde{\mathcal{A}}^*$–comodule structure. □

The objects of $\mathcal{U}^{\mathrm{bi.gr}}$ are naturally bigraded (which justifies the notation), by the following result, in which $\mathcal{V}_{\mathbb{F}}^{\star\star}$ denotes the category of $\mathbb{Z}\times\mathbb{Z}$–graded vector spaces.

Proposition 4.1.5 *The morphism $\overline{\mathcal{B}}\to\mathbb{F}[\xi_0,u]$ induces an exact functor of tensor abelian categories $\mathcal{U}^{\mathrm{bi.gr}}\to\mathcal{V}_{\mathbb{F}}^{\star\star}$.*

Notation 4.1.6 The bidegrees will be written as (a,b), where a denotes the u–degree and b denotes the ξ_0–degree.

Proposition 4.1.7 *There is a diagram of exact functors between abelian categories:*

$$\begin{tikzcd} & \mathcal{U}' \arrow[r] \arrow[d, hook, "\hat{\mathcal{O}}"'] & \mathcal{U}'' \arrow[d, hook] \\ \mathcal{U}(\mathcal{B}) \arrow[r, "\Psi"'] & \mathcal{U}^{\mathrm{bi.gr}} \arrow[r, "\Theta"'] & \mathcal{U} \end{tikzcd}$$

in which the horizontal morphisms indicate forgetful functors. The embeddings $\mathcal{U}'\hookrightarrow\mathcal{U}^{\mathrm{bi.gr}}$ and $\mathcal{U}''\hookrightarrow\mathcal{U}$ are fully faithful and admit retractions $\mathcal{U}^{\mathrm{bi.gr}}\to\mathcal{U}'$ and $\mathcal{U}\to\mathcal{U}''$ respectively.

Proof The exact functors are induced by the corestriction functors which are associated to the canonical morphisms of the respective bialgebras (cf Lemma 3.3.3). The retractions are given by the respective right adjoint functors. □

Remark 4.1.8

(1) The category $\mathcal{U}''$ is related to the full sub-category $\mathcal{U}'$ of $\mathcal{U}$ which identifies with the objects which are concentrated in even degree, which has been used in the work of Lannes and Zarati on the category $\mathcal{U}$. In particular, there is an adjunction, $\mathcal{O}\colon\mathcal{U}'\rightleftarrows\mathcal{U}:\widetilde{\mathcal{O}}$, where $\mathcal{O}$ denotes the forgetful functor and $\widetilde{\mathcal{O}}$ its right adjoint. The category $\mathcal{U}''$ splits as a product of two copies of $\mathcal{U}'$, corresponding respectively to elements in even (resp. odd) degrees.

(2) The category $\mathcal{U}(\mathcal{B})$ allows for the action of the element w of negative total degree.

(3) The category $\mathcal{U}^{\mathrm{bi.gr}}$ sheds light on the structure of $\mathcal{U}$; it is also of interest in studying unstable modules over the motivic Steenrod algebra, since motivic cohomology is naturally bigraded by the topological degree and the twist (or weight).

Example 4.1.9 The object $H:=\Lambda(y)\otimes\mathbb{F}[x]$ has the structure of an object in $\mathcal{U}(\mathcal{B})$ and hence of an object in $\mathcal{U}^{\mathrm{bi.gr}}$, by corestriction. Thus, for each non-negative integer m, $H^{\otimes m}$ defines an object of $\mathcal{U}(\mathcal{B})$ and therefore of $\mathcal{U}^{\mathrm{bi.gr}}$, by corestriction.

4.2 Simple comodules

The simple objects of the abelian categories $\mathcal{U}$, $\mathcal{U}^{\mathrm{bi.gr}}$, $\mathcal{U}'$, $\mathcal{U}''$ are understood via the following result.

Proposition 4.2.1 *A simple object M in one of the abelian categories $\mathcal{U}$, $\mathcal{U}^{\mathrm{bi.gr}}$, $\mathcal{U}'$, $\mathcal{U}''$ has underlying vector spaces of total dimension one, with comodule structure which corresponds to the grading.*

Example 4.2.2 The bigraded vector space $\mathbb{F}(a,b)$ of total dimension one, concentrated in bidegree (a,b) where a,b are non-negative integers has the natural structure of an object of $\mathcal{U}^{\mathrm{bi.gr}}$.

Remark 4.2.3 The bigraded vector space $\mathbb{F}(a,b)$ does not in general have the structure of an object of $\mathcal{U}(\mathcal{B})$. The simple objects of the category $\mathcal{U}(\mathcal{B})$ are constructed from the simple objects of the category $\mathit{Comod}^{\mathrm{gr}}_{\mathcal{C}}$, where $\mathcal{C}$ denotes the quotient bialgebra of $\mathcal{B}$ with underlying algebra $\mathbb{F}[\xi_0,u]\otimes\Lambda(\tau_0,w)\in \mathit{Bialg}^{\mathbb{Z}/2}_{\mathbb{F}}$.

4.3 Suspension functors

The simple objects of the categories $\mathcal{U}^{\mathrm{bi.gr}},\mathcal{U},\mathcal{U}',\mathcal{U}''$ define suspension functors by forming the tensor product.

Definition 4.3.1 Let a,b,n be non-negative integers.

(1) The suspension functor $\Sigma^{(a,b)}$ of bidegree (a,b) is the functor defined by the tensor product $\mathbb{F}(a,b)\otimes -\colon \mathcal{U}^{\mathrm{bi.gr}}\to\mathcal{U}^{\mathrm{bi.gr}}$.

(2) The suspension functor $\Sigma^n\colon \mathcal{U}\to\mathcal{U}$ is the functor defined by the tensor product $\mathbb{F}(n)\otimes -\colon \mathcal{U}\to\mathcal{U}$.

The forgetful functor $\Theta\colon \mathcal{U}^{\mathrm{bi.gr}}\to\mathcal{U}$ sends $\mathbb{F}(a,b)$ to the vector space $\mathbb{F}(a+2b)$ of total dimension one, concentrated in degree $a+2b$. The following lemma is clear.

Lemma 4.3.2 *Let a,b be non-negative integers, then there is a commutative diagram*

$$\begin{tikzcd}
\mathcal{U}^{\mathrm{bi.gr}} \arrow[r, "\Sigma^{(a,b)}"] \arrow[d, "\Theta"'] & \mathcal{U}^{\mathrm{bi.gr}} \arrow[d, "\Theta"] \\
\mathcal{U} \arrow[r, "\Sigma^{a+2b}"'] & \mathcal{U}.
\end{tikzcd}$$

Part II The functorial viewpoint

5 Functors and bialgebras

The bialgebra $\mathcal{B}$ is the endomorphism bialgebra of the free graded-commutative algebra on the $\mathbb{Z}/2$–graded vector space $\langle x, y\rangle$; forgetting the algebra structure, the free graded-commutative algebra can be considered as a functor from $\mathbb{F}$–vector spaces to $\mathbb{F}$–vector spaces. The relation between the category of unstable modules and the category of functors between $\mathbb{F}$–vector spaces follows from the comparison between the bialgebra of endomorphisms of the additive group and the bialgebra of endomorphisms of the free graded-commutative algebra functor, which is presented in Theorem 5.4.2.

5.1 Functors

The category $\mathcal{F}$ of functors from finite-dimensional $\mathbb{F}$–vector spaces to $\mathbb{F}$–vector spaces is an abelian tensor category with enough projectives and enough injectives, given respectively by Yoneda's lemma and its dual. An object of $\mathcal{F}$ (a functor) is *finite* if it has a finite composition series and is *polynomial* if it is polynomial in the sense of Eilenberg–MacLane (see Kuhn [6]); these two conditions are equivalent for functors which take finite-dimensional values. A functor is *analytic* if it is the colimit of its finite subobjects; the full subcategory of analytic functors is denoted $\mathcal{F}_\omega$.

The divided power functors, Γ^n, defined by $V \mapsto (V^{\otimes n})^{\mathfrak{S}_n}$, the symmetric power functors, S^n, and the exterior power functors, Λ^n, are finite functors which are of fundamental importance to the theory. The embedding theorem of [6] is interpreted as follows.

Theorem 5.1.1 (Kuhn [6]) *For F a finite functor in $\mathcal{F}$, there exists a finite set of non-negative integers $\{n_i\}$ and a surjection $\bigoplus_i \Gamma^{n_i} \twoheadrightarrow F$.*

An important observation which is used in the proof is the following:

Lemma 5.1.2 *Suppose that $\Gamma^a \twoheadrightarrow F_1$, $\Gamma^b \twoheadrightarrow F_2$ are surjections in $\mathcal{F}$, then there exists a surjection $\Gamma^N \twoheadrightarrow F_1 \otimes F_2$ in $\mathcal{F}$, for some non-negative integer N.*

An extensive array of calculations of Ext groups in the category $\mathcal{F}$ have been performed (see Franjou–Friedlander–Scorichenko–Suslin [4], for example). For the purposes of this paper, the following elementary calculation is important.

Lemma 5.1.3 *For a, b non-negative integers,*

(1) $\mathrm{Hom}_{\mathcal{F}}(\Gamma^b, \Lambda^a) = \begin{cases} \mathbb{F} & b = \sum_{i=1}^{a} p^{n_i}, n_i < n_{i+1} \\ 0 & \textit{otherwise;} \end{cases}$

(2) *for* $p > 2$, $\mathrm{Hom}_{\mathcal{F}}(\Lambda^a, \Gamma^b) = \begin{cases} \mathbb{F} & a = b, a \in \{0, 1\} \\ 0 & \textit{otherwise.} \end{cases}$

This result, together with the calculation of $\mathrm{Hom}_{\mathcal{F}}(\Gamma^*, \Gamma^*)$ (see [6]), is used in conjunction with the exponential property of the functors Γ^*, Λ^*.

5.2 Generalities on exponential functors

A functor $E \in \mathcal{F}$ is exponential if there exists a binatural isomorphism $E(V \oplus W) \cong E(V) \otimes E(W)$, for $V, W \in \mathcal{V}_{\mathbb{F}}^{\mathrm{fd}}$. A graded functor is exponential if it satisfies this property with respect to the graded tensor product[5].

Example 5.2.1 The injective functor I_V, for $V \in \mathcal{V}_{\mathbb{F}}^{\mathrm{fd}}$, defined by $I_V : W \mapsto \mathbb{F}^{\mathrm{Hom}(W,V)}$ is an exponential functor.

Remark 5.2.2

(1) The tensor product of two (ungraded) exponential functors is an exponential functor; in the graded case, one obtains a bigraded exponential functor.

(2) The structure of an exponential functor E induces a canonical product μ: $E \otimes E \to E$ and coproduct Δ: $E \to E \otimes E$.

Example 5.2.3 For $\mathbb{F}$ a prime field of odd characteristic, the graded functor Λ^* is a graded exponential functor; the associated product and coproduct are graded commutative, since Koszul signs intervene. More generally, for the topological application of this paper, it is necessary to consider the bigraded exponential functor which corresponds to $\Gamma^* \otimes \Lambda^*$; the associated product and coproduct are graded commutative in the usual sense.

