

## The action of the primitive Steenrod–Milnor operations on the modular invariants

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We compute the action of the primitive Steenrod–Milnor operations on generators of algebras of invariants of subgroups of general linear group  $GL_n = GL(n, \mathbb{F}_p)$  in the polynomial algebra with  $p$  an odd prime number.

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### 1 Introduction

Let  $p$  be an odd prime number. Denote by  $GL_n = GL(n, \mathbb{F}_p)$  the general linear group over the prime field  $\mathbb{F}_p$  and  $T_n$  the Sylow  $p$ –subgroup of  $GL_n$  consisting of all upper triangular matrices with 1 on the main diagonal. For any integer  $d$ ,  $1 \leq d \leq p - 1$ , we set

$$SL_n^d = \{\omega \in GL_n; (\det \omega)^d = 1\}.$$

It is easy to see that  $SL_n^d$  is a subgroup of  $GL_n$  and  $SL_n^{p-1} = GL_n$ . Each subgroup of  $GL_n$  acts on  $P_n = E(x_1, \dots, x_n) \otimes \mathbb{F}_p(y_1, \dots, y_n)$  in the usual manner. Here and in what follows,  $E(\dots)$  and  $\mathbb{F}_p(\dots)$  are the exterior and polynomial algebras over  $\mathbb{F}_p$  generated by the indicated variables. We grade  $P_n$  by assigning  $\dim x_i = 1$  and  $\dim y_i = 2$ .

Dickson [1] showed that the invariant algebra  $\mathbb{F}_p(y_1, \dots, y_n)^{GL_n}$  is a polynomial algebra generated by the Dickson invariants  $Q_{n,s}$ ,  $0 \leq s < n$ . Huynh Mui [6; 7] computed the invariant algebras  $P_n^{T_n}$  and  $P_n^{SL_n^d}$  for  $d = 1, p - 1, (p - 1)/2$ . He proved that  $P_n^{T_n}$  is generated by  $V_m$ ,  $1 \leq m \leq n$ ,  $M_{m,s_1, \dots, s_k}$ ,  $0 \leq s_1 < \dots < s_k < m \leq n$  and that  $P_n^{SL_n^d}$  is generated by  $L_n^d$ ,  $Q_{n,s}$ ,  $1 \leq s < n$ ,  $M_{n,s_1, \dots, s_k}^{(d)}$ ,  $0 \leq s_1 < \dots < s_k < n$ . Here  $V_m, M_{n,s_1, \dots, s_k}^{(d)}$  are Mui invariants and  $L_n^d, Q_{n,s}$  are Dickson invariants (see Section 2). Note that  $M_{n,s_1, \dots, s_k}^{(1)} = M_{n,s_1, \dots, s_k}$ .

Let  $\mathcal{A}(p)$  be the mod  $p$  Steenrod algebra and let  $\tau_s, \xi_i$  be the Milnor elements of dimensions  $2p^s - 1, 2p^i - 2$  respectively in the dual algebra  $\mathcal{A}(p)^*$  of  $\mathcal{A}(p)$ . In [5], Milnor showed that as an algebra,

$$\mathcal{A}(p)^* = E(\tau_0, \tau_1, \dots) \otimes \mathbb{F}_p(\xi_1, \xi_2, \dots).$$

Then  $\mathcal{A}(p)^*$  has a basis consisting of all monomials

$$\tau_S \xi^R = \tau_{s_1} \dots \tau_{s_k} \xi_1^{r_1} \dots \xi_m^{r_m},$$

with  $S = (s_1, \dots, s_k)$ ,  $0 \leq s_1 < \dots < s_k$ ,  $R = (r_1, \dots, r_m)$ ,  $r_i \geq 0$ . Let  $\text{St}^{S,R} \in \mathcal{A}(p)$  denote the dual of  $\tau_S \xi^R$  with respect to that basis. Then  $\mathcal{A}(p)$  has a basis consisting of all operations  $\text{St}^{S,R}$ . For  $S = \emptyset, R = (r)$ ,  $\text{St}^{\emptyset,(r)}$  is nothing but the Steenrod operation  $P^r$ . So, we call  $\text{St}^{S,R}$  the Steenrod–Milnor operation of type  $(S, R)$ .

We have the Cartan formula

$$\text{St}^{S,R}(uv) = \sum_{\substack{S_1 \cup S_2 = S \\ R_1 + R_2 = R}} (-1)^{(\dim u + \ell(S_1))\ell(S_2)} (S : S_1, S_2) \text{St}^{S_1,R_1}(u) \text{St}^{S_2,R_2}(v),$$

where  $R_1 = (r_{1i}), R_2 = (r_{2i}), R_1 + R_2 = (r_{1i} + r_{2i}), S_1 \cap S_2 = \emptyset, u, v \in P_n, \ell(S_i)$  is the length of  $S_i$  and

$$(S : S_1, S_2) = \text{sign} \begin{pmatrix} s_1 & \dots & s_h & s_{h+1} & \dots & s_k \\ s_{11} & \dots & s_{1h} & s_{21} & \dots & s_{2r} \end{pmatrix},$$

with  $S_1 = (s_{11}, \dots, s_{1h}), s_{11} < \dots < s_{1h}, S_2 = (s_{21}, \dots, s_{2r}), s_{21} < \dots < s_{2r}$  (see Mui [7]).

We denote  $\text{St}_u = \text{St}^{(u),(0)}, \text{St}^{\Delta_i} = \text{St}^{\emptyset,\Delta_i}$ , where  $\Delta_i = (0, \dots, 1, \dots, 0)$  with 1 at the  $i$ -th place. In [7], Huỳnh Mùi proved that as a coalgebra,

$$\mathcal{A}(p) = \Lambda(\text{St}_0, \text{St}_1, \dots) \otimes \Gamma(\text{St}^{\Delta_1}, \text{St}^{\Delta_1}, \dots).$$

Here,  $\Lambda(\text{St}_0, \text{St}_1, \dots)$  (resp.  $\Gamma(\text{St}^{\Delta_1}, \text{St}^{\Delta_2}, \dots)$ ) denotes the exterior (resp. polynomial) Hopf algebra with divided powers generated by the primitive Steenrod–Milnor operations  $\text{St}_0, \text{St}_1, \dots$  (resp.  $\text{St}^{\Delta_1}, \text{St}^{\Delta_2}, \dots$ ).

The Steenrod algebra  $\mathcal{A}(p)$  acts on  $P_n$  by means of the Cartan formula together with the relations

$$\begin{aligned} \text{(i)} \quad \text{St}^{S,R} x_k &= \begin{cases} x_k, & S = \emptyset, R = (0), \\ y_k^{p^u}, & S = (u), R = (0), \\ 0, & \text{otherwise,} \end{cases} \\ \text{(ii)} \quad \text{St}^{S,R} y_k &= \begin{cases} y_k, & S = \emptyset, R = (0), \\ y_k^{p^i}, & S = \emptyset, R = \Delta_i, \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

for  $k = 1, 2, \dots, n$  (see Steenrod and Epstein [10] and Sum [13]). Since this action commutes with the action of  $GL_n$ , it induces actions of  $\mathcal{A}(p)$  on  $P_n^{T_n}$  and  $P_n^{SL_n^d}$ .

