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On behavior of the fifth algebraic transfer

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In this paper, we show that Singer's fifth transfer is not an epimorphism in degree 11. More precisely, it does not detect the element $P(h_2) \in \mathrm{Ext}_{\mathcal{A}}^{5,16}(\mathbb{F}_2, \mathbb{F}_2)$.

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1 Introduction and statement of results

Throughout the paper, the homology is taken with coefficients in $\mathbb{F}_2$. Let $\mathbb{V}_k$ denote a k–dimensional $\mathbb{F}_2$–vector space, and $PH_*(B\mathbb{V}_k)$ the primitive subspace consisting of all elements in $H_*(B\mathbb{V}_k)$, which are annihilated by every positive-degree operation in the mod 2 Steenrod algebra, $\mathcal{A}$. The general linear group $GL_k := GL(\mathbb{V}_k)$ acts regularly on $\mathbb{V}_k$ and therefore on the homology and cohomology of $B\mathbb{V}_k$. Since the two actions of $\mathcal{A}$ and GL_k upon $H^*(B\mathbb{V}_k)$ commute with each other, there are inherited actions of GL_k on $\mathbb{F}_2 {\otimes}_{\mathcal{A}} H^*(B\mathbb{V}_k)$ and $PH_*(B\mathbb{V}_k)$. In [6], W Singer defined the algebraic transfer

$$\mathrm{Tr}_k\colon \mathbb{F}_2 {\otimes}_{GL_k} PH_d(B\mathbb{V}_k) \to \mathrm{Ext}_{\mathcal{A}}^{k,k+d}(\mathbb{F}_2, \mathbb{F}_2)$$

as an algebraic version of the geometrical transfer $\mathrm{tr}_k\colon \pi_*^S((B\mathbb{V}_k)_+) \to \pi_*^S(S^0)$ to the stable homotopy groups of spheres.

It has been proved that Tr_k is an isomorphism for $k = 1, 2$ by Singer [6] and for $k = 3$ by Boardman [1]. Among other things, these data together with the fact that $\mathrm{Tr} = \bigoplus_k \mathrm{Tr}_k$ is an algebra homomorphism [6] show that Tr_k is highly nontrivial.

Therefore, the algebraic transfer is expected to be a useful tool in the study of the mysterious cohomology of the Steenrod algebra, $\mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$. In [4], Hưng established an attractive relationship between the algebraic transfer, the classical conjecture on spherical classes, and the so-called "hit" problem.

Further, in [6], Singer gave computations to show that Tr_4 is an isomorphism in a range of degrees and recognized that Tr_5 is not an epimorphism in degree 9. Then, he set up the following conjecture.

Conjecture 1.1 (Singer [6]) *Tr_k is a monomorphism for every k.*

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Recently, Bruner–Ha–Hưng showed in [3] that Tr_4 does not detect the family $\{g_i \mid i \geq 0\}$. Furthermore, Hưng proved in [5] that for every $k \geq 5$, there are infinitely many degrees in which Tr_k is not an isomorphism. Remarkably, it has not been known whether the algebraic transfer fails to be a monomorphism or fails to be an epimorphism for $k > 5$. Therefore, Singer's conjecture is still open.

The aim of this paper is to investigate the behavior of Tr_5 in degree 11. We prove the following theorem.

Theorem 1.2 *The element* $P(h_2) \in \mathrm{Ext}_{\mathcal{A}}^{5,16}(\mathbb{F}_2, \mathbb{F}_2)$ *is not in the image of the algebraic transfer* Tr_5*.*

Let $P_k := H^*(B\mathbb{V}_k)$ be the polynomial algebra of k variables, each of degree 1. Then, the domain of Tr_k, $\mathbb{F}_2 \otimes_{GL_k} PH_*(B\mathbb{V}_k)$, is dual to $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)^{GL_k}$. In order to prove Theorem 1.2, it suffices to show the following.

Proposition 1.3 $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{11}^{GL_5} = 0$.

Although our result does not give an answer to Singer's conjecture, it gives one more degree where the fifth algebraic transfer fails to be an epimorphism.

It should be noted that, R Bruner generously informed us that by using computer, he showed that $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{11}$ is a 315–dimensional $\mathbb{F}_2$–vector space, and that its GL_5–invariant is zero. In this paper, we prove the proposition by using some convenient generators for $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{11}$, which do not form a basis of the vector space.

The paper is divided into four sections. Section 2 deals with the computation of minimal $\mathcal{A}$–generators for the polynomial algebra P_5 in degree 11. Then, we prove Proposition 1.3 and Theorem 1.2 in Section 3 and Section 4 respectively.

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2 Computation of the indecomposables of P_5 in degree 11.

From now on, let us write $x = x_1$, $y = x_2$, $z = x_3$, $t = x_4$, $u = x_5$ and denote the monomial $x^a y^b z^c t^d u^e$ by (a, b, c, d, e) for abbreviation.

Lemma 2.1 *The* $\mathbb{F}_2$*–vector space* $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{11}$ *is generated by (the classes represented by) the following monomials and their permutations:*

$$\begin{matrix} (7,3,1,0,0), & (7,2,1,1,0), & (7,1,1,1,1), & (5,3,1,1,1) \\ (5,3,3,0,0), & (5,3,2,1,0), & (3,3,3,1,1), & (4,3,2,1,1). \end{matrix}$$

Proof The monomials in the third column are *spikes* in the meaning of W M Singer [7] that their exponents are all of the form $2^n - 1$ for some n. It is well known that spikes do not appear in the expression of $\mathrm{Sq}^i Y$ for any i positive and any monomial Y, since the powers x^{2^n-1} are not hit in the one variable case. Note that the elements in the first and second columns are respectively monomials which depend only on three and four variables. The last two column's monomials depend on exactly five variables.

