The action of the primitive Steenrod–Milnor operations on the modular invariants

NGUYỄN SUM

We compute the action of the primitive Steenrod–Milnor operations on generators of algebras of invariants of subgroups of general linear group $GL_n = GL(n, \mathbb{F}_p)$ in the polynomial algebra with p an odd prime number.

55S10; 55P47, 55Q45, 55T15

1 Introduction

Let p be an odd prime number. Denote by $GL_n = GL(n, \mathbb{F}_p)$ the general linear group over the prime field $\mathbb{F}_p$ and T_n the Sylow p–subgroup of GL_n consisting of all upper triangular matrices with 1 on the main diagonal. For any integer d, $1 \le d \le p-1$, we set

$$SL_n^d = \{\omega \in GL_n;\ (\det \omega)^d = 1\}.$$

It is easy to see that SL_n^d is a subgroup of GL_n and $SL_n^{p-1} = GL_n$. Each subgroup of GL_n acts on $P_n = E(x_1, \dots, x_n) \otimes \mathbb{F}_p(y_1, \dots, y_n)$ in the usual manner. Here and in what follows, $E(., \dots, .)$ and $\mathbb{F}_p(., \dots, .)$ are the exterior and polynomial algebras over $\mathbb{F}_p$ generated by the indicated variables. We grade P_n by assigning $\dim x_i = 1$ and $\dim y_i = 2$.

Dickson [1] showed that the invariant algebra $\mathbb{F}_p(y_1, \dots, y_n)^{GL_n}$ is a polynomial algebra generated by the Dickson invariants $Q_{n,s}$, $0 \le s < n$. Huỳnh Mùi [6; 7] computed the invariant algebras $P_n^{T_n}$ and $P_n^{SL_n^d}$ for $d = 1, p-1, (p-1)/2$. He proved that $P_n^{T_n}$ is generated by V_m, $1 \le m \le n$, $M_{m,s_1,\dots,s_k}$, $0 \le s_1 < \dots < s_k < m \le n$ and that $P_n^{SL_n^d}$ is generated by L_n^d, $Q_{n,s}$, $1 \le s < n$, $M^{(d)}_{n,s_1,\dots,s_k}$, $0 \le s_1 < \dots < s_k < n$. Here $V_m, M^{(d)}_{n,s_1,\dots,s_k}$ are Mùi invariants and $L_n^d, Q_{n,s}$ are Dickson invariants (see Section 2). Note that $M^{(1)}_{n,s_1,\dots,s_k} = M_{n,s_1,\dots,s_k}$.

Let $\mathcal{A}(p)$ be the mod p Steenrod algebra and let τ_s, ξ_i be the Milnor elements of dimensions $2p^s-1$, $2p^i-2$ respectively in the dual algebra $\mathcal{A}(p)^*$ of $\mathcal{A}(p)$. In [5], Milnor showed that as an algebra,

$$\mathcal{A}(p)^* = E(\tau_0, \tau_1, \dots) \otimes \mathbb{F}_p(\xi_1, \xi_2, \dots).$$

Published: 14 November 2007 DOI: 10.2140/gtm.2007.11.349

Then $\mathcal{A}(p)^*$ has a basis consisting of all monomials

$$\tau_S\xi^R = \tau_{s_1}\dots\tau_{s_k}\xi_1^{r_1}\dots\xi_m^{r_m},$$

with $S=(s_1,\dots,s_k)$, $0\le s_1<\dots<s_k$, $R=(r_1,\dots,r_m)$, $r_i\ge 0$. Let $\mathrm{St}^{S,R}\in\mathcal{A}(p)$ denote the dual of $\tau_S\xi^R$ with respect to that basis. Then $\mathcal{A}(p)$ has a basis consisting of all operations $\mathrm{St}^{S,R}$. For $S=\varnothing, R=(r)$, $\mathrm{St}^{\varnothing,(r)}$ is nothing but the Steenrod operation P^r. So, we call $\mathrm{St}^{S,R}$ the Steenrod–Milnor operation of type (S,R).

We have the Cartan formula

$$\mathrm{St}^{S,R}(uv)=\sum_{\substack{S_1\cup S_2=S\\ R_1+R_2=R}}(-1)^{(\dim u+\ell(S_1))\ell(S_2)}(S:S_1,S_2)\,\mathrm{St}^{S_1,R_1}(u)\,\mathrm{St}^{S_2,R_2}(v),$$

where $R_1=(r_{1i})$, $R_2=(r_{2i})$, $R_1+R_2=(r_{1i}+r_{2i})$, $S_1\cap S_2=\varnothing$, $u,v\in P_n$, $\ell(S_i)$ is the length of S_i and

$$(S:S_1,S_2)=\mathrm{sign}\begin{pmatrix}s_1 & \dots & s_h & s_{h+1} & \dots & s_k\\ s_{11} & \dots & s_{1h} & s_{21} & \dots & s_{2r}\end{pmatrix},$$

with $S_1=(s_{11},\dots,s_{1h}), s_{11}<\dots<s_{1h}$, $S_2=(s_{21},\dots,s_{2r}), s_{21}<\dots<s_{2r}$ (see Mui [7]).

We denote $\mathrm{St}_u=\mathrm{St}^{(u),(0)}$, $\mathrm{St}^{\Delta_i}=\mathrm{St}^{\varnothing,\Delta_i}$, where $\Delta_i=(0,\dots,1,\dots,0)$ with 1 at the i–th place. In [7], Huỳnh Mùi proved that as a coalgebra,

$$\mathcal{A}(p)=\Lambda(\mathrm{St}_0,\mathrm{St}_1,\dots)\otimes\Gamma(\mathrm{St}^{\Delta_1},\mathrm{St}^{\Delta_1},\dots).$$

Here, $\Lambda(\mathrm{St}_0,\mathrm{St}_1,\dots)$ (resp. $\Gamma(\mathrm{St}^{\Delta_1},\mathrm{St}^{\Delta_2},\dots)$) denotes the exterior (resp. polynomial) Hopf algebra with divided powers generated by the primitive Steenrod–Milnor operations $\mathrm{St}_0,\mathrm{St}_1,\dots$ (resp. $\mathrm{St}^{\Delta_1},\mathrm{St}^{\Delta_2},\dots$).

The Steenrod algebra $\mathcal{A}(p)$ acts on P_n by means of the Cartan formula together with the relations

(i) $$\mathrm{St}^{S,R}x_k=\begin{cases}x_k, & S=\varnothing,\ R=(0),\\ y_k^{p^u}, & S=(u),\ R=(0),\\ 0, & \text{otherwise},\end{cases}$$

(ii) $$\mathrm{St}^{S,R}y_k=\begin{cases}y_k, & S=\varnothing,\ R=(0),\\ y_k^{p^i}, & S=\varnothing,\ R=\Delta_i,\\ 0, & \text{otherwise},\end{cases}$$

for $k=1,2,\dots,n$ (see Steenrod and Epstein [10] and Sum [13]). Since this action commutes with the action of GL_n, it induces actions of $\mathcal{A}(p)$ on $P_n^{T_n}$ and $P_n^{SL_n^d}$.