The action of  $\text{St}^{S,R}$  on the modular invariants of subgroups of general linear group has been studied by many authors. This action for  $S = \emptyset$ ,  $R = (r)$  was explicitly determined by Hưng and Minh [2], Kechagias [3], Madsen and Milgram [4] and Sum [13]. Smith and Switzer [9], Wilkerson [14] and Neusel [8] have studied the action of  $\text{St}^{\Delta^i}$  on the Dickson invariants.

The purpose of the paper is to compute the action of the primitive Steenrod–Milnor operations on generators of  $P_n^{T_n}$  and  $P_n^{SL_n^d}$ .

The rest of the paper contains three sections. In Section 2, we recall some needed information on the invariant theory and compute the action of the primitive Steenrod–Milnor operations on the determinant invariants. In Section 3, we compute the action of the primitive Steenrod–Milnor operations on Dickson and Mui invariants. Finally, we give in Section 4 some formulae from which we can describe our results in terms of Dickson and Mui invariants.

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## 2 Preliminaries

**Definition 2.1** Let  $(e_{k+1}, \dots, e_m)$ ,  $0 \leq k < m \leq n$ , be a sequence of nonnegative integers. Following Dickson [1] and Mui [6], we define

$$[k; e_{k+1}, \dots, e_m] = \frac{1}{k!} \begin{vmatrix} x_1 & \cdots & x_m \\ \vdots & \cdots & \vdots \\ x_1 & \cdots & x_m \\ y_1^{p^{e_{k+1}}} & \cdots & y_m^{p^{e_{k+1}}} \\ \vdots & \cdots & \vdots \\ y_1^{p^{e_m}} & \cdots & y_m^{p^{e_m}} \end{vmatrix}.$$

The precise meaning of the right hand side is given in [6]. For  $k = 0$ , we write

$$[0; e_1, \dots, e_m] = [e_1, \dots, e_m] = \det(y_i^{p^{e_j}}).$$

In particular, we set

$$\begin{aligned} L_{m,s} &= [0, 1, \dots, \hat{s}, \dots, m], \quad 0 \leq s \leq m \leq n, \\ L_m &= L_{m,m} = [0, 1, \dots, m-1], \\ M_{m,s_1, \dots, s_k} &= [k; 0, \dots, \hat{s}_1, \dots, \hat{s}_k, \dots, m-1], \end{aligned}$$

for  $0 \leq s_1 < \dots < s_k < m \leq n$ . Each  $[k; e_{k+1}, \dots, e_m]$  is an invariant of  $SL_m^1$  and  $[e_1, \dots, e_m]$  is divisible by  $L_m$ . Then, Dickson invariants  $Q_{n,s}$  and Mui invariants  $M_{n,s_1, \dots, s_k}^{(d)}$  and  $V_m$  are defined by

$$Q_{n,s} = L_{n,s}/L_n, \quad M_{n,s_1, \dots, s_k}^{(d)} = M_{n,s_1, \dots, s_k} L_n^{d-1} \quad \text{and} \quad V_m = L_m/L_{m-1}.$$

Here, by convention,  $L_0 = [\emptyset] = 1$ .

Now we prepare some data in order to prove our main results.

**Lemma 2.2** *Suppose  $e_\ell \neq e_j$  for  $\ell \neq j$ ,  $u \geq 0$ . Then we have*

$$\text{St}_u[k; e_{k+1}, \dots, e_n] = \begin{cases} (-1)^{k-1} [k-1; u, e_{k+1}, \dots, e_n], & k > 0, \\ 0, & k = 0. \end{cases}$$

**Proof** Let  $I$  be a subset of  $\{1, \dots, n\}$  and let  $I'$  be its complement in  $\{1, 2, \dots, n\}$ . Writing  $I = \{i_1, i_2, \dots, i_k\}$  and  $I' = \{i_{k+1}, i_{k+2}, \dots, i_n\}$  with  $i_1 < i_2 < \dots < i_k$  and  $i_{k+1} < i_{k+2} < \dots < i_n$ . We set

$$\begin{aligned} x_I &= x_{i_1} x_{i_2} \dots x_{i_k}, \\ [e_{k+1}, e_{k+2}, \dots, e_n]_I &= [e_{k+1}, e_{k+2}, \dots, e_n](y_{i_{k+1}}, y_{i_{k+2}}, \dots, y_{i_n}) \\ \sigma_I &= \begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix} \in \Sigma_n, \end{aligned}$$

where  $\Sigma_n$  is the symmetric group on  $n$  letters. Using the Laplace development, we have

$$[k; e_{k+1}, e_{k+2}, \dots, e_n] = \sum_I \text{sign } \sigma_I x_I [e_{k+1}, e_{k+2}, \dots, e_n]_I.$$

From the relation (ii), we see that  $\text{St}_u[e_{k+1}, e_{k+2}, \dots, e_n]_I = 0$ . Then, using the Cartan formula, we get

$$(1) \quad \text{St}_u[k; e_{k+1}, e_{k+2}, \dots, e_n] = \sum_I \text{sign } \sigma_I \text{St}_u(x_I)[e_{k+1}, e_{k+2}, \dots, e_n]_I.$$

In [7, 5.2], Mui showed that

$$\text{St}_u(x_I) = (-1)^{k-1} [k-1; u](x_{i_1}, x_{i_2}, \dots, x_{i_k}, y_{i_1}, y_{i_2}, \dots, y_{i_k}).$$

Hence, using (1) and the Laplace development we obtain the lemma. □

**Lemma 2.3** *Suppose  $e_\ell \neq e_j$  for  $\ell \neq j$ ,  $e_{k+1} < e_j$  for  $j > k + 1$ . Then we have*

$$\text{St}^{\Delta_i}[k; e_{k+1}, \dots, e_n] = \begin{cases} [k; i, e_{k+2}, \dots, e_n], & e_{k+1} = 0, \\ 0, & e_{k+1} > 0. \end{cases}$$

**Proof** Suppose  $e_{k+1} > 0$ . From the relations (i) and (ii) and the Cartan formula, we easily obtain

$$\text{St}^{\Delta_i} x_\ell = 0, \quad \text{St}^{\Delta_i} y_\ell^{p^{e_j}} = p^{e_j} y_\ell^{p^{e_j+p^i-1}} = 0,$$

for  $\ell = 1, 2, \dots, n$  and  $j = k + 1, k + 2, \dots, n$ . From this, we get

$$\text{St}^{\Delta_i} [k; e_{k+1}, \dots, e_n] = 0.$$

If  $e_{k+1} = 0$  then  $\text{St}^{\Delta_i} y_\ell^{p^{e_j}} = 0$ , for  $\ell = 1, 2, \dots, n$  and  $j = k + 2, \dots, n$ , and

$$\text{St}^{\Delta_i} y_\ell^{p^{e_{k+1}}} = \text{St}^{\Delta_i} y_\ell = y_\ell^{p^i}.$$

Hence, using the Laplace development and the Cartan formula, we obtain

$$\text{St}^{\Delta_i} [k; e_{k+1}, e_{k+2}, \dots, e_n] = [k; i, e_{k+2}, \dots, e_n]. \quad \square$$

To make the paper self-contained, we give here a proof for the following theorem, which will be needed in the next section.