Consider the projection $P_5 \to \mathbb{F}_2 \otimes_{\mathcal{A}} P_5$. We show that under this projection, all monomials in degree 11 not listed in the Lemma go to zero except for the following six and permutations

$$\begin{aligned} (6,3,1,1) &\mapsto (5,3,2,1) + (5,3,1,2) \\ (4,3,3,1) &\mapsto (2,3,5,1) + (2,5,3,1) \\ (3,3,3,2) &\mapsto (2,3,5,1) + (2,5,3,1) + (3,2,5,1) + (5,2,3,1) + (3,5,2,1) \\ &\quad + (5,3,2,1) \\ (5,2,2,1,1) &\mapsto (3,4,2,1,1) + (3,2,4,1,1) \\ (6,2,1,1,1) &\mapsto (3,4,2,1,1) + (3,2,4,1,1) + (3,4,1,2,1) + (3,2,1,4,1) \\ &\quad + (3,4,1,1,2) + (3,2,1,1,4) \\ (3,3,2,2,1) &\mapsto (5,3,1,1,1) + (3,5,1,1,1) + (4,3,1,1,2) + (3,4,1,1,2). \end{aligned}$$

As the action of the Steenrod algebra on P_5 commutes with that of the general linear group GL_5, without loss of generality, we need only to consider monomials (a, b, c, d, e) in degree 11 of P_5 with $a \geq b \geq c \geq d \geq e$. We have the following five cases.

Case 1 The monomial (a, b, c, d, e) depends only on one variable, $(a, b, c, d, e) = (a, 0, 0, 0, 0)$ with $a \neq 0$. There is only one such a monomial in degree 11 of P_5, namely $(11, 0, 0, 0, 0)$. It is hit because

$$(11, 0, 0, 0, 0) = \mathrm{Sq}^4(7, 0, 0, 0, 0).$$

Case 2 The monomial (a,b,c,d,e) depends on exactly two variables, $(a,b,c,d,e)=(a,b,0,0,0)$, where a and b are nonzero. It is also hit, as we have

$$\begin{aligned}
(10,1,0,0,0)&=\mathrm{Sq}^4(6,1,0,0,0)\\
(9,2,0,0,0)&=\mathrm{Sq}^4(5,2,0,0,0)\\
(8,3,0,0,0)&=\mathrm{Sq}^4(4,3,0,0,0)\\
(7,4,0,0,0)&=\mathrm{Sq}^4(5,2,0,0,0)+\mathrm{Sq}^2(7,2,0,0,0)\\
(6,5,0,0,0)&=\mathrm{Sq}^4(4,3,0,0,0)+\mathrm{Sq}^2(6,3,0,0,0).
\end{aligned}$$

Case 3 The monomial (a,b,c,d,e) depends exactly on three variables $(a,b,c,d,e)=(a,b,c,0,0)$, where a, b and c are nonzero. This should be one of the following monomials:

$$\begin{aligned}
&(7,3,1,0,0),(5,3,3,0,0)\\
&(9,1,1,0,0),(8,2,1,0,0),(7,2,2,0,0),(6,4,1,0,0),(6,3,2,0,0),(5,4,2,0,0)\\
&(5,5,1,0,0),(4,4,3,0,0).
\end{aligned}$$

The first two monomials are listed in the lemma. The last eight monomials are killed by the Steenrod algebra, since we have

$$\begin{aligned}
(9,1,1,0,0)&=\mathrm{Sq}^4(5,1,1,0,0)\\
(8,2,1,0,0)&=\mathrm{Sq}^4(4,2,1,0,0)\\
(7,2,2,0,0)&=\mathrm{Sq}^1(7,2,1,0,0)+\mathrm{Sq}^4(4,2,1,0,0)\\
(6,4,1,0,0)&=\mathrm{Sq}^2(6,2,1,0,0)+\mathrm{Sq}^4(4,2,1,0,0)\\
(6,3,2,0,0)&=\mathrm{Sq}^1(6,3,1,0,0)+\mathrm{Sq}^2(6,2,1,0,0)+\mathrm{Sq}^4(4,2,1,0,0)\\
(5,4,2,0,0)&=\mathrm{Sq}^1(5,4,1,0,0)+\mathrm{Sq}^2(6,2,1,0,0)+\mathrm{Sq}^4(4,2,1,0,0),
\end{aligned}$$

and

$$\begin{aligned}
(5,5,1,0,0)&=(6,4,1,0,0)+(6,3,2,0,0)+(5,4,2,0,0)+\mathrm{Sq}^2(5,3,1,0,0)\\
(4,4,3,0,0)&=(4,2,5,0,0)+\mathrm{Sq}^2(4,2,3,0,0).
\end{aligned}$$

Case 4 The monomial (a,b,c,d,e) depends exactly on four variables, $(a,b,c,d,e)=(a,b,c,d,0)$, where a, b, c and d are non zero. This should be one of the following monomials:

$$\begin{aligned}
&(7,2,1,1,0),(5,3,2,1,0)\\
&(8,1,1,1,0),(6,2,2,1,0),(5,2,2,2,0),(4,4,2,1,0),(4,3,2,2,0),(5,4,1,1,0)\\
&(6,3,1,1,0),(4,3,3,1,0),(3,3,3,2,0).
\end{aligned}$$

The first two monomials are listed in the lemma. The next six monomials are killed by the Steenrod algebra, since we have

$$\begin{aligned}
(8,1,1,1,0) &= \mathrm{Sq}^4(4,1,1,1,0)\\
(6,2,2,1,0) &= \mathrm{Sq}^5(3,1,1,1,0) + \mathrm{Sq}^4(4,1,1,1,0)\\
(5,2,2,2,0) &= (6,2,2,1,0) + \mathrm{Sq}^1(5,2,2,1,0)\\
(4,4,2,1,0) &= \mathrm{Sq}^4(2,2,2,1,0) + \mathrm{Sq}^2(2,2,4,1,0)\\
(4,3,2,2,0) &= (4,4,2,1,0) + \mathrm{Sq}^1(4,3,2,1,0)\\
(5,4,1,1,0) &= (4,4,2,1,0) + (4,4,1,2,0) + (3,4,2,2,0) + \mathrm{Sq}^2(3,4,1,1,0).
\end{aligned}$$

The last three monomials $(6,3,1,1,0), (4,3,3,1,0), (3,3,3,2,0)$ can be expressed in terms of the monomials $(7,2,1,1,0), (5,3,2,1,0)$ and their permutations. Indeed, we get the following equalities

$$\begin{aligned}
(6,3,1,1,0) &= (5,3,2,1,0) + (5,3,1,2,0) + (5,4,1,1,0) + \mathrm{Sq}^1(5,3,1,1,0)\\
(4,3,3,1,0) &= (2,3,5,1,0) + (2,5,3,1,0) + (2,4,4,1,0) + (2,3,4,2,0)\\
&\quad + (2,4,3,2,0) + \mathrm{Sq}^2(2,3,3,1,0)\\
(3,3,3,2,0) &= (4,3,3,1,0) + (3,4,3,1,0) + (3,3,4,1,0) + \mathrm{Sq}^1(3,3,3,1,0).
\end{aligned}$$

