Lectures on the stable homotopy of BG

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This paper is a survey of the stable homotopy theory of BG for G a finite group. It is based on a series of lectures given at the Summer School associated with the Topology Conference at the Vietnam National University, Hanoi, August 2004.

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Let G be a finite group. Our goal is to study the stable homotopy of the classifying space BG completed at some prime p. For ease of notation, we shall always assume that any space in question has been p–completed. Our fundamental approach is to decompose the stable type of BG into its various summands. This is useful in addressing many questions in homotopy theory especially when the summands can be identified with simpler or at least better known spaces or spectra. It turns out that the summands of BG appear at various levels related to the subgroup lattice of G. Moreover since we are working at a prime, the modular representation theory of automorphism groups of p–subgroups of G plays a key role. These automorphisms arise from the normalizers of these subgroups exactly as they do in p–local group theory. The end result is that a complete stable decomposition of BG into indecomposable summands can be described (Theorem 6) and its stable homotopy type can be characterized algebraically (Theorem 7) in terms of simple modules of automorphism groups.

This paper is a slightly expanded version of lectures given at the International School of the Hanoi Conference on Algebraic Topology, August 2004. The author is extremely grateful to Mike Hill for taking notes and producing a TeX document which formed the basis of the present work. Additional comments and references have been added to make the exposition reasonably self-contained. Many of the results of this paper were obtained jointly with my longtime collaborator, John Martino.

1 Preliminaries

1.1 What do we mean by "stable homotopy"

Given a pointed space $(X,*)$, let $\Sigma X = X \times I / X \times \{0,1\} \cup \{*\} \times I$ denoted the reduced suspension. We can represent this pictorially as:

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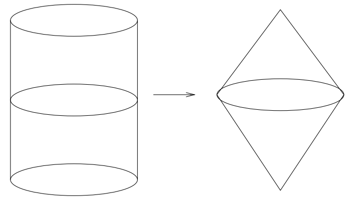

One can also quickly check that $\Sigma S^n = S^{n+1}$, where S^n is the n–sphere.

Now, $\widetilde{H}_*(\Sigma X) = \widetilde{H}_{*-1}(X)$, and for any space Y, $[\Sigma X, Y]$ is a group. The composition is as defined for homotopy groups: if we have two homotopy classes $[f], [g]$, then we let $[f] \cdot [g]$ denote the composite $[\nabla \circ (f, g) \circ \pi']$, where $\pi' \colon \Sigma X \to \Sigma X \vee \Sigma X$ is the "pinch" map defined by

$$[(x,t)] \mapsto \begin{cases} \big([(x,2t)], *\big) & 0 < t < 1/2 \\ \big(*, [(x, 2t-1)]\big) & 1/2 \leq t \leq 1 \end{cases}$$

and $\nabla \colon \Sigma X \vee \Sigma X \to \Sigma X$ is the "fold" map. The fact that we can add maps gives the theory a very different, more algebraic flavor, than that of unstable homotopy theory. Moreover, the same proof as for ordinary homotopy groups shows that $[\Sigma^2 X, Y]$ is an abelian group.

For us, "stable" just means that we can suspend any number of times, even an infinite number, as needed. More precisely let $QY = \operatorname{colim} \Omega^n \Sigma^n Y$. Then we define the stable homotopy classes of maps $\{X, Y\} = [X, QY]$. If X is a finite complex $\{X, Y\} = \operatorname{colim}[\Sigma^n X, \Sigma^n Y]$. For a discussion of spectra, see Adams [1].

1.2 Classifying spaces

Let G be a finite group. Define EG to be a free, contractible G–space, and let BG denote the quotient EG/G. The contractibility of EG shows us that BG is a space with a single nontrivial homotopy group: $\pi_1(BG) = G$. We give two explicit constructions of EG and then give an application.

1.2.1 Milnor's definition We define

$$EG = \bigcup_n \underbrace{G * \cdots * G}_{n},$$

where $*$ denotes the join of two spaces, which we take to be the suspension of the smash product. Since the join includes a suspension, the greater the number of copies of G being joined, the higher the connectivity, and so EG is contractible.

As an example, we take $G = \mathbb{Z}/2$. In this case,

$$E\mathbb{Z}/2 = \bigcup_n \underbrace{\mathbb{Z}/2 * \cdots * \mathbb{Z}/2}_n = \bigcup_n S^{n-1} = S^\infty,$$

and the $\mathbb{Z}/2$ action is the usual diagonal reflection action, and in this case, the quotient $B\mathbb{Z}/2$ is just $\mathbb{R}P^\infty$.

1.2.2 Simplicial model We can think of a group as a category with a single object and whose morphisms are the elements of the group. We can now pull in the categorical construction of the nerve, and this will give us a model for BG.

First we recall briefly the definition of a simplicial set. A more complete reference is May [12]. Let Δ be the category whose objects are the sets $0, \ldots, n$ for all n and whose morphisms are nondecreasing maps. A *simplicial set* is a contravariant functor from Δ to the category of sets. We can think of a simplicial set as a collection of sets indexed by the natural numbers together with a large family of structure maps called faces and degeneracies which satisfy certain properties, modeled dually on the inclusion of faces in the standard simplices in $\mathbb{R}^n$.

To any simplicial set S, we can associate a topological space, the geometric realization, $|S|$. Loosely speaking, this is defined by putting a copy of the standard n–simplex in for every element of S_n and gluing them all together via the face and degeneracy maps.

