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Young tableaux and the Steenrod algebra

G WALKER
R M W WOOD

The purpose of this paper is to forge a direct link between the hit problem for the action of the Steenrod algebra $\mathcal{A}$ on the polynomial algebra $\mathbf{P}(n) = \mathbb{F}_2[x_1, \ldots, x_n]$, over the field $\mathbb{F}_2$ of two elements, and semistandard Young tableaux as they apply to the modular representation theory of the general linear group $GL(n, \mathbb{F}_2)$. The cohits $\mathbf{Q}^d(n) = \mathbf{P}^d(n)/\mathbf{P}^d(n) \cap \mathcal{A}^+(\mathbf{P}(n))$ form a modular representation of $GL(n, \mathbb{F}_2)$ and the hit problem is to analyze this module. In certain generic degrees d we show how the semistandard Young tableaux can be used to index a set of monomials which span $\mathbf{Q}^d(n)$. The hook formula, which calculates the number of semistandard Young tableaux, then gives an upper bound for the dimension of $\mathbf{Q}^d(n)$. In the particular degree d where the Steinberg module appears for the first time in $\mathbf{P}(n)$ the upper bound is exact and $\mathbf{Q}^d(n)$ can then be identified with the Steinberg module.

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1 Introduction

Young tableaux form a combinatorial device for constructing representations of the general linear group $GL(n)$ of $n \times n$ non-singular matrices and its subgroup Σ_n of permutation matrices, both in the classical case, over the field of complex numbers, and in the modular case, where the characteristic of the field divides the order of the group (see Fulton [7], James–Kerber [8], MacDonald [14], Sagan [19] and Stanley [21]). The group $GL(n, \mathbb{F}_2)$, over the field $\mathbb{F}_2$ of two elements, acts naturally on the polynomial algebra $\mathbf{P}(n) = \mathbb{F}_2[x_1, \ldots, x_n]$ by matrix substitution and the homogeneous polynomials $\mathbf{P}^d(n)$ of degree d form a representation space. The modular representation theory of subgroups of $GL(n, \mathbb{F}_2)$, acting in this way on $\mathbf{P}(n)$, is important in understanding the nature of the *hit problem* for the action of the mod 2 Steenrod algebra $\mathcal{A}$ on $\mathbf{P}(n)$. The problem is to find a minimal generating set for $\mathbf{P}(n)$ as an $\mathcal{A}$–module (see Boardman [2], Janfada–Wood [9; 10], Kameko [12; 13], Peterson [18] and Wood [24; 26; 27; 28; 29]). The Steenrod squaring operators Sq^k generate $\mathcal{A}$ as an algebra and act as $GL(n, \mathbb{F}_2)$–module maps from $\mathbf{P}^d(n)$ to $\mathbf{P}^{d+k}(n)$. A polynomial h is *hit* if it can be written as a finite sum $h = \sum_{k>0} \Theta_k(f_k)$ for elements Θ_k of positive grading

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in $\mathcal{A}$ and suitable polynomials f_k. Equivalently, $h = \sum_{k>0} \mathrm{Sq}^k(g_k)$ for suitable polynomials g_k.

The hit problem can be viewed in terms of finding a vector space basis for the quotient $\mathbf{Q}(n)$ of $\mathbf{P}(n)$ by the hit elements. This quotient is a $GL(n, \mathbb{F}_2)$–module. It is clear that polynomials in $\mathbf{P}^d(n)$ which represent non-trivial elements in an irreducible composition factor of $\mathbf{P}^d(n)$, occurring for the first time in degree d, cannot be hit, otherwise there would be a Steenrod operation linking the composition factor with an earlier occurrence. This goes some way to explain the interrelationship between modular representation theory and the Steenrod algebra. From the point of view of representation theory, the quotient $\mathbf{Q}(n)$ is a repository for the irreducible modular representations of $GL(n, \mathbb{F}_2)$ and from the point of view of the Steenrod algebra, first occurrences of irreducible representations contribute to a generating set of the $\mathcal{A}$–module $\mathbf{P}(n)$.

We shall explain a direct connection between Young tableaux and generators for the $\mathcal{A}$–module $\mathbf{P}(n)$. In general, there are too many Young tableaux to solve the hit problem precisely but in certain degrees the number of semistandard Young tableaux does give the correct minimal number of generators. The following is a sample result from the more general Theorem 3.15.

Theorem 1.1 *In any minimal generating set for the $\mathcal{A}$–module $\mathbf{P}(n)$, there are $2^{\binom{n}{2}}$ elements in degree $d = 2^n - n - 1$. In this degree monomial generators in $\mathbf{P}(n)$ may be chosen in bijective correspondence with the semistandard Young tableaux associated with the partition $(n-1, n-2, \dots, 1)$ of the number $\binom{n}{2}$. Furthermore, these generators provide representatives for an additive basis for the first occurrence in degree d of the Steinberg representation of $GL(n, \mathbb{F}_2)$, viewed as the quotient of $\mathbf{P}^d(n)$ by the hit elements.*

For instance, taking the case $n = 3$, there are eight semistandard Young tableaux as exhibited below.

Example 1.2

$$\begin{smallmatrix}1&1\\2&\end{smallmatrix}\quad \begin{smallmatrix}1&1\\3&\end{smallmatrix}\quad \begin{smallmatrix}1&2\\2&\end{smallmatrix}\quad \begin{smallmatrix}1&2\\3&\end{smallmatrix}\quad \begin{smallmatrix}2&2\\3&\end{smallmatrix}\quad \begin{smallmatrix}2&3\\3&\end{smallmatrix}\quad \begin{smallmatrix}1&3\\2&\end{smallmatrix}\quad \begin{smallmatrix}1&3\\3&\end{smallmatrix}$$

The corresponding monomial generators in $\mathbf{P}^4(3)$, equivalently representative monomials of a vector space basis for $\mathbf{Q}^4(3)$, are

$$x_1^3x_2,\quad x_1^3x_3,\quad x_1x_2^3,\quad x_1x_2^2x_3,\quad x_2^3x_3,\quad x_2x_3^3,\quad x_1x_2x_3^2,\quad x_1x_3^3.$$

The fact that the first occurrence of the Steinberg representation is in degree $2^n - n - 1$ is well known (Mitchell–Priddy [16], Minh–Tri [15] and Walker–Wood [23]). The result of Theorem 1.1 may be paraphrased by saying that the Steinberg representation is the only irreducible representation of $GL(n, \mathbb{F}_2)$ to contribute to a minimal generating set for the $\mathcal{A}$–module $\mathbf{P}(n)$ in this degree.