**Theorem 2.4** (Sum [12]) *Let  $(e_1, e_2, \dots, e_n)$  be a sequence of nonnegative integers and  $0 \leq k < n$ . We have*

$$\begin{aligned} & [e_1, e_2, \dots, e_{n-1}, e_n + n - 1] \\ (2) \quad & = \sum_{s=0}^{n-2} (-1)^{n+s} [e_1, e_2, \dots, e_{n-1}, e_n + s] Q_{n-1,s}^{p^{e_n}} + [e_1, e_2, \dots, e_{n-1}] V_n^{p^{e_n}}, \\ (3) \quad & [k; e_{k+1}, \dots, e_{n-1}, e_n + n] = \sum_{s=0}^{n-1} (-1)^{n+s-1} [k; e_{k+1}, \dots, e_{n-1}, e_n + s] Q_{n,s}^{p^{e_n}}. \end{aligned}$$

**Proof** We recall Mui’s formula in [6],

$$[k; e_{k+1}, \dots, e_n] = \sum_{0 \leq s_1 < \dots < s_k} (-1)^{k(k-1)/2} (-1)^{s_1 + \dots + s_k} M_{n,s_1, \dots, s_k} [s_1, \dots, s_k, e_{k+1}, \dots, e_n] / L_n.$$

Hence, it suffices to prove the theorem for  $k = 0$ .

The proof of the theorem proceeds by induction on  $n$ . It is easy to see that the theorem holds for  $n = 1$ . Suppose  $n \geq 2$  and the theorem holds for  $n - 1$ .

Using the Laplace development and the inductive hypothesis, we have

$$\begin{aligned}
 & [e_1, \dots, e_{n-1}, e_n + n - 1] \\
 &= \sum_{t=1}^{n-1} (-1)^{n+t} [e_1, \dots, \hat{e}_t, \dots, e_{n-1}, e_n + n - 1] y_n^{p^{e_t}} + [e_1, \dots, e_{n-1}] y_n^{p^{e_n+n-1}} \\
 &= \sum_{t=1}^{n-1} (-1)^{n+t} \left( \sum_{s=0}^{n-2} (-1)^{n+s} [e_1, \dots, \hat{e}_t, \dots, e_{n-1}, e_n + s] Q_{n-1,s}^{p^{e_n}} \right) y_n^{p^{e_t}} \\
 & \qquad \qquad \qquad + [e_1, \dots, e_{n-1}] y_n^{p^{e_n+n-1}} \\
 &= \sum_{s=0}^{n-2} (-1)^{n+s} \left( \sum_{t=1}^{n-1} (-1)^{n+t} [e_1, \dots, \hat{e}_t, \dots, e_{n-1}, e_n + s] y_n^{p^{e_t}} \right) Q_{n-1,s}^{p^{e_n}} \\
 & \qquad \qquad \qquad + [e_1, \dots, e_{n-1}] y_n^{p^{e_n+n-1}} \\
 &= \sum_{s=0}^{n-2} (-1)^{n+s} [e_1, \dots, e_{n-1}, e_n + s] Q_{n-1,s}^{p^{e_n}} \\
 & \qquad \qquad \qquad + [e_1, \dots, e_{n-1}] \sum_{s=0}^{n-1} (-1)^{n+s-1} Q_{n-1,s}^{p^{e_n}} y_n^{p^{e_n+s}}.
 \end{aligned}$$

Since  $V_n = \sum_{s=0}^{n-1} (-1)^{n+s-1} Q_{n-1,s} y_n^{p^s}$ , the relation (2) holds for  $n$ .

Now we prove the relation (3) for  $n$ . By a direct calculation using (2) and the relation  $Q_{n,s} = Q_{n-1,s-1}^p + Q_{n-1,s} V_n^{p-1}$ , we get

$$\begin{aligned}
 & [e_1, e_2, \dots, e_{n-1}, e_n + n] \\
 &= \sum_{s=1}^{n-1} (-1)^{n+s-1} [e_1, \dots, e_{n-1}, e_n + s] Q_{n-1,s-1}^{p^{e_n+1}} + [e_1, \dots, e_{n-1}] V_n^{p^{e_n+1}} \\
 &= \sum_{s=1}^{n-1} (-1)^{n+s-1} [e_1, \dots, e_{n-1}, e_n + s] Q_{n,s}^{p^{e_n}} \\
 & \qquad \qquad \qquad - [e_1, \dots, e_{n-1}, e_n + n - 1] V_n^{(p-1)p^{e_n}} \\
 & \qquad \qquad \qquad + \left( \sum_{s=1}^{n-2} (-1)^{n+s} [e_1, \dots, e_{n-1}, e_n + s] Q_{n-1,s}^{p^{e_n}} + [e_1, \dots, e_{n-1}] V_n^{p^{e_n}} \right) V_n^{(p-1)p^{e_n}}.
 \end{aligned}$$

Combining this equality and the relation (2) we obtain

$$[e_1, e_2, \dots, e_{n-1}, e_n + n] = \sum_{s=1}^{n-1} (-1)^{n+s-1} [e_1, \dots, e_{n-1}, e_n + s] Q_{n,s}^{p^{en}} - (-1)^n [e_1, \dots, e_{n-1}, e_n] Q_{n-1,0}^{p^{en}} V_n^{(p-1)p^{en}}.$$

Since  $Q_{n,0} = Q_{n-1,0} V_n^{p-1}$ , the relation (3) holds for  $n$ .

This completes the proof of Theorem 2.4. □

### 3 Main results

Observe that using the Cartan formula and the relations (i) and (ii), we obtain  $\text{St}_u x = 0$  for either  $x = Q_{n,s}$  or  $x = V_n$ . So, in this section we only compute  $\text{St}^{\Delta_i} x$  for  $x = Q_{n,s}, V_n, M_{n,s_1, \dots, s_k}^{(d)}$  and  $\text{St}_u M_{n,s_1, \dots, s_k}^{(d)}$ .