To any category $\mathcal{C}$, we can associate a simplicial set, the *nerve*, $N\mathcal{C}_*$. The k–simplices of $N\mathcal{C}$ are the k–tuples of composable morphisms in $\mathcal{C}$. The face maps are induced by the various ways to compose adjacent maps (or to forget the ends), and the degeneracies comes from inserting the identity map in various places. We define the classifying space $B\mathcal{C}$ to be the geometric realization of the nerve. With a little work, one can quickly show that this construction is functorial.

In the case $\mathcal{C} = G$, a finite group, $N\mathcal{C}_k = G^k$, since all morphisms are composable. We then get a model of BG by taking the geometric realization.

This construction has some very nice advantages over the previous one, and to show this, we need a small proposition.

Proposition 1 *If $F_0, F_1\colon \mathcal{C} \to \mathcal{C}'$, and H is a natural transformation from F_0 to F_1, then $BF_0 \simeq BF_1$ as maps $B\mathcal{C} \to B\mathcal{C}'$, and the homotopy is given by BH on $B(\mathcal{C} \times \{0 \to 1\}) = B\mathcal{C} \times [0,1]$.*

This immediately gives us an important result about conjugation.

Corollary 1 *Let $x \in G$, and let $C_x(g) = x^{-1}gx$ denote conjugation by x. We then have $BC_x \simeq \mathrm{Id}_{BG}$.*

Proof There is a natural transformation between C_x, viewed as an endofunctor of G, and the identity functor given by "multiplication by x":

$$\begin{tikzcd} e \arrow[r, "C_x(g)"] \arrow[d, "x"'] & e \arrow[d, "x"] \\ e \arrow[r, "g"'] & e \end{tikzcd}$$

where e denotes the single object in the category. In other words, the morphism $x\colon e \to e$ is a natural transformation between C_x and the identity, and the result follows. □

It is this simple corollary which gives us the basic connection between group theory and the homotopy theory of classifying spaces.

1.3 Group cohomology

The space EG allows us to define group homology and cohomology. The singular chains $C_*(EG)$ is a $\mathbb{Z}[G]-$free resolution of $\mathbb{Z}$.

Definition For any $G-$module M, let $H_*(G;M) = H(C_*(EG) \otimes_{\mathbb{Z}[G]} M)$, and let $H^*(G;M) = H(\mathrm{Hom}_{\mathbb{Z}[G]}(C_*(EG), M)$.

Note in particular that if $M = \mathbb{Z}$, the trivial $G-$module, then $H_*(G;M) = H_*(BG)$ and similarly for cohomology. If M is not trivial, then $H_*(G;M)$ can be similarly related to $H_*(BG)$ but with twisted coefficients.

In what follows all cohomology is taken with simple coefficients in $\mathbb{F}_p$.

2 Stable splittings

Suppose that $\Sigma BG = X_1 \vee \cdots \vee X_N$. If we can do this, then for any generalized cohomology theory E,

$$E^*(BG) = E^{*+1}(\Sigma BG) = \bigoplus E^{*+1} X_i.$$

In general, this is a simpler object to study. We want now to find ways to relate the X_i to G itself.

2.1 Summands via idempotent self-maps

Let $e\colon X \to X$ for some pointed space X. If $e^2 \simeq e$, we call e a *homotopy idempotent*. We now form the mapping telescope $eX = \mathrm{Tel}(X, e)$ which is the homotopy colimit of the diagram $X \xrightarrow{e} X$. More explicitly, we start with the disjoint union

$$\coprod_{n \geq 0} (X \times [2n, 2n+1])$$

and identify $(x, 2n+1)$ with $(e(x), 2n+2)$ and pinch $(*, t)$ to a point. In this case,

$$\pi_*(eX) = \operatorname{colim} \pi_*(X) = e_*\pi_*(X),$$

where the structure maps in the limit are e_*. A similar statement holds for homology.

If X is a suspension, then we can add and subtract maps, and in particular, we can form a map $X \to eX \vee (1-e)X$ whenever e is a homotopy idempotent. From the above comments, this is an equivalence. Our next task is then to find idempotents in $[X, X]$. In the case of $X = BG$ or ΣBG, we shall get the first layer of these from algebra, using the $\mathrm{Aut}(G)$ action on $[BG, BG]$.

Rather than looking at homotopy classes of maps, we'll look at stable homotopy classes of self maps $\{BG, BG\}$. Under composition, this has the structure of a ring. The group of stable homotopy self-maps carries an action of $\mathrm{Aut}(G)$ via the map which sends $\alpha \in \mathrm{Aut}(G)$ to the stable class of $B\alpha$. This therefore extends to a map of rings from $\mathbb{Z}[\mathrm{Aut}(G)] \to \{BG, BG\}$. If we can find idempotents in $\mathbb{Z}[\mathrm{Aut}(G)]$, then we can push them forward to stable homotopy idempotents. Since we are interested in working one prime at a time and since idempotent theory is easier for completed rings, we shall assume BG is completed at p and consider the induced map $\mathbb{Z}_p[\mathrm{Aut}(G)] \to \{BG, BG\}$. If G is a p–group then BG is already p–complete.

We start by reducing mod p, since any idempotent $e \in \mathbb{F}_p[\mathrm{Aut}(G)]$ lifts to an idempotent in $\mathbb{Z}_p[\mathrm{Aut}(G)]$. Moreover, if we have a primitive orthogonal idempotent decomposition $1 = e_1 + \cdots + e_n$ in $\mathbb{F}_p[\mathrm{Aut}(G)]$, where $e_ie_j = 0$ for $i \neq j$, $e_i^2 = e_i$ then this lifts to a decomposition of the same form in $\mathbb{Z}_p[\mathrm{Aut}(G)]$.