The action of $\mathrm{St}^{S,R}$ on the modular invariants of subgroups of general linear group has been studied by many authors. This action for $S = \varnothing$, $R = (r)$ was explicitly determined by Hưng and Minh [2], Kechagias [3], Madsen and Milgram [4] and Sum [13]. Smith and Switzer [9], Wilkerson [14] and Neusel [8] have studied the action of St^{Δ_i} on the Dickson invariants.

The purpose of the paper is to compute the action of the primitive Steenrod–Milnor operations on generators of $P_n^{T_n}$ and $P_n^{SL_n^d}$.

The rest of the paper contains three sections. In Section 2, we recall some needed information on the invariant theory and compute the action of the primitive Steenrod–Milnor operations on the determinant invariants. In Section 3, we compute the action of the primitive Steenrod–Milnor operations on Dickson and Mùi invariants. Finally, we give in Section 4 some formulae from which we can describe our results in terms of Dickson and Mùi invariants.

Acknowledgements The author is grateful to the referee for his valuable comments on the first manuscript of this paper.

2 Preliminaries

Definition 2.1 Let $(e_{k+1}, \dots, e_m)$, $0 \le k < m \le n$, be a sequence of nonnegative integers. Following Dickson [1] and Mùi [6], we define

$$[k; e_{k+1}, \dots, e_m] = \frac{1}{k!} \begin{vmatrix} x_1 & \cdots & x_m \\ \vdots & \cdots & \vdots \\ x_1 & \cdots & x_m \\ y_1^{p^{e_{k+1}}} & \cdots & y_m^{p^{e_{k+1}}} \\ \vdots & \cdots & \vdots \\ y_1^{p^{e_m}} & \cdots & y_m^{p^{e_m}} \end{vmatrix}.$$

The precise meaning of the right hand side is given in [6]. For $k = 0$, we write

$$[0; e_1, \dots, e_m] = [e_1, \dots, e_m] = \det(y_i^{p^{e_j}}).$$

In particular, we set

$$\begin{aligned} L_{m,s} &= [0, 1, \dots, \widehat{s}, \dots, m],\ 0 \le s \le m \le n, \\ L_m = L_{m,m} &= [0, 1, \dots, m-1], \\ M_{m,s_1,\dots,s_k} &= [k; 0, \dots, \widehat{s}_1, \dots, \widehat{s}_k, \dots, m-1], \end{aligned}$$

for $0 \le s_1 < \ldots < s_k < m \le n$. Each $[k; e_{k+1}, \ldots, e_m]$ is an invariant of SL_m^1 and $[e_1, \ldots, e_m]$ is divisible by L_m. Then, Dickson invariants $Q_{n,s}$ and Mùi invariants $M^{(d)}_{n,s_1,\ldots,s_k}$ and V_m are defined by

$$Q_{n,s} = L_{n,s}/L_n, \quad M^{(d)}_{n,s_1,\ldots,s_k} = M_{n,s_1,\ldots,s_k} L_n^{d-1} \quad \text{and} \quad V_m = L_m/L_{m-1}.$$

Here, by convention, $L_0 = [\varnothing] = 1$.

Now we prepare some data in order to prove our main results.

Lemma 2.2 *Suppose $e_\ell \ne e_j$ for $\ell \ne j$, $u \ge 0$. Then we have*

$$\mathrm{St}_u[k; e_{k+1}, \ldots, e_n] = \begin{cases} (-1)^{k-1}[k-1; u, e_{k+1}, \ldots, e_n], & k > 0, \\ 0, & k = 0. \end{cases}$$

Proof Let I be a subset of $\{1, \ldots, n\}$ and let I' be its complement in $\{1, 2, \ldots, n\}$. Writing $I = \{i_1, i_2, \ldots, i_k\}$ and $I' = \{i_{k+1}, i_{k+2}, \ldots, i_n\}$ with $i_1 < i_2 < \ldots < i_k$ and $i_{k+1} < i_{k+2} < \ldots < i_n$. We set

$$\begin{aligned} x_I &= x_{i_1} x_{i_2} \ldots x_{i_k}, \\ [e_{k+1}, e_{k+2}, \ldots, e_n]_I &= [e_{k+1}, e_{k+2}, \ldots, e_n](y_{i_{k+1}}, y_{i_{k+2}}, \ldots, y_{i_n}) \\ \sigma_I &= \begin{pmatrix} 1 & 2 & \ldots & n \\ i_1 & i_2 & \ldots & i_n \end{pmatrix} \in \Sigma_n, \end{aligned}$$

where Σ_n is the symmetric group on n letters. Using the Laplace development, we have

$$[k; e_{k+1}, e_{k+2}, \ldots, e_n] = \sum_I \text{sign } \sigma_I x_I [e_{k+1}, e_{k+2}, \ldots, e_n]_I.$$

From the relation (ii), we see that $\mathrm{St}_u[e_{k+1}, e_{k+2}, \ldots, e_n]_I = 0$. Then, using the Cartan formula, we get

$$\mathrm{St}_u[k; e_{k+1}, e_{k+2}, \ldots, e_n] = \sum_I \text{sign } \sigma_I \, \mathrm{St}_u(x_I)[e_{k+1}, e_{k+2}, \ldots, e_n]_I. \tag{1}$$

In [7, 5.2], Mùi showed that

$$\mathrm{St}_u(x_I) = (-1)^{k-1}[k-1; u](x_{i_1}, x_{i_2}, \ldots, x_{i_k}, y_{i_1}, y_{i_2}, \ldots, y_{i_k}).$$

Hence, using (1) and the Laplace development we obtain the lemma. □

Lemma 2.3 *Suppose $e_\ell \ne e_j$ for $\ell \ne j$, $e_{k+1} < e_j$ for $j > k+1$. Then we have*

$$\mathrm{St}^{\Delta_i}[k; e_{k+1}, \ldots, e_n] = \begin{cases} [k; i, e_{k+2}, \ldots, e_n], & e_{k+1} = 0, \\ 0, & e_{k+1} > 0. \end{cases}$$

Proof Suppose $e_{k+1} > 0$. From the relations (i) and (ii) and the Cartan formula, we easily obtain

$$\mathrm{St}^{\Delta_i}\, x_\ell = 0, \quad \mathrm{St}^{\Delta_i}\, y_\ell^{p^{e_j}} = p^{e_j}\, y_\ell^{p^{e_j}+p^i-1} = 0,$$

for $\ell = 1, 2, \ldots, n$ and $j = k+1, k+2, \ldots, n$. From this, we get

$$\mathrm{St}^{\Delta_i}[k; e_{k+1}, \ldots, e_n] = 0.$$

If $e_{k+1} = 0$ then $\mathrm{St}^{\Delta_i}\, y_\ell^{p^{e_j}} = 0$, for $\ell = 1, 2, \ldots, n$ and $j = k+2, \ldots, n$, and

$$\mathrm{St}^{\Delta_i}\, y_\ell^{p^{e_{k+1}}} = \mathrm{St}^{\Delta_i}\, y_\ell = y_\ell^{p^i}.$$

Hence, using the Laplace development and the Cartan formula, we obtain

$$\mathrm{St}^{\Delta_i}[k; e_{k+1}, e_{k+2}, \ldots, e_n] = [k; i, e_{k+2}, \ldots, e_n]. \qquad \square$$

To make the paper self-contained, we give here a proof for the following theorem, which will be needed in the next section.