**Theorem 3.1** For any integers  $i, n, s$  with  $0 \leq s < n$  and  $i \geq 1$ , we have

$$\text{St}^{\Delta_i} Q_{n,s} = (-1)^n [0, 1, \dots, \hat{s}, \dots, n-1, i] L_n^{p-2}.$$

**Proof** Since  $L_{n,s} = L_n Q_{n,s}$ , using the Cartan formula, we get

$$(4) \quad \text{St}^{\Delta_i} L_{n,s} = L_n \text{St}^{\Delta_i} Q_{n,s} + Q_{n,s} \text{St}^{\Delta_i} L_n.$$

According to Lemma 2.3, we have

$$\text{St}^{\Delta_i} L_{n,s} = \begin{cases} [i, 1, 2, \dots, \hat{s}, \dots, n], & s > 0, \\ 0, & s = 0. \end{cases}$$

In particular,  $\text{St}^{\Delta_i} L_n = [i, 1, 2, \dots, n-1]$ .

If  $s = 0$  then  $\text{St}^{\Delta_i} L_{n,s} = 0$  and

$$\begin{aligned} \text{St}^{\Delta_i} L_n &= [i, 1, 2, \dots, n-1] \\ &= (-1)^{n-1} [1, 2, \dots, n-1, i]. \end{aligned}$$

Combining (4) and the above equalities, we get

$$\begin{aligned} \text{St}^{\Delta_i} Q_{n,0} &= -(\text{St}^{\Delta_i} L_n) Q_{n,0} / L_n \\ &= (-1)^n [1, 2, \dots, n-1, i] Q_{n,0} / L_n. \end{aligned}$$

Since  $Q_{n,0} = L_n^{p-1}$ , the theorem holds.

If  $s > 0$  then  $\text{St}^{\Delta_i} L_n = [i, 1, 2, \dots, n - 1]$  and  $\text{St}^{\Delta_i} L_{n,s} = [i, 1, 2, \dots, \hat{s}, \dots, n]$ . Hence, using Theorem 2.4, we get

$$\begin{aligned} \text{St}^{\Delta_i} L_{n,s} &= \sum_{t=0}^{n-1} (-1)^{n-1+t} [i, 1, 2, \dots, \hat{s}, \dots, n-1, t] Q_{n,t} \\ &= (-1)^{n-1} [i, 1, 2, \dots, \hat{s}, \dots, n-1, 0] Q_{n,0} \\ &\quad + (-1)^{n-1+s} [i, 1, 2, \dots, \hat{s}, \dots, n-1, s] Q_{n,s} \\ &= [i, 1, 2, \dots, n-1] Q_{n,s} - [i, 0, 1, \dots, \hat{s}, \dots, n-1] Q_{n,0}. \end{aligned}$$

Combining (4), the above equalities and the relation  $Q_{n,0} = L_n^{p-1}$ , we get

$$\begin{aligned} \text{St}^{\Delta_i} Q_{n,s} &= (\text{St}^{\Delta_i} L_{n,s} - Q_{n,s} \text{St}^{\Delta_i} L_n) / L_n \\ &= -[i, 0, 1, 2, \dots, \hat{s}, \dots, n-1] Q_{n,0} / L_n \\ &= (-1)^n [0, 1, 2, \dots, \hat{s}, \dots, n-1, i] L_n^{p-2}. \quad \square \end{aligned}$$

The following was proved in Smith and Switzer [9] by another method.

**Corollary 3.2** (Smith–Switzer [9]) *For any integers  $i, n, s$  with  $0 \leq s < n$  and  $1 \leq i \leq n$ , we have*

$$\text{St}^{\Delta_i} Q_{n,s} = \begin{cases} (-1)^{s-1} Q_{n,0}, & i = s > 0, \\ (-1)^n Q_{n,0} Q_{n,s}, & i = n, \\ 0, & \text{otherwise.} \end{cases}$$

**Proof** Suppose  $i = s$ . According to Theorem 3.1, we have

$$\begin{aligned} \text{St}^{\Delta_s} Q_{n,s} &= (-1)^n [0, 1, \dots, \hat{s}, \dots, n-1, s] L_n^{p-2} \\ &= (-1)^{s-1} [0, 1, \dots, n-1] L_n^{p-2} \\ &= (-1)^{s-1} L_n^{p-1} = (-1)^{s-1} Q_{n,0}. \end{aligned}$$

If  $i < n$  and  $i \neq s$  then  $[0, 1, \dots, \hat{s}, \dots, n-1, i] = 0$ . Hence,  $\text{St}^{\Delta_i} Q_{n,s} = 0$ .

If  $i = n$  then

$$\begin{aligned} \text{St}^{\Delta_n} Q_{n,s} &= (-1)^n [0, 1, \dots, \hat{s}, \dots, n-1, n] L_n^{p-2} \\ &= (-1)^n L_{n,s} L_n^{p-2} \\ &= (-1)^n L_n^{p-1} Q_{n,s} \\ &= (-1)^n Q_{n,0} Q_{n,s}. \end{aligned}$$

The corollary follows. □



Now, we show that our formula in Theorem 3.1 implies Wilkerson’s formula in [14]. To do this, we need the following.

**Proposition 3.3** (Sum [12]) *Let  $(e_{k+1}, e_{k+2}, \dots, e_n)$  be a sequence of nonnegative integers with  $0 \leq k < n$  and  $e_\ell \neq e_j$  for  $\ell \neq j$ . Then*

$$P^r[k; e_{k+1}, e_{k+2}, \dots, e_n] = \begin{cases} [k; e_{k+1} + \varepsilon_{k+1}, e_{k+2} + \varepsilon_{k+2}, \dots, e_n + \varepsilon_n], \\ \quad \text{if } r = \sum_{j=k+1}^n \varepsilon_j p^{e_j} \text{ with } \varepsilon_j \in \{0, 1\}, \\ 0, \quad \text{otherwise.} \end{cases}$$

This proposition can easily be proved by using the Laplace development, the Cartan formula and the relations (i) and (ii).

From the formula in Theorem 3.1, one gets Wilkerson’s formula as follows.

**Theorem 3.4** (Wilkerson [14]) *For any integers  $0 \leq s < n \leq i$ , we have*

$$\text{St}^{\Delta_{i+1}} Q_{n,s} = P^{p^i} \text{St}^{\Delta_i} Q_{n,s}.$$

**Proof** Applying Theorem 3.1, the Cartan formula and Proposition 3.3, we get

$$\begin{aligned} P^{p^i} \text{St}^{\Delta_i} Q_{n,s} &= (-1)^n P^{p^i} ([0, 1, \dots, \hat{s}, \dots, n-1, i] L_n^{p-2}) \\ &= (-1)^n \sum_r P^r ([0, 1, \dots, \hat{s}, \dots, n-1, i]) P^{p^i-r} (L_n^{p-2}), \end{aligned}$$

where the sum runs over all

$$r = \varepsilon_0 p^0 + \varepsilon_1 p^1 + \dots + \varepsilon_{s-1} p^{s-1} + \varepsilon_{s+1} p^{s+1} + \dots + \varepsilon_{n-1} p^{n-1} + \varepsilon_i p^i$$

with  $\varepsilon_j \in \{0, 1\}$  for any  $j$  and  $r \leq p^i$ .