In the next section we explain how to associate monomials with tableaux and, more generally, we translate some of the traditional language used in the combinatorial theory of tableaux into the language of *block technology*, which is appropriate for handling the action of the Steenrod algebra. In particular, we introduce a combinatorial procedure, called *splicing*, which is used to replace a block by a formal sum of *semistandard blocks*. This is analogous to the *straightening* process for bringing Young tableaux into standard form in the context of group rings, see Fulton [7]. In Section 3 it is shown how splicing can be realized by the action of the Steenrod algebra and Theorem 3.15 is proved. In Section 4 we show how Theorem 1.1 follows by considering the special case of the Steinberg representation, using the hook formula to count the number of semistandard Young tableaux.

In general, the hook formula shows that for a fixed $n \geq 2$ and increasing d the number of semistandard Young tableaux increases, whereas the dimension of $\mathbf{Q}^d(n)$ is known to be bounded in d for a given n, see Carlisle–Wood [4]. It would be interesting to find a more restrictive condition on semistandard Young tableaux which cuts down the number of generators of $\mathbf{Q}^d(n)$, at least in the row-regular case, to a number bounded in d which estimates more closely the dimension of the cohits. It would also be interesting to investigate the dual hit problem and identify a basis for the kernel of the down Steenrod action in terms of the combinatorics of Young tableaux and the relationship with the ring of lines as described in Alghamdi–Crabb–Hubbuck [1] and Crabb–Hubbuck [5]. At the end of Section 3 we give an example to show the limitations of the main theorem. In the last section we explain briefly how Theorem 1.1 can be extended to other irreducible representations of $GL(n, \mathbb{F}_2)$ having a certain affinity to the Steinberg representation.

2 Binary blocks and Young tableaux

There are two frequently used numerical functions in the context of the hit problem. One is the α–function $\alpha(d)$ of a positive integer d, which counts the number of digits 1 in the binary expansion of d, and the other is the μ–function $\mu(d)$, which is the smallest number k for which d can be partitioned in exponential form $d = \sum_{i=1}^{k}(2^{\lambda_i} - 1)$. We extend the definitions to cover $\alpha(0) = \mu(0) = 0$. In general, the exponential partition of a number d, with a given value of $\mu(d)$, is not unique. For example $\mu(17) = 3$ and $17 = 15 + 1 + 1 = 7 + 7 + 3$.

We are concerned with two types of partitions of numbers: the exponential partition of d as in the definition of the μ–function and the ordinary partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of the number $|\lambda| = \lambda_1 + \cdots + \lambda_n$, where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$. The *length* of the partition is the number of non-zero parts λ_i. In this article we reserve n for the number of variables in $\mathbf{P}^d(n)$ and restrict attention to partitions of length not greater than n. In combinatorics it is customary to illustrate the partition λ by a *Ferrers diagram*, which is an array of boxes in echelon shape with λ_i boxes in the i th row. The positions of the boxes are the *nodes* of the Ferrers diagram. A *Young tableau* is a Ferrers diagram in which each box is filled with a positive integer. In particular, filling each box with the digit 1 produces an array of the form

$$F = \begin{array}{l} 1 \ldots 1 \ldots 1 \ldots 1 \\ 1 \ldots 1 \ldots 1 \\ \ldots \\ 1 \ldots 1 \end{array}$$

with λ_i contiguous digits 1 in the i th row. We shall call this a *Ferrers block* and interpret it in terms of the exponential partition $d = \sum_{i=1}^{k} (2^{\lambda_i} - 1)$, where now the rows of F are the reverse binary expansions of the numbers $2^{\lambda_i} - 1$ as read from left to right. More generally, a *binary block* is a $(0, 1)$–array associated with a monomial $f = x_1^{d_1} \ldots x_n^{d_n}$, whose entries are the digits, in reversed binary expansion, of the exponents d_i. Blocks were introduced in Carlisle–Wood [4] as a graphical device for keeping track of the action of Steenrod squares on monomials and have been used in several places to exhibit minimal sets of monomial generators (see Janfada–Wood [10]). A formal sum of blocks corresponds to a polynomial over $\mathbb{F}_2$ (ordinary addition of matrices is not used in this article). If we are working in $\mathbf{P}(n)$ then the number of rows in a block is n. In particular a Ferrers block may have zero rows at the bottom. On the other hand the number c of columns in a block is not determined by the corresponding monomial. We adopt the convention of regarding two row-vectors of nonnegative integers as equivalent if they differ by trailing zeros and we omit trailing zeros when convenient. In particular the empty vector is identified with a vector of zero entries. The convention is extended to arrays except that an empty row in indicated by a leading 0. This is necessary to keep track of the positions of missing variables in a monomial and maintain the number of rows at n. Under these conventions we have a bijective correspondence between blocks with n rows and monomials in $\mathbf{P}(n)$. The blocks associated with the monomials in Example 1.2 are given in the following list.

Example 2.1

$$\begin{array}{l} 1\,1 \\ 1 \\ 0 \end{array} \quad \begin{array}{l} 1\,1 \\ 0 \\ 1 \end{array} \quad \begin{array}{l} 1 \\ 1\,1 \\ 0 \end{array} \quad \begin{array}{l} 1 \\ 0\,1 \\ 1 \end{array} \quad \begin{array}{l} 0 \\ 1\,1 \\ 1 \end{array} \quad \begin{array}{l} 0 \\ 1 \\ 1\,1 \end{array} \quad \begin{array}{l} 1 \\ 1 \\ 0\,1 \end{array} \quad \begin{array}{l} 1 \\ 0 \\ 1\,1 \end{array}$$

In the context of the hit problem, a *spike* in $\mathbf{P}(n)$ is a monomial of the form $x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}$, where each d_i has the form $2^{\lambda_i} - 1$. The corresponding block is a row permutation of the Ferrers block, with appropriate 0–rows inserted.

A more compact way of designating a $(0,1)$–array F is to form the corresponding array Y of nonnegative integers by the following rule. For each number j let L denote the list of row positions occupied by a digit 1 in the jth column of F, counting from the top row down. Then the ith element of L occupies position (i,j) in Y. If a column of F has no digits 1 then the corresponding column of Y has zero entries, keeping in mind the trailing zero convention for rows of the array. By construction, the non-zero entries in a column of Y are strictly increasing.