Theorem 2.4 (Sum [12]) *Let $(e_1, e_2, \ldots, e_n)$ be a sequence of nonnegative integers and $0 \le k < n$. We have*

$$\begin{aligned}&[e_1, e_2, \ldots, e_{n-1}, e_n + n - 1] \\ (2) \qquad &= \sum_{s=0}^{n-2} (-1)^{n+s}[e_1, e_2, \ldots, e_{n-1}, e_n + s] Q_{n-1,s}^{p^{e_n}} + [e_1, e_2, \ldots, e_{n-1}] V_n^{p^{e_n}},\end{aligned}$$

$$(3) \quad [k; e_{k+1}, \ldots, e_{n-1}, e_n + n] = \sum_{s=0}^{n-1} (-1)^{n+s-1}[k; e_{k+1}, \ldots, e_{n-1}, e_n + s] Q_{n,s}^{p^{e_n}}.$$

Proof We recall Mùi's formula in [6],

$$[k; e_{k+1}, \ldots, e_n] = (-1)^{k(k-1)/2} \sum_{0 \le s_1 < \ldots < s_k} (-1)^{s_1+\ldots+s_k} M_{n,s_1,\ldots,s_k}[s_1, \ldots, s_k, e_{k+1}, \ldots, e_n]/L_n.$$

Hence, it suffices to prove the theorem for $k = 0$.

The proof of the theorem proceeds by induction on n. It is easy to see that the theorem holds for $n = 1$. Suppose $n \ge 2$ and the theorem holds for $n - 1$.

Using the Laplace development and the inductive hypothesis, we have

$$\begin{aligned}
&[e_1, \dots, e_{n-1}, e_n + n - 1] \\
&\quad = \sum_{t=1}^{n-1} (-1)^{n+t} [e_1, \dots, \hat{e}_t, \dots, e_{n-1}, e_n + n - 1] y_n^{p^{e_t}} + [e_1, \dots, e_{n-1}] y_n^{p^{e_n + n-1}} \\
&\quad = \sum_{t=1}^{n-1} (-1)^{n+t} \Big(\sum_{s=0}^{n-2} (-1)^{n+s} [e_1, \dots, \hat{e}_t, \dots, e_{n-1}, e_n + s] Q_{n-1,s}^{p^{e_n}} \Big) y_n^{p^{e_t}} \\
&\qquad\qquad + [e_1, \dots, e_{n-1}] y_n^{p^{e_n + n-1}} \\
&\quad = \sum_{s=0}^{n-2} (-1)^{n+s} \Big(\sum_{t=1}^{n-1} (-1)^{n+t} [e_1, \dots, \hat{e}_t, \dots, e_{n-1}, e_n + s] y_n^{p^{e_t}} \Big) Q_{n-1,s}^{p^{e_n}} \\
&\qquad\qquad + [e_1, \dots, e_{n-1}] y_n^{p^{e_n + n-1}} \\
&\quad = \sum_{s=0}^{n-2} (-1)^{n+s} [e_1, \dots, e_{n-1}, e_n + s] Q_{n-1,s}^{p^{e_n}} \\
&\qquad\qquad + [e_1, \dots, e_{n-1}] \sum_{s=0}^{n-1} (-1)^{n+s-1} Q_{n-1,s}^{p^{e_n}} y_n^{p^{e_n+s}}.
\end{aligned}$$

Since $V_n = \sum_{s=0}^{n-1} (-1)^{n+s-1} Q_{n-1,s} y_n^{p^s}$, the relation (2) holds for n.

Now we prove the relation (3) for n. By a direct calculation using (2) and the relation $Q_{n,s} = Q_{n-1,s-1}^p + Q_{n-1,s} V_n^{p-1}$, we get

$$\begin{aligned}
&[e_1, e_2, \dots, e_{n-1}, e_n + n] \\
&\quad = \sum_{s=1}^{n-1} (-1)^{n+s-1} [e_1, \dots, e_{n-1}, e_n + s] Q_{n-1,s-1}^{p^{e_n+1}} + [e_1, \dots, e_{n-1}] V_n^{p^{e_n+1}} \\
&\quad = \sum_{s=1}^{n-1} (-1)^{n+s-1} [e_1, \dots, e_{n-1}, e_n + s] Q_{n,s}^{p^{e_n}} \\
&\qquad\qquad - [e_1, \dots, e_{n-1}, e_n + n - 1] V_n^{(p-1)p^{e_n}} \\
&\qquad + \Big(\sum_{s=1}^{n-2} (-1)^{n+s} [e_1, \dots, e_{n-1}, e_n + s] Q_{n-1,s}^{p^{e_n}} + [e_1, \dots, e_{n-1}] V_n^{p^{e_n}} \Big) V_n^{(p-1)p^{e_n}}.
\end{aligned}$$

Combining this equality and the relation (2) we obtain

$$[e_1,e_2,\dots,e_{n-1},e_n+n]=\sum_{s=1}^{n-1}(-1)^{n+s-1}[e_1,\dots,e_{n-1},e_n+s]Q_{n,s}^{p^{e_n}} \\ -(-1)^n[e_1,\dots,e_{n-1},e_n]Q_{n-1,0}^{p^{e_n}}V_n^{(p-1)p^{e_n}}.$$

Since $Q_{n,0}=Q_{n-1,0}V_n^{p-1}$, the relation (3) holds for n.

This completes the proof of Theorem 2.4. □

3 Main results

Observe that using the Cartan formula and the relations (i) and (ii), we obtain $\mathrm{St}_u\,x=0$ for either $x=Q_{n,s}$ or $x=V_n$. So, in this section we only compute $\mathrm{St}^{\Delta_i}\,x$ for $x=Q_{n,s},V_n,M^{(d)}_{n,s_1,\dots,s_k}$ and $\mathrm{St}_u\,M^{(d)}_{n,s_1,\dots,s_k}$.

Theorem 3.1 *For any integers i,n,s with $0\le s<n$ and $i\ge 1$, we have*

$$\mathrm{St}^{\Delta_i}\,Q_{n,s}=(-1)^n[0,1,\dots,\hat{s},\dots,n-1,i]L_n^{p-2}.$$

Proof Since $L_{n,s}=L_nQ_{n,s}$, using the Cartan formula, we get

$$\mathrm{St}^{\Delta_i}\,L_{n,s}=L_n\,\mathrm{St}^{\Delta_i}\,Q_{n,s}+Q_{n,s}\,\mathrm{St}^{\Delta_i}\,L_n. \tag{4}$$

According to Lemma 2.3, we have

$$\mathrm{St}^{\Delta_i}\,L_{n,s}=\begin{cases}[i,1,2,\dots,\hat{s},\dots,n], & s>0,\\ 0, & s=0.\end{cases}$$

In particular, $\mathrm{St}^{\Delta_i}\,L_n=[i,1,2,\dots,n-1]$.