Exponentiality facilitates calculations of $\mathrm{Hom}_{\mathcal{F}}$: for E an exponential functor and F, G functors in $\mathcal{F}$, where E, F, G take finite-dimensional values, there is a natural isomorphism of vector spaces

$$\mathrm{Hom}_{\mathcal{F}}(E, F \otimes G) \cong \mathrm{Hom}_{\mathcal{F}}(E, F) \otimes \mathrm{Hom}_{\mathcal{F}}(E, G).$$

In the graded case, the right hand side has to be treated as a graded tensor product.

[5]For the remainder of this section, 'exponential functor' is used to indicate either the graded or the ungraded version - the context should make the meaning clear.

5.3 Endomorphisms of exponential functors

A graded exponential functor is of finite type if each component is a finite functor. The endomorphism ring $\mathrm{End}_{\mathcal{F}}(E)$ of a graded exponential functor is a bigraded vector space, which has additional structure.

Lemma 5.3.1 *Let E be a graded exponential functor of finite type and let $\mathrm{End}_{\mathcal{F}}(E)^*$ denote the bigraded dual. Then the following statements hold.*

(1) $\mathrm{End}_{\mathcal{F}}(E)^*$ *has the structure of a bialgebra.*

(2) *There is a natural coaction $\psi\colon E \to E\hat{\otimes}\mathrm{End}_{\mathcal{F}}(E)^*$ which satisfies the following properties:*
   (a) *ψ is multiplicative.*
   (b) *the diagonal $\Delta\colon E \to E \otimes E$ is a morphism of $\mathrm{End}_{\mathcal{F}}(E)^*$–comodules, where $E \otimes E$ is given the tensor product comodule structure.*

This result extends to the graded commutative setting.

5.4 Fundamental examples

The graded exponential functor $S^\bullet$ and the bigraded exponential functor $S^\bullet \otimes \Lambda^\bullet$ respectively[6] provide the examples of importance to the theory of unstable modules.

Example 5.4.1

(1) The graded exponential functor $S^\bullet$ is of finite type and there is a natural coaction

$$S^\bullet \to S^\bullet\hat{\otimes}\mathrm{End}_{\mathcal{F}}(S^\bullet)^*.$$

(2) Let $\mathbb{F}$ be a prime field of odd characteristic. The bigraded exponential functor $\Lambda^\bullet \otimes \Gamma^\bullet$ is of finite type and there is a natural coaction

$$S^\bullet \otimes \Lambda^\bullet \to (S^\bullet \otimes \Lambda^\bullet)\hat{\otimes}\mathrm{End}_{\mathcal{F}}(S^\bullet \otimes \Lambda^\bullet)^*.$$

Theorem 5.4.2 *For $\mathbb{F}$ a prime field of odd characteristic, there is a natural morphism $\mathcal{B} \to \mathrm{End}_{\mathcal{F}}(S^\bullet \otimes \Lambda^\bullet)^*$ of $\mathbb{Z}/2$–bialgebras, which is an isomorphism.*

Proof The morphism exists by the universal property of $\mathcal{B}$ which is implicit in the definition. It is straightforward to verify that it is an isomorphism. □

Remark 5.4.3 For $\mathbb{F}$ a prime field of odd characteristic, $\mathrm{Hom}_{\mathcal{F}}(\Lambda^m, \Gamma^n)$ is trivial unless either $m=n=0$ or $m=n=1$. The vector space $\mathrm{Hom}_{\mathcal{F}}(\Lambda^1, \Gamma^1)$ has dimension one and the generator w of $\mathcal{B}$ is dual to a generator of this vector space.

[6]The notation $E^\bullet$ is used here to avoid confusion with vector space duality.

6 Representation categories

This section recalls and extends the results of Kuhn [6] on representation categories which are relevant to the study of the category of unstable modules. The main result, Theorem 6.3.1, identifies the category $\mathcal{U}(\mathcal{B})$ as a representation category. The functors r', $r_{\mathcal{B}}$ introduced in this section are used in the analysis of the projective and injective objects of $\mathcal{U}^{\text{bi-gr}}$ in Section 7.

6.1 Generalities

Throughout this section, the following hypothesis is supposed to hold on the pair of categories $\mathcal{C}, \mathcal{S}$.

Hypothesis 6.1.1 The category $\mathcal{C}$ is abelian and contains all small inductive limits, which are exact. The category $\mathcal{S} \hookrightarrow \mathcal{C}$ is a full small subcategory of $\mathcal{C}$ with objects $\{S_i\}$ indexed over a set $\mathcal{I}$.

Kuhn [6] defines the representation category defined by the category $\mathcal{S}$, $\mathrm{Rep}(\mathcal{S}^{\mathrm{op}})$, as the multi-object version of the category of representations (left modules) of the ring $\mathrm{End}(S)^{\mathrm{op}}$, for S an object of $\mathcal{C}$. The standard example of an object of $\mathrm{Rep}(\mathcal{S}^{\mathrm{op}})$ is given by the $\mathcal{I}$–indexed object $r_{\mathcal{S}}(X)_i := \mathrm{Hom}_{\mathcal{C}}(S_i, X)$ for X an object of $\mathcal{C}$.

There is an adjunction of categories:

$$l_{\mathcal{S}}\colon \mathrm{Rep}(\mathcal{S}^{\mathrm{op}}) \rightleftarrows \mathcal{C} : r_{\mathcal{S}},$$

in which the functor $r_{\mathcal{S}}$ is defined as above.

The one-sided Morita equivalence result of Kuhn is basic to the theory:

Theorem 6.1.2 (Kuhn [6, Theorem 2.1]) *The following statements are equivalent.*

(1) *$\mathcal{S}$ generates $\mathcal{C}$.*

(2) *$l_{\mathcal{S}}$ is exact and $r_{\mathcal{S}}$ is fully faithful.*

(3) *$\mathcal{C}$ has enough injectives and, for all injectives I, J in $\mathcal{C}$, $r_{\mathcal{S}}(I), r_{\mathcal{S}}(J)$ are injective and the functor $r_{\mathcal{S}}$ induces an isomorphism*

$$\mathrm{Hom}_{\mathcal{C}}(I, J) \cong \mathrm{Hom}_{\mathrm{Rep}(\mathcal{S}^{\mathrm{op}})}(r_{\mathcal{S}}(I), r_{\mathcal{S}}(J)).$$

Moreover, if these conditions are satisfied, the adjunction counit $l_{\mathcal{S}} r_{\mathcal{S}} \to 1_{\mathcal{C}}$ is a natural equivalence.

A set of projective generators of the category $\mathrm{Rep}(\mathcal{S}^{\mathrm{op}})$ is given by $\{r_{\mathcal{S}}(S) | S \in \mathrm{Object}(\mathcal{S})\}$, by Yoneda's lemma. In the case that $\mathcal{C}$ is a k–linear category over a field k, and under a locally finite-type hypothesis, there is a dual description of a set of injective cogenerators.

Notation 6.1.3 For $\mathcal{S}, \mathcal{C}$ as above, where $\mathcal{C}$ is a k–linear category over a field k, let $\rho\colon \mathcal{C} \to \mathrm{Rep}(\mathcal{S}^{\mathrm{op}})$ denote the functor

$$X \mapsto \mathrm{Hom}_{\mathcal{C}}(X, -)^*$$

where $*$ denotes vector space duality and the right hand side is regarded as a contravariant functor on $\mathcal{S}$.

Proposition 6.1.4 *For $\mathcal{C}$ a k–linear category over a field k such that the vector space $\mathrm{Hom}(S, T)$ is of finite dimension, for each pair of objects (S, T) of $\mathcal{S}$, the category $\mathrm{Rep}(\mathcal{S}^{\mathrm{op}})$ has set of injective cogenerators $\{\rho(S) | S \in \mathrm{Object}(\mathcal{S})\}$.*

Proof The proof is straightforward, using vector space duality to reduce to the Yoneda lemma. □

6.2 Restriction and extension for representation categories

There are restriction and extension functors for representation categories, which are associated to two small subcategories of an abelian category. This is relevant to the study of the category of unstable modules over the Steenrod algebra when passing from the category of objects concentrated in even degree to the full category of unstable modules (see Corollary 6.3.4 below).

Let $\mathcal{S}, \mathcal{C}$ satisfy Hypothesis 6.1.1 and suppose moreover that there are inclusions of full subcategories $\mathcal{S} \subset \mathcal{T} \subset \mathcal{C}$, where $\mathcal{T}$ has a set of objects indexed over a set $\mathcal{J}$.

There are abelian representation categories $\mathrm{Rep}(\mathcal{S}^{\mathrm{op}}), \mathrm{Rep}(\mathcal{T}^{\mathrm{op}})$ and canonical adjunctions

$$l_{\mathcal{S}}\colon \mathrm{Rep}(\mathcal{S}^{\mathrm{op}}) \rightleftarrows \mathcal{C} : r_{\mathcal{S}} \qquad\qquad l_{\mathcal{T}}\colon \mathrm{Rep}(\mathcal{T}^{\mathrm{op}}) \rightleftarrows \mathcal{C} : r_{\mathcal{T}},$$

where the functor $r_{\mathcal{S}}$ is induced by $\mathrm{Hom}_{\mathcal{C}}(\mathcal{S}, -)$ and similarly for $r_{\mathcal{T}}$.

Proposition 6.2.1 *The inclusion functor $\mathcal{S} \hookrightarrow \mathcal{T}$ induces an exact restriction functor $\mathsf{Res}\colon \mathrm{Rep}(\mathcal{T}^{\mathrm{op}}) \to \mathrm{Rep}(\mathcal{S}^{\mathrm{op}})$. Moreover, there is an adjunction*

$$\mathsf{K}\colon \mathrm{Rep}(\mathcal{S}^{\mathrm{op}}) \rightleftarrows \mathrm{Rep}(\mathcal{T}^{\mathrm{op}}) : \mathsf{Res}$$

where the functor Res is defined by restriction and the functor K is induced by Kan extension. The functor Res is exact and the functor K is right exact.

The Kan functor induces a commutative diagram

$$\begin{tikzcd} \mathrm{Rep}(\mathcal{S}^{\mathrm{op}}) \arrow[r, "l_{\mathcal{S}}"] \arrow[d, "\mathsf{K}"'] & \mathcal{C} \arrow[d, equal] \\ \mathrm{Rep}(\mathcal{T}^{\mathrm{op}}) \arrow[r, "l_{\mathcal{T}}"'] & \mathcal{C}. \end{tikzcd}$$

Proof The functor Res is the evident restriction functor; the functor K is defined by the Kan extension of the functor which associates to $r_{\mathcal{S}}(S)$ the object $r_{\mathcal{T}}(S)$.