Case 5 The monomial (a,b,c,d,e) depends exactly on five variables, $(a,b,c,d,e) = (a,b,c,d,e)$, where a, b, c, d and e are nonzero. This should be one of the following monomials:

$$\begin{gathered}
(7,1,1,1,1), (5,3,1,1,1), (3,3,3,1,1), (4,3,2,1,1)\\
(4,4,1,1,1), (4,2,2,2,1), (3,2,2,2,2)\\
(5,2,2,1,1), (6,2,1,1,1), (3,3,2,2,1).
\end{gathered}$$

The first four monomials are listed in the lemma. The next three monomials are hit by the Steenrod algebra, since we have

$$\begin{aligned}
(4,4,1,1,1) &= \mathrm{Sq}^2(4,2,1,1,1) + \mathrm{Sq}^2(2,4,1,1,1) + \mathrm{Sq}^4(2,2,1,1,1)\\
(4,2,2,2,1) &= \mathrm{Sq}^4(2,2,1,1,1) + \mathrm{Sq}^2(2,4,1,1,1) + \mathrm{Sq}^1(4,2,1,1,2)\\
(3,2,2,2,2) &= (4,2,2,2,1) + \mathrm{Sq}^1(3,2,2,2,1).
\end{aligned}$$

The last three monomials $(5,2,2,1,1), (6,2,1,1,1), (3,3,2,2,1)$ are expressed in terms of the monomials $(5,3,1,1,1), (4,3,2,1,1)$ and their permutations. Indeed

$$\begin{aligned}(5,2,2,1,1) &= (3,4,2,1,1)+(3,2,4,1,1)+(4,2,2,2,1)+(4,2,2,1,2)\\ &\quad +(3,2,2,2,2)+\mathrm{Sq}^2(3,2,2,1,1)\\ (6,2,1,1,1) &= (5,2,2,1,1)+(5,2,1,2,1)+(5,2,1,1,2)+\mathrm{Sq}^1(5,2,1,1,1)\\ (3,3,2,2,1) &= (5,3,1,1,1)+(3,5,1,1,1)+(4,4,1,1,1)+(4,3,1,1,2)\\ &\quad +(3,4,1,1,2)+\mathrm{Sq}^2(3,3,1,1,1)\\ &\quad +\mathrm{Sq}^1(3,3,1,1,2)+\mathrm{Sq}^1(4,3,1,1,1)+\mathrm{Sq}^1(3,4,1,1,1).\end{aligned}$$

The lemma is proved. □

We denote by A, B, C, D, E, F, G, H the families of all permutations of the following monomials respectively

$$\begin{gathered}(7,3,1,0,0), (5,3,3,0,0), (7,2,1,1,0), (5,3,2,1,0)\\ (7,1,1,1,1), (3,3,3,1,1), (5,3,1,1,1), (4,3,2,1,1).\end{gathered}$$

For X one of the families A, B, C, D, E, F, G, H, let $\mathcal{L}(X)$ be the vector subspace of $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{11}$ spanned by all the elements of the family X. Further, set $\mathcal{L}(G,H) = \mathcal{L}(G) + \mathcal{L}(H)$.

Lemma 2.2 *Every $p \in (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{11}$ can be expressed uniquely as a sum*

$$p = p_A + p_B + p_C + p_D + p_E + p_F + p_{(G,H)},$$

where $p_X \in \mathcal{L}(X)$ for $X \in \{A, B, C, D, E, F\}$ and $p_{(G,H)} \in \mathcal{L}(G,H)$.

Proof By Lemma 2.1, if $p \in (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{11}$ then p can be expressed as a sum of elements in $\mathcal{L}(A), \mathcal{L}(B), \mathcal{L}(C), \mathcal{L}(D), \mathcal{L}(E), \mathcal{L}(F)$ and in $\mathcal{L}(G,H)$. In order to prove the uniqueness of the expression we now suppose that there is a linear relation

$$p_A + p_B + p_C + p_D + p_E + p_F + p_{(G,H)} = 0$$

in $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{11}$, where $p_X \in \mathcal{L}(X)$ for $X \in \{A, B, C, D, E, F\}$ and $X = (G,H)$. We need to show $p_A = p_B = p_C = p_D = p_E = p_F = p_{(G,H)} = 0$ in $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{11}$. First, we note that $p_A = p_E = p_F = 0$, as p_A, p_E, p_F are expressed in terms of the spikes, which do not appear in the expression of $\mathrm{Sq}^i(Y)$ for any i positive and any monomial Y. Hence

$$p_B + p_C + p_D + p_{(G,H)} = 0.$$

Consider the homomorphism $\pi_{tu}\colon \mathbb{F}_2 \otimes_{\mathcal{A}} P_5 \to \mathbb{F}_2 \otimes_{\mathcal{A}} P_3$ induced by the projection $P_5 \to P_5/(t,u) \cong P_3$. Under this homomorphism, the image of the above linear relation is $\pi_{tu}(p_B) = 0$. Using all the projections from P_5 to its quotients by the ideals generated by any pairs of the five variables x, y, z, t, u, we get $p_B = 0$. Hence

$$p_C + p_D + p_{(G,H)} = 0.$$

Next, we consider the homomorphism $\pi_u\colon \mathbb{F}_2\otimes_{\mathcal{A}} P_5 \to \mathbb{F}_2\otimes_{\mathcal{A}} P_4$ induced by the projection $P_5 \to P_5/(u) \cong P_4$. Let π_u act on both sides of the above equality, we get

$$\pi_u(p_C) + \pi_u(p_D) = 0,$$

where $\pi_u(p_C)$ is a linear combination of permutations of element $(7,2,1,1)$. As 7 and 1 are of the form $2^n - 1$, the monomial $(7,2,1,1)$ appears only as a term in $\mathrm{Sq}^i(a,b,c,d)$ for $i = 1$ and $(a,b,c,d) = (7,1,1,1)$ as follows

$$\mathrm{Sq}^1(7,1,1,1) = (8,1,1,1) + (7,2,1,1) + (7,1,2,1) + (7,1,1,2).$$

So, $\pi_u(p_C)$ contains $(7,2,1,1)$ as a term if and only if it also contains $(7,1,2,1) + (7,1,1,2)$. A consequence of the above expression of $\mathrm{Sq}^1(7,1,1,1)$ is

$$(7,2,1,1) + (7,1,2,1) + (7,1,1,2) = 0,$$

since $(8,1,1,1) = \mathrm{Sq}^4(4,1,1,1)$. Thus, $\pi_u(p_C) = 0$, and therefore $\pi_u(p_D) = 0$.