If $s=0$ then $\mathrm{St}^{\Delta_i}\,L_{n,s}=0$ and

$$\mathrm{St}^{\Delta_i}\,L_n=[i,1,2,\dots,n-1] \\ =(-1)^{n-1}[1,2,\dots,n-1,i].$$

Combining (4) and the above equalities, we get

$$\mathrm{St}^{\Delta_i}\,Q_{n,0}=-(\mathrm{St}^{\Delta_i}\,L_n)Q_{n,0}/L_n \\ =(-1)^n[1,2,\dots,n-1,i]Q_{n,0}/L_n.$$

Since $Q_{n,0}=L_n^{p-1}$, the theorem holds.

If $s > 0$ then $\mathrm{St}^{\Delta_i} L_n = [i, 1, 2, \dots, n-1]$ and $\mathrm{St}^{\Delta_i} L_{n,s} = [i, 1, 2, \dots, \widehat{s}, \dots, n]$. Hence, using Theorem 2.4, we get

$$\begin{aligned}\mathrm{St}^{\Delta_i} L_{n,s} &= \sum_{t=0}^{n-1} (-1)^{n-1+t} [i, 1, 2, \dots, \widehat{s}, \dots, n-1, t] Q_{n,t} \\ &= (-1)^{n-1} [i, 1, 2, \dots, \widehat{s}, \dots, n-1, 0] Q_{n,0} \\ &\quad + (-1)^{n-1+s} [i, 1, 2, \dots, \widehat{s}, \dots, n-1, s] Q_{n,s} \\ &= [i, 1, 2, \dots, n-1] Q_{n,s} - [i, 0, 1, \dots, \widehat{s}, \dots, n-1] Q_{n,0}.\end{aligned}$$

Combining (4), the above equalities and the relation $Q_{n,0} = L_n^{p-1}$, we get

$$\begin{aligned}\mathrm{St}^{\Delta_i} Q_{n,s} &= (\mathrm{St}^{\Delta_i} L_{n,s} - Q_{n,s}\, \mathrm{St}^{\Delta_i} L_n)/L_n \\ &= -[i, 0, 1, 2, \dots, \widehat{s}, \dots, n-1] Q_{n,0}/L_n \\ &= (-1)^n [0, 1, 2, \dots, \widehat{s}, \dots, n-1, i] L_n^{p-2}.\end{aligned}$$ □

The following was proved in Smith and Switzer [9] by another method.

Corollary 3.2 (Smith–Switzer [9]) *For any integers i, n, s with $0 \le s < n$ and $1 \le i \le n$, we have*

$$\mathrm{St}^{\Delta_i} Q_{n,s} = \begin{cases} (-1)^{s-1} Q_{n,0}, & i = s > 0, \\ (-1)^n Q_{n,0} Q_{n,s}, & i = n, \\ 0, & \text{otherwise.} \end{cases}$$

Proof Suppose $i = s$. According to Theorem 3.1, we have

$$\begin{aligned}\mathrm{St}^{\Delta_s} Q_{n,s} &= (-1)^n [0, 1, \dots, \widehat{s}, \dots, n-1, s] L_n^{p-2} \\ &= (-1)^{s-1} [0, 1, \dots, n-1] L_n^{p-2} \\ &= (-1)^{s-1} L_n^{p-1} = (-1)^{s-1} Q_{n,0}.\end{aligned}$$

If $i < n$ and $i \ne s$ then $[0, 1, \dots, \widehat{s}, \dots, n-1, i] = 0$. Hence, $\mathrm{St}^{\Delta_i} Q_{n,s} = 0$.

If $i = n$ then
$$\begin{aligned}\mathrm{St}^{\Delta_n} Q_{n,s} &= (-1)^n [0, 1, \dots, \widehat{s}, \dots, n-1, n] L_n^{p-2} \\ &= (-1)^n L_{n,s} L_n^{p-2} \\ &= (-1)^n L_n^{p-1} Q_{n,s} \\ &= (-1)^n Q_{n,0} Q_{n,s}.\end{aligned}$$

The corollary follows. □

Now, we show that our formula in Theorem 3.1 implies Wilkerson's formula in [14]. To do this, we need the following.

Proposition 3.3 (Sum [12]) *Let $(e_{k+1}, e_{k+2}, \dots, e_n)$ be a sequence of nonnegative integers with $0 \le k < n$ and $e_\ell \ne e_j$ for $\ell \ne j$. Then*

$$P^r[k; e_{k+1}, e_{k+2}, \dots, e_n] = \begin{cases} [k; e_{k+1}+\varepsilon_{k+1}, e_{k+2}+\varepsilon_{k+2}, \dots, e_n+\varepsilon_n], \\ \qquad \text{if } r = \sum_{j=k+1}^{n} \varepsilon_j p^{e_j} \text{ with } \varepsilon_j \in \{0,1\}, \\ 0, \qquad \text{otherwise.} \end{cases}$$

This proposition can easily be proved by using the Laplace development, the Cartan formula and the relations (i) and (ii).

From the formula in Theorem 3.1, one gets Wilkerson's formula as follows.

Theorem 3.4 (Wilkerson [14]) *For any integers $0 \le s < n \le i$, we have*

$$\mathrm{St}^{\Delta_{i+1}} Q_{n,s} = P^{p^i} \mathrm{St}^{\Delta_i} Q_{n,s}.$$

Proof Applying Theorem 3.1, the Cartan formula and Proposition 3.3, we get

$$\begin{aligned} P^{p^i} \mathrm{St}^{\Delta_i} Q_{n,s} &= (-1)^n P^{p^i}([0, 1, \dots, \hat{s}, \dots, n-1, i] L_n^{p-2}) \\ &= (-1)^n \sum_r P^r([0, 1, \dots, \hat{s}, \dots, n-1, i]) P^{p^i - r}(L_n^{p-2}), \end{aligned}$$

where the sum runs over all

$$r = \varepsilon_0 p^0 + \varepsilon_1 p^1 + \dots + \varepsilon_{s-1} p^{s-1} + \varepsilon_{s+1} p^{s+1} + \dots + \varepsilon_{n-1} p^{n-1} + \varepsilon_i p^i$$

with $\varepsilon_j \in \{0,1\}$ for any j and $r \le p^i$.

If $\varepsilon_i = 0$ then $r < p^0 + p^1 + \dots + p^{n-1}$ and

$$\begin{aligned} 2(p^i - r) &> 2(p^i - (p^0 + p^1 + \dots + p^{n-1})) \\ &= 2(p^i - p^n + 1 + (p-2)(p^0 + p^1 + \dots + p^{n-1})) \\ &> 2(p-2)(p^0 + p^1 + \dots + p^{n-1}) = \dim L_n^{p-2}. \end{aligned}$$

This implies $P^{p^i - r}(L_n^{p-2}) = 0$.

Since $r \le p^i$, if $\varepsilon_i = 1$ then $\varepsilon_j = 0$ for $j \ne i$ and $r = p^i$. Hence, using the above equalities and Proposition 3.3, we obtain

$$\begin{aligned} P^{p^i}\,\mathrm{St}^{\Delta_i}\,Q_{n,s} &= (-1)^n P^{p^i}([0,1,\dots,\hat{s},\dots,n-1,i])L_n^{p-2} \\ &= (-1)^n[0,1,\dots,\hat{s},\dots,n-1,i+1]L_n^{p-2} \\ &= \mathrm{St}^{\Delta_{i+1}}\,Q_{n,s}. \end{aligned}$$ □

Next, we compute the action of St^{Δ_i} on Mùi invariants.