Definition 2.2 An array is *column strict* if the non-zero entries of any column are strictly increasing from top down. The process of assigning the column-strict array Y to the block B is called the *column-position correspondence* and is denoted by $Y = \mathrm{cp}(F)$.

Given a column-strict array Y with no entry larger than n, then it is clear how to constitute the block F with n rows so that $\mathrm{cp}(F) = Y$. The column-position correspondence is therefore bijective between blocks and column-strict arrays. It is easy to see that the blocks of Example 2.1 and the arrays of Example 1.2 are related by the cp correspondence and this in turn establishes the correspondence with the monomials in Example 1.2.

We shall now translate some of the traditional language of Young tableaux (see Fulton [7], Macdonald [14], Sagan [19] and Stanley [21]) into the language of block technology (see Janfada–Wood [10]). The *ω–vector* of a block F is the vector $\omega(F) = (\omega_1, \ldots, \omega_c)$ of column sums of F. The *α–vector* of F is the vector $\alpha(F) = (\alpha_1, \ldots, \alpha_n)$ of row sums. In the case of the Ferrers block associated with the partition λ we have $\alpha(F) = \lambda$ and $\omega(F) = \lambda'$, the conjugate of λ. The *degree* d of a block F, or associated array $\mathrm{cp}(F)$, means the degree of the corresponding monomial, and this is a function of the ω–vector given by $d = \sum_{j>0} \omega_j 2^{j-1}$. In terms of $\mathrm{cp}(F)$, the jth entry ω_j of the ω–vector is the number of non-zero entries in the jth column of $\mathrm{cp}(F)$.

Of particular interest in this article are the monomials with *descending ω–*vectors, meaning that $\omega_j \geq \omega_{j+1}$ for $j \geq 1$, keeping in mind the trailing zeros convention for vectors. All the blocks in Example 2.1 are of this type with ω–vector $(2,1)$. If a block F has a descending ω–vector then it is easy to see that the corresponding column-position array $Y = \mathrm{cp}(F)$ is a column-strict Young tableau. In some parts of the literature column-strict is included in the definition of a Young tableau. One can easily check that the ith entry of $\alpha(F)$ is the number of repetitions of i in Y.

It follows that a monomial with descending ω–vector has a uniquely associated column-strict Young tableau via the column-position correspondence. In combinatorics a column-strict Young tableau is called *semistandard* if the rows are weakly increasing. The Young tableaux in Example 1.2 are semistandard.

The following lemma, which is straightforward to prove, summarizes the situation so far.

Lemma 2.3 *Working in* $\mathbf{P}(n)$*, the column-position correspondence sets up a bijection between monomials with descending* ω*–vectors and column-strict Young tableaux, with entries taken from the set* $\{1,\dots,n\}$*, based on Ferrers blocks with* n *rows. A semistandard tableau* $\mathrm{cp}(F)$ *corresponds to a block* F *with the property that* $\omega(F[i])$ *is descending for each* i *in the range* $1 \le i \le n$*, where* $F[i]$ *denotes the block formed by taking the first* i *rows of* F*.*

In the light of this lemma it is appropriate to make the following definition.

Definition 2.4 A block F with n rows is *semistandard* if $\omega(F[i])$ is descending for each subblock of $F[i]$ for $1 \le i \le n$.

The ultimate aim of this article is to find a generating set for the $\mathcal{A}$–module $\mathbf{P}(n)$ among semistandard blocks at least in certain degrees. We shall call a degree d *row-regular for* n if it has an exponential partition $d = \sum_{i=1}^{n}(2^{\lambda_i}-1)$, where the partition $\lambda = (\lambda_1,\dots,\lambda_n)$ satisfies the condition $\lambda_1 > \cdots > \lambda_n \ge 0$. In this case d has a unique exponential partition of length $n-1$ or n. For such a degree we have $\mu(d) = n$ or $\mu(d) = n-1$, but not all degrees with these μ–values are row-regular for n. Up to permutation of rows there is only one spike in $\mathbf{P}^d(n)$ when d is row-regular, and therefore just one associated Ferrers block and just one descending ω–vector. All monomials in $\mathbf{P}^d(n)$ with this ω–vector have corresponding column-strict Young tableaux with the same underlying Ferrers diagram.

Later proofs will require induction on certain partial order relations on monomials and corresponding blocks. These are constructed from total order relations on ω–vectors. We shall highlight two of these.

Definition 2.5 Let $L = (a_1, a_2, \dots, a_s)$ and $M = (b_1, b_2, \dots, b_s)$ be two vectors of non-negative integers. We write $L >_l M$ and read 'greater than in left order' if $a_1 > b_1$ or $a_i = b_i$ for $1 \le i < t \le s$ and $a_t > b_t$. We also write $L >_r M$ and read 'greater than in right order' if $a_s < b_s$ or $a_i = b_i$ for $1 \le t < i \le s$ and $a_t < b_t$.

To compare ω–vectors we allow trailing zeros to equalize length. As usual, in either ordering we write $L < M$ to mean $M > L$.

Both right and left orderings are total and induce partial orderings on blocks by ordering their ω–vectors. We shall also talk about the left and right ordering of blocks. The *reverse* lexicographic order is chosen in the right order case to provide consistency with the action of the Steenrod algebra, as we shall see later in Lemma 3.8.

The following statement is a simple numerical fact about unique descending ω–vectors that will be required later.

Proposition 2.6 *If $\mathbf{P}^d(n)$ admits a unique descending ω–vector ω, then, for any block B in $\mathbf{P}^d(n)$ with $\omega(B) >_l \omega$, the first number t for which $\omega_t(B) > \omega_t$ also satisfies the conditions $t > 1$ and $\omega_{t-1}(B) < \omega_t(B)$.*

For example, $\mathbf{P}^8(3)$ has the unique ω–vector $(2,1,1)$, with Ferrers block F, and the left greater block B as shown below.

Example 2.7

$$F = \begin{smallmatrix} 1 & 1 & 1 \\ 1 & & \\ 0 & & \end{smallmatrix} \qquad B = \begin{smallmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{smallmatrix}$$

We see that $\omega_1(B) = \omega_1(F)$ and $\omega_2(B) > \omega_2(F)$. Also $\omega_1(B) < \omega_2(B)$.

A familiar process in the combinatorics of Young tableaux is *straightening* which is a device for maneuvering a Young tableau into an equivalent sum of semistandard Young tableaux in the context of group rings. We shall now explain an analogous process for blocks which we shall later relate to the action of the Steenrod algebra on polynomials. The idea is to maneuver a block in $\mathbf{P}(n)$ into a formal sum of semistandard blocks. We adopt the usual notation $F_{i,j}$ for the (i,j)th entry of the block F.