The commutativity of the given diagram follows from the fact that both functors $\mathrm{Rep}(\mathcal{S}^{\mathrm{op}}) \to \mathcal{C}$ are left adjoint to the functor $r_{\mathcal{S}}\colon \mathcal{C} \to \mathrm{Rep}(\mathcal{S}^{\mathrm{op}})$. □

Lemma 6.2.2 *Under the hypotheses of Proposition 6.2.1, there is a natural transformation $l_{\mathcal{S}}\mathsf{Res} \to l_{\mathcal{T}}$ of functors from $\mathrm{Rep}(\mathcal{T}^{\mathrm{op}})$ to $\mathcal{C}$.*

Proof The adjunction unit $1 \to r_{\mathcal{T}} l_{\mathcal{T}}$ induces a natural transformation $\mathsf{Res} \to \mathsf{Res} r_{\mathcal{T}} l_{\mathcal{T}}$, by composition with Res, and the functor $\mathsf{Res} r_{\mathcal{T}} l_{\mathcal{T}}$ is naturally equivalent to $r_{\mathcal{S}} l_{\mathcal{T}}$. The required natural transformation is given by adjunction. □

Proposition 6.2.3 *Under the hypotheses of Proposition 6.2.1, suppose that the objects of $\mathcal{S}$ generate $\mathcal{C}$ then the following properties hold.*

(1) *The functors $l_{\mathcal{S}}\colon \mathrm{Rep}(\mathcal{S}^{\mathrm{op}}) \to \mathcal{C}$, $l_{\mathcal{T}}\colon \mathrm{Rep}(\mathcal{T}^{\mathrm{op}}) \to \mathcal{C}$ are exact.*

(2) *The natural transformation $l_{\mathcal{S}}\mathsf{Res} \to l_{\mathcal{T}}$ is a natural equivalence.*

(3) *The functors K, Res induce an equivalence of categories*

$$\mathrm{Rep}(\mathcal{S}^{\mathrm{op}})/\ker(l_{\mathcal{S}}) \cong \mathrm{Rep}(\mathcal{T}^{\mathrm{op}})/\ker(l_{\mathcal{T}}).$$

Proof The first statement follows from Theorem 6.1.2, since the hypothesis on $\mathcal{S}$ implies that the objects of $\mathcal{T}$ generate $\mathcal{C}$.

For the second statement, consider the natural transformation $l_{\mathcal{S}}\mathsf{Res} \to l_{\mathcal{T}}$. The functors $l_{\mathcal{S}}\mathsf{Res}$ and $l_{\mathcal{T}}$ are exact and send coproducts to coproducts, under the hypotheses of the Proposition. Hence, by forming projective resolutions, it is sufficient to show that the natural transformation is an equivalence on a set of projective generators of $\mathrm{Rep}(\mathcal{T}^{\mathrm{op}})$. The Yoneda lemma implies that the objects $r_{\mathcal{T}}(T)$, for T objects of $\mathcal{T}$, form such a set of projective generators. Hence, it is sufficient to show that the natural transformation $l_{\mathcal{S}}\mathsf{Res} r_{\mathcal{T}} \to l_{\mathcal{T}} r_{\mathcal{T}}$ induced by composition with $r_{\mathcal{T}}$, is an equivalence. Under the hypotheses of the Proposition, both functors above are naturally equivalent to the identity of $\mathcal{C}$, by Theorem 6.1.2; the verification that the above morphism is a natural equivalence is an adjunction argument, which is left to the reader.

The final statement is a corollary of the identification, provided by Kuhn, of $\mathcal{C}$ with the respective localized representation categories. □

Example 6.2.4 The hypotheses of the Proposition are necessary: for example, let $\mathcal{C}$ be the category $\mathcal{F}$ and let $\mathcal{S}$, $\mathcal{T}$ be the full subcategories with sets of objects $\{\Gamma^1\}$ and $\{\Gamma^n | n \geq 0\}$ respectively. The category $\mathrm{Rep}(\mathcal{S}^{\mathrm{op}})$ is equivalent to the category of vector spaces and it is straightforward to see that the natural transformation $l_{\mathcal{S}}\mathsf{Res} \to l_{\mathcal{T}}$ is not a natural equivalence.

6.3 Representation categories related to unstable modules

Kuhn showed that the category of unstable modules over the $\mathbb{F}_2$–Steenrod algebra is equivalent to the representation category associated to the set of objects $\{\Gamma^*\}$ in the category of functors $\mathcal{F}$ defined with respect to $\mathbb{F}_2$–vector spaces. This result extends to give a description of $\mathcal{U}'$ in the case of odd characteristic, which is equivalent to the representation category $\mathrm{Rep}(\mathcal{S}^{\mathrm{op}})$ for the full subcategory $\mathcal{S}$ of $\mathcal{F}$ with set of objects $\{\Gamma^*\}$.

This does not extend to a description of the category $\mathcal{U}$; however, the following holds:

Theorem 6.3.1 *$\mathcal{U}(\mathcal{B})$ is equivalent to the representation category $\mathrm{Rep}(\mathcal{T}^{\mathrm{op}})$ for the full subcategory $\mathcal{T}$ of $\mathcal{F}$ with set of objects $\{\Gamma^{*_1} \otimes \Lambda^{*_2}\}$.*

Proof This is an immediate consequence of Theorem 5.4.2. □

Corollary 6.3.2 *For n a non-negative integer, the object $r_{\mathcal{B}}(I_{\mathbb{F}^n})$ of $\mathcal{U}(\mathcal{B})$ is injective.*

The theorem provides adjunctions

$$l_{\mathcal{B}}\colon \mathcal{U}(\mathcal{B}) \rightleftarrows \mathcal{F} :r_{\mathcal{B}} \qquad\qquad l'\colon \mathcal{U}' \rightleftarrows \mathcal{F} :r'.$$

In terms of the general framework introduced in the previous sections, the category $\mathcal{C}$ corresponds to the full subcategory $\mathcal{F}_\omega$ of analytic functors in $\mathcal{F}$ (see Henn–Lannes–Schwartz [5] and Kuhn [6]).

Notation 6.3.3 For a full subcategory $\mathcal{S} \hookrightarrow \mathcal{C}$ with set of objects $\mathbb{S}$, write $\mathrm{Rep}(\{\mathbb{S}\}^{\mathrm{op}})$ for the associated representation category $\mathrm{Rep}(\mathcal{S}^{\mathrm{op}})$.

Corollary 6.3.4 *There is a diagram of functors, which is commutative up to natural equivalence:*

$$\begin{tikzcd}
\mathcal{U}(\mathcal{B}) \cong \mathrm{Rep}(\{\Gamma^* \otimes \Lambda^*\}^{\mathrm{op}}) \arrow[r, "l_{\mathcal{B}}"] \arrow[d, shift left, "\mathsf{Res}"] & \mathcal{F} \arrow[d, equal] \\
\mathcal{U}' \cong \mathrm{Rep}(\{\Gamma^*\}^{\mathrm{op}}) \arrow[r, "l'"'] \arrow[u, shift left, "\mathsf{K}"] & \mathcal{F},
\end{tikzcd}$$

in which $l_{\mathcal{B}}, l'$ are the exact left adjoints of the adjunction between $\mathcal{F}$ and the respective representation categories. Moreover, the adjoint functors $(\mathsf{K}, \mathsf{Res})$ induce an equivalence of categories

$$\mathrm{Rep}(\{\Gamma^* \otimes \Lambda^*\}^{\mathrm{op}})/\ker(l_{\mathcal{B}}) \cong \mathrm{Rep}(\{\Gamma^*\}^{\mathrm{op}})/\ker(l') \cong \mathcal{F}_{\omega}.$$

Proof The result follows from Proposition 6.2.3. □

Part III Further properties

7 Projective and injective objects

The definition of $\mathcal{U}^{\mathrm{bi.gr}}$ as a category of comodules allows the construction of injective cogenerators and projective generators in terms of cotensor products. For non-negative integers a, b, the category $\mathcal{U}^{\mathrm{bi.gr}}$ contains projective objects $F(a, b)$, analogues of the Massey–Peterson free unstable modules $F(n) \in \mathcal{U}$, and injective objects $J(a, b)$, analogues of the Brown–Gitler modules $J(n) \in \mathcal{U}$. The structure of the objects $F(a, b)$ in the category $\mathcal{U}^{\mathrm{bi.gr}}$ sheds light on the structure of the objects $F(n)$ in the category of unstable modules. (The reader is referred to Schwartz [10] for the traditional approach to the projective and injective objects in $\mathcal{U}$).

Throughout this section, let $\mathbb{F}$ be the prime field of characteristic p, where p is an odd prime. The results of Appendix A, together with the material on graded comodules introduced in Section 4, will be used in this section without further comment.

7.1 The comodules $F(a, b)$ and $J(a, b)$

Recall that $\mathcal{V}_{\mathbb{F}}^{\star}$ denotes the category of $\mathbb{Z}$–graded vector spaces and $\mathcal{V}_{\mathbb{F}}^{\star\star}$ denotes the category of $\mathbb{Z} \times \mathbb{Z}$–graded vector spaces.

Let $\mathbb{F}(n) \in \mathcal{V}_{\mathbb{F}}^{\star}$ denote the graded vector space of total dimension one concentrated in degree n and let $\mathbb{F}(a, b) \in \mathcal{V}_{\mathbb{F}}^{\star\star}$ denote the bigraded vector space of total dimension one concentrated in bidegree (a, b). There is an exact functor $\mathcal{V}_{\mathbb{F}}^{\star\star} \to \mathcal{V}_{\mathbb{F}}^{\star}$ which commutes with colimits and which is therefore determined by $\mathbb{F}(a,b) \mapsto \mathbb{F}(a+2b)$, for all pairs of integers (a,b).

There is a commutative diagram of exact functors:

$$\begin{tikzcd}
\mathcal{U}(\mathcal{B}) \arrow[r, "\Psi"] \arrow[dr, "\gamma(\mathcal{B})"'] & \mathcal{U}^{\mathrm{bi.gr}} \arrow[r, "\Theta"] \arrow[d, "\gamma(\overline{\mathcal{B}})"] & \mathcal{U} \arrow[d, "\gamma"] & \mathcal{U}'' \arrow[l] \arrow[d, "\gamma'"] \\
 & \mathcal{V}_{\mathbb{F}}^{\star\star} \arrow[r] & \mathcal{V}_{\mathbb{F}}^{\star} \arrow[r, equal] & \mathcal{V}_{\mathbb{F}}^{\star},
\end{tikzcd}$$

in which the arrows labelled by variants of γ are the gradings induced by corestriction functors.

Proposition 7.1.1 *The functors $\gamma\colon \mathcal{U} \to \mathcal{V}_{\mathbb{F}}^{\star}$, $\gamma'\colon \mathcal{U}' \to \mathcal{V}_{\mathbb{F}}^{\star}$, $\gamma(\overline{\mathcal{B}})\colon \mathcal{U}^{\mathrm{bi.gr}} \to \mathcal{V}_{\mathbb{F}}^{\star\star}$, $\gamma(\mathcal{B})\colon \mathcal{U}(\mathcal{B}) \to \mathcal{V}_{\mathbb{F}}^{\star\star}$ admit both left and right adjoints.*

Proof The results (cf Appendix A.2) on the existence of right and left adjoints to the corestriction functor extend to these categories *Comod*$^{\mathrm{gr}}$ of graded right comodules. □

The following Corollary is immediate.