Theorem 3.5 *For any positive integers i, n, we have*

$$\mathrm{St}^{\Delta_i}\,V_n = (-1)^{n-1}[0,1,\dots,n-2,i]L_{n-1}^{p-2}.$$

Proof Since $L_n = L_{n-1}V_n$, applying the Cartan formula, we get

$$\mathrm{St}^{\Delta_i}\,L_n = L_{n-1}\,\mathrm{St}^{\Delta_i}\,V_n + V_n\,\mathrm{St}^{\Delta_i}\,L_{n-1}. \tag{5}$$

Using Lemma 2.3 and Theorem 2.4, we have

$$\begin{aligned} \mathrm{St}^{\Delta_i}\,L_{n-1} &= [i,1,2,\dots,n-2], \\ \mathrm{St}^{\Delta_i}\,L_n &= [i,1,2,\dots,n-2,n-1] \\ &= \sum_{s=0}^{n-2}(-1)^{n+s}[i,1,2,\dots,n-2,s]Q_{n-1,s} + [i,1,2,\dots,n-2]V_n \\ &= (-1)^n[i,1,2,\dots,n-2,0]Q_{n-1,0} + [i,1,2,\dots,n-2]V_n. \end{aligned}$$

Combining (5), the above equalities and the relation $Q_{n-1,0} = L_{n-1}^{p-1}$, we get

$$\begin{aligned} \mathrm{St}^{\Delta_i}\,V_n &= (\mathrm{St}^{\Delta_i}\,L_n - V_n\,\mathrm{St}^{\Delta_i}\,L_{n-1})/L_{n-1} \\ &= (-1)^n[i,1,2,\dots,n-2,0]Q_{n-1,0}/L_{n-1} \\ &= (-1)^{n-1}[0,1,2,\dots,n-2,i]L_{n-1}^{p-2}. \end{aligned}$$ □

Corollary 3.6 *For any integers $0 < i \le n$, we have*

$$\mathrm{St}^{\Delta_i}\,V_n = \begin{cases} 0, & i < n-1, \\ (-1)^{n-1}Q_{n-1,0}V_n, & i = n-1, \\ (-1)^{n-1}Q_{n-1,0}(Q_{n-1,n-2}^p V_n + V_n^p), & i = n. \end{cases}$$

Proof If $i < n-1$ then $[0, 1, \dots, n-2, i] = 0$. Hence, $\mathrm{St}^{\Delta_i}\, V_n = 0$.

For $i = n-1$, we have $[0, 1, 2, \dots, n-2, n-1] = L_n = L_{n-1} V_n$. Hence, from Theorem 3.5, we get

$$\mathrm{St}^{\Delta_{n-1}}\, V_n = (-1)^{n-1} L_{n-1}^{p-1} V_n = (-1)^{n-1} Q_{n-1,0} V_n.$$

Let $i = n$. A direct computation shows

$$\begin{aligned}[0, 1, \dots, n-2, n] = L_{n,n-1} &= L_n Q_{n,n-1}\\ &= L_{n-1} V_n (Q_{n-1,n-2}^p + V_n^{p-1}).\end{aligned}$$

From the above equalities, Theorem 3.5 and the relation $L_{n-1}^{p-1} = Q_{n-1,0}$, we obtain

$$\mathrm{St}^{\Delta_n}\, V_n = (-1)^{n-1} Q_{n-1,0} (Q_{n-1,n-2}^p V_n + V_n^p).$$

The corollary follows. $\square$

Theorem 3.7 *Set $s_0 = 0$. Then $\mathrm{St}^{\Delta_i}\, M_{n,s_1,\dots,s_k}^{(d)}$ equals*

$$\begin{cases} (-1)^{s_t - t} M_{n,s_0,\dots,\hat{s}_t,\dots,s_k}^{(d)}, & s_1 > 0, i = s_t, 1 \le t \le k,\\ (-1)^{n-1} (d-1) M_{n,s_1,\dots,s_k} [1, 2, \dots, n-1, i] L_n^{d-2}, & i \ge n, s_1 = 0,\\ (-1)^{n-1} \big((-1)^k [k; 1, \dots, \hat{s}_1, \dots, \hat{s}_k, \dots, n-1, i] L_n^{d-1} & \\ \quad + (d-1) M_{n,s_1,\dots,s_k} [1, 2, \dots, n-1, i] L_n^{d-2}\big), & i \ge n, s_1 > 0,\\ 0, & \text{otherwise.} \end{cases}$$

Proof Applying Lemma 2.2, we have

$$\mathrm{St}^{\Delta_i}\, M_{n,s_1,\dots,s_k} = \begin{cases} [k; i, 1, \dots, \hat{s}_1, \dots, \hat{s}_k, \dots, n-1], & s_1 > 0,\\ 0, & s_1 = 0. \end{cases}$$

If $i = s_t$ then $[k; i, 1, \dots, \hat{s}_1, \dots, \hat{s}_k, \dots, n-1] = (-1)^{s_t - t} M_{n,s_0,\dots,\hat{s}_t,\dots,s_k}$.

If $i \ge n$ then

$$[k; i, 1, \dots, \hat{s}_1, \dots, \hat{s}_k, \dots, n-1] = (-1)^{n-k-1} [k; 1, \dots, \hat{s}_1, \dots, \hat{s}_k, \dots, n-1, i].$$

Thus the theorem is proved for $d = 1$.

For $d > 1$, using Lemma 2.2 and the Cartan formula, we have

$$\begin{aligned} \mathrm{St}^{\Delta_i}\, L_n^{d-1} &= (d-1) L_n^{d-2}\, \mathrm{St}^{\Delta_i}\, L_n,\\ \mathrm{St}^{\Delta_i}\, L_n &= (-1)^{n-1} [1, 2, \dots, n-1, i],\\ \mathrm{St}^{\Delta_i}\, M_{n,s_1,\dots,s_k}^{(d)} &= \mathrm{St}^{\Delta_i} (M_{n,s_1,\dots,s_k}) L_n^{d-1} + (d-1) M_{n,s_1,\dots,s_k} L_n^{d-2}\, \mathrm{St}^{\Delta_i}\, L_n. \end{aligned}$$

Combining the above equalities we obtain the theorem. □

Theorem 3.8 *For* $1 \le d \le p-1$, *we have*

$$\mathrm{St}_u\, M^{(d)}_{n,s_1,\dots,s_k} = \begin{cases} (-1)^{k+s_t-t} M^{(d)}_{n,s_1,\dots,\widehat{s}_t,\dots,s_k}, & u = s_t, \\ (-1)^{n-1}[k-1;0,\dots,\widehat{s}_1,\dots,\widehat{s}_k,\dots,n-1,u]L_n^{d-1}, & u \ge n, \\ 0, & \text{otherwise.} \end{cases}$$

Proof Since $M_{n,s_1,\dots,s_k} = [k;0,\dots,\widehat{s}_1,\dots,\widehat{s}_k,\dots,n-1]$, applying Lemma 2.2, we get

$$\mathrm{St}_u\, M_{n,s_1,\dots,s_k} = (-1)^{k-1}[k-1;u,0,\dots,\widehat{s}_1,\dots,\widehat{s}_k,\dots,n-1].$$

If $0 \le u \le n-1$ then

$$[k-1;u,0,\dots,\widehat{s}_1,\dots,\widehat{s}_k,\dots,n-1] = \begin{cases} (-1)^{s_t-t+1} M_{n,s_1,\dots,\widehat{s}_t,\dots,s_k}, & u = s_t, \\ 0, & \text{otherwise.} \end{cases}$$

If $u > n-1$ then we have

$$[k-1;u,0,\dots,\widehat{s}_1,\dots,\widehat{s}_k,\dots,n-1] = (-1)^{n-k}[k-1;0,\dots,\widehat{s}_1,\dots,\widehat{s}_k,\dots,n-1,u].$$

The theorem is proved for $d = 1$.