If  $\varepsilon_i = 0$  then  $r < p^0 + p^1 + \dots + p^{n-1}$  and

$$\begin{aligned} 2(p^i - r) &> 2(p^i - (p^0 + p^1 + \dots + p^{n-1})) \\ &= 2(p^i - p^n + 1 + (p-2)(p^0 + p^1 + \dots + p^{n-1})) \\ &> 2(p-2)(p^0 + p^1 + \dots + p^{n-1}) = \dim L_n^{p-2}. \end{aligned}$$

This implies  $P^{p^i-r} (L_n^{p-2}) = 0$ .

Since  $r \leq p^i$ , if  $\varepsilon_i = 1$  then  $\varepsilon_j = 0$  for  $j \neq i$  and  $r = p^i$ . Hence, using the above equalities and Proposition 3.3, we obtain

$$\begin{aligned} P^{p^i} \text{St}^{\Delta_i} Q_{n,s} &= (-1)^n P^{p^i} ([0, 1, \dots, \widehat{s}, \dots, n-1, i]) L_n^{p-2} \\ &= (-1)^n [0, 1, \dots, \widehat{s}, \dots, n-1, i+1] L_n^{p-2} \\ &= \text{St}^{\Delta_{i+1}} Q_{n,s}. \end{aligned} \quad \square$$

Next, we compute the action of  $\text{St}^{\Delta_i}$  on Mui invariants.

**Theorem 3.5** *For any positive integers  $i, n$ , we have*

$$\text{St}^{\Delta_i} V_n = (-1)^{n-1} [0, 1, \dots, n-2, i] L_{n-1}^{p-2}.$$

**Proof** Since  $L_n = L_{n-1} V_n$ , applying the Cartan formula, we get

$$(5) \quad \text{St}^{\Delta_i} L_n = L_{n-1} \text{St}^{\Delta_i} V_n + V_n \text{St}^{\Delta_i} L_{n-1}.$$

Using Lemma 2.3 and Theorem 2.4, we have

$$\begin{aligned} \text{St}^{\Delta_i} L_{n-1} &= [i, 1, 2, \dots, n-2], \\ \text{St}^{\Delta_i} L_n &= [i, 1, 2, \dots, n-2, n-1] \\ &= \sum_{s=0}^{n-2} (-1)^{n+s} [i, 1, 2, \dots, n-2, s] Q_{n-1,s} + [i, 1, 2, \dots, n-2] V_n \\ &= (-1)^n [i, 1, 2, \dots, n-2, 0] Q_{n-1,0} + [i, 1, 2, \dots, n-2] V_n. \end{aligned}$$

Combining (5), the above equalities and the relation  $Q_{n-1,0} = L_{n-1}^{p-1}$ , we get

$$\begin{aligned} \text{St}^{\Delta_i} V_n &= (\text{St}^{\Delta_i} L_n - V_n \text{St}^{\Delta_i} L_{n-1}) / L_{n-1} \\ &= (-1)^n [i, 1, 2, \dots, n-2, 0] Q_{n-1,0} / L_{n-1} \\ &= (-1)^{n-1} [0, 1, 2, \dots, n-2, i] L_{n-1}^{p-2}. \end{aligned} \quad \square$$

**Corollary 3.6** *For any integers  $0 < i \leq n$ , we have*

$$\text{St}^{\Delta_i} V_n = \begin{cases} 0, & i < n-1, \\ (-1)^{n-1} Q_{n-1,0} V_n, & i = n-1, \\ (-1)^{n-1} Q_{n-1,0} (Q_{n-1,n-2}^p V_n + V_n^p), & i = n. \end{cases}$$

**Proof** If  $i < n - 1$  then  $[0, 1, \dots, n - 2, i] = 0$ . Hence,  $\text{St}^{\Delta_i} V_n = 0$ .

For  $i = n - 1$ , we have  $[0, 1, 2, \dots, n - 2, n - 1] = L_n = L_{n-1} V_n$ . Hence, from Theorem 3.5, we get

$$\text{St}^{\Delta_{n-1}} V_n = (-1)^{n-1} L_{n-1}^{p-1} V_n = (-1)^{n-1} Q_{n-1,0} V_n.$$

Let  $i = n$ . A direct computation shows

$$\begin{aligned} [0, 1, \dots, n - 2, n] &= L_{n,n-1} = L_n Q_{n,n-1} \\ &= L_{n-1} V_n (Q_{n-1,n-2}^p + V_n^{p-1}). \end{aligned}$$

From the above equalities, Theorem 3.5 and the relation  $L_{n-1}^{p-1} = Q_{n-1,0}$ , we obtain

$$\text{St}^{\Delta_n} V_n = (-1)^{n-1} Q_{n-1,0} (Q_{n-1,n-2}^p V_n + V_n^p).$$

The corollary follows. □

**Theorem 3.7** Set  $s_0 = 0$ . Then  $\text{St}^{\Delta_i} M_{n,s_1,\dots,s_k}^{(d)}$  equals

$$\begin{cases} (-1)^{s_t-t} M_{n,s_0,\dots,\hat{s}_t,\dots,s_k}^{(d)}, & s_1 > 0, i = s_t, 1 \leq t \leq k, \\ (-1)^{n-1} (d-1) M_{n,s_1,\dots,s_k} [1, 2, \dots, n-1, i] L_n^{d-2}, & i \geq n, s_1 = 0, \\ (-1)^{n-1} ((-1)^k [k; 1, \dots, \hat{s}_1, \dots, \hat{s}_k, \dots, n-1, i] L_n^{d-1} \\ \quad + (d-1) M_{n,s_1,\dots,s_k} [1, 2, \dots, n-1, i] L_n^{d-2}), & i \geq n, s_1 > 0, \\ 0, & \text{otherwise.} \end{cases}$$

**Proof** Applying Lemma 2.2, we have

$$\text{St}^{\Delta_i} M_{n,s_1,\dots,s_k} = \begin{cases} [k; i, 1, \dots, \hat{s}_1, \dots, \hat{s}_k, \dots, n-1], & s_1 > 0, \\ 0, & s_1 = 0. \end{cases}$$

If  $i = s_t$  then  $[k; i, 1, \dots, \hat{s}_1, \dots, \hat{s}_k, \dots, n-1] = (-1)^{s_t-t} M_{n,s_0,\dots,\hat{s}_t,\dots,s_k}$ .

If  $i \geq n$  then

$$[k; i, 1, \dots, \hat{s}_1, \dots, \hat{s}_k, \dots, n-1] = (-1)^{n-k-1} [k; 1, \dots, \hat{s}_1, \dots, \hat{s}_k, \dots, n-1, i].$$

Thus the theorem is proved for  $d = 1$ .