Definition 2.8 Let F be a block with n rows and let k,t be integers with $1 \le k \le n$ and $t \ge 0$. Assume that, for a certain pair of non-intersecting sets S,T, each containing k numbers between 1 and n, F has entries $F_{i,t+2} = 1$ and $F_{i,t+1} = 0$ for $i \in S$ and $F_{i,t+2} = 0$ and $F_{i,t+1} = 1$ for $i \in T$. Let $G(S,T)$ be the matrix formed from F by leaving all entries unchanged except in columns $t+1$ and $t+2$, where $G_{i,t+2} = 0$ and $G_{i,t+1} = 1$ for $i \in S$ and $G_{i,t+2} = 1$ and $G_{i,t+1} = 0$ for $i \in T$. The process of replacing F by the formal sum of the blocks $G(S,T)$ for S fixed and all possible T is called *k–splicing* of F at *column position* $t+2$ and *row positions* S.

To put it briefly, splicing replaces the block F with the formal sum of all the blocks $G(S,T)$ formed from F by pulling a selection of k digits 1 in column $t+2$ back one place into zero positions, and pushing a non-overlapping collection of k digits in column $t+1$ forward one place into zero positions. Only two adjacent columns of the block are altered, so effectively splicing is a process carried out on a 2–column block implanted as adjacent columns in a larger matrix. Of course, even when the first part of the procedure is possible, it may not always be possible to carry out the second part, in which case we define the result to be 0. It should be noted that the α and ω–vectors of each $G(S,T)$ are the same as those of F.

In the following example there is only one way of carrying out 2–splicing of the matrix B and the result is the sum of blocks C.

Example 2.9

$$B = \begin{matrix} 0\,1 \\ 1\,0 \\ 0\,1 \\ 1\,0 \\ 1\,0 \end{matrix} \quad C = \begin{matrix} 1\,0 \\ 0\,1 \\ 1\,0 \\ 0\,1 \\ 1\,0 \end{matrix} \; + \; \begin{matrix} 1\,0 \\ 0\,1 \\ 1\,0 \\ 1\,0 \\ 0\,1 \end{matrix} \; + \; \begin{matrix} 1\,0 \\ 1\,0 \\ 1\,0 \\ 0\,1 \\ 0\,1 \end{matrix}$$

Here $S=\{1,3\}$ and there are three possible choices of T corresponding to picking two rows from the list $\{2,4,5\}$.

We are now ready to establish the combinatorial part of our main theorem.

Theorem 2.10 *By iterated splicing, any block with descending ω–vector can be replaced by a formal sum of semistandard blocks.*

Proof Let F be a block with n rows and descending ω–vector. We argue by induction on rows, working from the bottom row upwards. We recall that for any block F the subblock of the first i rows of F is denoted by $F[i]$. As the inductive step, assume that, for some number r, F has been replaced by a formal sum of blocks G such that the ω–vectors $\omega(G[i])$ of the subblocks are descending for all i satisfying $1 \le r+1 \le i \le n$. The start of the induction is $r=n-1$, since we are given that $\omega(F)$ is descending. If $r>0$ and $\omega(G[r])$ fails to be descending, then we can find a column position $t+1$ for $t \ge 0$ such that $\omega_{t+1}(G[r]) < \omega_{t+2}(G[r])$. Let S denote the set of row positions i in $G[r]$ for which $G[r]_{i,t+1}=0$ and $G[r]_{i,t+2}=1$, and suppose S has k elements. The effect of performing a k–splice of G at column position $t+2$ and row positions S is to produce a formal sum of matrices $H=H(S,T)$ with the properties

(i) $H[i]$ has descending ω–vector for $r+1 \le i \le n$,

(ii) $\omega(H[r]) >_l \omega(G[r])$.

Assuming these two facts for the moment, we see by (i) that the process of splicing can be continued at column positions where the subblocks at level r fail to be descending without disturbing the condition of descending ω–vectors for levels below row r. By (ii) this process must come to a stop since the ω–vectors are bounded above in left order. The process stops when all blocks H are such that $\omega(H[r])$ is descending and this completes the inductive step.

It remains to justify (i) and (ii). To obtain a typical block H from G we move k digits 1 of $G[r]$ from column $t+2$ back to column $t+1$ and, say, a digits 1 from column $t+1$ forward to column $t+2$. Since $\omega_{t+1}(G[r]) < \omega_{t+2}(G[r])$ we must have $a < k$. The other $k-a$ digits are moved from column $t+1$ to column $t+2$ below level r in G. Then $\omega_{t+1}(H[r]) = \omega_{t+1}(G[r]) + k - a$. It follows that $\omega_{t+1}(H[r]) - \omega_{t+1}(G[r]) > 0$ which proves (ii). Furthermore, let i be a number between $r+1$ and n and suppose b digits 1, in rows $r+1$ to i, move from column $t+1$ to column $t+2$. Then

$$\omega_{t+1}(H[i]) = \omega_{t+1}(G[i]) + k - a - b, \quad \omega_{t+2}(H[i]) = \omega_{t+2}(G[i]) - k + a + b.$$

Hence

$$\omega_{t+1}(H[i]) - \omega_{t+2}(H[i]) = \omega_{t+1}(G[i]) - \omega_{t+2}(G[i]) + 2(k - a - b).$$

By assumption we have

$$a + b \le k \text{ and } \omega_{t+1}(G[i]) \ge \omega_{t+2}(G[i]).$$

Hence $\omega_{t+1}(H[i]) \ge \omega_{t+2}(H[i])$ and since no other columns besides $t+1$ and $t+2$ have been disturbed, (i) follows. □

3 The hit problem for the Steenrod algebra

In this section we explain the action of the Steenrod algebra on polynomials and show how the combinatorial process of k–splicing can be realized by this action up to certain error terms. In favourable situations the error terms are hit, and this leads to our main theorem for generators in row-regular degrees. Background material on the hit problem can be found in Wood [24; 26; 27; 28; 29], Janfada–Wood [9; 10], Alghamdi–Crabb–Hubbuck [1] and Crabb–Hubbuck [5].

The Steenrod algebra $\mathcal{A}$ is a graded algebra generated by the Steenrod squares Sq^k in grading k, over the field $\mathbb{F}_2$, subject to the Adem relations (see Steenrod–Epstein [22]) and $\mathrm{Sq}^0 = 1$.