Since $\mathrm{St}_u\, L_n = 0$, using the Cartan formula, we get

$$\mathrm{St}_u(M^{(d)}_{n,s_1,\dots,s_k}) = \mathrm{St}_u(M_{n,s_1,\dots,s_k})L_n^{d-1}.$$

The theorem now follows from the above equalities. □

By the analogous argument as given in the proof of Theorem 3.4, we can show that the Wilkerson formula also holds for Mùi invariants.

Theorem 3.9 *For any integers* $i, u \ge n$, *we have*

$$\mathrm{St}^{\Delta_i}\, V_n = P^{p^{i-1}}\, \mathrm{St}^{\Delta_{i-1}}\, V_n,$$
$$\mathrm{St}^{\Delta_{i+1}}\, M^{(d)}_{n,s_1,\dots,s_k} = P^{p^i}\, \mathrm{St}^{\Delta_i}\, M^{(d)}_{n,s_1,\dots,s_k},$$
$$\mathrm{St}_{u+1}\, M^{(d)}_{n,s_1,\dots,s_k} = P^{p^u}\, \mathrm{St}_u\, M^{(d)}_{n,s_1,\dots,s_k}.$$

Remark 3.10 Using Theorem 2.4 and the above results, we can compute the action of the primitive Steenrod–Milnor operations on the modular invariants in terms of Dickson and Mùi invariants for $i, u \geq n$. For example, by a direct calculation, we easily obtain

$$\mathrm{St}^{\Delta_{n+1}} Q_{n,s} = (-1)^n Q_{n,0}(Q_{n,n-1}^p Q_{n,s} - Q_{n,s-1}^p),$$
$$\mathrm{St}^{\Delta_{n+2}} Q_{n,s} = (-1)^n Q_{n,0}(Q_{n,n-1}^{p^2+p} Q_{n,s} - Q_{n,n-2}^{p^2} Q_{n,s} + Q_{n,s-2}^{p^2} - Q_{n,s-1}^p Q_{n,n-1}^{p^2}).$$

Here, by convention, $Q_{n,t} = 0$ for $t < 0$.

$$\mathrm{St}^{\Delta_{n+1}} V_n = (-1)^{n-1} Q_{n-1,0}\big((Q_{n-1,n-2}^{p^2+p} - Q_{n-1,n-3}^{p^2})V_n + Q_{n-1,n-2}^{p^2} V_n^p + V_n^{p^2}\big),$$
$$\mathrm{St}_n\, M_{n,s_1,\ldots,s_k}^{(d)} = \sum_{t=1}^{k} (-1)^{n-1+k-t} M_{n,s_1,\ldots,\widehat{s}_t,\ldots,s_k}^{(d)} Q_{n,s_t},$$
$$\mathrm{St}^{\Delta_n}\, M_{n,s_1,\ldots,s_k}^{(d)} = (-1)^{n-1}\Big(\sum_{t=1}^{k} (-1)^t M_{n,s_0,\ldots,\widehat{s}_t,\ldots,s_k}^{(d)} Q_{n,s_t} + dM_{n,s_1,\ldots,s_k}^{(d)} Q_{n,0}\Big),$$

where $s_0 = 0$ and $s_1 > 0$. If $s_1 = 0$ then

$$\mathrm{St}^{\Delta_n}\, M_{n,s_1,\ldots,s_k}^{(d)} = (-1)^{n-1}(d-1) M_{n,s_1,\ldots,s_k}^{(d)} Q_{n,0}.$$

Furthermore, the computation of the action of the primitive Steenrod–Milnor operations on the modular invariants in terms of Dickson and Mùi invariants by the use of our results in this section is more convenient than that by using Wilkerson's formula. For example, to compute $\mathrm{St}^{\Delta_{n+2}} Q_{n,s}$ by using Wilkerson's formula, we need to compute $P^{p^{n+1}}(Q_{n,0}(Q_{n,n-1}^p Q_{n,s} - Q_{n,s-1}^p))$ in terms of Dickson invariants. But computing $P^{p^{n+1}}(Q_{n,0}(Q_{n,n-1}^p Q_{n,s} - Q_{n,s-1}^p))$ is more difficult than that of $[0, 1, \ldots, \widehat{s}, \ldots, n-1, n+2]$.

4 On the description of the determinant invariants in terms of Dickson and Mùi invariants

In this section, we study the problem of description of the determinant invariants in terms of Dickson and Mùi invariants. The explicit formulae for the determinant invariants in terms of Dickson and Mùi invariants are useful tools for computing the action of the cohomology operations on the modular invariants.

In general, it is difficult to give explicit formulae for this problem. In particular, for $n = 2, 3$, we can explicitly compute $[u, v], [u, v, w]$ in terms of Mùi invariants and $[u,v], [u,v,v+1]$ in terms of Dickson invariants, where u,v,w are nonnegative integers.

Note that the problem of description of $[u,v,w]$ in terms of Dickson invariants is complicated. It is still open.

Proposition 4.1 *For $0 \le u < v < w$, we have*

$$[u,v] = \sum_{s=u}^{v-1} V_1^{p^v - p^{s+1} + p^u} V_2^{p^s}, \tag{6}$$

$$[u,v,w] = \sum_{s=u}^{v-1} [u,s+1][v,w] L_2^{-p^{s+1}} V_3^{p^s} + \sum_{s=v}^{w-1} [u,v][s+1,w] L_2^{-p^{s+1}} V_3^{p^s}. \tag{7}$$

Proof The relation (6) is proved by induction on v. We prove (7) by induction on v,w. Applying Theorem 2.4, we can easily prove the following by induction on v

$$[u,v,v+1] = \sum_{s=u}^{v-1} [u,s+1] L_2^{p^v - p^{s+1}} V_3^{p^s}. \tag{8}$$

Since $L_2^{p^v} = [v,v+1]$, the relation (7) holds for $w = v+1$.