In the above argument, replacing the homomorphism π_u by any of $\pi_x, \pi_y, \pi_z, \pi_t$, and we get

$$p_C = p_D = p_{(G,H)} = 0. \qquad \square$$

The following Lemma is a consequence of Lemma 2.1 and Lemma 2.2.

Lemma 2.3 *There is a decomposition of F_2–vector spaces*

$$(\mathbb{F}_2\otimes_{\mathcal{A}} P_5)_{11} = \mathcal{L}(A) \oplus \mathcal{L}(B) \oplus \mathcal{L}(C) \oplus \mathcal{L}(D) \oplus \mathcal{L}(E) \oplus \mathcal{L}(F) \oplus \mathcal{L}(G,H).$$

3 GL_5–invariants of the indecomposables of P_5 in degree 11

The goal of this section is to prove the following proposition, which is also numbered as Proposition 1.3 in the introduction.

Proposition 3.1 $(\mathbb{F}_2\otimes_{\mathcal{A}} P_5)^{GL_5}_{11} = 0$.

Let S_5 be the symmetric group on 5 letters x, y, z, t, u. It is easy to see that $\mathcal{L}(A)$, $\mathcal{L}(B)$, $\mathcal{L}(C)$, $\mathcal{L}(D)$, $\mathcal{L}(E)$, $\mathcal{L}(F)$ and $\mathcal{L}(G,H)$ are all S_5–submodules. So the equality in Lemma 2.3

$$(\mathbb{F}_2\otimes_{\mathcal{A}} P_5)_{11} = \mathcal{L}(A) \oplus \mathcal{L}(B) \oplus \mathcal{L}(C) \oplus \mathcal{L}(D) \oplus \mathcal{L}(E) \oplus \mathcal{L}(F) \oplus \mathcal{L}(G,H)$$

is a decomposition of S_5–modules.

By Lemma 2.2, every $p \in (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{11}$ can be expressed uniquely as a sum

$$p = p_A + p_B + p_C + p_D + p_E + p_F + p_{(G,H)},$$

where $p_X \in \mathcal{L}(X)$ for $X \in \{A, B, C, D, E, F\}$ and $p_{(G,H)} \in \mathcal{L}(G, H)$. So, each term of the sum is an S_5–invariant.

For X one of the letters A, B, C, D, E, F, let x_i be the coefficient in the above expression of p of the i th monomial in the family X ordered lexicographically. Note that all monomials of families A, E and F are spikes. It is well known that spikes do not appear in the expression of $\mathrm{Sq}^i Y$ for any i positive and any monomial Y. Hence, the coefficient of any spike is zero in every linear relation in $\mathbb{F}_2 \otimes_{\mathcal{A}} P_5$. It implies that, in the expression of p, the coefficients of monomials in each of the families A, E, F are equal to each other.

Proposition 3.1 is proved by combining the following five lemmas.

Lemma 3.2 *If $p = p_A + p_B + p_C + p_D + p_E + p_F + p_{(G,H)}$ is the decomposition of $p \in (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{11}^{GL_5}$ as in Lemma 2.2, then $p_A = p_B = 0$.*

Proof With π_{tu} defined as in the proof of Lemma 2.2, we have

$$\pi_{tu}(p) = \pi_{tu}(p_A) + \pi_{tu}(p_B).$$

We have

$$\pi_{tu}(p_B) = b_1(5,3,3) + b_2(3,5,3) + b_3(3,3,5).$$

According to the argument given above, the coefficients a_i are equal each other. Set $a = a_i$ and we have

$$\pi_{tu}(p_A) = a[(7,3,1) + (7,1,3) + (3,7,1) + (3,1,7) + (1,7,3) + (1,3,7)].$$

We will show that $b_1 = b_2 = b_3$ and $a = 0$. Associated to the two variables x and y, let σ_{xy} be the transposition of x and y that keeps the other variables fixed.

As p is a GL_5–invariant in $\mathbb{F}_2 \otimes_{\mathcal{A}} P_5$, we have

$$\pi_{tu}(\sigma_{xy}(p) + p) = \pi_{tu}(0) = 0 \text{ in } \mathbb{F}_2 \otimes_{\mathcal{A}} P_3,$$

equivalently

$$\sigma_{xy}(\pi_{tu}(p)) + \pi_{tu}(p) = 0 \text{ in } \mathbb{F}_2 \otimes_{\mathcal{A}} P_3.$$

Combining $\pi_{tu}(p_A) = a[(7,3,1) + \text{symmetrized}]$ with the fact that the monomial $(7,3,1)$ is spike, we have

$$\sigma_{xy}(\pi_{tu}(p_A)) + \pi_{tu}(p_A) = 0.$$

From this, it follows that

$$\sigma_{xy}(\pi_{tu}(p_B)) + \pi_{tu}(p_B) = 0,$$

or equivalently

$$(b_1 + b_2)((5,3,3) + (3,5,3)) = 0.$$

However,

$$\begin{aligned}(5,3,3) + (3,5,3) + (3,3,5) = {} & \mathrm{Sq}^2(3,3,3) + \mathrm{Sq}^1(4,3,3) + \mathrm{Sq}^4(4,2,1) \\ & + \mathrm{Sq}^2(6,2,1) + \mathrm{Sq}^1(5,4,1) + \mathrm{Sq}^2(3,4,2).\end{aligned}$$

So, we get

$$(b_1 + b_2)(3,3,5) = 0.$$

On the other hand, the linear transformation $x \mapsto x + z, y \mapsto y, z \mapsto z$ sends $(3,3,5)$ to $(3,3,5) + (2,3,6) + (1,3,7) + (0,3,8) \sim (3,3,5) + (1,3,7)$. As the action of the Steenrod algebra commutes with linear maps, if $(3,3,5)$ is hit then so is $(1,3,7)$. This is impossible, because $(1,3,7)$ is a spike. Thus, $(3,3,5) \neq 0$ in $\mathbb{F}_2 \otimes_{\mathcal{A}} P_5$ and therefore $b_1 + b_2 = 0$, or $b_1 = b_2$. By similarity, using all transpositions of any pairs of the three variables x, y, z, we get $b_1 = b_2 = b_3$. Hence

$$\pi_{tu}(p_B) = b_1[(5,3,3) + (3,5,3) + (3,3,5)] = b_1.0 = 0.$$

By the symmetry of the variables, we also obtain $\pi_{ij}(p_B) = 0$, where (i,j) is any pair of the five variables x, y, z, t, u. Thus $p_B = 0$.