Proposition 3.1 *The Steenrod squares $\mathrm{Sq}^k, k \ge 0$, act on polynomials by linear transformations $\mathrm{Sq}^k \colon \mathbf{P}^d \to \mathbf{P}^{d+k}$, determined by the conditions,*

$$\mathrm{Sq}^1(x_i) = x_i^2, \quad \mathrm{Sq}^k(x_i) = 0 \text{ for } k > 1,$$

and the Cartan formulae for polynomials f, g

$$\mathrm{Sq}^k(fg) = \sum_{i=0}^{k} \mathrm{Sq}^i(f)\mathrm{Sq}^{k-i}(g).$$

The action of a general element of $\mathcal{A}$ is by addition of compositions of the Steenrod squares. The polynomial algebra $\mathbf{P}(n)$ is a graded left $\mathcal{A}$–module, where the grading is given by degree of polynomials. In principle a Steenrod square can be evaluated on a monomial by iterated use of the Cartan formulae. A more compact way of stating these formulae is in terms of the *total squaring operation*, which is the formal sum $SQ = 1 + \mathrm{Sq}^1 + \mathrm{Sq}^2 + \cdots$. Then SQ is multiplicative, that is, $SQ(fg) = SQ(f)SQ(g)$ for polynomials f, g and the Cartan formulae arise by comparing terms of degree k.

Definition 3.2 Two homogeneous polynomials f, g of the same degree are *equivalent modulo hits* if they satisfy the relation

$$f = g + \sum_{i>0} \mathrm{Sq}^i(h_i),$$

over $\mathbb{F}_2$, which we refer to as a *hit equation*. In particular, if $g = 0$ then f is hit. We write $f \cong g$ if $f - g$ is hit.

The hit problem is to find a minimal generating set for the $\mathcal{A}$–module $\mathbf{P}(n)$. Equivalently we want a vector space basis for the quotient $\mathbf{Q}^d(n)$ of $\mathbf{P}^d(n)$ by the hits in each degree d, frequently referred to as the *cohits*. Such a basis may be represented by a list of monomials of degree d, as in Example 1.2, where $\mathbf{Q}^4(3)$ has dimension 8.

The action of the Steenrod squares as described in Proposition 3.1, when applied to polynomials in an arbitrary number of variables, faithfully represents the Steenrod algebra in the sense that all relations in $\mathcal{A}$ can be detected by the action. Some elementary consequences for a homogeneous polynomial f are easy to prove by induction on degree.

Proposition 3.3 *If $k > \deg(f)$ then $\mathrm{Sq}^k(f) = 0$, and if $k = \deg(f)$ then $\mathrm{Sq}^k(f) = f^2$. If r is not divisible by 2^k then $\mathrm{Sq}^r(f^{2^k}) = 0$ while $\mathrm{Sq}^{s2^k}(f^{2^k}) = (\mathrm{Sq}^s(f))^{2^k}$.*

The second statement expresses the *fractal* nature of the Steenrod action. We shall frequently invoke it when considering the action of a Steenrod square on a monomial b in terms of its columnwise action on the associated block B.

There are some important facts about the Steenrod algebra which are not immediately obvious from its action on polynomials. The Steenrod algebra $\mathcal{A}$ is multiplicatively generated by the Steenrod squares Sq^{2^k} for $k \geq 0$. It admits a coproduct which makes $\mathcal{A}$ into a Hopf algebra (see Steenrod–Epstein [22]) with a *conjugation* operator χ. This is a grade-preserving anti-automorphism of order 2. As in Proposition 3.1 there are rules for working out conjugates of Steenrod squares on polynomials (see Walker–Wood [23]).

Proposition 3.4 *The action of the conjugate Steenrod squares on polynomials* $\chi(\mathrm{Sq}^k)\colon \mathbf{P}^d \to \mathbf{P}^{d+k}$ *are determined by*

$$\chi(\mathrm{Sq}^k)(x_i) = x_i^{2^k} \text{ if } k = 2^a - 1, a \geq 0, \text{ and zero otherwise,}$$

and the Cartan formula $\chi(SQ)(fg) = \chi(SQ)(f)\chi(SQ)(g)$ *for the total conjugate square* $\chi(SQ) = 1 + \chi(\mathrm{Sq}^1) + \chi(\mathrm{Sq}^2) + \cdots$.

There is one fact about the action of Sq^k and its conjugate $\chi(\mathrm{Sq}^k)$ on a product of distinct variables that we shall need at a later stage in relation to the splicing process. Let $\{y_1, \dots, y_m\}$ be a subset of the variables $\{x_1, \dots, x_n\}$.

Lemma 3.5 *For* $k \leq m$,

$$\mathrm{Sq}^k(y_1 \cdots y_m) = y_1 \cdots y_m \sum_{\{i_1,\cdots,i_k\}} y_{i_1} \cdots y_{i_k},$$

where the summation is taken over k*–element subsets of* $\{y_1, \dots, y_m\}$. *If* $k > m$ *then the result is zero.*

$$\chi(\mathrm{Sq}^k)(y_1 \cdots y_m) = \mathrm{Sq}^k(y_1 \cdots y_m) + f,$$

where every monomial in the polynomial f *has an exponent* ≥ 4.

Proof By Proposition 3.4 we have

$$\chi(SQ)(y_1 \cdots y_m) = \prod_{i=1}^{m} \chi(SQ)(y_i) = \prod_{i=1}^{m} (y_i + y_i^2 + y_i^4 + \cdots).$$

Hence

$$\chi(SQ)(y_1 \cdots y_m) = \prod_{i=1}^{m} (y_i + y_i^2) + f = SQ(y_1 \cdots y_m) + f,$$

where all monomials in f have an exponent ≥ 4. The result then follows by comparing terms of degree $m + k$. □

The following result has been significant in proving many results on the hit problem and is known as the χ*–trick* (Crossley [6] and Wood [24; 26; 27; 28; 29]).

Proposition 3.6 *For homogeneous polynomials u, v*

$$u\mathrm{Sq}^k(v) - v\chi(\mathrm{Sq}^k)(u) = \sum_{i>0} \mathrm{Sq}^i(v\chi(\mathrm{Sq}^{k-i})(u)).$$

Thus $u\mathrm{Sq}^k(v) \cong v\chi(\mathrm{Sq}^k)(u)$ and the statement extends by composition and addition of Steenrod operations to show that $u\Theta(v) \cong v\chi(\Theta)(u)$ for any element Θ in $\mathcal{A}$. The χ–trick is the analogue of integration by parts in calculus, when the Steenrod squares are interpreted as differential operators (see Wood [25]). An immediate application of the χ–trick is the following well known observation, used to prove the Peterson conjecture [24; 26; 27; 28; 29].