Example 1 If $G = \mathbb{Z}/p$, then $\mathrm{Aut}(G) = \mathbb{Z}/(p-1)$. If $p = 3$, then we can readily find two idempotents in $\mathbb{F}_3[\mathbb{Z}/2]$, namely $-1-e$ and $-1+e$, where e is the nontrivial element in $\mathbb{Z}/2$.

In general, there are $p-1$ primitives with idempotents given by

$$e_i = \prod_{j \neq i} \frac{\xi - a^i}{a^i - a^j}, \quad i = 0, \ldots, p-2$$

where ξ is the generator of $\mathbb{Z}/p-1$ and a is the element in $\mathbb{F}_p = \mathbb{Z}/p$ by which ξ acts.

Proposition 2 *Stably and p–completed,*

$$B\mathbb{Z}/p \simeq X_0 \vee \cdots \vee X_{p-2},$$

where $X_i = e_i B\mathbb{Z}/p$.

We will say more about X_0 below. One can also try to use the full ring of endomorphisms for an abelian p–group P. This approach has been thoroughly studied by Harris and Kuhn [5].

3 Transfer maps

Let $H \subset G$ be a subgroup of index $[G:H] = n$. If we take the quotient of EG by H, then we get BH, since EG is contractible and being G free forces it to be H free. We can further quotient by all of G to get a map $BH \to BG$ and the fiber of this map is G/H. In other words, we have an n–sheeted cover $BH \to BG$. The map $BH \to BG$ is also easily seen to be equivalent to B of the inclusion $H \to G$.

The transfer is a stable map which goes from BG back to BH. In cohomology, we can easily define it. Let $\pi\colon X \to Y$ be an n–sheeted cover. For each small enough simplex $\Delta \in C_*(Y)$, we can find n simplices in $C_*(X)$ lying over it. The transfer is the map in homology induced by

$$\Delta \mapsto \sum_{\Delta' \in \pi^{-1}(\Delta)} \Delta'.$$

If we compose now with the projection map, then it is clear that the composite is simply multiplication by n.

Actually getting a stable map requires a little more work. Write $G = \coprod \tau_i H$. Given an element τ_i, left multiplication by $g \in G$ sends it to $\tau_{\sigma(g)(i)} h_{i,g}$. This gives us a permutation representation $\sigma\colon G \to \Sigma_n$ and a homomorphism

$$\begin{aligned} G &\to H^n \rtimes \Sigma_n = \Sigma_n \wr H \\ g &\mapsto \big(h_{1,g}, \dots, h_{n,g}, \sigma(g)\big). \end{aligned}$$

We define the *transfer* to be the map adjoint to the composite

$$\begin{tikzcd} BG \arrow[r] \arrow[drrr] & B(\Sigma_n \wr H) \arrow[r, "="] & BH^n \times_{\Sigma_n} B\Sigma_n \arrow[r] & (QBH)^n \times_{\Sigma_n} B\Sigma_n \arrow[d, "\Theta"] \\ & & & QBH \end{tikzcd} ,$$

where Θ is the Dyer–Lashof map arising from the infinite loop structure of QBH. It is not difficult to see that in homology this map agrees with the previous definition for the covering $BH \to BG$. Actually the map we have defined is sometimes referred to as the reduced transfer. Let BG^+ denote BG with an added disjoint basepoint so that $BG^+ \simeq BG \vee S^0$. Then it is easy to extend this definition to a stable map tr: $BG^+ \to BH^+$ which is multiplication by $[G, H]$ on the bottom cell. For a detailed exposition see Kahn and Priddy [6]; another approach is given by Adams [1].

3.1 Properties of the transfer and corollaries

We have already seen homologically that the composite $BG \xrightarrow{\mathrm{tr}} BH \xrightarrow{Bi} BG$ is multiplication by the index $[G:H]$.

Corollary 2 *If $H \subset G$, and $[G:H]$ is prime to p, then BG is a stable summand of BH when completed at p.*

Proof Since $[G:H]$ is prime to p, it is a unit in $\mathbb{Z}_p$, and multiplication by it is an equivalence. Thus the transfer and inclusion give the splitting. □

Corollary 3 *Stably and completed at p, $B\Sigma_p$ is a stable summand of $B\mathbb{Z}/p$.*

Proposition 3 (Properties of the transfer) *We will write tr_H^G for the transfer with H considered as a subgroup of G.*

(1) *If $H = G$, then $\mathrm{tr}_H^G = \mathrm{Id}$.*

(2) *If $K \subset H \subset G$, then $\mathrm{tr}_K^G = \mathrm{tr}_H^G \circ \mathrm{tr}_K^H$.*

(3) *The transfer is natural with respect to maps of coverings.*

(4) *The "Double Coset Formula" holds: If $H, K \subset G$, write $G = \coprod KxH$ for some collection of $x \in G$. Let tr_x denote the transfer $BK \to B(K \cap x^{-1}Hx)$, and let i_x denote the inclusion $xKx^{-1} \cap H \to H$. Then if i_K is the map $BK \to BG$, we have*
$$\mathrm{tr}_H^G \circ i_K = \sum_x i_x \circ C_{x^{-1}} \circ \mathrm{tr}_x .$$

Lemma 1 *If G is an elementary abelian p–group, and $H \subsetneq G$, then the transfer induces the zero map in mod p cohomology.*

Proof The map i_H^* is surjective, since H sits inside G as a summand. Since
$$\mathrm{tr}_H^{G\,*} i_H^*(x) = [G:H]x = p^n x = 0,$$
we conclude that $\mathrm{tr}^* = 0$. □

Corollary 4 *If $V \subset G$ is elementary abelian, then*

$$i_V^* \circ \mathrm{tr}_V^{G*} = \sum_{w \in N(V)/V} C_w^*.$$

Proof Let $K=V$ in the double coset formula and vary x over coset representatives. □

It follows from Corollary 2 that if G_p is a Sylow p–subgroup, then BG is a stable summand of BG_p after p–completion. We now specialize to the case that $G_p = V$ is elementary abelian.