In order to prove $a = 0$, we consider the linear transformation, ω_{xy}, that sends x to $x + y$ and keeps the other variables fixed. As $p_B = 0$, we have $\pi_{tu}(p) = \pi_{tu}(p_A)$. From $\omega_{xy}(p) + p = 0$, it follows that

$$\omega_{xy}(\pi_{tu}(p_A)) + \pi_{tu}(p_A) = 0,$$

or equivalently

$$a[(5,3,3) + (3,5,3) + (1,7,3) + (3,7,1) + (1,3,7)] = 0.$$

Combining this with the fact that $(1,7,3)$ is a spike, we get $a = 0$. □

Lemma 3.3 *If $p = p_A + p_B + p_C + p_D + p_E + p_F + p_{(G,H)}$ is the decomposition of $p \in (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{11}^{GL_5}$ as in Lemma 2.2, then $p_C = p_D = 0$.*

Proof By Lemma 3.2, $p_A = p_B = 0$. As a consequence, $p = p_C + p_D + p_E + p_F + p_{(G,H)}$. Let $\pi_u\colon \mathbb{F}_2 \otimes_{\mathcal{A}} P_5 \to \mathbb{F}_2 \otimes_{\mathcal{A}} P_4$ be the homomorphism induced by the projection $P_5 \to P_5/(u) \cong P_4$ as in the proof of Lemma 2.2. We have

$$\pi_u(p) = \pi_u(p_C) + \pi_u(p_D),$$

where $\pi_{(u)}(p_C)$ and $\pi_{(u)}(p_D)$ are respectively certain linear combinations of permutations of the elements $(7,2,1,1)$ and $(5,3,2,1)$.

In the families $\pi_u(C), \pi_u(D)$, there are exactly three monomials (x,y,z,t) with $t = 7$, namely

$$(2,1,1,7), (1,2,1,7), (1,1,2,7).$$

We have $\mathrm{Sq}^1(1,1,1,7) = (2,1,1,7) + (1,2,1,7) + (1,1,2,7)$, and hence

$$(2,1,1,7) = (1,2,1,7) + (1,1,2,7) \text{ in } \mathbb{F}_2 \otimes_{\mathcal{A}} P_4.$$

So we get

$$\pi_u(p) = c_1(1,2,1,7) + c_2(1,1,2,7) + \text{ terms of the form } (x,y,z,t) \text{ with } t \neq 7.$$

Let ω_{xy} be the transposition of x and y as defined in the proof of Lemma 3.2. It is easily seen that

$$\begin{aligned}\omega_{xy}(c_1(1,2,1,7) + c_2(1,1,2,7)) = c_1(1,2,1,7) + c_2(1,1,2,7) + c_1(0,3,1,7)\\ + c_2(0,2,2,7).\end{aligned}$$

Combining this with the fact that $\omega_{xy}(\pi_u(p)) + \pi_u(p) = 0$, we obtain $c_1 = 0$, as $(0,3,1,7)$ is a spike.

By a similar argument using ω_{xz}, we get $c_2 = 0$. Hence $\pi_u(p_C) = 0$.

By the symmetry of the variables, we have

$$\pi_x(p_C) = \pi_y(p_C) = \pi_z(p_C) = \pi_t(p_C) = \pi_u(p_C) = 0.$$

As a consequence, we get $p_C = 0$.

Similarly, in order to prove $p_D = 0$ we need only to show that $\pi_u(p_D) = 0$. The family $\pi_u(D)$, which consists of all the permutations of the monomials $(5,3,2,1)$, has twenty-four elements. A direct calculation shows the following table.

monomial	ω_{xy}(monomial)+monomial	monomial	ω_{xy}(monomial)+monomial
(5,3,2,1)	(1,7,2,1)	(5,3,1,2)	(1,7,1,2)
(5,2,3,1)	(4,3,3,1)+(1,6,3,1)+(0,7,3,1)	(5,2,1,3)	(4,3,1,3)+(1,6,1,3)+(0,7,1,3)
(5,1,3,2)	(1,5,3,2)	(5,1,2,3)	(1,5,2,3)
(3,5,2,1)	(1,7,2,1)	(3,5,1,2)	(1,7,1,2)
(3,2,5,1)	(2,3,5,1)	(3,2,1,5)	(2,3,1,5)
(3,1,5,2)	(1,3,5,2)	(3,1,2,5)	(1,3,2,5)
(2,5,3,1)	(0,7,3,1)	(2,5,1,3)	(0,7,1,3)
(2,3,5,1)	0	(2,3,1,5)	0
(2,1,5,3)	(0,3,5,3)	(2,1,3,5)	(0,3,3,5)
(1,5,3,2)	0	(1,5,2,3)	0
(1,3,5,2)	0	(1,3,2,5)	0
(1,2,5,3)	(0,3,5,3)	(1,2,3,5)	(0,3,3,5).

Let $d_{(a,b,c,d)}$ be the coefficient of the monomial (a,b,c,d) in the expression of $\pi_u(p_D)$. Since $\pi_u(p_D)$ is a GL_4–invariant, we have $\omega_{xy}(\pi_u(p_D)) + \pi_u(p_D) = 0$ in $\mathbb{F}_2 \otimes_{\mathcal{A}} P_4$. Combining this and the above table we obtain

$$\begin{aligned}
&[d_{(5,3,2,1)} + d_{(3,5,2,1)}](1,7,2,1) + [d_{(5,3,1,2)} + d_{(3,5,1,2)}](1,7,1,2) = 0\\
&[d_{(5,2,3,1)} + d_{(2,5,3,1)}](0,7,3,1) + [d_{(5,2,1,3)} + d_{(2,5,1,3)}](0,7,1,3) = 0\\
&[d_{(2,1,5,3)} + d_{(1,2,5,3)}](0,3,5,3) + [d_{(2,1,3,5)} + d_{(1,2,3,5)}](0,3,3,5) = 0.
\end{aligned}$$

As $(0,7,3,1)$ and $(0,7,1,3)$ are spikes, we get

$$d_{(5,2,3,1)} = d_{(2,5,3,1)} \quad \text{and} \quad d_{(5,2,1,3)} = d_{(2,5,1,3)}.$$

Let ω_{xz} be the linear transformation which sends x to $x+z$ and keeps the other variables fixed. Applying ω_{xz} to the above first equality, we get

$$[d_{(5,3,2,1)} + d_{(3,5,2,1)}](0,7,3,1) + [d_{(5,3,1,2)} + d_{(3,5,1,2)}](0,7,2,2) = 0.$$

It implies $d_{(5,3,2,1)} = d_{(3,5,2,1)}$ and similarly $d_{(5,3,1,2)} = d_{(3,5,1,2)}$.