Corollary 7.1.2 *The categories $\mathcal{U}$, $\mathcal{U}'$, $\mathcal{U}^{\mathrm{bi.gr}}$, $\mathcal{U}(\mathcal{B})$ have enough projective objects and enough injective objects.*

The Brown–Gitler modules and the Massey–Peterson modules have the following definition from the comodule viewpoint. (cf the material of Appendix A.2 on cotensor products and duality).

Definition 7.1.3 For n a non-negative integer, define the following objects of $\mathcal{U}$, considered as the category of right $\widetilde{\mathcal{A}}^*$–comodules:

(1) $J(n) := \mathbb{F}(n)\Box_{\mathbb{F}[u]}\widetilde{\mathcal{A}}^*$;

(2) $F(n)$ the right $\widetilde{\mathcal{A}}^*$–comodule associated by duality to the left $\widetilde{\mathcal{A}}^*$–comodule $\widetilde{\mathcal{A}}^*\Box_{\mathbb{F}[u]}\mathbb{F}(n)$.

There is the analogous definition in the bigraded situation:

Definition 7.1.4 For a, b non-negative integers, define the following objects of $\mathcal{U}^{\mathrm{bi.gr}}$, the category of right $\overline{\mathcal{B}}$–comodules:

(1) $J(a,b) := \mathbb{F}(a,b)\Box_{\mathbb{F}[u,\xi_0]}\overline{\mathcal{B}}$;

(2) $F(a,b)$, the right comodule associated to the left $\overline{\mathcal{B}}$–comodule $\overline{\mathcal{B}}\square_{\mathbb{F}[u,\xi_0]}\mathbb{F}(a,b)$ by duality.

The definitions of the objects $F(a,b)$, $J(a,b)$ in terms of adjoints to the corestriction functor implies the following characterization, which is analogous to the characterization of the unstable modules $F(n), J(n)$.

Proposition 7.1.5 *For a,b non-negative integers, the object $F(a,b)$ is projective in $\mathcal{U}^{\text{bi.gr}}$, $J(a,b)$ is injective in $\mathcal{U}^{\text{bi.gr}}$ and, for M an object of $\mathcal{U}^{\text{bi.gr}}$, there are isomorphisms*

(1) $\mathrm{Hom}_{\mathcal{U}^{\text{bi.gr}}}(F(a,b),M)\cong M_{(a,b)}$;

(2) $\mathrm{Hom}_{\mathcal{U}^{\text{bi.gr}}}(M,J(a,b))\cong M^*_{(a,b)}$,

where $M_{(a,b)}$ denotes the homogeneous component of M in bidegree (a,b).

It is straightforward to verify the following connectivity result:

Lemma 7.1.6 *For a,b non-negative integers,*

(1) $F(a,b)_{(s,t)}=\begin{cases}0 & s<a \text{ or } t<b\\ \mathbb{F} & (s,t)=(a,b).\end{cases}$

(2) $J(a,b)_{(s,t)}=\begin{cases}0 & s>a \text{ or } t>b\\ \mathbb{F} & (s,t)=(a,b).\end{cases}$

The lemma implies that the objects $F(a,b)$ and $J(a,b)$ both have fundamental classes in bidegree (a,b), which are unique up to non-zero scalar multiple.

Lemma 7.1.7 *For a,b non-negative integers,*

(1) *the object $\Theta J(a,b)$ of $\mathcal{U}$ is a sub-module of the Brown–Gitler module $J(a+2b)$;*

(2) *the object $\Theta F(a,b)$ of $\mathcal{U}$ is a quotient of the free unstable module $F(a+2b)$.*

Proof Straightforward. □

This implies the following:

Lemma 7.1.8 *For a,b non-negative integers, there are isomorphisms:*

(1) $\mathrm{End}_{\mathcal{U}}(\Theta J(a,b))\cong\mathbb{F}$;

(2) $\mathrm{End}_{\mathcal{U}}(\Theta F(a,b))\cong\mathbb{F}$.

7.2 Properties of the projective objects of $\mathcal{U}^{\text{bi.gr}}$

The projective $F(a,b)$ is defined as the left comodule $\overline{\mathcal{B}}\square_{\mathbb{F}[u,\xi_0]}\mathbb{F}(a,b)$ and hence its underlying vector space depends only upon the right $\mathbb{F}[u,\xi_0]$–comodule structure of $\overline{\mathcal{B}}$. The following result reduces the study of these projective generators, via tensor products, to the cases where either a or b is zero.

Proposition 7.2.1 *For a,b non-negative integers, there is an isomorphism of objects $F(a,b)\cong F(a,0)\otimes F(0,b)$ in $\mathcal{U}^{\text{bi.gr}}$.*

Recall that $\mathbb{F}[\xi_j\,|\,j\geq 0]$ has a natural bialgebra structure and that there exists a morphism of bialgebras $\mathbb{F}[\xi_j\,|\,j\geq 0]\to\mathbb{F}[\xi_0]$, which induces a right $\mathbb{F}[\xi_0]$–comodule structure upon $\mathbb{F}[\xi_j\,|\,j\geq 0]$. The $\mathbb{Z}/2$–graded algebra $\mathbb{F}[u]\otimes\Lambda(\tau_i|i\geq 0)$ has the structure of a right $\mathbb{F}[u]$–comodule with respect to the multiplicative structure morphism which is induced by $u\mapsto u\otimes u$, $\tau_i\mapsto\tau_i\otimes u$.

Proposition 7.2.1 is a formal consequence of the following observation, using the properties of cotensor products over a tensor product of coalgebras.

Lemma 7.2.2 *As a right $\mathbb{F}[u,\xi_0]$–comodule, $\overline{\mathcal{B}}$ is isomorphic to the exterior tensor product of the right $\mathbb{F}[\xi_0]$–comodule $\mathbb{F}[\xi_j\,|\,j\geq 0]$ and of the right $\mathbb{F}[u]$–comodule $\mathbb{F}[u]\otimes\Lambda(\tau_i)$.*

Proof Immediate. □

Recall that there is a commutative diagram of functors

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "r_{\mathcal{B}}"] \arrow[dr, "\Psi r_{\mathcal{B}}"'] & \mathrm{Rep}(\{\Gamma^*\otimes\Lambda^*\}^{\mathrm{op}})\cong\mathcal{U}(\mathcal{B}) \arrow[d, "\Psi"] \\
 & \mathcal{U}^{\text{bi.gr}} \\
\mathcal{F} \arrow[r, "r'"'] \arrow[ur, "\widehat{\mathcal{O}}r'"] & \mathrm{Rep}(\{\Gamma^*\}^{\mathrm{op}})\cong\mathcal{U}', \arrow[u, "\widehat{\mathcal{O}}"']
\end{tikzcd}$$

in which Ψ and $\widehat{\mathcal{O}}$ denote the exact corestriction functors.

Proposition 7.2.3 *For a,b non-negative integers, there are natural isomorphisms in $\mathcal{U}^{\text{bi.gr}}$*

(1) $F(0,b)\cong\widehat{\mathcal{O}}r'(\Gamma^b)$

(2) $F(a,0) \cong \Psi r_{\mathcal{B}}(\Lambda^a)$.

Proof (Indications) It is straightforward to show that there is a surjection $F(0,b) \to \widehat{\mathcal{O}}r'(\Gamma^b)$, using the defining property of $F(0,b)$. This can be seen to be an isomorphism by comparing Poincaré series. The second statement admits a similar proof; namely, there is a surjection $F(a,0) \to \Psi r_{\mathcal{B}}(\Lambda^a)$ and the result follows by comparing Poincaré series. □

Remark 7.2.4

(1) It is not true in general that there is a surjection $F(0,b) \to \Psi r(\Gamma^b)$; the obstruction is the isomorphism $\Lambda^1 \to \Gamma^1$.

(2) The structure of the objects $F(0,b)$ is analogous to the structure of the projective generators $F(b)$ in the category of unstable modules at the prime two (see [10, Proposition 1.6.3 and Proposition 1.7.3]).

Lemma 7.2.5 *For a positive integer a, there is a non-trivial surjection $F(a,0) \to \Sigma^{(1,0)}F(a-1,0)$, which is unique up to non-trivial scalar multiple.*

Proof Straightforward. □

The description of the objects $F(a,0)$ given in Proposition 7.2.3 leads to the direct proof of the following Proposition.

Proposition 7.2.6 *For a positive integer a, there exists a finite filtration*

$$0 = g_{-1}F(a,0) \subset g_0 F(a,0) \subset \ldots g_a F(a,0) = F(a,0),$$

such that the filtration quotients are identified, for $0 \le j \le a$, by:

$$g_j F(a,0)/g_{j-1}F(a,0) \cong \Sigma^{(j,0)}\widehat{\mathcal{O}}r'(\Lambda^{a-j}).$$

In particular, there is a monomorphism $\widehat{\mathcal{O}}r'(\Lambda^a) \hookrightarrow F(a,0)$, which fits into a short exact sequence in $\mathcal{U}^{\mathrm{bi.gr}}$:

$$0 \to \widehat{\mathcal{O}}r'(\Lambda^a) \to F(a,0) \to \Sigma^{(1,0)}F(a-1,0) \to 0.$$

Proof (Indications) The filtration can be deduced by using the exponential property of the graded functor Λ^* in $\mathcal{F}$ and the fact that Λ^n is a simple object for each integer n. For the final statement, it is straightforward to identify the kernel of the morphism $F(a,0) \to \Sigma^{(1,0)}F(a-1,0)$. □

Notation 7.2.7 For a non-negative integer a, define $\Phi F(a,0)$ to be $F(a,0)$ for $a \in \{0,1\}$ and, for $a>1$, via the short exact sequence in $\mathcal{U}^{\mathrm{bi.gr}}$:

$$0 \to \Phi F(a,0) \to F(a,0) \to \Sigma^{(2,0)} F(a-2,0) \to 0.$$

(The convention is adopted that Φ is associated to the double suspension $\Sigma^{(2,0)}$ and to the single suspension $\Sigma^{(0,1)}$).

Lemma 7.2.8 *For $a>1$ an integer, there is a short exact sequence in $\mathcal{U}^{\mathrm{bi.gr}}$:*

$$0 \to \widehat{O}r'(\Lambda^a) \to \Phi F(a,0) \to \Sigma^{(1,0)} \widehat{O}r'(\Lambda^{a-1}) \to 0.$$

Proof The result follows from the final statement of Proposition 7.2.6. □

7.3 Properties of the projective objects in $\mathcal{U}$

The results of the previous section allow the description of the projective generators of $\mathcal{U}$ in terms of the forgetful functor $\Theta\colon \mathcal{U}^{\mathrm{bi.gr}} \to \mathcal{U}$ and the projective objects $F(a,0), F(0,b) \in \mathcal{U}^{\mathrm{bi.gr}}$. Proposition 7.3.3 expresses $F(n)$ in terms of an equalizer diagram; the result is essentially a formal consequence of the definition of the corestriction functor $\mathcal{U}^{\mathrm{bi.gr}} \to \mathcal{U}$.