For  $d > 1$ , using Lemma 2.2 and the Cartan formula, we have

$$\begin{aligned} \text{St}^{\Delta_i} L_n^{d-1} &= (d-1) L_n^{d-2} \text{St}^{\Delta_i} L_n, \\ \text{St}^{\Delta_i} L_n &= (-1)^{n-1} [1, 2, \dots, n-1, i], \\ \text{St}^{\Delta_i} M_{n,s_1,\dots,s_k}^{(d)} &= \text{St}^{\Delta_i} (M_{n,s_1,\dots,s_k}) L_n^{d-1} + (d-1) M_{n,s_1,\dots,s_k} L_n^{d-2} \text{St}^{\Delta_i} L_n. \end{aligned}$$

Combining the above equalities we obtain the theorem. □

**Theorem 3.8** For  $1 \leq d \leq p - 1$ , we have

$$\text{St}_u M_{n,s_1,\dots,s_k}^{(d)} = \begin{cases} (-1)^{k+s_t-t} M_{n,s_1,\dots,\widehat{s}_t,\dots,s_k}^{(d)}, & u = s_t, \\ (-1)^{n-1} [k - 1; 0, \dots, \widehat{s}_1, \dots, \widehat{s}_k, \dots, n - 1, u] L_n^{d-1}, & u \geq n, \\ 0, & \text{otherwise.} \end{cases}$$

**Proof** Since  $M_{n,s_1,\dots,s_k} = [k; 0, \dots, \widehat{s}_1, \dots, \widehat{s}_k, \dots, n - 1]$ , applying Lemma 2.2, we get

$$\text{St}_u M_{n,s_1,\dots,s_k} = (-1)^{k-1} [k - 1; u, 0, \dots, \widehat{s}_1, \dots, \widehat{s}_k, \dots, n - 1].$$

If  $0 \leq u \leq n - 1$  then

$$[k - 1; u, 0, \dots, \widehat{s}_1, \dots, \widehat{s}_k, \dots, n - 1] = \begin{cases} (-1)^{s_t-t+1} M_{n,s_1,\dots,\widehat{s}_t,\dots,s_k}, & u = s_t, \\ 0, & \text{otherwise.} \end{cases}$$

If  $u > n - 1$  then we have

$$\begin{aligned} [k - 1; u, 0, \dots, \widehat{s}_1, \dots, \widehat{s}_k, \dots, n - 1] \\ = (-1)^{n-k} [k - 1; 0, \dots, \widehat{s}_1, \dots, \widehat{s}_k, \dots, n - 1, u]. \end{aligned}$$

The theorem is proved for  $d = 1$ .

Since  $\text{St}_u L_n = 0$ , using the Cartan formula, we get

$$\text{St}_u (M_{n,s_1,\dots,s_k}^{(d)}) = \text{St}_u (M_{n,s_1,\dots,s_k}) L_n^{d-1}.$$

The theorem now follows from the above equalities. □

By the analogous argument as given in the proof of Theorem 3.4, we can show that the Wilkerson formula also holds for Mũi invariants.

**Theorem 3.9** For any integers  $i, u \geq n$ , we have

$$\begin{aligned} \text{St}^{\Delta_i} V_n &= P^{p^{i-1}} \text{St}^{\Delta_{i-1}} V_n, \\ \text{St}^{\Delta_{i+1}} M_{n,s_1,\dots,s_k}^{(d)} &= P^{p^i} \text{St}^{\Delta_i} M_{n,s_1,\dots,s_k}^{(d)}, \\ \text{St}_{u+1} M_{n,s_1,\dots,s_k}^{(d)} &= P^{p^u} \text{St}_u M_{n,s_1,\dots,s_k}^{(d)}. \end{aligned}$$

**Remark 3.10** Using Theorem 2.4 and the above results, we can compute the action of the primitive Steenrod–Milnor operations on the modular invariants in terms of Dickson and Mui invariants for  $i, u \geq n$ . For example, by a direct calculation, we easily obtain

$$\begin{aligned} \text{St}^{\Delta_{n+1}} Q_{n,s} &= (-1)^n Q_{n,0} (Q_{n,n-1}^p Q_{n,s} - Q_{n,s-1}^p), \\ \text{St}^{\Delta_{n+2}} Q_{n,s} &= (-1)^n Q_{n,0} (Q_{n,n-1}^{p^2+p} Q_{n,s} - Q_{n,n-2}^{p^2} Q_{n,s} + Q_{n,s-2}^{p^2} - Q_{n,s-1}^p Q_{n,n-1}^{p^2}). \end{aligned}$$

Here, by convention,  $Q_{n,t} = 0$  for  $t < 0$ .

$$\begin{aligned} \text{St}^{\Delta_{n+1}} V_n &= (-1)^{n-1} Q_{n-1,0} ((Q_{n-1,n-2}^{p^2+p} - Q_{n-1,n-3}^{p^2}) V_n + Q_{n-1,n-2}^{p^2} V_n^p + V_n^{p^2}), \\ \text{St}_n M_{n,s_1,\dots,s_k}^{(d)} &= \sum_{t=1}^k (-1)^{n-1+k-t} M_{n,s_1,\dots,\hat{s}_t,\dots,s_k}^{(d)} Q_{n,s_t}, \\ \text{St}^{\Delta_n} M_{n,s_1,\dots,s_k}^{(d)} &= (-1)^{n-1} \left( \sum_{t=1}^k (-1)^t M_{n,s_0,\dots,\hat{s}_t,\dots,s_k}^{(d)} Q_{n,s_t} + d M_{n,s_1,\dots,s_k}^{(d)} Q_{n,0} \right), \end{aligned}$$

where  $s_0 = 0$  and  $s_1 > 0$ . If  $s_1 = 0$  then

$$\text{St}^{\Delta_n} M_{n,s_1,\dots,s_k}^{(d)} = (-1)^{n-1} (d-1) M_{n,s_1,\dots,s_k}^{(d)} Q_{n,0}.$$

Furthermore, the computation of the action of the primitive Steenrod–Milnor operations on the modular invariants in terms of Dickson and Mui invariants by the use of our results in this section is more convenient than that by using Wilkerson’s formula. For example, to compute  $\text{St}^{\Delta_{n+2}} Q_{n,s}$  by using Wilkerson’s formula, we need to compute  $P^{p^{n+1}}(Q_{n,0}(Q_{n,n-1}^p Q_{n,s} - Q_{n,s-1}^p))$  in terms of Dickson invariants. But computing  $P^{p^{n+1}}(Q_{n,0}(Q_{n,n-1}^p Q_{n,s} - Q_{n,s-1}^p))$  is more difficult than that of  $[0, 1, \dots, \hat{s}, \dots, n-1, n+2]$ .

## 4 On the description of the determinant invariants in terms of Dickson and Mui invariants

In this section, we study the problem of description of the determinant invariants in terms of Dickson and Mui invariants. The explicit formulae for the determinant invariants in terms of Dickson and Mui invariants are useful tools for computing the action of the cohomology operations on the modular invariants.

In general, it is difficult to give explicit formulae for this problem. In particular, for  $n = 2, 3$ , we can explicitly compute  $[u, v], [u, v, w]$  in terms of Mui invariants

and  $[u, v], [u, v, v + 1]$  in terms of Dickson invariants, where  $u, v, w$  are nonnegative integers.