Similarly, it follows from the third equality that

$$d_{(2,1,5,3)} = d_{(1,2,5,3)} \quad \text{and} \quad d_{(2,1,3,5)} = d_{(1,2,3,5)}.$$

It is easy to see that the symmetric group on the four letters $\{5,3,2,1\}$ is generated by the transpositions $(5,3), (5,2), (2,1)$. Combining this with the above equalities, it implies that all coefficients $d_{(a,b,c,d)}$ are the same. Let us denote this common coefficient by d. We have

$$\begin{aligned}\omega_{xy}(\pi_u(p_D)) + \pi_u(p_D) = d[&(4,3,3,1) + (1,6,3,1) + (4,3,1,3) + (1,6,1,3)\\ &+(1,5,3,2) + (1,5,2,3) + (2,3,5,1) + (2,3,1,5)\\ &+(1,3,5,2) + (1,3,2,5)].\end{aligned}$$

As shown in the proof of Lemma 2.1, we get

$$\begin{aligned}(4,3,3,1) &= (2,3,5,1) + (2,5,3,1)\\ (1,6,3,1) &= (2,5,3,1) + (1,5,3,2)\\ (4,3,1,3) &= (2,3,1,5) + (2,5,1,3)\\ (1,6,1,3) &= (2,5,1,3) + (1,5,2,3).\end{aligned}$$

Hence, the above equality is reduced to

$$d[(1,3,5,2) + (1,3,2,5)] = 0.$$

Applying ω_{xt} to this relation, we get $d[(0,3,5,3)] = 0$. It implies $d = 0$, since we have shown that $(0,3,5,3)$ is nonzero.

So $\pi_u(p_D) = 0$ and therefore $p_D = 0$. □

Lemma 3.4 *If $p = p_A + p_B + p_C + p_D + p_E + p_F + p_{(G,H)}$ is the decomposition of $p \in (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{11}^{GL_5}$ as in Lemma 2.2, then $p_E = 0$.*

Proof According to the above two lemmas, $p = p_E + p_F + p_{(G,H)}$.

As $(7,1,1,1,1)$ is a spike, the coefficients of its all permutations in the expression of $p \in (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{11}^{GL_5}$ are equal to each other. We denote this common coefficient by e.

So, p_E can be written in the form

$$p_E = e[(7,1,1,1,1) + (1,7,1,1,1) + (1,1,7,1,1) + (1,1,1,7,1) + (1,1,1,1,7)],$$

where $e \in \mathbb{F}_2$.

In the families E, F, G, H there is exactly one monomial with u=7, namely $(1,1,1,1,7)$. Let σ be the linear transformation that sends x to $x+z$, y to $y+z$ and keeps the other variables fixed.

An easy computation shows

$$\sigma(1,1,1,1,7) = (1,1,1,1,7) + (1,0,2,1,7) + (0,1,2,1,7) + (0,0,3,1,7).$$

Note that the images under σ of the other monomials of the families E, F, G, H in the expression of p do not contain the spike $(0,0,3,1,7)$.

So, $\sigma(p) + p$ contains $e(0,0,3,1,7)$ as a term. It implies $\alpha = 0$, and therefore $p_E = 0$. $\square$

Lemma 3.5 *If* $p = p_A + p_B + p_C + p_D + p_E + p_F + p_{(G,H)}$ *is the decomposition of* $p \in (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{11}^{GL_5}$ *as in Lemma 2.2, then* $p_F = 0$.

Proof According to the above three lemmas, we have $p = p_F + p_{(G,H)}$.

By the same argument given in the previous lemma, as $(3,3,3,1,1)$ is a spike, the coefficients of its all permutations in the expression of $p \in (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{11}^{GL_5}$ are equal each other. We denote this common coefficient by f.

In the family F, G, H, there are exactly two monomials with $z = 3, t = 3, u = 1$, namely

$$(3,1,3,3,1), (1,3,3,3,1).$$

As p is a GL_5–invariant in $\mathbb{F}_2 \otimes_{\mathcal{A}} P_5$, we have particularly

$$\omega_{xy}(p) + p = 0.$$

A routine computation shows

$$\begin{aligned}
&\omega_{xy}(3,1,3,3,1) + (3,1,3,3,1) = (2,2,3,3,1) + (1,3,3,3,1) + (0,4,3,3,1)\\
&\omega_{xy}(1,3,3,3,1) + (1,3,3,3,1) = (0,4,3,3,1).
\end{aligned}$$

Note that the images under ω_{xy} of the other monomials of the families F, G, H in the expression of p do not contain the spike $(1,3,3,3,1)$.

Thus, $\omega_{xy}(p) + p$ contains $f(1,3,3,3,1)$ as a term. This implies $f = 0$ and therefore $p_F = 0$. $\square$

Lemma 3.6 *If* $p = p_A + p_B + p_C + p_D + p_E + p_F + p_{(G,H)}$ *is the decomposition of* $p \in (\mathbb{F}_2 \otimes_{\mathcal{A}} P_5)_{11}^{GL_5}$ *as in Lemma 2.2, then* $p_{(G,H)} = 0$.

Proof According to the above four lemma, we have $p = p_{(G,H)}$. Recall that $p_{(G,H)}$ is expressed in terms of the elements of the families G and H.

The proof is divided into 2 steps.

Step 1 Let K be the family of all variable permutations of monomial $(3,3,2,2,1)$. We will show that p can be expressed in terms of the elements of the family K.

The elements in family G are divided into pairs by twisting the variables whose exponents are 5 and 3.