Let $w = v+2$. By a direct computation using Theorem 2.4 and (8), we have

$$\begin{aligned}[u,v,v+2] &= [u,v,v+1] Q_{2,1}^{p^v} + [u,v] V_3^{p^v} \\ &= \sum_{s=u}^{v-1} [u,s+1] L_2^{p^v - p^{s+1}} V_3^{p^s} Q_{2,1}^{p^v} + [u,v] V_3^{p^v}.\end{aligned}$$

We observe that $(L_2 Q_{2,1})^{p^v} = [v,v+2]$, $L_2^{p^{v+1}} = [v+1,v+2]$. Hence, the relation (7) holds for $w = v+2$. Suppose that (7) holds for w and $w+1$. It is easy to see that

$$[w+1,w] Q_{2,0}^{p^w} = -L_2^{p^{w+1}}.$$

Hence, using Theorem 2.4 and the inductive hypothesis, we get

$$\begin{aligned}[u,v,w+2] &= [u,v,w+1] Q_{2,1}^{p^w} - [u,v,w] Q_{2,0}^{p^w} + [u,v] V_3^{p^w} \\ &= \Big(\sum_{s=u}^{v-1} [u,s+1][v,w+1] L_2^{-p^{s+1}} V_3^{p^s} \\ &\qquad + \sum_{s=v}^{w} [u,v][s+1,w+1] L_2^{-p^{s+1}} V_3^{p^s} \Big) Q_{2,1}^{p^w}\end{aligned}$$

$$
\begin{aligned}
&-\Big(\sum_{s=u}^{v-1}[u,s+1][v,w]L_2^{-p^{s+1}}V_3^{p^s}\\
&\quad+\sum_{s=v}^{w-1}[u,v][s+1,w]L_2^{-p^{s+1}}V_3^{p^s}\Big)Q_{2,0}^{p^w}+[u,v]V_3^{p^w}\\
=&\sum_{s=u}^{v-1}[u,s+1]\big([v,w+1]Q_{2,1}^{p^w}-[v,w]Q_{2,0}^{p^w}\big)L_2^{-p^{s+1}}V_3^{p^s}\\
&\quad+\sum_{s=v}^{w}[u,v]\big([s+1,w+1]Q_{2,1}^{p^w}-[s+1,w]Q_{2,0}^{p^w}\big)L_2^{-p^{s+1}}V_3^{p^s}.
\end{aligned}
$$

This equality and Theorem 2.4 imply the relation (7) for $w+2$, completing the proof. □

Now, we compute $[u,v]$ in terms of L_2 and $Q_{2,1}$.

Let $\alpha_i(a)$ denote the i–th coefficient in p–adic expansion of a nonnegative integer a. That means

$$a=\alpha_0(a)p^0+\alpha_1(a)p^1+\alpha_2(a)p^2+\dots,$$

for $0\le\alpha_i(a)<p, i\ge 0$. We set $\alpha_i(a)=0$ for $i<0$.

Denote by $I(u,v)$ the set of all integers a satisfying

$$
\begin{aligned}
&\alpha_i(a)+\alpha_{i+1}(a)\le 1, && \text{for any } i,\\
&\alpha_i(a)=0, && \text{for either } i<u \text{ or } i\ge v-2.
\end{aligned}
$$

The following was proved in Sum [11] for $p=2$.

Proposition 4.2 *Under the above notation, we have*

$$[u,v]=\sum_{a\in I(u,v)}(-1)^a L_2^{p^u+p(p-1)a}Q_{2,1}^{\frac{p^{v-1}-p^u}{p-1}-(p+1)a}.$$

Proof The proof is by induction on v. Obviously, $I(u,u+1)=I(u,u+2)=\{0\}$ and $[u,u+1]=L_2^{p^u}$, $[u,u+2]=L_2^{p^u}Q_{2,1}^{p^u}$. Hence, the proposition follows with $v=u+1$ and $v=u+2$. From the definition of the set $I(u,v)$, we obtain

(9) $$I(u,v+2)=I(u,v+1)\cup(p^{v-1}+I(u,v)),$$

where $p^{v-1}+I(u,v)=\{p^{v-1}+a\ ;\ a\in I(u,v)\}$.

Combining Theorem 2.4, the inductive hypothesis and the relation $Q_{2,0} = L_2^{p-1}$, we get

$$\begin{aligned}
[u, v+2] &= [u, v+1]Q_{2,1}^{p^v} - [u, v]Q_{2,0}^{p^v} \\
&= \Big(\sum_{a \in I(u,v+1)} (-1)^a L_2^{p^u + p(p-1)a} Q_{2,1}^{\frac{p^v - p^u}{p-1} - (p+1)a} \Big) Q_{2,1}^{p^v} \\
&\quad - \Big(\sum_{a \in I(u,v)} (-1)^a L_2^{p^u + p(p-1)a} Q_{2,1}^{\frac{p^{v-1} - p^u}{p-1} - (p+1)a} \Big) Q_{2,0}^{p^v} \\
&= \sum_{a \in I(u,v+1)} (-1)^a L_2^{p^u + p(p-1)a} Q_{2,1}^{\frac{p^{v+1} - p^u}{p-1} - (p+1)a} \\
&\quad + \sum_{a \in I(u,v)} (-1)^{p^{v-1} + a} L_2^{p^u + p(p-1)(p^{v-1}+a)} Q_{2,1}^{\frac{p^{v+1} - p^u}{p-1} - (p+1)(p^{v-1}+a)}.
\end{aligned}$$

From this equality and (9), we see that the proposition is true for $v+2$, so the proof is completed. □

Now, we compute $[u, v, v+1]$ in terms of $L_3, Q_{3,1}, Q_{3,2}$.

Denote by $J(u, v)$ the set of all integers a satisfying

$$\begin{aligned}
&\alpha_i(a) \le 1 \quad \text{and} \quad \alpha_i(a) + \alpha_{i+1}(a) + \alpha_{i+2}(a) \le 2, && \text{for any } i, \\
&\alpha_i(a) = 0, && \text{for either } i < u \text{ or } i \ge v-2.
\end{aligned}$$

It is easy to see that for any $a \in J(u, v)$, there exists uniquely an expansion

$$a = a_0 + p^{i_1} + p^{i_1+1} + a_1 + \ldots + p^{i_k} + p^{i_k+1} + a_k,$$

with $i_0 = u-3 < i_1 < \ldots < i_k < i_{k+1} = v-1, i_{j+1} - i_j \ge 3$ and $a_j \in I(i_j+3, i_{j+1})$ for $0 \le j \le k$.

We define the functions $b_{u,v}, c_{u,v} \colon J(u, v) \to \mathbb{Z}$ by setting

$$\begin{aligned}
b_{u,v}(a) &= \frac{p^{v-1} - p^u}{p-1} - (p+1)a + p(p^{i_1} + \ldots + p^{i_k}), \\
c_{u,v}(a) &= a_0 + a_1 + \ldots + a_k.
\end{aligned}$$

Proposition 4.3 *Under the above notation, we have*

$$[u, v, v+1] = \sum_{a \in J(u,v)} (-1)^a L_3^{p^u + p(p-1)a} Q_{3,1}^{b_{u,v}(a)} Q_{3,2}^{c_{u,v}(a)}.$$

The proof of the proposition is based on some lemmas.