Note that the problem of description of  $[u, v, w]$  in terms of Dickson invariants is complicated. It is still open.

**Proposition 4.1** For  $0 \leq u < v < w$ , we have

$$(6) \quad [u, v] = \sum_{s=u}^{v-1} V_1^{p^v - p^{s+1} + p^u} V_2^{p^s},$$

$$(7) \quad [u, v, w] = \sum_{s=u}^{v-1} [u, s + 1][v, w] L_2^{-p^{s+1}} V_3^{p^s} + \sum_{s=v}^{w-1} [u, v][s + 1, w] L_2^{-p^{s+1}} V_3^{p^s}.$$

**Proof** The relation (6) is proved by induction on  $v$ . We prove (7) by induction on  $v, w$ . Applying Theorem 2.4, we can easily prove the following by induction on  $v$

$$(8) \quad [u, v, v + 1] = \sum_{s=u}^{v-1} [u, s + 1] L_2^{p^v - p^{s+1}} V_3^{p^s}.$$

Since  $L_2^{p^v} = [v, v + 1]$ , the relation (7) holds for  $w = v + 1$ .

Let  $w = v + 2$ . By a direct computation using Theorem 2.4 and (8), we have

$$\begin{aligned} [u, v, v + 2] &= [u, v, v + 1] Q_{2,1}^{p^v} + [u, v] V_3^{p^v} \\ &= \sum_{s=u}^{v-1} [u, s + 1] L_2^{p^v - p^{s+1}} V_3^{p^s} Q_{2,1}^{p^v} + [u, v] V_3^{p^v}. \end{aligned}$$

We observe that  $(L_2 Q_{2,1})^{p^v} = [v, v + 2]$ ,  $L_2^{p^{v+1}} = [v + 1, v + 2]$ . Hence, the relation (7) holds for  $w = v + 2$ . Suppose that (7) holds for  $w$  and  $w + 1$ . It is easy to see that

$$[w + 1, w] Q_{2,0}^{p^w} = -L_2^{p^{w+1}}.$$

Hence, using Theorem 2.4 and the inductive hypothesis, we get

$$\begin{aligned} [u, v, w + 2] &= [u, v, w + 1] Q_{2,1}^{p^w} - [u, v, w] Q_{2,0}^{p^w} + [u, v] V_3^{p^w} \\ &= \left( \sum_{s=u}^{v-1} [u, s + 1][v, w + 1] L_2^{-p^{s+1}} V_3^{p^s} \right. \\ &\quad \left. + \sum_{s=v}^w [u, v][s + 1, w + 1] L_2^{-p^{s+1}} V_3^{p^s} \right) Q_{2,1}^{p^w} \end{aligned}$$

$$\begin{aligned}
 & - \left( \sum_{s=u}^{v-1} [u, s+1][v, w] L_2^{-p^{s+1}} V_3^{p^s} \right. \\
 & \quad \left. + \sum_{s=v}^{w-1} [u, v][s+1, w] L_2^{-p^{s+1}} V_3^{p^s} \right) Q_{2,0}^{p^w} + [u, v] V_3^{p^w} \\
 &= \sum_{s=u}^{v-1} [u, s+1] ([v, w+1] Q_{2,1}^{p^w} - [v, w] Q_{2,0}^{p^w}) L_2^{-p^{s+1}} V_3^{p^s} \\
 & \quad + \sum_{s=v}^w [u, v] ([s+1, w+1] Q_{2,1}^{p^w} - [s+1, w] Q_{2,0}^{p^w}) L_2^{-p^{s+1}} V_3^{p^s}.
 \end{aligned}$$

This equality and Theorem 2.4 imply the relation (7) for  $w + 2$ , completing the proof.  $\square$

Now, we compute  $[u, v]$  in terms of  $L_2$  and  $Q_{2,1}$ .

Let  $\alpha_i(a)$  denote the  $i$ -th coefficient in  $p$ -adic expansion of a nonnegative integer  $a$ . That means

$$a = \alpha_0(a)p^0 + \alpha_1(a)p^1 + \alpha_2(a)p^2 + \dots,$$

for  $0 \leq \alpha_i(a) < p, i \geq 0$ . We set  $\alpha_i(a) = 0$  for  $i < 0$ .

Denote by  $I(u, v)$  the set of all integers  $a$  satisfying

$$\begin{aligned}
 & \alpha_i(a) + \alpha_{i+1}(a) \leq 1, \quad \text{for any } i, \\
 & \alpha_i(a) = 0, \quad \text{for either } i < u \text{ or } i \geq v - 2.
 \end{aligned}$$

The following was proved in Sum [11] for  $p = 2$ .

**Proposition 4.2** *Under the above notation, we have*

$$[u, v] = \sum_{a \in I(u, v)} (-1)^a L_2^{p^u + p(p-1)a} Q_{2,1}^{\frac{p^{v-1} - p^u}{p-1} - (p+1)a}.$$

**Proof** The proof is by induction on  $v$ . Obviously,  $I(u, u + 1) = I(u, u + 2) = \{0\}$  and  $[u, u + 1] = L_2^{p^u}$ ,  $[u, u + 2] = L_2^{p^u} Q_{2,1}^{p^u}$ . Hence, the proposition follows with  $v = u + 1$  and  $v = u + 2$ . From the definition of the set  $I(u, v)$ , we obtain

$$(9) \quad I(u, v + 2) = I(u, v + 1) \cup (p^{v-1} + I(u, v)),$$

where  $p^{v-1} + I(u, v) = \{p^{v-1} + a; a \in I(u, v)\}$ .

Combining Theorem 2.4, the inductive hypothesis and the relation  $Q_{2,0} = L_2^{p-1}$ , we get

$$\begin{aligned} [u, v + 2] &= [u, v + 1]Q_{2,1}^{p^v} - [u, v]Q_{2,0}^{p^v} \\ &= \left( \sum_{a \in I(u, v+1)} (-1)^a L_2^{p^u + p(p-1)a} Q_{2,1}^{\frac{p^v - p^u}{p-1} - (p+1)a} \right) Q_{2,1}^{p^v} \\ &\quad - \left( \sum_{a \in I(u, v)} (-1)^a L_2^{p^u + p(p-1)a} Q_{2,1}^{\frac{p^{v-1} - p^u}{p-1} - (p+1)a} \right) Q_{2,0}^{p^v} \\ &= \sum_{a \in I(u, v+1)} (-1)^a L_2^{p^u + p(p-1)a} Q_{2,1}^{\frac{p^{v+1} - p^u}{p-1} - (p+1)a} \\ &\quad + \sum_{a \in I(u, v)} (-1)^{p^{v-1} + a} L_2^{p^u + p(p-1)(p^{v-1} + a)} Q_{2,1}^{\frac{p^{v+1} - p^u}{p-1} - (p+1)(p^{v-1} + a)}. \end{aligned}$$

From this equality and (9), we see that the proposition is true for  $v + 2$ , so the proof is completed. □

Now, we compute  $[u, v, v + 1]$  in terms of  $L_3, Q_{3,1}, Q_{3,2}$ .