Theorem 1 *If $V \subset G$ is an elementary abelian Sylow p–subgroup, then*

(1) $H^*(G) \cong H^*(V)^W$, *where $W = N_G(V)/V$ is the Weyl group.*

(2) $BN_G(V) \to BG$ *is an $H\mathbb{Z}/p$–equivalence, even unstably.*

Proof The second result follows immediately from the first.

For the first part, note that we always have $H^*(G) \subset H^*(V)$. Since conjugation acts as the identity on $H^*(G)$, we must have

$$H^*(G) \subset H^*(V)^{N_G(V)}.$$

Since conjugation by V is trivial in cohomology, $H^*(V)^{N_G(V)} = H^*(V)^W$.

From Corollary 4, we know that the composite

$$H^*(V) \xrightarrow{\mathrm{tr}^*} H^*(G) \xrightarrow{i^*} H^*(V)^W$$

is just $\sum_{w \in W} C_w$. Since $|W|$ is prime to p, it is invertible in $\mathbb{Z}_p$, and $e = \sum C_w/|W|$ is an idempotent invariant under the action of W. Conversely, all invariants arise in this way, since on the subalgebra of W–invariants, the composite is just multiplication by $|W|$ and is therefore invertible. This in particular shows that i^* is surjective, and the result follows. □

Example 2 For $G = \Sigma_p$, the Sylow p–subgroups are $\mathbb{Z}/p$, and $N_{\Sigma_p}(C_p)/C_p = \mathbb{Z}/(p-1)$. Now $H^*(\Sigma_p) = H^*(\mathbb{Z}/p)^W$. The group W acts on the cohomology $H^*(\mathbb{Z}/p) = E(x_1) \otimes \mathbb{F}_p[y_2]$ as multiplication by a generator of $\mathbb{F}_p^\times$ on x_1 and $y_2 = \beta(x_1)$. The fixed point algebra is then generated by $x_1 y_2^{p-2}$ and y_2^{p-1} as an unstable algebra over the Steenrod algebra. This shows that

$$H^*(\Sigma_p; \mathbb{F}_p) = \begin{cases} \mathbb{Z}/p & * = 0, -1 \mod p \\ 0 & \text{otherwise.} \end{cases}$$

These dimensions explain why the Steenrod operations occur where they do, just as a similar computation for the map $B(\mathbb{Z}/p \times \mathbb{Z}/p) \to B(\mathbb{Z}/p \wr \mathbb{Z}/p) \to B\Sigma_{p^2}$ yields the Adem relations.

With more work one can show a generalization of Theorem 1.

Theorem 2 (Harris–Kuhn [5]) *If $P \subset G$ is a Sylow p–subgroup, and P is an abelian p–group, then*

(1) $H^*(G) \cong H^*(P)^W = H^*(N_G(P))$.

(2) $BN_G(P) \to BG$ *is an $H\mathbb{F}_p$–equivalence.*

3.2 Modular representation theory

If p divides the order of the automorphism group, then the representation theory of $\mathrm{Aut}(G)$ over $\mathbb{F}_p$ lies in the realm of modular representation theory and hence becomes more complicated. We demonstrate this with some basic examples of increasing trickiness.

Example 3 Let $G = V_2 = \mathbb{Z}/2 \times \mathbb{Z}/2$, and take $p = 2$. Now

$$\mathrm{Aut}(V_2) = GL_2(\mathbb{F}_2) = \Sigma_3$$

has order divisible by 2, and we can find simple generators

$$\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \tau = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In $R = \mathbb{F}_2[\mathrm{Aut}(V_2)]$ the element

$$f_1 = 1 + \sigma + \sigma^3$$

is an idempotent, since $f_1^2 = 3f_1$. Since $\tau\sigma\tau = \sigma^2$, f_1 and $f_2 = 1 - f_1$ are central idempotents. With a small bit of work, one can show the following.

Proposition 4 *$R \cong Rf_1 \times Rf_2$, and $Rf_1 = E(\gamma)$, where $\gamma = \sum_{g \in GL} g$ and $Rf_2 = M_2(\mathbb{F}_2)$.*

We can lift f_1 and f_2 to idempotents e_1 and e_2 in $\{BV_2, BV_2\}$, so we conclude that

$$BV_2 = e_1 BV_2 \vee e_2 BV_2.$$

The first summand we can identify, as it is clearly the same as B of the semi-direct product

$$(\mathbb{Z}/2 \times \mathbb{Z}/2) \rtimes \mathbb{Z}/3 = A_4.$$

Now f_2 can be written as the sum of two idempotents F_1 and F_2, where

$$F_1 = (1 + \tau\sigma)(1 + \tau) \quad \text{and} \quad F_2 = (1 + \tau)(1 + \tau\sigma).$$

These can also be lifted to idempotents in $\mathbb{Z}_2[\mathrm{Aut}(V)]$, so BV_2 splits further as

$$BV_2 = BA_4 \vee \widetilde{e}_1 BV_2 \vee \widetilde{e}_2 BV_2. \tag{1}$$

Finally, $\widetilde{e}_1 BV_2 \simeq \widetilde{e}_2 BV_2$, since we have a sequence

$$F_1 R \xrightarrow{1+\tau} F_2 R \xrightarrow{1+\tau\sigma} F_1 R \xrightarrow{1+\tau} F_2 R$$

in which the composite of any two successive arrows is the identity. It is known that

$$\widetilde{e}_1 BV_2 = L(2) \vee B\mathbb{Z}/2,$$

where $L(2)$ is a spectrum that is related to Steenrod operations of length two. We therefore have the following result of Mitchell [13]:

$$BV_2 = BA_4 \vee (L(2) \vee \mathbb{R}P^\infty) \vee (L(2) \vee \mathbb{R}P^\infty),$$

in which the summands are indecomposable.