Lemma 4.4 *For $0 \le u < v$,*

$$J(u,v+3) = J(u,v+2) \cup (p^v + J(u,v+1)) \cup (p^v + p^{v-1} + J(u,v)).$$

Here, for $x \in \mathbb{Z}$ and $A \subset \mathbb{Z}$, we write $x + A = \{x + a \ ; \ a \in A\}$.

$$\begin{aligned}
b_{u,v+3}(a) &= p^{v+1} + b_{u,v+2}(a), & \\
c_{u,v+3}(a) &= c_{u,v+2}(a), & \text{for } a \in J(u,v+2), \\
b_{u,v+3}(p^v + a) &= b_{u,v+1}(a), & \\
c_{u,v+3}(p^v + a) &= p^v + c_{u,v+1}(a), & \text{for } a \in J(u,v+1), \\
b_{u,v+3}(p^v + p^{v-1} + a) &= b_{u,v}(a), & \\
c_{u,v+3}(p^v + p^{v-1} + a) &= c_{u,v}(a), & \text{for } a \in J(u,v).
\end{aligned}$$

This lemma can easily be proved by computing directly from the definitions of $J(u,v)$, $b_{u,v}$ and $c_{u,v}$.

Lemma 4.5 *For any $0 \le u < v$, we have*

$$[u,v+3,v+4] = [u,v+2,v+3]Q_{3,1}^{p^{v+1}} - [u,v+1,v+2]Q_{3,0}^{p^{v+1}}Q_{3,2}^{p^v} + [u,v,v+1]Q_{3,0}^{p^{v+1}+p^v}.$$

Proof A direct calculation using Theorem 2.4 gives

$$\begin{aligned}
[u,v+3,v+4] &= [u,v+2,v+3]Q_{2,0}^{p^{v+2}} + [u,v+3]V_3^{p^{v+2}} \\
&= [u,v+2,v+3](Q_{3,1}^{p^{v+1}} - Q_{2,1}^{p^{v+1}}V_3^{(p-1)p^{v+1}}) \\
&\quad + ([u,v+2]Q_{2,1}^{p^{v+1}} - [u,v+1]Q_{2,0}^{p^{v+1}})V_3^{p^{v+2}} \\
&\qquad\qquad (\text{since } Q_{3,1} = Q_{2,0}^p + Q_{2,1}V_3^{p-1}) \\
&= [u,v+2,v+3]Q_{3,1}^{p^{v+1}} \\
&\quad - ([u,v+1,v+2]Q_{2,0}^{p^{v+1}} + [u,v+2]V_3^{p^{v+1}})Q_{2,1}^{p^{v+1}}V_3^{(p-1)p^{v+1}} \\
&\quad + [u,v+2]Q_{2,1}^{p^{v+1}}V_3^{p^{v+2}} - [u,v+1]Q_{2,0}^{p^{v+1}}V_3^{p^{v+2}} \\
&= [u,v+2,v+3]Q_{3,1}^{p^{v+1}} \\
&\quad - [u,v+1,v+2]Q_{2,0}^{p^{v+1}}V_3^{(p-1)p^{v+1}}(Q_{2,1}^{p^{v+1}} + V_3^{(p-1)p^v}) \\
&\quad + ([u,v+1,v+2] - [u,v+1]V_3^{p^v})Q_{2,0}^{p^{v+1}}V_3^{(p-1)(p^{v+1}+p^v)}.
\end{aligned}$$

Using Theorem 2.4 and the relations $Q_{3,2} = Q_{2,1}^p + V_3^{p-1}$, $Q_{3,0} = Q_{2,0}V_3^{p-1}$, we obtain the lemma. □

Proof of Proposition 4.3 The proof is by induction on v. For $v = u+1, u+2, u+3$ the proposition is obvious. Suppose that it is true for $v, v+1, v+2$. Using Lemma 4.5, the inductive hypothesis and the relation $Q_{3,0} = L_3^{p-1}$, we get

$$\begin{aligned}[u, v+3, v+4] = &\sum_{a\in J(u,v+2)} (-1)^a L_3^{p^u+p(p-1)a} Q_{3,1}^{p^{v+1}+b_{u,v+2}(a)} Q_{3,2}^{c_{u,v+2}(a)} \\ &+ \sum_{a\in J(u,v+1)} (-1)^{p^v+a} L_3^{p^u+p(p-1)(p^v+a)} Q_{3,1}^{b_{u,v+1}(a)} Q_{3,2}^{p^v+c_{u,v+1}(a)} \\ &+ \sum_{a\in J(u,v)} (-1)^{p^v+p^{v-1}+a} L_3^{p^u+p(p-1)(p^v+p^{v-1}+a)} Q_{3,1}^{b_{u,v}(a)} Q_{3,2}^{c_{u,v}(a)}.\end{aligned}$$

Combining this equality and Lemma 4.4, we see that the proposition holds for $v+3$. Hence, the proposition is proved. □

References

[1] **L E Dickson**, *A fundamental system of invariants of the general modular linear group with a solution of the form problem*, Trans. Amer. Math. Soc. 12 (1911) 75–98 MR1500882

[2] **N H V Hưng**, **P A Minh**, *The action of the mod p Steenrod operations on the modular invariants of linear groups*, Vietnam J. Math. 23 (1995) 39–56 MR1367491

[3] **N E Kechagias**, *The Steenrod algebra action on generators of rings of invariants of subgroups of* $\mathrm{GL}_n(\mathbb{Z}/p\mathbb{Z})$, Proc. Amer. Math. Soc. 118 (1993) 943–952 MR1152986

[4] **I Madsen**, **R J Milgram**, *The classifying spaces for surgery and cobordism of manifolds*, Annals of Mathematics Studies 92, Princeton University Press, Princeton, N.J. (1979) MR548575

[5] **J Milnor**, *The Steenrod algebra and its dual*, Ann. of Math. (2) 67 (1958) 150–171 MR0099653

[6] **H Mùi**, *Modular invariant theory and cohomology algebras of symmetric groups*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 22 (1975) 319–369 MR0422451

[7] **H Mùi**, *Cohomology operations derived from modular invariants*, Math. Z. 193 (1986) 151–163 MR852916

[8] **M D Neusel**, *Inverse invariant theory and Steenrod operations*, Mem. Amer. Math. Soc. 146 (2000) x+158 MR1693799

[9] **L Smith**, **R M Switzer**, *Realizability and nonrealizability of Dickson algebras as cohomology rings*, Proc. Amer. Math. Soc. 89 (1983) 303–313 MR712642

[10] **N E Steenrod**, **D B A Epstein**, *Cohomology operations*, Annals of Mathematics Studies 50, Princeton University Press, Princeton, N.J. (1962) MR0145525

[11] **N Sum**, *On the action of the Steenrod–Milnor operations on the modular invariants of linear groups*, Japan. J. Math. (N.S.) 18 (1992) 115–137 MR1173832

[12] **N Sum**, *On the action of the Steenrod algebra on the modular invariants of special linear group*, Acta Math. Vietnam. 18 (1993) 203–213 MR1292080

[13] **N Sum**, *Steenrod operations on the modular invariants*, Kodai Math. J. 17 (1994) 585–595 MR1296929Workshop on Geometry and Topology (Hanoi, 1993)

[14] **C Wilkerson**, *A primer on the Dickson invariants*, from: "Proceedings of the Northwestern Homotopy Theory Conference (Evanston, Ill., 1982)", Contemp. Math. 19, Amer. Math. Soc., Providence, RI (1983) 421–434 MR711066

Department of Mathematics, University of Quynhon
170 An Duong Vuong, Quynhon, Binhdinh, Vietnam

ngnsum@yahoo.com

Received: 15 November 2004 Revised: 28 August 2005