Consider two monomials $(5,3,1,1,1), (3,5,1,1,1)$ in one of the pairs. With ω_{xy} as defined in the proof of Lemma 3.2, we have

$$\omega_{xy}(5,3,1,1,1) = (5,3,1,1,1) + (4,4,1,1,1) + (1,7,1,1,1) + (0,8,1,1,1)$$

$$\omega_{xy}(3,5,1,1,1) = (3,5,1,1,1) + (2,6,1,1,1) + (1,7,1,1,1) + (0,8,1,1,1).$$

Further, $(1,7,1,1,1)$ does not appear in the expressions of the images under ω_{xy} of any other elements in G, H. As p is a GL_5–invariant, it satisfies

$$\omega_{xy}(p) + p = 0 \text{ in } W\mathbb{F}_2 \otimes_{\mathcal{A}} P_5.$$

However, $(1,7,1,1,1)$ is a spike, which does not appear in the expression of $\mathrm{Sq}^i Y$ for any i positive and any monomial Y. So, the coefficients of the monomials $(5,3,1,1,1)$ and $(3,5,1,1,1)$ in the expression of p are equal each other.

On the other hand, by using by $\mathrm{Sq}^2(3,3,1,1,1) + \mathrm{Sq}^1(4,3,1,1,1) + \mathrm{Sq}^1(3,4,1,1,1)$ we get

$$(5,3,1,1,1) + (3,5,1,1,1) = (3,3,2,2,1) + (3,3,1,2,2) + (3,3,2,1,2).$$

Then, in the expression of p, the sum of monomials in family G can be written as a sum of monomials in family K.

Next, we consider in the expression of p the sum of monomials in the family H.

First, we consider the set of monomials of the forms $(4,3,c,d,e)$ and $(3,4,c,d,e)$ in the family H. Then, (c,d,e) is a permutation of $(2,1,1)$. We will show that the sum of the monomials in this set occurring in the expression of p equals to the sum of some monomials in the family K.

We have

$$(3,4,2,1,1) = (4,3,2,1,1) + (3,3,2,2,1) + (3,3,2,1,2)$$

as $(3,4,2,1,1) = (4,3,2,1,1) + (3,3,2,2,1) + (3,3,2,1,2) + \mathrm{Sq}^1(3,3,2,1,1)$.

Similarly,

$$(3,4,1,2,1) = (4,3,1,2,1) + (3,3,2,2,1) + (3,3,1,2,2)$$
$$(3,4,1,1,2) = (4,3,1,1,2) + (3,3,1,2,2) + (3,3,2,1,2).$$

We also have

$$(4,3,1,1,2) = (4,3,2,1,1) + (4,3,1,2,1),$$

because $(4,3,1,1,2) = (4,3,2,1,1) + (4,3,1,2,1) + (4,4,1,1,1) + \mathrm{Sq}^1(4,3,1,1,1)$.

Let h_1 and h_2 be the coefficients respectively of the monomials $(4,3,2,1,1)$ and $(4,3,1,2,1)$ in an expression of p . Then

$$p = h_1(4,3,2,1,1) + h_2(4,3,1,2,1) + \text{ other terms.}$$

On the other hand, p is a GL_5–invariant, so

$$\omega_{xy}(p) + p = 0.$$

We have

$$\begin{aligned}\omega_{xy}(4,3,2,1,1) &= (4,3,2,1,1) + (0,7,2,1,1)\\ \omega_{xy}(4,3,1,2,1) &= (4,3,1,2,1) + (0,7,1,2,1),\end{aligned}$$

and the images under ω_{xy} of any other monomials in the expression of p do not contain the monomials $(0,7,2,1,1), (0,7,1,2,1), (0,7,1,1,2)$ as terms. Thus,

$$\omega_{xy}(p) + p = h_1(0,7,2,1,1) + h_2(0,7,1,2,1) + \text{ other terms not in } C.$$

So, we get

$$h_1(0,7,2,1,1) + h_2(0,7,1,2,1) = 0.$$

Applying ω_{ut}, which sends t to $t+u$ and keeps the other variables fixed, to this equality, we obtain

$$h_1(0,7,2,2,0) + h_2(0,7,1,3,0) = 0.$$

This implies $h_2 = 0$, as $(0,7,1,3,0)$ is a spike. Similarly, we have $h_1 = 0$.

We have shown that in the expression of p, the sum of monomials of the forms $(4,3,c,d,e)$ and $(3,4,c,d,e)$ in H can be written in terms of monomials in the family K.

Because of the symmetry of the variables, the above argument also works for the sum of monomials in H in the expression of p

Step 2 We will show that if $p \in \mathcal{L}(K)$ is a ' GL_5–invariant, then p equals zero.

Note that if $p \in \mathcal{L}(K)$, then it is expressed in the terms of the variables permutations of the monomial $(3,3,2,2,1)$. Let $k_{(a,b,c,d,e)}$ be the coefficient of the monomial (a,b,c,d,e) in an expression of p. Because of the symmetry of the variables, in order to prove $p = 0$ we need only to prove $k_{(2,2,3,3,1)} = 0$.

There are exactly three monomials of the form $(a,b,c,3,1)$ in K, namely

$$(3,2,2,3,1), (2,3,2,3,1), (2,2,3,3,1).$$

Let σ be the transformation defined in the proof of Lemma 3.4, which sends x to $x+z$, y to $y+z$ and fixes the other variables.

A routine computation shows

$$\begin{aligned}\sigma(3,2,2,3,1) &= (3,2,2,3,1)+(3,0,4,3,1)+(2,2,3,3,1)+(2,0,5,3,1)\\ &\quad +(1,2,4,3,1)+(1,0,6,3,1)+(0,2,5,3,1)+(0,0,7,3,1)\\ \sigma(2,3,2,3,1) &= (2,3,2,3,1)+(0,3,4,3,1)+(2,2,3,3,1)+(0,2,5,3,1)\\ &\quad +(2,1,4,3,1)+(0,1,6,3,1)+(2,0,5,3,1)+(0,0,7,3,1)\\ \sigma(2,2,3,3,1) &= (2,2,3,3,1)+(0,2,5,3,1)+(2,0,5,3,1)+(0,0,7,3,1).\end{aligned}$$