Example 4 Let $G = D_8$, the dihedral group of order 8. One can show that $\mathrm{Aut}(D_8) = D_8$, so this is a two group.

Lemma 2 *If G is a p–group, then $\mathbb{F}_p[G]$ has only one simple module, the trivial one.*

The lemma follows from the fact that the augmentation ideal is nilpotent in this case.

In the case $G = D_8$, the lemma shows that we have only one idempotent in $\mathbb{F}_2[\mathrm{Aut}(G)]$, namely the element 1. Nevertheless, we have the following splitting [13]:

$$BD_8 = BPSL_2(\mathbb{F}_7) \vee (L(2) \vee \mathbb{R}P^\infty) \vee (L(2) \vee \mathbb{R}P^\infty).$$

Using ring theory, we can get a more direct relationship between the structure of $R = \mathbb{F}_p[\mathrm{Aut}(G)]$ and idempotents. Let J be the Jacobson radical of R, namely the elements annihilating all simple R–modules or equivalently the intersection of all maximal ideals. Ring theory tells us that R/J is semisimple and therefore splits as a product of matrix rings over division algebras:

$$R/J = M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k).$$

The simple R–modules are just the columns of the various matrix rings. Lifting the idempotents from this decomposition, we obtain in R a primitive orthogonal decomposition

$$1 = \sum_j e_j = n_1 e_1 + \cdots + n_k e_k.$$

For the earlier Example 3 of V_2, the Jacobson radical is the ideal generated by the element γ, and so

$$\mathbb{F}_2[\mathrm{Aut}(V_2)]/J = \mathbb{F}_2 \times M_2(\mathbb{F}_2),$$

giving us the correct number of factors for the decomposition of BV_2 in (1). To obtain the complete decomposition of (1) one must use the full endomorphism group $\mathrm{End}(V_2)$.

Example 5 Next, we look at a more complicated example, $V_3 = (\mathbb{Z}/2)^3$. In this case, $R = \mathbb{F}_2[GL_3(\mathbb{F}_2)]$ has 4 simple modules.

Module	$\mathbb{F}_2$	V_3	V_3^*	St_3
Dimension	1	3	3	8

where ${V_3}^*$ is the contragredient module and St_3 is the Steinberg module (described explicitly in Example 6 below). This means that we have a splitting

$$BV_3 = (\widetilde{e}_1 BV_3) \vee 3(\widetilde{e}_2 BV_3) \vee 3(\widetilde{e}_3 BV_3) \vee 8(\widetilde{e}_4 BV_3).$$

As before, we can identify the first summand. In this case, it is the same as B of the group $\mathbb{F}_8^\times \rtimes \mathrm{Gal}(\mathbb{F}_8 / \mathbb{F}_2)$ obtained by taking the semidirect product of $V_3 = \mathbb{F}_8$ with $\mathbb{Z}/7 \rtimes \mathbb{Z}/3 = \mathrm{Gal}(\mathbb{F}_8 / \mathbb{F}_2)$.

Generalizing the example of the dihedral group D_8 (Example 4), we consider $P = D_{2^n}$, the dihedral group of order 2^n. We have the following splitting theorem:

Theorem 3 (Mitchell–Priddy [13]) *$BD_{2^n} = BPSL_2(\mathbb{F}_q) \vee 2L(2) \vee 2B\mathbb{Z}/2$, where $q = p^k$ for p odd and D_{2^n} is a Sylow 2–subgroup of $PSL_2(\mathbb{F}_q)$.*

This condition translates to saying that $n = \nu_2(\frac{q^2-1}{2})$ is the order of 2 in $(q^2-1)/2$.

We can also explain the existence of the summands $2L(2) \vee 2B\mathbb{Z}/2$. There are two nonconjugate copies of $\mathbb{Z}/2 \times \mathbb{Z}/2$ sitting inside D_{2^n}, and the summands in question appear via transfers from BD_{2^n} to B of these subgroups, followed by projection onto the summands.

4 The Segal Conjecture and its consequences

While studying the idempotents in the ring $\{BG, BG\}$ via $\mathrm{Aut}(G)$ provides a good bit of information about the splittings of BG, if $G = P$ is a p–group then the Segal conjecture completely determines the ring $\{BG, BG\}$, so it is to this that we turn.

Let $A(P, P)$ be the Grothendieck ring of $P \times P$ sets which are free on the right. The sum is given by disjoint union, and the product is the product over P. If $P_0 \subset P$ is a subgroup, and $\rho\colon P_0 \to P$ is a homomorphism, then we can define elements of $A(P, P)$ by

$$P \times_\rho P = P \times P/\sim$$

where $(xp_0, y) \sim (x, \rho(p_0)y)$, $p_0 \in P_0$. As a group, $A(P, P)$ is a free abelian group on these transitive sets. There is a homomorphism

$$\alpha\colon A(P, P) \to \{BP, BP\}$$

defined by $$\alpha(P \times_\rho P) = (BP_+ \xrightarrow{\mathrm{tr}_{P_0}^{P}} BP_{0+} \xrightarrow{B\rho} BP_+).$$

Upon completion this map is essentially an isomorphism. More precisely $A(P, P)$ contains an ideal $\widetilde{A}(P, P)$ which is a free abelian group on the classes $P \times_\rho P - (P/P_0 \times P)$ and we have the following theorem.