Proposition 3.7 *Let u and v be homogeneous polynomials such that $\deg(u) < \mu(\deg(v))$. Then uv^2 is hit.*

The proof follows by writing $v^2 = \mathrm{Sq}^d(v)$, where d is the degree of v, applying the χ–trick, and then the fact that the excess of $\chi(\mathrm{Sq}^d)$ is $\mu(d)$. The condition $\deg(u) < \mu(d)$ and the definition of excess (see Steenrod–Epstein [22]) implies $\chi(\mathrm{Sq}^d)(u) = 0$.

We shall find it convenient to switch back and forth between blocks and monomials where appropriate. To avoid repetition we adopt the temporary convention of using upper case letters for blocks and their lower case versions for corresponding monomials. A vertical partition of a block $B = FG$ corresponds to the monomial $b = fg^{2^t}$ if F has t columns. If $t = 0$ then F is empty (corresponding to the monomial 1). If $H = \sum H_k$ is a formal sum of blocks then FH is the formal sum $\sum FH_k$. From the Cartan formula of Proposition 3.1 and the fractal nature of the action of Steenrod squares, as explained in Proposition 3.3, we can study the action of Sq^m columnwise on blocks. For example, corresponding to $B = FG$ we have

$$\mathrm{Sq}^m(b) = \sum \mathrm{Sq}^p(f)(\mathrm{Sq}^q(g))^{2^t},$$

where the summation is over all $p, q \geq 0$ with $p + 2^t q = m$. Splitting a block into its columns as $B = B_1 B_2 \dots B_t$ leads to the formula

$$\mathrm{Sq}^m(b) = \sum \mathrm{Sq}^{p_1}(b_1)(\mathrm{Sq}^{p_2}(b_2))^2 \dots (\mathrm{Sq}^{p_t}(b_t))^{2^t},$$

where the summation is taken over all solutions in non-negative integers p_i of the equation $p_1 + 2p_2 + \cdots + 2^t p_t = m$. In the light of Lemma 3.5, describing the action of a Steenrod square on a product of distinct variables, it is easy to see that the typical action of a Steenrod square on a block moves digits 1 from one column to the next column on the right in the same row, with the knock-on effect of binary addition if digits superimpose. In particular we deduce the following fact about the order relations introduced in Section 1.

Lemma 3.8 *Let* $\mathrm{Sq}^k(F) = F_1 + \cdots + F_s$*, for* $k \geq 1$*, be a formal sum of distinct blocks. Then* $F_i < F$ *in both the left and right orderings for* $1 \leq i \leq s$*.*

We shall now interpret Lemma 3.5 in block language and use the χ–trick to show how k–splicing in the second column of a 2–column block can be realized by the action of the Steenrod algebra modulo certain error terms.

Let C be a 2–column block. Let R be the set of row positions where there is a digit 1 in C. Partition R into three subsets U, V, W as follows. For $i \in U$ we require $C_{i,1} = C_{i,2} = 1$, and for $i \in V$ we require $C_{i,1} = 0$ and $C_{i,2} = 1$, and for $i \in W$ we require $C_{i,1} = 1$ and $C_{i,2} = 0$. Now select a subset S of k elements of V and let B be the 2–column block with zero entries except for $B_{i,1} = 1$ for $i \in S$. Let A be the block formed from C by deleting the digits in positions $C_{i,2}$ for $i \in S$. The following diagrams illustrate an example where

$$U = \{1\}, \quad V = \{2,3,4\}, \quad W = \{5,6,7\}, \quad S = \{2,3\}, \quad k = 2.$$

Example 3.9

$$C = \begin{matrix} 1\,1 \\ 0\,1 \\ 0\,1 \\ 0\,1 \\ 1\,0 \\ 1\,0 \\ 1\,0 \end{matrix} \quad A = \begin{matrix} 1\,1 \\ 0\,0 \\ 0\,0 \\ 0\,1 \\ 1\,0 \\ 1\,0 \\ 1\,0 \end{matrix} \quad B = \begin{matrix} 0\,0 \\ 1\,0 \\ 1\,0 \\ 0\,0 \\ 0\,0 \\ 0\,0 \\ 0\,0 \end{matrix}$$

Then $c = a\mathrm{Sq}^k(b) \cong b\chi(\mathrm{Sq}^k)(a)$ by the χ–trick. Now monomials with exponents ≥ 4 correspond to blocks which are right lower than any 2–column block. Hence by Lemma 3.5 the effect of $\chi(\mathrm{Sq}^k)$ on A is the same as Sq^k on A modulo right lower blocks. Furthermore, any effect arising from Sq^k via the Cartan formula on the second column of A also produces right lower blocks, as does the action of Sq^k on the first column on any row in the set U by the knock-on effect of binary addition. On the other hand, the effect of Sq^k concentrated on the rows of the set W is to produce exactly the result of the k–splicing process on A. Consequently $b\chi(\mathrm{Sq}^k)(a)$ produces the effect of k–splicing C modulo the error terms as described. This is summarized in the following statement.

Proposition 3.10 *Let* C *be a* 2*–column block and let* C' *be the sum of* 2*–column blocks arising from a* k*–splicing process of* C *at column* 2*. Then* $C \cong C'$ *modulo blocks which are right lower than any* 2*–column block.*

Now we need to investigate what happens when a block is implanted as adjacent columns in a larger block.

Lemma 3.11 *Let $B = FCG$ be a vertical splitting of a block and suppose $C \cong C' + R$, where C' is a sum of blocks of the same size as C, and R is a sum of blocks each of which is right lower than C. Let $B' = FC'G$. Then $B \cong B' + F'H + FK$, where F' is a sum of blocks of the same size as F, each of which is left lower than F, and K is a sum of blocks each of which is right lower than CG. In particular B is equivalent to B' modulo blocks which are either left or right lower than B.*

Proof Substituting R for C in B immediately produces blocks which may overlap with G but certainly have the form FK, as stated in the proposition. We may therefore assume that $R = 0$. In terms of corresponding monomials we have $b = fc^{2^t} g^{2^s}$, where t is the number of columns in F and $s - t$ the number of columns in C. Then $C \cong C'$ and there is a hit equation $c = c' + \sum_{k>0} \mathrm{Sq}^k h_k$. By the fractal property in Proposition 3.3, we have the hit equation $c^{2^t} = (c')^{2^t} + \sum_{k>0} \mathrm{Sq}^{2^t k}(h_k^{2^t})$. Applying the χ–trick in Proposition 3.6 to $u = fg^{2^s}$ and $v = h_k^{2^t}$ for each k in turn and then adding, we see that $b - b' \cong \Theta(u)v$ for some positively graded element Θ in the Steenrod algebra. Then by the Cartan formula, Θ must have a positive action either on f or g^{2^s} which means that, in the language of blocks, by Lemma 3.8, either F or G is moved to a sum of lower blocks in either order. The result follows. □

An immediate corollary of Lemma 3.11 and Proposition 3.10 is the following result.