Further, the images under σ of the other terms in the expression of p do not contain $(0,0,7,3,1)$ as a term, because the exponents of t and u in these monomials are not respectively 3 and 1. So, $\sigma(p)+p$ contains $(k_{(3,2,2,3,1)}+k_{(2,3,2,3,1)}+k_{(2,2,3,3,1)})(0,0,7,3,1)$ as a term. Moreover, as p is a GL_5–invariant, $\sigma(p)+p=0$. It implies

$$k_{(3,2,2,3,1)}+k_{(2,3,2,3,1)}+k_{(2,2,3,3,1)}=0.$$

On the other hand, consider the set of monomials of the form $(a,b,2,d,1)$ and $(a,b,1,d,2)$ in the family K. Then, (a,b,d) is a permutation of $(3,3,2)$. We have

$$\begin{aligned}\omega_{xy}(3,3,2,2,1)+(3,3,2,2,1) &= (2,4,2,2,1)+(1,5,2,2,1)+(0,6,2,2,1)\\ \omega_{xy}(3,2,2,3,1)+(3,2,2,3,1) &= (2,3,2,3,1)+(1,4,2,3,1)+(0,5,2,3,1)\\ \omega_{xy}(2,3,2,3,1)+(2,3,2,3,1) &= (0,5,2,3,1).\end{aligned}$$

Let ω_{yt} be the transformation that sends y to $y+t$ and keeps the other variables fixed. Apply ω_{yt} to $\omega_{xy}(p)+p$, we have

$$\omega_{yt}(0,5,2,3,1)=(0,5,2,3,1)+(0,4,2,4,1)+(0,1,2,7,1)+(0,0,2,8,1).$$

It is easy to see that the actions of ω_{xy} and ω_{yt} on the monomial do not change the exponents of z and u. Combining this with the fact that the exponents of z and u in the other monomials are not respectively 2 and 1, it implies $\omega_{yt}(\omega_{xy}(p)+p)$ contains $(k_{(3,2,2,3,1)}+k_{(2,3,2,3,1)})(0,1,2,7,1)$ as a term.

Similarly, $\omega_{yt}(\omega_{xy}(p)+p)$ contains $(k_{(3,2,1,3,2)}+k_{(2,3,1,3,2)})(0,1,1,7,2)$ as a term.

Further, both the exponents of z and u in the other monomials are not equal to 1. So, their image under the action ω_{yt},ω_{xy} does not contain the monomial $(0,2,1,7,1)$.

Hence

$$\begin{aligned}\omega_{yt}(\omega_{xy}(p)+p) = {} & (k_{(3,2,2,3,1)}+k_{(2,3,2,3,1)})(0,1,2,7,1)\\ & +(k_{(3,2,1,3,2)}+k_{(2,3,1,3,2)})(0,1,1,7,2)\\ & + \text{other term is not in } \pi_x(C).\end{aligned}$$

As $\omega_{xy}(p)+p=0,$ we have

$$(k_{(3,2,2,3,1)}+k_{(2,3,2,3,1)})(0,1,2,7,1)+(k_{(3,2,1,3,2)}+k_{(2,3,1,3,2)})(0,1,1,7,2)=0.$$

As shown in the proof of Lemma 3.3, this implies

$$k_{(3,2,2,3,1)}+k_{(2,3,2,3,1)}=0.$$

As a consequence

$$k_{(2,2,3,3,1)}=0. \qquad \square$$

4 The fifth algebraic transfer is not an epimorphism

The target of this section is to prove the following theorem, which is also numbered as Theorem 1.2 in the introduction.

Theorem 4.1 *The element* $P(h_2)\in \mathrm{Ext}_{\mathcal{A}}^{5,16}(\mathbb{F}_2,\mathbb{F}_2)$ *is not in the image of the algebraic transfer* $\mathrm{Tr}_5\colon \mathbb{F}_2\otimes_{GL_5} PH_{11}(B\mathbb{V}_5)\to \mathrm{Ext}_{\mathcal{A}}^{5,16}(\mathbb{F}_2,\mathbb{F}_2)$.

Proof According to Proposition 3.1, we have

$$(\mathbb{F}_2\otimes_{\mathcal{A}} P_5)_{11}^{GL_5}=0.$$

As $\mathbb{F}_2\otimes_{GL_5} PH_*(B\mathbb{V}_5)$ is dual to $(\mathbb{F}_2\otimes_{\mathcal{A}} P_5)^{GL_5}$, we get

$$\mathbb{F}_2\otimes_{GL_5} PH_{11}(B\mathbb{V}_5)=0.$$

It is well known (see, for example, M C Tangora [8] and R R Bruner [2]) that the element $P(h_2)$ is nonzero in $\mathrm{Ext}_{\mathcal{A}}^{5,16}(\mathbb{F}_2,\mathbb{F}_2)$. So, the fifth algebraic transfer

$$\mathrm{Tr}_5\colon \mathbb{F}_2\otimes_{GL_5} PH_{11}(B\mathbb{V}_5)\to \mathrm{Ext}_{\mathcal{A}}^{5,16}(\mathbb{F}_2,\mathbb{F}_2)$$

does not detect the nonzero element $P(h_2)$. $\square$

References

- [1] **J M Boardman**, *Modular representations on the homology of powers of real projective space*, from: "Algebraic topology (Oaxtepec, 1991)", Contemp. Math. 146, Amer. Math. Soc., Providence, RI (1993) 49–70 MR1224907
- [2] **R R Bruner**, *The cohomology of the mod 2 Steenrod algebra: A computer calculation*, WSU Research Report 37 (1997)
- [3] **R R Bruner**, **L M Hà**, **N H V Hưng**, *On the behavior of the algebraic transfer*, Trans. Amer. Math. Soc. 357 (2005) 473–487 MR2095619
- [4] **N H V Hưng**, *Spherical classes and the algebraic transfer*, Trans. Amer. Math. Soc. 349 (1997) 3893–3910 MR1433119
- [5] **N H V Hưng**, *The cohomology of the Steenrod algebra and representations of the general linear groups*, Trans. Amer. Math. Soc. 357 (2005) 4065–4089 MR2159700
- [6] **W M Singer**, *The transfer in homological algebra*, Math. Z. 202 (1989) 493–523 MR1022818
- [7] **W M Singer**, *On the action of Steenrod squares on polynomial algebras*, Proc. Amer. Math. Soc. 111 (1991) 577–583 MR1045150
- [8] **M C Tangora**, *On the cohomology of the Steenrod algebra*, Math. Z. 116 (1970) 18–64 MR0266205

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