Theorem 4 (Carlsson [3]; Lewis–May–McClure [7]) *The map α induces a ring isomorphism*

$$\widetilde{\alpha}\colon \widetilde{A}(P, P) \otimes \mathbb{Z}_p \to \{BP, BP\}.$$

Corollary 5 (1) *$\{BP, BP\}$ is a finitely generated, free $\mathbb{Z}_p$–module.*

(2) *BP splits as a finite wedge $X_1 \vee \dots X_n$, where the X_i are indecomposable p–complete spectra, unique up to order and equivalence.*

Proof The first is immediate. For the second, after tensoring with $\mathbb{F}_p$ we have a finite dimensional $\mathbb{F}_p$–algebra. This means that $1 = \sum e_i$ is a decomposition into primitive idempotents unique up to order and conjugation. □

Corollary 6 *Given a finite p–group P, there exist finitely many stable homotopy types of BG with P a Sylow p–subgroup of G.*

Proof We have already seen that if P is a Sylow p–subgroup of G, then we have a splitting $BP \simeq BG \vee \mathrm{Rest}$. The finiteness result of the Corollary 5 gives this one. □

Corollary 7 *Each summand of BP is also an infinite complex.*

Proof By the first part of Corollary 1, we know that $\{BP, BP\}$ is torsion free. If X were both a finite complex and a summand of BP, then $BP \to X \to BP$, the projection followed by the inclusion, would be a torsion free map. However, if X is a finite complex, then the identity map of X has torsion, since X is p–complete, so the torsion free composite must as well. □

Theorem 5 (Nishida [14]) *Given G, G' finite groups with $BG \simeq BG'$ stably at p, then the Sylow p–subgroups of G and G' are isomorphic.*

We shall derive this from a more general result in Section 6.

4.1 Analysis of indecomposable summands

We will now assume that X is an indecomposable summand of BP for P a fixed p–group.

Definition X *originates* in BP if it does not occur as a summand of BQ for any $Q \subsetneq P$. X is a *dominant summand* of BP if it originates in BP.

The notion of dominant summand is due to Nishida. As an example, for BV_2 the dominant summands are BA_4 and $L(2)$.

It is also clear that every X must originate in some subgroup Q of P.

Now let $J(P)$ be the ideal in $\{BP, BP\}$ generated by the maps which factor through BQ for some p-group Q such that $|Q| < |P|$. In other words, these are the maps which arise from transitive sets for which P_0 is a proper subgroup or if $P_0 = P$ from proper (ie nonsurjective) endomorphisms $\rho\colon P_0 \to P$. Since every summand $X = eBP$ for some idempotent $e \in \{BP, BP\}$, X is dominant if and only if $e \notin J(P)$. Furthermore we have an isomorphism of rings

$$\mathbb{Z}_p[\mathrm{Out}(P)] \xrightarrow{i} \{BP, BP\} \xrightarrow{\pi} \{BP, BP\}/J(P) = \mathbb{Z}_p[\mathrm{Out}(P)].$$

This follows by remembering that $\{BP, BP\}$ is generated by transfers followed by homomorphisms. If the subgroup for the transfer is proper, or if the homomorphism is not surjective then this map is in $J(P)$, so all that we have left over are the automorphisms of P. This gives us the following equivalence:

$$\{\text{Homotopy types of dominant summands of } BP\} \to \{\text{Isomorphism classes of simple } \mathbb{F}_p[\mathrm{Out}(P)]\text{–modules}\}$$

given by $eBP \mapsto e_0\,\mathbb{F}_p[\mathrm{Out}(Q)]$, where e_0 is determined as follows: since e is primitive we can find a primitive idempotent $\widetilde{e} \in \mathbb{Z}_p[\mathrm{Out}(P)]$ such that $i(\widetilde{e}) = e \mod J$. Then e_0 is the mod p reduction of $\widetilde{e}$.

Let S be a simple $\mathbb{F}_p[\mathrm{Out}(Q)]$ module and let $X_{Q,S}$ be its corresponding dominant summand of BQ.

Theorem 6 (Mitchell–Priddy [9]; Benson–Feshbach [2]) *There is a complete stable decomposition unique up to order and equivalence of factors*

$$BP = \bigvee_{Q,S} n_{Q,S} X_{Q,S}$$

where Q runs over the subgroups of P, S runs over the simple $\mathbb{F}_p[\mathrm{Out}(Q)]$ modules, and $n_{Q,S}$ is the multiplicity of $X_{Q,S}$ in BP.

4.2 Principal dominant summand

Among all dominant summands, there is a distinguished one corresponding to the trivial module. If we decompose $1 \in \mathbb{F}_p[\mathrm{Aut}(P)]$ into primitive orthogonal idempotents, then we can consider the image of them under the augmentation ring map $\epsilon\colon \mathbb{F}_p[\mathrm{Aut}(P)] \to F_p$ defined by sending all $g \in \mathrm{Aut}(P)$ to $1 \in \mathbb{F}_p$. Since $e_i^2 = e_i$, these must map to either 0 or 1 under the augmentation map. Additionally, exactly one must map to 1, since the augmentation sends 1 to 1, and $e_i e_j = 0$ for $i \neq j$. We denote by e_0 the idempotent that maps to 1 and say that it is the *principal* idempotent. The corresponding summand will be denoted X_0 and called the *principal dominant summand*.