Proposition 3.12 *Let $B = FCG$ be a partitioned block, where C has two columns in positions $t+1, t+2$. Let C' be the sum of the 2–column blocks arising out of a k–splicing process at column $t+2$ and let $B' = FC'G$. Then $B \cong B'$ modulo blocks of the form $F'H$, where F' is left lower than F, and blocks FK, where K is right lower than CG. In particular $B \cong B'$ modulo blocks which are either left or right lower than B.*

Of course, if for some choice of k, the first stage of k–splicing is not possible, then the above statement is void. On the other hand, if k can be chosen in such a way that k–splicing produces the zero result, then Proposition 3.12 says that B is reducible modulo hits to blocks which are either left or right lower than B in the specified way. This leads to the following result.

Proposition 3.13 *Let B in $\mathbf{P}^k(n)$ be a block whose ω–vector is not descending, so that $\omega_{t+1}(B) < \omega_{t+2}(B)$ for some value of t. We can write $B = FCG$, where F has t columns and C is a 2–column block with $\omega_1(C) < \omega_2(C)$. Then B is hit modulo blocks of the form $F'H$, where F' is left lower than F, and blocks FK, where K is right lower than CG. In particular B is hit modulo blocks which are either left or right lower than B.*

Proof The condition $\omega_1(C) < \omega_2(C)$ ensures k–splicing of C in the second column is possible and the largest such k produces the zero result. □

We shall now exploit the above results in a situation where we can control the error terms. Parts of the next proposition, originating in Singer's work [20], are known in more generality (see Carlisle–Wood [4] and Mothebe [17]) but for the sake of completeness we include proofs of these particular cases.

Proposition 3.14 *Assume that $\mathbf{P}^d(n)$ admits a unique descending ω–vector ω. Let B be a block in $\mathbf{P}^d(n)$.*

(i) *If $\omega(B) <_l \omega$ then B is hit.*

(ii) *If $\omega(B) <_r \omega$ then B is hit.*

(iii) *There is a generating set of blocks B for $\mathbf{Q}^d(n)$ with $\omega(B) = \omega$.*

Proof We start with the proof of (i). Let B be a block in $\mathbf{P}^d(n)$ with $\omega(B) <_l \omega$. There is a first column position $t < n$ from the left where $\omega_t(B) < \omega_t$. Consider the vertical splitting $B = FCG$ where F has $t-1$ columns (empty if $t = 1$), C has one column in position t, and G (non-empty) has the rest of the columns of B. Then $\mu(\deg(G)) > \deg(C)$, otherwise we can create a block $B' = FCG'$, where G' is a spike with fewer rows than C. Then $\omega(B')$ is descending and $\omega(B') <_l \omega$, contrary to the assumption that ω is the unique descending ω–vector. It follows from Proposition 3.7 that CG is hit. Then by the arguments used in previous propositions we see that $B \cong F'H$, where F' is a sum of blocks of the same size as F and lower than F in the left order. For a typical such block B'' we have $\omega_s(B'') < \omega_s$ for some $s < t$. Iteration of the process must come to a stop at or before $t = 1$ when the result is zero. Hence B is hit.

To prove (ii) we may as well start with a block B for which $\omega(B) <_r \omega$ and $\omega(B) >_l \omega$. Let t be the first number such that $\omega_{t+2}(B) > \omega_{t+2}$. Then by Proposition 2.6 we can write $B = FCG$ where F (possibly empty) has t columns and C has two columns with $\omega_1(C) < \omega_2(C)$ as in Proposition 3.13. According to this proposition B is equivalent to a sum of blocks of the form $F'H$, where F' is left lower than F, and blocks FK, where K is right lower than CG. But then $\omega(F'H)$ is left lower than ω and therefore $F'H$ is hit by part (i). Hence B is equivalent to a sum of blocks right lower than B. In particular their ω–vectors are right lower than ω and the process can therefore be iterated. The procedure must come to a stop since we cannot have an infinite chain of right lower blocks. The process ends when the result is zero, and this proves that B is hit.

The proof of (iii) follows the same line of argument as the proof of (ii), except that the process stops when the ω–vectors of the blocks reach ω. □

We now state and prove the main result.

Theorem 3.15 *Suppose that* $\mathbf{P}^d(n)$ *admits a unique descending* ω*–vector. Then the cohits* $\mathbf{Q}^d(n)$ *are spanned by the semistandard blocks.*

Proof By part (iii) of Proposition 3.14 we can start with a spanning set for $\mathbf{Q}^d(n)$ consisting of blocks B having the unique descending ω–vector. By Proposition 3.12 and Proposition 3.14 we can replace B by the result of any k–splicing modulo hits. The proof is then complete by Theorem 2.10. □

The limitation of the above approach in the non-regular case, where there is more than one descending ω–vector, is illustrated by the example $\mathbf{P}^7(4)$. Here there are two descending ω–vectors $(1,1,1) < (3,2)$, the least and greatest in either of the order relations. The other possible ω–vectors are $(3,0,1)$ and $(1,3)$ which lie between these extremes.

Example 3.16 Consider the following block C with $\omega(C) = (1,3)$.

$$C = \begin{array}{ll} 1 & \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{array} \quad E = \begin{array}{lll} 0 & 0 & 1 \\ 1 & & \\ 1 & & \\ 1 & & \end{array} \quad F = \begin{array}{ll} 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & \end{array} \quad G = \begin{array}{ll} 0 & 1 \\ 0 & 1 \\ 1 & \\ 0 & 1 \end{array} \quad H = \begin{array}{ll} 0 & 1 \\ 1 & \\ 0 & 1 \\ 0 & 1 \end{array}$$

Now 3–splicing of C in the second column has zero effect but the Steenrod realization has error term E with $\omega(E) = (3,0,1)$. Hence $C \cong E$. Similarly, 1–splicing E in the third column produces the equivalence $E \cong F + G + H$. So iterated splicing has produced the relation

$$C + F + G + H \cong 0,$$

involving blocks with ω–vector $(1,3)$. However, it can be shown that C is not equivalent to a combination of blocks with ω–vectors $(1,1,1)$ or $(3,2)$.