This analysis is used to provide a filtration of the objects $F(n)$, which is given in Theorem 7.3.8; the associated graded has an explicit description which can be regarded as being a natural extension of the analysis of the structure of the objects $F(n)$ at the prime two.

Lemma 7.3.1 *For n a non-negative integer and a,b non-negative integers such that $n=a+2b$, there is a non-trivial morphism $\mu_{a,b}\colon F(n) \to \Theta F(a,b)$, unique up to non-zero scalar multiple, which is surjective.*

Proof The existence of the surjection is given by Lemma 7.1.7; the fact that it is unique up to non-zero scalar multiple follows from the fact that $\Theta F(a,b)$ is of dimension one in degree $a+2b$, which implies that $\mathrm{Hom}_{\mathcal{U}}(F(n), \Theta F(a,b)) \cong \mathbb{F}$. □

Recall that there are surjections in $\mathcal{U}^{\mathrm{bi.gr}}$, for integers $a \geq 2$ and $b \geq 0$:

$$F(a,0) \to \Sigma^{(2,0)} F(a-2,0)$$
$$F(0,b+1) \to \Sigma^{(0,1)} F(0,b).$$

These induce surjections

$$l_{a,b}\colon \Theta F(a,b) \to \Sigma^2 \Theta F(a-2,b)$$
$$r_{a-2,b+1}\colon \Theta F(a-2,b+1) \to \Sigma^2 \Theta F(a-2,b)$$

in the category $\mathcal{U}$.

Lemma 7.3.2 *For integers $a \geq 2, b \geq 1$ satisfying $a+2b=n$, the following diagram commutes up to non-zero scalar multiple*

$$\begin{tikzcd}
F(n) \arrow[r, "\mu_{a-2,b+1}"] \arrow[d, "\mu_{a,b}"'] & \Theta F(a-2,b+1) \arrow[d, "r_{a-2,b+1}"] \\
\Theta F(a,b) \arrow[r, "l_{a,b}"'] & \Sigma^2 \Theta F(a-2,b).
\end{tikzcd}$$

Proof This result follows immediately from Lemma 7.1.6. □

Proposition 7.3.3 *For n a non-negative integer, the object $F(n)$ identifies with the equalizer of the diagram*

$$\bigoplus_{a+2b=n} \Theta F(a,b) \overset{l_{a,b}}{\underset{r_{a,b}}{\rightrightarrows}} \bigoplus_{c+2d=n} \Theta F(c,d).$$

In particular, the morphism

$$\bigoplus_{a+2b=n} \mu_{a,b} \colon F(n) \to \bigoplus_{a+2b=n} \Theta F(a,b)$$

is a monomorphism.

The Proposition is a formal consequence of the definition of the functor $\mathcal{U}^{\mathrm{bi.gr}} \to \mathcal{U}$; it can be proved explicitly using the identification provided by Lemma 7.3.4 below. Recall that the bialgebra $\overline{\mathcal{B}}$ has the structure of a right $\mathbb{F}[u]$–comodule via the corestriction associated to the morphism of Hopf algebras $\mathbb{F}[u,\xi_0] \to \mathbb{F}[u]$ given by $\xi_0 \mapsto u^2$.

Lemma 7.3.4 *For n a non-negative integer,*

(1) *the surjection $\overline{\mathcal{B}} \to \widetilde{\mathcal{A}}^*$ induces a surjection $\overline{\mathcal{B}} \square_{\mathbb{F}[u]} \mathbb{F}(n) \twoheadrightarrow \widetilde{\mathcal{A}}^* \square_{\mathbb{F}[u]} \mathbb{F}(n)$, which is a morphism of left $\widetilde{\mathcal{A}}^*$–comodules, where $\overline{\mathcal{B}} \square_{\mathbb{F}[u]} \mathbb{F}(n)$ is given the corestricted structure;*

(2) *the left $\overline{\mathcal{B}}$–comodule $\overline{\mathcal{B}} \square_{\mathbb{F}[u]} \mathbb{F}(n)$ is isomorphic to $\bigoplus_{a+2b=n} \overline{\mathcal{B}} \square_{\mathbb{F}[u,\xi_0]} \mathbb{F}(a,b)$.*

Proof The surjectivity of the morphism in (1) follows from the right exactness of $-\square_{\mathbb{F}[u]} \mathbb{F}(n)$, which reflects a coflatness property. The proof of the isomorphism of (2) is straightforward. □

Proposition 7.3.3 gives rise to a filtration of the object $F(n)$ in the category $\mathcal{U}$ as follows. The choice of the filtration is motivated by the consideration of the structure of the object $F(2)$ in the category $\mathcal{U}$, for which there is a short exact sequence:

$$0 \to \Theta\Phi F(2,0) \to F(2) \to \Theta F(0,1) \to 0.$$

Notation 7.3.5 For $j \geq -1$ an integer, let $f_j F(n)$ denote the subobject of $F(n) \in \mathcal{U}$ defined by the kernel of the morphism

$$F(n) \to \bigoplus_{\substack{a+2b=n \\ b \geq j+1}} \Theta F(a,b).$$

This defines an increasing filtration of $F(n)$, with $f_{-1} F(n)$ zero and $f_j F(n) = F(n)$ for $2j \geq n$.

The filtration quotients can be identified explicitly as follows. In the Lemma below, the morphism $l_{a,b}$ is taken to be zero if the integer a is in $\{0,1\}$.

Lemma 7.3.6 *For $a, b \geq 0$ integers, the kernel of the morphism $l_{a,b}\colon \Theta F(a,b) \to \Sigma^2\Theta F(a-2,b)$ identifies with the object $\Theta\{\Phi F(a,0) \otimes F(0,b)\}$.*

Proof The lemma follows immediately from the definition of the objects $\Phi F(a,0)$. □

Lemma 7.3.7 *For a, b non-negative integers such that $a + 2b = n$, the morphism $\mu_{a,b}$ induces a monomorphism*

$$f_b F(n)/f_{b-1} F(n) \hookrightarrow \Theta\{\Phi F(a,0) \otimes F(0,b)\}.$$

Proof This is an immediate consequence of the definition of the filtration and of Lemma 7.3.6. □

Theorem 7.3.8 *For n a positive integer, the object $F(n) \in \mathcal{U}$ has a finite increasing filtration $\{f_b F(n)\}$ such that the filtration quotients are of the form*

$$f_b F(n)/f_{b-1} F(n) \cong \Theta\{\Phi F(a,0) \otimes F(0,b)\},$$

where a, b are non-negative integers such that $a + 2b = n$.

Proof (Indications) The theorem is proved by showing that the monomorphisms defined in Lemma 7.3.7 are isomorphisms. This can be proved by analysing the monomial basis of $F(n)$ in conjunction with Proposition 7.3.3; indeed, it is sufficient to use a comparison of Poincaré series. □

A related analysis of the structure of $F(n)$ is given in Schwartz [11], from a different viewpoint. The advantage of the approach given above is that it leads immediately to the following description of the filtration quotients in a finite filtration of $F(n)$.

Corollary 7.3.9 *For n a positive integer, $F(n)$ has a finite filtration with associated graded*

$$\bigoplus_{\substack{a+2b=n\\ a\geq 0}} \mathcal{O}r'(\Lambda^a \otimes \Gamma^b) \oplus \bigoplus_{\substack{a+2b=n\\ a\geq 1}} \Sigma\mathcal{O}r'(\Lambda^{a-1} \otimes \Gamma^b).$$

Proof The Corollary follows from an analysis of the unstable modules $\Theta(\Phi F(a,0) \otimes F(0,b))$, by Theorem 7.3.8. Lemma 7.2.8 implies that there is a short exact sequence

$$0 \to \hat{\mathcal{O}}r'(\Lambda^a) \otimes F(0,b) \to \Phi F(a,0) \otimes F(0,b) \to \Sigma^{(1,0)}\hat{\mathcal{O}}r'(\Lambda^{a-1}) \otimes F(0,b) \to 0$$

and $F(0,b)$ identifies with $\hat{\mathcal{O}}r'(\Gamma^b)$, by Proposition 7.2.3. Moreover, exponentiality implies that there is an identification of $\hat{\mathcal{O}}r'(\Lambda^a) \otimes \hat{\mathcal{O}}r'(\Gamma^b)$ with $\hat{\mathcal{O}}r'(\Lambda^a \otimes \Gamma^b)$ and likewise for the term involving $a-1$. The result follows by applying the functor Θ, using the identification $\mathcal{O} = \Theta\hat{\mathcal{O}}$. □

7.4 The injective cogenerators of $\mathcal{U}^{\text{bi.gr}}$ and $\mathcal{U}$

In this section, the injective cogenerators $J(a,b)$ are analysed. It is shown that the classical results concerning the structure of the Brown–Gitler modules over an odd prime p arise from the category $\mathcal{U}^{\text{bi.gr}}$. In particular, a description of the injective cogenerators is given in terms of the functor $\Psi\rho\colon \mathcal{F} \to \mathcal{U}^{\text{bi.gr}}$.

Proposition 7.4.1 *For a, b non-negative integers, there are canonical isomorphisms in $\mathcal{U}^{\text{bi.gr}}$:*

(1) $J(a,0) \cong \mathbb{F}(a,0)$

(2) $J(a,b) \cong J(a,0) \otimes J(0,b)$.

Proof The result is a formal consequence of the observation that, as a left $\mathbb{F}[u,\xi_0]$–comodule, the bialgebra $\overline{\mathcal{B}}$ is the exterior tensor product of the algebra $\mathbb{F}[u]$, considered as a left $\mathbb{F}[u]$–comodule, and the algebra $\mathbb{F}[\xi_j \,|\, j \geq 0] \otimes \Lambda(\tau_i | i \geq 0)$, which has the structure of a left $\mathbb{F}[\xi_0]$–comodule. □

The Proposition implies that the problem of understanding the structure of the objects $J(a,b)$ can be reduced to understanding the objects $J(0,b)$ in $\mathcal{U}^{\text{bi.gr}}$, up to suspension.

Lemma 7.4.2 *For b a non-negative integer, there is a canonical monomorphism $\Sigma^{(0,1)} J(0,b) \hookrightarrow J(0,b+1)$ in the category $\mathcal{U}^{\mathrm{bi.gr}}$, unique up to non-zero scalar multiplication.*

Proof Straightforward. □

Recall that the functor $\rho\colon \mathcal{F} \to \mathrm{Rep}(\{\Gamma^* \otimes \Lambda^*\}^{\mathrm{op}}) \cong \mathcal{U}(\mathcal{B})$ is defined by $F \mapsto \mathrm{Hom}_{\mathcal{F}}(F,-)^*$. There is the composite functor

$$\Psi\rho\colon \mathcal{F} \to \mathcal{U}^{\mathrm{bi.gr}},$$

where Ψ is the forgetful functor $\mathcal{U}(\mathcal{B}) \to \mathcal{U}^{\mathrm{bi.gr}}$. The bigraded exponential property of the functors $\Gamma^* \otimes \Lambda^*$ implies the following variant of a standard result.