Proposition 5 (Nishida [14]) *X_0 is a summand of BG for all G with P a Sylow p–subgroup.*

For two examples, for $B\mathbb{Z}/p$, the principal dominant summand is $B\Sigma_p$, and for BV_2, it is BA_4.

4.3 Ring of universally stable elements

For a fixed p–group P, we define the ring of universally stable elements as

$$I(P) = \bigcap_{\substack{G \supset P \text{as} \\ \text{a Sylow} \\ p\text{–subgroup}}} \mathrm{Im}\big(H^*(G) \to H^*(P)\big).$$

Theorem 7 (Evens–Priddy [4]) *$H^*(P)$ is a finite module over $I(P)$. This implies that $H^*(P)$ is a finitely generated algebra over $I(P)$ of the same Krull dimension, which in turn implies $I(P)$ is a finitely generated $\mathbb{F}_p$–algebra.*

This follows from Quillen's theorem [16].

Proposition 6 [4] *If E is an elementary abelian p–group of rank n, then $I(E) = H^*(E)^{GL(E)}$, except when $p = n = 2$, where it is $H^*(BA_4)$.*

In general, $I(P)$ is not realizable as the cohomology of a spectrum. In certain familiar cases, however, it is not only realizable but also connected to the cohomology of the principal dominant summand.

Proposition 7 [4] *We have the following rings of universally stable elements.*

$$\begin{array}{ll} I(D_{2^n}) = H^*(PSL_2(\mathbb{F}_q)) & n = \nu_2\left(\frac{q^2-1}{2}\right) \\ I(Q_{2^{n+1}}) = H^*(SL_2(\mathbb{F}_q)) & n = \nu_2\left(\frac{q^2-1}{2}\right) \\ I(SD_{2^n}) = H^*(SL_3(\mathbb{F}_q)) & n = \nu_2((q^2-1)(q+1)),\ \ q = 3 \mod 4 \end{array}$$

For all of these groups, X_0 is B of the group shown.

5 Summands of supergroups

Recall that if X is an indecomposable summand of BQ, then we say it is dominant if it is not a summand of BQ' for any subgroup $Q' \subsetneq Q$. If P is a p–group, and $Q \subset P$ is a subgroup with X a dominant summand of BQ, we can ask when X is a summand of BP.

Example 6 We recall the definition of the Steinberg module. Define $e_{\mathrm{St}_n} \in R = \mathbb{F}_p[GL_n(\mathbb{F}_p)]$ by

$$e_{\mathrm{St}_n} = \frac{1}{[GL_n(\mathbb{F}_p) : U_n]}\left(\sum_{b \in B_n, \sigma \in \Sigma_n} (-1)^\sigma b\sigma\right),$$

where B_n is the Borel subgroup of upper triangular matrices and U_n is the unipotent subgroup thereof. This element is idempotent and primitive, and if we let $\mathrm{St}_n = e_{\mathrm{St}_n} R$, then St_n is a simple, projective module of dimension $p^{\binom{n}{2}}$.

If V is an elementary abelian group of dimension n, then

$$e_{\mathrm{St}_n} BV = L(n) \vee L(n-1),$$

where $L(n) = \Sigma^{-n}\big(Sp^{p^n}(S^0)/Sp^{p^{n-1}}(S^0)\big)$ and $Sp^n(X) = X^n/\Sigma_n$ is the symmetric product [13].

Proposition 8 [15] *If P is a p–subgroup of rank n (ie n is largest number such that $(\mathbb{Z}/p)^n$ is a subgroup of P), then $L(n)$ is a summand of BP if and only if P contains a self-centralizing elementary abelian subgroup of rank n.*

Example 7 For $p = 2$ and low values of n, we have the following examples.

$n = 2$ $L(2)$ is a summand of BD_{2^n}.

$n = 1$ $L(1)$ is actually $\mathbb{R}P^\infty$, and we have seen already instances when this occurs as a summand.

$n = 1$ As a "non-example", $L(1)$ is not a summand of BQ_8, since the $\mathbb{Z}/2 \subset Q_8$ is central.

Now recall the principal summand X_0. Let $PS(X_0, t)$ be its Poincaré series in mod–p cohomology. By the dimension of X_0 we mean the order of the pole of $PS(X_0, t)$ at $t = 0$.

Proposition 9 (Martino–Priddy [10]) *X_0 has dimension n in BP if the rank of P is n.*

5.1 Generalized Dickson invariants

For the Dickson invariants, we normally start with the algebra $H^*(E)^{GL(E)}$. If E is an elementary abelian p–group of rank n, then we can form the composite

$$E \xrightarrow{\text{Reg}} \Sigma_E \to U(p^n),$$

where Reg is the regular representation of E acting on itself. In cohomology, the Chern classes $c_{p^n - p^i}$ map to the Dickson invariants c_i. These Dickson invariants carve out a polynomial invariant subalgebra of $H^*(BE)^{GL(E)}$ of dimension n.

For a general p–group P, we can formally mirror the above construction. Let ρ denote the composite complex representation

$$\rho\colon P \xrightarrow{\text{Reg}} \Sigma_{|P|} \to U(|P|),$$

and define the (generalized) Dickson invariants of P to be the image under ρ^* of

$$\mathbb{F}_p[c_{p^s(p^n - p^i)} : i = 0, \dots, n-1\,;\ s = |P|/p^n] \subset H^*(U(|P|)$$

where $s = |P|/p^n$. This obviously forms a subalgebra of $H^*(BP)$ which we will denote $D(P)$.