Denote by  $J(u, v)$  the set of all integers  $a$  satisfying

$$\begin{aligned} \alpha_i(a) \leq 1 \quad \text{and} \quad \alpha_i(a) + \alpha_{i+1}(a) + \alpha_{i+2}(a) \leq 2, \quad \text{for any } i, \\ \alpha_i(a) = 0, \quad \text{for either } i < u \text{ or } i \geq v - 2. \end{aligned}$$

It is easy to see that for any  $a \in J(u, v)$ , there exists uniquely an expansion

$$a = a_0 + p^{i_1} + p^{i_1+1} + a_1 + \dots + p^{i_k} + p^{i_k+1} + a_k,$$

with  $i_0 = u - 3 < i_1 < \dots < i_k < i_{k+1} = v - 1, i_{j+1} - i_j \geq 3$  and  $a_j \in I(i_j + 3, i_{j+1})$  for  $0 \leq j \leq k$ .

We define the functions  $b_{u,v}, c_{u,v}: J(u, v) \rightarrow \mathbb{Z}$  by setting

$$\begin{aligned} b_{u,v}(a) &= \frac{p^{v-1} - p^u}{p-1} - (p+1)a + p(p^{i_1} + \dots + p^{i_k}), \\ c_{u,v}(a) &= a_0 + a_1 + \dots + a_k. \end{aligned}$$

**Proposition 4.3** *Under the above notation, we have*

$$[u, v, v + 1] = \sum_{a \in J(u, v)} (-1)^a L_3^{p^u + p(p-1)a} Q_{3,1}^{b_{u,v}(a)} Q_{3,2}^{c_{u,v}(a)}.$$



The proof of the proposition is based on some lemmas.

**Lemma 4.4** For  $0 \leq u < v$ ,

$$J(u, v + 3) = J(u, v + 2) \cup (p^v + J(u, v + 1)) \cup (p^v + p^{v-1} + J(u, v)).$$

Here, for  $x \in \mathbb{Z}$  and  $A \subset \mathbb{Z}$ , we write  $x + A = \{x + a; a \in A\}$ .

$$\begin{aligned} b_{u,v+3}(a) &= p^{v+1} + b_{u,v+2}(a), \\ c_{u,v+3}(a) &= c_{u,v+2}(a), && \text{for } a \in J(u, v + 2), \\ b_{u,v+3}(p^v + a) &= b_{u,v+1}(a), \\ c_{u,v+3}(p^v + a) &= p^v + c_{u,v+1}(a), && \text{for } a \in J(u, v + 1), \\ b_{u,v+3}(p^v + p^{v-1} + a) &= b_{u,v}(a), \\ c_{u,v+3}(p^v + p^{v-1} + a) &= c_{u,v}(a), && \text{for } a \in J(u, v). \end{aligned}$$

This lemma can easily be proved by computing directly from the definitions of  $J(u, v)$ ,  $b_{u,v}$  and  $c_{u,v}$ .

**Lemma 4.5** For any  $0 \leq u < v$ , we have

$$\begin{aligned} [u, v + 3, v + 4] &= [u, v + 2, v + 3]Q_{3,1}^{p^{v+1}} \\ &\quad - [u, v + 1, v + 2]Q_{3,0}^{p^{v+1}} Q_{3,2}^{p^v} + [u, v, v + 1]Q_{3,0}^{p^{v+1}+p^v}. \end{aligned}$$

**Proof** A direct calculation using Theorem 2.4 gives

$$\begin{aligned} [u, v + 3, v + 4] &= [u, v + 2, v + 3]Q_{2,0}^{p^{v+2}} + [u, v + 3]V_3^{p^{v+2}} \\ &= [u, v + 2, v + 3](Q_{3,1}^{p^{v+1}} - Q_{2,1}^{p^{v+1}} V_3^{(p-1)p^{v+1}}) \\ &\quad + ([u, v + 2]Q_{2,1}^{p^{v+1}} - [u, v + 1]Q_{2,0}^{p^{v+1}})V_3^{p^{v+2}} \\ &\quad \quad \quad \text{(since } Q_{3,1} = Q_{2,0}^p + Q_{2,1}V_3^{p-1}) \\ &= [u, v + 2, v + 3]Q_{3,1}^{p^{v+1}} \\ &\quad - ([u, v + 1, v + 2]Q_{2,0}^{p^{v+1}} + [u, v + 2]V_3^{p^{v+1}})Q_{2,1}^{p^{v+1}} V_3^{(p-1)p^{v+1}} \\ &\quad + [u, v + 2]Q_{2,1}^{p^{v+1}} V_3^{p^{v+2}} - [u, v + 1]Q_{2,0}^{p^{v+1}} V_3^{p^{v+2}} \\ &= [u, v + 2, v + 3]Q_{3,1}^{p^{v+1}} \\ &\quad - [u, v + 1, v + 2]Q_{2,0}^{p^{v+1}} V_3^{(p-1)p^{v+1}} (Q_{2,1}^{p^{v+1}} + V_3^{(p-1)p^v}) \\ &\quad + ([u, v + 1, v + 2] - [u, v + 1]V_3^{p^v})Q_{2,0}^{p^{v+1}} V_3^{(p-1)(p^{v+1}+p^v)}. \end{aligned}$$

Using Theorem 2.4 and the relations  $Q_{3,2} = Q_{2,1}^p + V_3^{p-1}$ ,  $Q_{3,0} = Q_{2,0}V_3^{p-1}$ , we obtain the lemma.  $\square$

**Proof of Proposition 4.3** The proof is by induction on  $v$ . For  $v = u + 1, u + 2, u + 3$  the proposition is obvious. Suppose that it is true for  $v, v + 1, v + 2$ . Using Lemma 4.5, the inductive hypothesis and the relation  $Q_{3,0} = L_3^{p-1}$ , we get

$$\begin{aligned} [u, v + 3, v + 4] = & \sum_{a \in J(u, v+2)} (-1)^a L_3^{p^u + p(p-1)a} Q_{3,1}^{p^{v+1} + b_{u, v+2}(a)} Q_{3,2}^{c_{u, v+2}(a)} \\ & + \sum_{a \in J(u, v+1)} (-1)^{p^v + a} L_3^{p^u + p(p-1)(p^v + a)} Q_{3,1}^{b_{u, v+1}(a)} Q_{3,2}^{p^v + c_{u, v+1}(a)} \\ & + \sum_{a \in J(u, v)} (-1)^{p^v + p^{v-1} + a} L_3^{p^u + p(p-1)(p^v + p^{v-1} + a)} Q_{3,1}^{b_{u, v}(a)} Q_{3,2}^{c_{u, v}(a)}. \end{aligned}$$

Combining this equality and Lemma 4.4, we see that the proposition holds for  $v + 3$ . Hence, the proposition is proved.  $\square$

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