Proposition 7.4.3 *For F, G objects of $\mathcal{F}$, there is a natural isomorphism in $\mathcal{U}^{\mathrm{bi.gr}}$:*

$$\Psi\rho(F \otimes G) \cong \Psi\rho(F) \otimes \Psi\rho(G),$$

where the tensor product on the right hand side denotes the bigraded tensor product.

The following result gives an elegant description of the objects $J(0,n)$.

Proposition 7.4.4 *For n a non-negative integer, there is an isomorphism $J(0,n) \cong \Psi\rho(\Gamma^n)$ in $\mathcal{U}^{\mathrm{bi.gr}}$. In particular, in bidegree (a,b), there is an isomorphism of vector spaces*

$$J(0,n)_{a,b} \cong \mathrm{Hom}_{\mathcal{F}}(\Gamma^n, \Lambda^a \otimes \Gamma^b)^*.$$

Proof (Indications) The result follows from the identification of the category $\mathcal{U}(\mathcal{B})$ as the representation category $\mathrm{Rep}(\{\Gamma^* \otimes \Lambda^*\}^{\mathrm{op}})$, the definition of the functor Ψ, together with the observation that the morphism $\Lambda^1 \to \Gamma^1$ does not intervene in the calculation of $\mathrm{Hom}_{\mathcal{F}}(\Gamma^n, \Lambda^* \otimes \Gamma^*)$. □

Corollary 7.4.5 *There is a surjection $J(0,1) \xrightarrow{\epsilon(0,1)} \mathbb{F}(1,0)$, which is unique up to non-zero scalar multiple.*

Proof Proposition 7.4.1 implies that $\mathbb{F}(1,0) \cong J(1,0)$, hence it is sufficient to show that the object $J(0,1)$ has dimension one in bidegree $(1,0)$, using the representing property of $J(1,0)$; this follows from Proposition 7.4.4. □

Remark 7.4.6 The analogous description does not hold for $J(a,0)$, in general; for $\mathrm{Hom}_{\mathcal{F}}(\Lambda^a, \Gamma^* \otimes \Lambda^*)$ has total dimension two, for a positive integer a, since the diagonal morphism $\Lambda^a \to \Lambda^1 \otimes \Lambda^{a-1}$ induces a morphism $\Lambda^a \to \Gamma^1 \otimes \Lambda^{a-1}$, using the isomorphism $\Lambda^1 \cong \Gamma^1$.

This result gives the following information on the structure of the objects of the form $\Psi\rho(F)$ in the category $\mathcal{U}^{\text{bi.gr}}$.

Corollary 7.4.7 *For $F \in \mathcal{F}$ a finite functor, there exists an exact sequence in $\mathcal{U}^{\text{bi.gr}}$*

$$\bigoplus_j J(0, c_j) \to \bigoplus_i J(0, b_i) \twoheadrightarrow \Psi\rho(F) \to 0$$

for finite sets of non-negative integers $\{b_i\}$, $\{c_j\}$.

Proof Theorem 5.1.1 implies that there exist finite sets of integers $\{b_i\}$, $\{c_j\}$ and an exact sequence $\bigoplus_j \Gamma^{c_j} \to \bigoplus_i \Gamma^{b_i} \to F \to 0$. The result follows since ρ is right exact. □

Notation 7.4.8 Let $\rho'\colon \mathcal{F} \to \text{Rep}(\{\Gamma^*\}^{\text{op}})$ denote the functor ρ associated to $\{\Gamma^*\}$, as in Notation 6.1.3.

Recall that there is a natural embedding $\mathcal{U}' \hookrightarrow \mathcal{U}^{\text{bi.gr}}$ which is denoted by $\widehat{\mathcal{O}}$. There is a natural embedding $\widehat{\mathcal{O}}\rho'(\Gamma^n) \hookrightarrow J(0,n)$ in $\mathcal{U}^{\text{bi.gr}}$, which corresponds to the beginning of a filtration with associated graded described explicitly by the following result.

Proposition 7.4.9 *For a non-negative integer n, the object $J(0,n)$ has a finite filtration with associated graded*

$$\bigoplus \Sigma^{(d,0)}\widehat{\mathcal{O}}\rho'(\Gamma^{n-d}),$$

where d ranges over the set of integers $0 \le d \le n$ which can be expressed as a sum of at most n pairwise distinct powers of p.

Proof The result is deduced by using the exponential property of Γ^* to obtain a direct sum decomposition at the level of vector spaces, which is the associated graded to a filtration in $\mathcal{U}^{\text{bi.gr}}$. □

There are analogues of the Mahowald exact sequences in the category $\mathcal{U}^{\text{bi.gr}}$ (cf [10, Proposition 2.3.4]). To state the result, define the following morphisms which are induced by the Verschiebung $\Gamma^{np} \xrightarrow{\mathcal{V}} \Gamma^n$. (Recall that the Verschiebung is the morphism which is dual to the Frobenius pth power morphism).

Definition 7.4.10 For n a positive integer,

(1) let $\mathcal{V}_n\colon J(0,np) \to J(0,n)$ denote the surjective morphism which is induced under the functor $\Psi\rho\colon \mathcal{F} \to \mathcal{U}^{\text{bi.gr}}$ by the Verschiebung $\Gamma^{np} \xrightarrow{\mathcal{V}} \Gamma^n$;

(2) let $\mathcal{V}'_n\colon J(0,np+1) \to \mathbb{F}(1,0)\otimes J(0,n) \cong \Sigma^{(1,0)} J(0,n)$ denote the composite morphism

$$J(0,np+1) \to J(0,1)\otimes J(0,n) \xrightarrow{\epsilon(0,1)\otimes 1} \mathbb{F}(1,0)\otimes J(0,n)$$

where the first morphism is induced under $\Psi\rho$ by the composite

$$\Gamma^{np+1} \xrightarrow{\Delta} \Gamma^1 \otimes \Gamma^{np} \xrightarrow{1\otimes\mathcal{V}} \Gamma^1\otimes\Gamma^n,$$

in which Δ denotes the coproduct.

Proposition 7.4.11 *For n a positive integer, there are short exact sequences in $\mathcal{U}^{\mathrm{bi.gr}}$:*

(1) $0 \to \Sigma^{(0,1)} J(0,np-1) \to J(0,np) \xrightarrow{\mathcal{V}_n} J(0,n) \to 0$;

(2) $0 \to \Sigma^{(0,1)} J(0,np) \to J(0,np+1) \xrightarrow{\mathcal{V}'_n} \Sigma^{(1,0)} J(0,n) \to 0$.

If m is a positive integer such that $m \not\equiv 0,1 \mod p$, then the canonical monomorphism

$$\Sigma^{(0,1)} J(0,m-1) \cong J(0,m)$$

is an isomorphism.

Proof (Indications) The result can be proved by a calculation in terms of the description of $J(0,*)$ as $\Psi\rho(\Gamma^*)$. □

Remark 7.4.12 The consideration of the bigraded situation naturally gives rise to the suspension which appears in the second Mahowald exact sequence (see [10, Proposition 2.3.4]), which corresponds to the presence of the Bockstein operation.

Proposition 7.4.13 *For n a positive integer, there are canonical isomorphisms*

(1) $J(2n) \cong \Theta J(0,n)$

(2) $J(2n+1) \cong \Theta J(1,n)$,

where $\Theta\colon \mathcal{U}^{\mathrm{bi.gr}} \to \mathcal{U}$ denotes the canonical forgetful functor.

Proof (Indications) There are many ways of approaching the proof of this result. For example, use the canonical (up to non-zero scalar multiple) monomorphisms $\Theta\colon J(a,b) \hookrightarrow J(a+2b)$. It is straightforward to use the Mahowald exact sequences to compare Poincaré series in the relevant cases of the Proposition. □

8 New proofs of fundamental results

The analysis of the categories $\mathcal{U}(\mathcal{B})$, $\mathcal{U}^{\mathrm{bi.gr}}$ in relation to the category $\mathcal{U}$ given in this paper yields direct proofs of the foundational results of the theory of nil-localization of the category of unstable modules in odd characteristic, generalizing the approach of Kuhn available over the field $\mathbb{F}_2$.

8.1 Injectivity of $H^*(B\mathbb{F}^n)$ in $\mathcal{U}$

This section is devoted to giving a self-contained proof of the injectivity of the object $H^*(B\mathbb{F}^n) \cong \widetilde{\Theta} r_{\mathcal{B}}(I_{\mathbb{F}^n})$ of $\mathcal{U}$ Theorem 6.1.2 applied to the representation category $\mathrm{Rep}(\{\Gamma^* \otimes \Lambda^*\}^{\mathrm{op}})$. The proof relies on the analysis of the injective cogenerators of the categories $\mathcal{U}, \mathcal{U}^{\mathrm{bi.gr}}$ in Section 7.

Proposition 8.1.1 *Let a be a non-negative integer, then the object $\widetilde{\Theta}\rho(\Gamma^a)$ is injective in $\mathcal{U}$.*

Proof The injective cogenerators of $\mathcal{U}$ are considered in Section 7.4; in particular, combining Proposition 7.4.4 and Proposition 7.4.13 implies that the object $\widetilde{\Theta}\rho(\Gamma^{a_i})$ identifies with an injective object in $\mathcal{U}$. □

Lemma 8.1.2 *Let n be a non-negative integer, then there exists a monomorphism in $\mathcal{U}(\mathcal{B})$*

$$r_{\mathcal{B}}(I_{\mathbb{F}^n}) \cong H^{\otimes n} \hookrightarrow \prod_{i\geq 0} \rho(\Gamma^{a_i}),$$

where the sequence of integers a_i can be taken to have limit ∞ as i goes to ∞.

Proof The structure theory of $\mathcal{U}' \cong \mathrm{Rep}(\{\Gamma^*\}^{\mathrm{op}})$ implies that there exists a morphism $\alpha\colon r_{\mathcal{B}}(I_{\mathbb{F}^n}) \cong H^{\otimes n} \to \prod_{i\geq 0}\rho(\Gamma^{a_i})$ in $\mathcal{U}(\mathcal{B})$ such that the restriction $\mathsf{Res}(\alpha)$ in $\mathcal{U}'$ is a monomorphism.

Theorem 5.1.1 implies that α is injective, as follows; suppose that (a,b) is a pair of non-negative integers, then Lemma 5.1.2 implies readily that there is a surjection $\phi\colon \Gamma^T \to \Lambda^a \otimes \Gamma^b$, for an integer T. The morphism α is injective in bidegree $(0,T)$, by hypothesis; the surjection ϕ implies that the morphism α is injective in bidegree (a,b). □

Theorem 8.1.3 *For a non-negative integer n, the object $H^*(B\mathbb{F}^n)$ is injective in $\mathcal{U}$.*

Proof There is a monomorphism $r_{\mathcal{B}}(I_{\mathbb{F}^n}) \cong H^{\otimes n} \hookrightarrow \prod_{i\geq 0} \rho(\Gamma^{a_i})$ in $\mathcal{U}(\mathcal{B})$, by the previous Lemma. The injectivity of $r_{\mathcal{B}}(I_{\mathbb{F}^n})$ in $\mathcal{U}(\mathcal{B})$ (Corollary 6.3.2) yields a retraction.