Proposition 10 *If P is a p–group of rank n, then*

(1) *$D(P)$ is a polynomial ring of dimension n.*

(2) *$D(P) \subset H^*(BP)^{\mathrm{Out}(P)}$.*

Proof This is easy to see. For the first part, we use the fact that the composite of the inclusion of E, an elementary abelian subgroup of rank n, into P followed by the regular representation map to $\Sigma_{|P|}$ is the same as p^s times the regular representation map of E. By the naturality of cohomology, the classes $c_{p^s(p^n-p^i)}$ pull back to the classes $(c_{p^n-p^i})^{p^s}$.

For the second part, given an automorphism f of P, we can form a commutative square

$$\begin{tikzcd} P \arrow[r] \arrow[d, "f"'] & \Sigma_P \arrow[d, "C_f"] \\ P \arrow[r] & \Sigma_P \end{tikzcd}$$

As before conjugation induces the identity in the cohomology of the classifying space; this shows that the pullback of any classes coming from $H^*(\Sigma_P)$ lies in the invariants of $H^*(P)$ under $\mathrm{Out}(P)$. □

6 Stable classifications of BG at p

The following result gives a classification of the stable type of BG in terms of its p–subgroups Q and associated $\mathrm{Out}(Q)$ modules. Let $\mathrm{Rep}(Q,G) = \mathrm{Hom}(Q,G)/G$ and $\mathrm{Inj}(Q,G) \subset \mathrm{Rep}(Q,G)$ be the classes of injections. Let $\mathrm{Cen}(Q,G) \subset \mathrm{Inj}(Q,G)$ be represented by monomorphisms $\alpha\colon Q \to G$ such that $C_G(\mathrm{Im}\alpha)/Z(\mathrm{Im}\alpha)$ is a p'–group.

Theorem 8 (Martino–Priddy [11]) *Let G and G' be finite groups. The following are equivalent:*

(1) *$BG \simeq BG'$ stably at p.*

(2) *For every finite p–group Q, there is an isomorphism of* $\mathrm{Out}(Q)$*–modules*

$$\mathbb{F}_p[\mathrm{Cen}(Q,G)] \cong \mathbb{F}_p[\mathrm{Cen}(Q,G')].$$

Note The proof of a related classification result of [11] contains an error. See [8] for a correction.

A stable equivalence $BG \simeq BG'$ at p induces an isomorphism

$$\mathbb{F}_p[\mathrm{Inj}(Q,G)] \cong \mathbb{F}_p[\mathrm{Inj}(Q,G')]$$

of $\mathrm{Out}(Q)$ modules. From this we can easily derive Nishida's result, Theorem 5:

Corollary 8 *If $BG \simeq BG'$ stably at p, then $P \cong P'$ where P, P' are respective Sylow p–subgroups.*

Proof *We have*

$$0 \neq \mathbb{F}_p[\mathrm{Inj}(P,G)] \cong \mathbb{F}_p[\mathrm{Inj}(P,G')].$$

This implies $P \subset P'$. Reversing P and P' gives the desired conclusion $P \cong P'$. □

Corollary 9 *If $BG \simeq BG'$ stably at p, then G and G' have the same number of conjugacy classes of p–subgroups of order $|Q|$ for all Q.*

Proof It is easy to see that

$$\mathbb{F}_p[\mathrm{Inj}(Q,G)] = \bigoplus_{\substack{(Q_1),\ Q_1 \cong Q \\ Q_1 \subset G}} \mathbb{F}_p[\mathrm{Out}(Q)] \otimes_{\mathbb{F}_p[W_G(Q_1)]} \mathbb{F}_p,$$

where
$$W_G(Q_1) = N_G(Q_1)/C_G(Q_1)\cdot Q_1$$

is the "Weyl group" of Q_1. From the third part of the theorem, we have an isomorphism

$$\bigoplus_{\substack{(Q_1),\ Q_1 \cong Q \\ Q_1 \subset G}} \mathbb{F}_p[\mathrm{Out}(Q)] \otimes_{\mathbb{F}_p[W_G(Q_1)]} \mathbb{F}_p \cong \bigoplus_{\substack{(Q_1),\ Q_1 \cong Q \\ Q_1 \subset G'}} \mathbb{F}_p[\mathrm{Out}(Q)] \otimes_{\mathbb{F}_p[W_G(Q_1)]} \mathbb{F}_p,$$

and if we apply to this the functor $\mathbb{F}_p \otimes_{\mathbb{F}_p[\mathrm{Out}(Q)]} (\cdot)$, then we conclude that

$$\bigoplus_{\substack{(Q_1),\ Q_1 \subset G \\ Q_1 \cong Q}} \mathbb{F}_p \cong \bigoplus_{\substack{(Q_1),\ Q_1 \subset G' \\ Q_1 \cong Q}} \mathbb{F}_p\,. \qquad \square$$

Definition Let H, K be subgroups of G. We say that H and K are *pointwise conjugate* in G if there is a bijection of sets $H \xrightarrow{\phi} K$ such that $\phi(h) = g(h)hg(h)^{-1}$ for some $g(h) \in G$ depending on h. This is equivalent to the statement that

$$|H \cap (g)| = |K \cap (g)|$$

for all $g \in G$.

Corollary 10 *Assume that G and G' have normal Sylow p–subgroups P and P' respectively. Then $BG \simeq BG'$ stably at p if and only if there is an isomorphism $P \xrightarrow{\phi} P'$ such that $W_G(P)$ is pointwise conjugate to $\phi^{-1}W_G(P')\phi$.*

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