Applying the functor $\widetilde{\Theta}\colon \mathcal{U}(\mathcal{B}) \to \mathcal{U}$ yields a split monomorphism in $\mathcal{U}$. The functor $\widetilde{\Theta}$ commutes with products and the objects $\widetilde{\Theta}\rho(\Gamma^{a_i})$ are injective in $\mathcal{U}$, by Proposition 8.1.1, hence the result follows. □

8.2 Nilclosure for unstable modules

This section makes explicit the structure of a nil-closed unstable module, when the field p has odd characteristic. Theorem 8.1.3 implies that there is an exact functor $l\colon \mathcal{U} \to \mathcal{F}$ and an adjunction

$$l\colon \mathcal{U} \rightleftarrows \mathcal{F} :r,$$

where the functor $l\colon \mathcal{U} \to \mathcal{F}$ is defined to be the functor which sends the object M of $\mathcal{U}$ to the functor $\mathbb{F}^n \mapsto \mathrm{Hom}_{\mathcal{U}}(M, H^*(B\mathbb{F}^n))'$, where $'$ denotes the profinite dual.

Notation 8.2.1 Let $\widetilde{\Theta}\colon \mathcal{U}(\mathcal{B}) \to \mathcal{U}$ denote the exact corestriction functor, which identifies with the composite $\mathcal{U}(\mathcal{B}) \to \mathcal{U}^{\mathrm{bi.gr}} \to \mathcal{U}$.

The definition of the categories as comodule categories implies the following result.

Lemma 8.2.2 *The functor $\widetilde{\Theta}\colon \mathcal{U}(\mathcal{B}) \to \mathcal{U}$ admits a left adjoint $\lambda\colon \mathcal{U} \to \mathcal{U}(\mathcal{B})$.*

The object $H^*(B\mathbb{F}^n)$ of $\mathcal{U}$ identifies with the object $\widetilde{\Theta} r_{\mathcal{B}} I_{\mathbb{F}^n}$, hence the following result follows, using the identification of the functor $l_{\mathcal{B}}$ which is provided by the theory of Section 6.

Corollary 8.2.3 *The adjunction $l\colon \mathcal{U} \rightleftarrows \mathcal{F} :r$ is the composite of the adjunctions $\lambda\colon \mathcal{U} \rightleftarrows \mathcal{U}(\mathcal{B}) :\widetilde{\Theta}$ and $l_{\mathcal{B}}\colon \mathcal{U}(\mathcal{B}) \rightleftarrows \mathcal{F} :r_{\mathcal{B}}$.*

Remark 8.2.4 The theory of [5] includes the identification of the kernel of the functor l as the full subcategory of nilpotent unstable modules, which is omitted above.

Using the terminology of localization of abelian categories as in [5], Corollary 8.2.3 implies the following result.

Corollary 8.2.5 *An object of $\mathcal{U}$ is nil-closed if and only it is isomorphic to an object of the form $\widetilde{\Theta} r_{\mathcal{B}}(F)$, for F an object of $\mathcal{F}$.*

Example 8.2.6 The object $\Theta F(a,0)$ is a nil-closed object in $\mathcal{U}$, whereas $\Theta F(0,b)$ is not nil-closed, for $b \geq 1$. It follows that the object $\Theta F(a,b)$ is not nil-closed in $\mathcal{U}$, for $b \geq 1$. There is a canonical monomorphism $\Theta F(a,b) \hookrightarrow \widetilde{\Theta} r_{\mathcal{B}}(\Lambda^a \otimes \Gamma^b)$, which represents the nil-closure, where $\widetilde{\Theta}\colon \mathcal{U}(\mathcal{B}) \to \mathcal{U}$ is the corestriction functor.

Part IV Motivic vistas at the prime 2

9 Bigraded and ordinary unstable modules at the prime two

The construction of a category of bigraded unstable modules $\mathcal{U}^{\text{bi.gr}}$ at the prime two proceeds as in the odd characteristic case by considering the endomorphisms of the primitively-generated Hopf algebra $\mathbb{F}[y]/y^2 \otimes \mathbb{F}[x]$, where $\mathbb{F}$ is taken to denote $\mathbb{F}_2$ throughout this section.

There is no direct analogue of the exact functor Θ to the category of unstable modules; the categories are related through a category $\mathcal{U}_{\mathcal{M}}$, which is a perturbation of the category of unstable modules; the category $\mathcal{U}_{\mathcal{M}}$ can be related to modules over the motivic Steenrod algebra (Voevodsky [14; 13]). In particular, motivic cohomology is a bigraded cohomology theory, hence the presence of the bigrading is essential.

This section outlines the construction of $\mathcal{U}_{\mathcal{M}}$ and its relation to $\mathcal{U}$; some of the results are implicit in the work of Yagita [15; 16].

9.1 The category $\mathcal{U}_{\mathcal{M}}$

The underlying category used in this section is the category of modules over the polynomial algebra $\mathbb{F}[\tau]$. In the study of motivic cohomology, this corresponds to the fact that the coefficient ring is not concentrated in a single degree.

The category $\mathcal{U}_{\mathcal{M}}$ is constructed by considering endomorphisms of the algebra $H_{\mathcal{M}}$ which is defined as follows.

Notation 9.1.1 Let $H_{\mathcal{M}}$ denote the commutative $\mathbb{F}[\tau]$–algebra $\mathbb{F}[\tau][x, y]/y^2 = \tau x$.

The algebra $H_{\mathcal{M}}$ is to be considered as a perturbation of the algebra $\mathbb{F}[y]$ and the generators y, x are given independent gradings; this imposes the requirement that the generator τ has non-zero grading. In order to encode the non-trivial action on the underlying ring $\mathbb{F}[\tau]$, one is obliged to work with comodules over an affine category scheme. The latter is the algebraic object which represents a small category; it is related to the more familiar notion of a Hopf algebroid (cf Ravenel [9, Appendix]) in the same way that a bialgebra is related to a Hopf algebra.

Remark 9.1.2 An affine category scheme in the category of $\mathbb{F}$–algebras is given by a pair of commutative $\mathbb{F}$–algebras (A, Γ) together with structure morphisms of $\mathbb{F}$–algebras $\eta_L, \eta_R\colon A \rightrightarrows \Gamma$, $\epsilon\colon \Gamma \to A$, $\Delta\colon \Gamma \to \Gamma \otimes_A \Gamma$ which satisfy a suitable subset of the axioms for a Hopf algebroid.

In the current situation, the algebra A corresponds to $\mathbb{F}[\tau]$ and the comodules considered are $\mathbb{F}[\tau]$–modules.

Remark 9.1.3 The comodules considered in this section need not have underlying module which is finitely-generated over the algebra $\mathbb{F}[\tau]$. For this reason, the comodule structures considered are defined with respect to the completed tensor product, which will correspond to the underlying grading in applications. The necessary details are left to be supplied by the reader.

Proposition 9.1.4 *There exists an affine category scheme $(\mathbb{F}[\tau], \mathcal{D})$ in the category of commutative $\mathbb{F}$–algebras, where*

$$\mathcal{D} \cong \mathbb{F}[\tau, t, u, \xi_i, \tau_j | i, j \geq 0]/(u^2 = t\xi_0, \tau_j^2 = t\tau\xi_{j+1}))$$

and the structure morphisms are given by $\eta_L\colon \tau \mapsto \tau$, $\eta_R\colon \tau \mapsto t\tau$, ϵ sends t, u, ξ_0 to 1 and all other generators to zero. The coproduct $\Delta\colon \mathcal{D} \to \mathcal{D} \otimes_{\mathbb{F}[\tau]} \mathcal{D}$ is determined by

$$u \mapsto u \otimes u \qquad\qquad t \mapsto t \otimes t$$

$$\xi_k \mapsto \sum_{i+j=k} \xi_j^{2^i} \otimes \xi_i \qquad\qquad \tau_k \mapsto \sum_{i+j=k} \xi_j^{2^i} \otimes \tau_i + \tau_k \otimes u.$$

Moreover, the $\mathbb{F}[\tau]$–algebra $H_{\mathcal{M}}$ has the structure of a right $(\mathbb{F}[\tau], \mathcal{D})$–comodule, with structure morphism $\psi\colon H_{\mathcal{M}} \to H_{\mathcal{M}} \widehat{\otimes}_{\mathbb{F}[\tau]} \mathcal{D}$ which is the morphism of algebras determined by

$$\tau \mapsto \tau t \qquad\qquad y \mapsto y \otimes u + \sum_{j \geq 0} x^{2^j} \otimes \tau_j \qquad\qquad x \mapsto \sum_{j \geq 0} x^{2^j} \otimes \xi_j.$$

Proof The proof is analogous to that of Proposition 3.2.4. This requires the verification that the only relations imposed by the relation $y^2 = \tau x$ are the relations $u^2 = t\xi_0$ (which corresponds to the imposed grading upon t) and the relations $\tau_j^2 = t\tau\xi_{j+1}$. □

Remark 9.1.5 The above result should be compared with the calculation of the dual of the algebraic motivic Steenrod algebra [14] over the coefficient ring $\mathbb{F}[\tau, \rho]$, when ρ is set to zero. The modification above is that the 'grouplike' elements t, u, ξ_0 are not taken to be 1.

Proposition 9.1.6 *There is a unique affine category scheme structure upon*

$$(\mathbb{F}[\tau], \mathbb{F}[\tau, t, u, \xi_0]/u^2 = t\xi_0)$$

such that the surjective morphism of $\mathbb{F}$–algebras $\mathcal{D} \to \mathbb{F}[\tau, t, u, \xi_0]/u^2 = t\xi_0$, defined by sending the generators ξ_{i+1}, τ_i to zero for $i \geq 1$, induces a morphism of affine category schemes

$$(\mathbb{F}[\tau], \mathcal{D}) \to (\mathbb{F}[\tau], \mathbb{F}[\tau, t, u, \xi_0]/u^2 = t\xi_0).$$

Proof Straightforward. □

Remark 9.1.7 The category of right comodules over the affine category scheme $(\mathbb{F}[\tau], \mathbb{F}[\tau, t, u, \xi_0]/u^2 = t\xi_0)$ can be identified with a category of bigraded $\mathbb{F}[\tau]$–modules, where the generator τ has non-zero bidegree.

In the motivic setting, the usual convention is to bigrade so that u–degree 1 corresponds to bidegree $(1, 1)$, ξ_0–degree 1 corresponds to bidegree $(2, 1)$ and t–degree 1 corresponds to bidegree $(0, 1)$.

The definition of the category of graded comodules given in Section 4.1 generalizes to the current context.

Definition 9.1.8 Let $\mathcal{U}_{\mathcal{M}}$ denote the category of graded right $(\mathbb{F}[\tau], \mathcal{D})$–comodules.

Proposition 9.1.9 *The category $\mathcal{U}_{\mathcal{M}}$ is a tensor abelian category.*

9.2 Relating $\mathcal{U}_{\mathcal{M}}$ to unstable modules

There are standard base change constructions for affine category schemes; namely, if (A, Γ) is an affine category scheme in the category of commutative $\mathbb{F}$–algebras and $f\colon A \to B$ is a morphism of commutative $\mathbb{F}$–algebras, then base change yields an affine category scheme $(B, B \otimes_A \Gamma \otimes_A B)$. The functor on the category of right A–modules, $M \mapsto M \otimes_A B$ extends to a functor from the category of right (A, Γ)–comodules to the category of right $(B, B \otimes_A \Gamma \otimes_A B)$–comodules, which is exact if the functor $M \mapsto M \otimes_A B$ is exact.

The base change constructions apply to the two choices of augmentation $\epsilon_1, \epsilon_0\colon \mathbb{F}[\tau] \rightrightarrows \mathbb{F}$ given respectively by $\tau \mapsto 1, \tau \mapsto 0$. To avoid confusion, the induced bialgebras will be written respectively $\epsilon_1^*\mathcal{D}$ and $\epsilon_0^*\mathcal{D}$ (noting that an $\mathbb{F}$–affine category scheme of the form $(\mathbb{F}, \Gamma)$ is a bialgebra).

Lemma 9.2.1 *The bialgebra $\epsilon_1^*\mathcal{D}$ is isomorphic to the bialgebra $\widetilde{\mathcal{A}}^*$.*

Proof The underlying $\mathbb{F}$–algebra of $\epsilon_1^*\mathcal{D}$ is generated by the elements $\{t, u, \xi_i, \tau_i\}$ subject to the induced relations. It is straightforward to verify that the assignment $\tau \mapsto 1$ implies the identity $t = 1$. The result follows readily. $\square$

Definition 9.2.2 Let $\mathcal{U}^{\text{bi.gr},t}$ denote the category of graded comodules over $\epsilon_0^*\mathcal{D}$.

Remark 9.2.3 The category $\mathcal{U}^{\text{bi.gr},t}$ is not the precise analogue of the category $\mathcal{U}^{\text{bi.gr}}$, since the grouplike element t is present in the bialgebra $\epsilon_0^*\mathcal{D}$ and there is the relation $u^2 = t\xi_0$.

Proposition 9.2.4 *The base change functors induce a diagram of functors between tensor abelian categories*

$$\begin{tikzcd} \mathcal{U}_{\mathcal{M}} \arrow[r, "\epsilon_0^*"] \arrow[d, "\epsilon_1^*"'] & \mathcal{U}^{\text{bi.gr},t} \\ \mathcal{U} & \end{tikzcd}$$

in which ϵ_1^* is exact and ϵ_0^* is right exact.

Proof The functors are induced by base change, using Lemma 9.2.1 to identify the category of comodules over $\epsilon_1^*\mathcal{D}$ with $\mathcal{U}$. The exactness properties correspond to the exactness properties of the respective functors $-\otimes_{\mathbb{F}[\tau]}\mathbb{F}$. $\square$

Proposition 9.2.5 *There is an adjunction*

$$\epsilon_1^*\colon \mathcal{U}_{\mathcal{M}} \rightleftarrows \mathcal{U} : \mu.$$

Moreover, the underlying $\mathbb{F}[\tau]$–module of μM, for M an unstable module, is τ–torsion free.

Proof (Indications) The existence of the right adjoint to ϵ_1^* is formal, using the existence of a set of projective generators for the category $\mathcal{U}_{\mathcal{M}}$. The τ–torsion statement is a consequence of the surjectivity of the morphism between projective generators which represents multiplication by τ. $\square$

Remark 9.2.6 The existence of μ and its fundamental properties is implicit in the work of Yagita [15] on the motivic cohomology of classifying spaces of finite groups.

Part V Appendix

Appendix A Comodules

Throughout this section, the base ring is taken to be a field, $\mathbb{F}$; all algebras are taken to be unital and associative and all coalgebras counital and coassociative. In particular, a bialgebra will have underlying algebra which is unital and augmented.

A.1 Duality for comodules and modules

The following results summarize the elementary properties of duality in the non-graded case. The category of left modules over an algebra A is written ${}_A\mathsf{Mod}$ and the category of right comodules over a coalgebra C is written $Comod_C$; the evident left/right mirror image categories are denoted in the obvious way.

Proposition A.1.1 (Milnor–Moore [8, Section 3]) *Let $\mathbb{F}$ be a field and let A be an algebra, C a coalgebra and B a bialgebra over $\mathbb{F}$, for which the underlying $\mathbb{F}$–vector spaces are of finite dimension. The following statements hold:*

(1) *the dual A^* has a natural coalgebra structure;*

(2) *the dual C^* has a natural algebra structure;*

(3) *the canonical morphisms $A \to A^{**}$ and $C \to C^{**}$ are isomorphisms of algebras and coalgebras respectively;*

(4) *the dual space B^* has the structure of a bialgebra and the canonical morphism $B \to B^{**}$ is an isomorphism of bialgebras;*

(5) *the bialgebra B has the structure of a Hopf algebra if and only if the dual B^* has the structure of a Hopf algebra;*

(6) *there are equivalences of categories: ${}_A\mathsf{Mod} \equiv Comod_{A^*}$ and $\mathsf{Mod}_A \equiv_{A^*} Comod$.*

The following result is a straightforward application of vector space duality.

Proposition A.1.2 *Let A be a finite dimensional $\mathbb{F}$–algebra and let C be a finite dimensional $\mathbb{F}$–coalgebra. The following statements hold:*

(1) *for M a left (respectively right) A–module, the dual vector space M^* has a natural left (resp. right) A^*–comodule structure;*

(2) *for N a left (respectively right) C–comodule, the dual vector space N^* has a natural left (resp. right) C^*–module structure.*

The dual M^* of a left module M over an $\mathbb{F}$–algebra A has the structure of a right A–module; under a finiteness hypothesis, there is an analogue for coalgebras.

Proposition A.1.3 *Let C be an $\mathbb{F}$–coalgebra of finite dimension and let N be a left C–comodule, then the dual vector space N^* has the structure of a right C–comodule.*

Proof The right C–comodule structure is adjoint to the left C^*–module structure which is given by the dual $C^* \otimes N^* \to N^*$ to the comodule structure morphism. □

A.2 Corestriction and cotensor products

The following standard result corresponds to the fact that the category of modules over a ring is abelian.

Proposition A.2.1 *The category of right (respectively left) comodules over a $\mathbb{F}$–coalgebra is an abelian category.*

The cotensor product of two comodules over a coalgebra is formally dual to the definition of the tensor product of two modules over an algebra.

Definition A.2.2 Let C be a $\mathbb{F}$–coalgebra and let M be a right C–comodule and let N be a left C–comodule; the cotensor product $M \square_C N$ is the kernel of the morphism $\psi_M \otimes 1 - 1 \otimes \psi_N \colon M \otimes N \to M \otimes C \otimes N$, where ψ_M, ψ_N denote the respective structure morphisms of M, N.

Lemma A.2.3 *For $M \in Comod_C$ a right C–comodule, there is a natural isomorphism of $\mathbb{F}$–vector spaces $M \cong M \square_C C$.*

Proof This fundamental result is a consequence of the counital axiom for the coalgebra C. □

The following result is standard.

Proposition A.2.4 *Let C be an $\mathbb{F}$–coalgebra and let N be a left C–comodule, then the cotensor product $-\square_C N \colon Comod_C \to \mathcal{V}_{\mathbb{F}}$ is a left exact functor.*

Example A.2.5 Let C denote the underlying coalgebra of the $\mathbb{F}$–Hopf algebra $\mathbb{F}[u, u^{-1}]$ and let $\mathbb{F}(n)$ denote the left C–comodule $\mathbb{F}$ with structure morphism $\mathbb{F} \to C \otimes \mathbb{F}$, $1 \mapsto x^n \otimes 1$, for some integer n. The functor $-\square_C \mathbb{F}(n)$ is an exact functor from the category of right C–comodules to the category of vector spaces. This functor corresponds to the projection of a graded vector space onto the component of degree n.

The following result has an obvious generalization to the graded finite-type case, when a suitable connectivity hypothesis is imposed on the graded modules (cf [8, Proposition 3.2], where all graded modules are taken to be connective in a suitable sense).

Lemma A.2.6 *Let C be an $\mathbb{F}$–coalgebra of finite dimension, let M be a right C–comodule and let N be a left C–comodule, such that M, N are of finite dimension. There is an isomorphism of vector spaces $(M \square_C N)^* \cong M^* \otimes_{C^*} N^*$.*

Definition A.2.7 For $f\colon D \to C$ a morphism of $\mathbb{F}$–coalgebras, the corestriction functor $f_*\colon Comod_D \to Comod_C$ is the functor which sends $M \in Comod_D$ to the right C–comodule $M \in Comod_C$, with structure morphism the composite

$$M \to M \otimes D \xrightarrow{1 \otimes f} M \otimes C.$$

The corestriction functor $f_*\colon {}_D Comod \to {}_C Comod$ is defined analogously.

Proposition A.2.8 *Let $f : D \to C$ be a morphism of $\mathbb{F}$–coalgebras, then the corestriction functor $f_*\colon Comod_D \to Comod_C$ admits a right adjoint $f^!\colon Comod_C \to Comod_D$ given by $N \in Comod_C \mapsto N \square_C D$, where the D–comodule structure on $N \square_C D$ is induced by the coproduct $D \to D \otimes D$.*

Proof The adjunction morphisms are induced by the following constructions. For M a right D–comodule, there is a canonical morphism $M \to (f_* M) \square_C D$ which is induced by the structure morphism $M \to M \otimes D$. For N a right C–comodule, the counit of the adjunction is the morphism $N \square_C D \to N \square_C C \cong C$ which is induced by f, where the isomorphism is provided by Lemma A.2.3. □

The corestriction functor admits a left adjoint when the coalgebras satisfy suitable finite-type hypotheses. The ungraded version of the result is the following:

Proposition A.2.9 *Let $f : D \to C$ be a morphism of $\mathbb{F}$–coalgebras, where D, C are of finite dimension. The corestriction functor $f_*\colon Comod_D \to Comod_C$ admits a left adjoint $f^*\colon Comod_C \to Comod_D$ given by $N \mapsto D^* \otimes_{C^*} N$, where N is regarded as a left C^*–module and the right D–comodule structure is adjoint to the extended left D^*–module structure.*

Remark A.2.10 Suppose that the right C–comodule N is of finite dimension, where $f\colon D \to C$ satisfies the hypotheses of the Proposition, then there is an isomorphism of right D–comodules $(D \square_C N^*)^* \cong D^* \otimes_{C^*} N$. This result generalizes to the context of graded comodules if the graded objects N, C, D are all of finite type and the dual algebras D^*, C^* are connective (trivial in sufficiently negative dimensions).

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