This example contrasts with the case $n = 3$, where a basis for the cohits can be taken with descending ω–vectors. The complete solution of the hit problem in the case $n = 4$ has been given by Kameko [11] in a format which analyzes the hit problem one ω–vector at a time. The vector space $\mathbf{Q}^\omega(n)$ is formed by taking the quotient of the subspace of $\mathbf{P}^d(n)$ generated by monomials with ω–vector $\le \omega$ by the hits and the subspace generated by monomials with ω–vector $< \omega$ in left order. Much of the above work can be applied to $\mathbf{Q}^\omega(n)$ when ω is the least descending ω–vector in degree d (which is the same in either order).

4 The Steinberg representation

The degree $d = 2^n - n - 1$ is row-regular for n and $d = \sum_{i=1}^{n}(2^{n-i} - 1)$ is the unique exponential partition of d into n parts, with unique descending ω–vector $(n-1, n-2, \dots, 1, 0)$ and Ferrers block F. The corresponding partition is also $\lambda = (n-1, n-2, \dots, 1, 0)$. The number of semistandard Young tableaux, and therefore semistandard blocks, is given in Fulton [7, page 55] by the *hook formula*

$$d_\lambda(m) = \prod_{(i,j)\in\lambda} \frac{m+j-i}{h(i,j)},$$

for the Ferrers diagram of λ, filled with numbers from the set $\{1, \dots, m\}$, where $h(i,j)$ denotes the hook length of the node in the Ferrers diagram at position (i,j), that is, the number of nodes to the right and below the given position in the Ferrers diagram including the position itself.

In our application, $m = n$ and $h(i,j) = 2(n-i-j)+1$ giving $d_\lambda(n) = 2^{\binom{n}{2}}$, the dimension of the Steinberg representation of $GL(n, \mathbb{F}_2)$ (see Mitchell–Priddy [16]). Theorem 3.15 shows that the dimension of the vector space of cohits $\mathbf{Q}^d(n)$ is bounded by $2^{\binom{n}{2}}$. The remarks in Section 1 about the first occurrence of an irreducible representation then finally establish Theorem 1.1.

For $m < n$, the Weyl module for $GL(n, \mathbb{F}_2)$ corresponding to the partition $\lambda = (m-1, m-2, \dots, 1, 0, \dots, 0)$ is irreducible, and has dimension $d_\lambda(n)$. By Carlisle–Kuhn [3, Theorem 1.1], the first occurrence as a composition factor is in degree $d = 2^{m+1} - 1 - m$. The work above can then be applied to $\mathbf{Q}^\omega(n)$, when ω is the least descending ω–vector in degree d, to show that $\dim(\mathbf{Q}^\omega(n)) = d_\lambda(n)$.

References

[1] **M A Alghamdi**, **M C Crabb**, **J R Hubbuck**, *Representations of the homology of BV and the Steenrod algebra I*, from: "Adams Memorial Symposium on Algebraic Topology 2 (Manchester, 1990)", London Math. Soc. Lecture Note Ser. 176, Cambridge Univ. Press, Cambridge (1992) 217–234 MR1232208

[2] **J M Boardman**, *Modular representations on the homology of powers of real projective space*, from: "Algebraic topology (Oaxtepec, 1991)", Contemp. Math. 146, Amer. Math. Soc., Providence, RI (1993) 49–70 MR1224907

[3] **D Carlisle**, **N J Kuhn**, *Subalgebras of the Steenrod algebra and the action of matrices on truncated polynomial algebras*, J. Algebra 121 (1989) 370–387 MR992772

[4] **D P Carlisle**, **R M W Wood**, *The boundedness conjecture for the action of the Steenrod algebra on polynomials*, from: "Adams Memorial Symposium on Algebraic Topology, 2 (Manchester, 1990)", London Math. Soc. Lecture Note Ser. 176, Cambridge Univ. Press, Cambridge (1992) 203–216 MR1232207

[5] **M C Crabb**, **J R Hubbuck**, *Representations of the homology of BV and the Steenrod algebra. II*, from: "Algebraic topology: new trends in localization and periodicity (Sant Feliu de Guíxols, 1994)", Progr. Math. 136, Birkhäuser, Basel (1996) 143–154 MR1397726

[6] **M D Crossley**, *H^*V is of bounded type over $\mathcal{A}_p$*, from: "Group representations: cohomology, group actions and topology (Seattle, WA, 1996)", Proc. Sympos. Pure Math. 63, Amer. Math. Soc., Providence, RI (1998) 183–190 MR1603151

[7] **W Fulton**, *Young tableaux*, London Mathematical Society Student Texts 35, Cambridge University Press, Cambridge (1997) MR1464693

[8] **G James**, **A Kerber**, *The representation theory of the symmetric group*, Encyclopedia of Mathematics and its Applications 16, Addison-Wesley Publishing Co., Reading, MA (1981) MR644144

[9] **A S Janfada**, **R M W Wood**, *The hit problem for symmetric polynomials over the Steenrod algebra*, Math. Proc. Cambridge Philos. Soc. 133 (2002) 295–303 MR1912402

[10] **A S Janfada**, **R M W Wood**, *Generating $H^*(\mathrm{BO}(3), \mathbb{F}_2)$ as a module over the Steenrod algebra*, Math. Proc. Cambridge Philos. Soc. 134 (2003) 239–258 MR1972137

[11] **M Kameko**, *Generators of the cohomology of BV_4*, preprint

[12] **M Kameko**, *Products of projective spaces as Steenrod modules*, PhD thesis, Johns Hopkins University (1990)

[13] **M Kameko**, *Generators of the cohomology of BV_3*, J. Math. Kyoto Univ. 38 (1998) 587–593 MR1661173

[14] **I G Macdonald**, *Symmetric functions and Hall polynomials*, second edition, Oxford Mathematical Monographs, Oxford University Press, Oxford (1995) MR1354144

[15] **P A Minh**, **T T Tri**, *The first occurrence for the irreducible modules of general linear groups in the polynomial algebra*, Proc. Amer. Math. Soc. 128 (2000) 401–405 MR1676308

[16] **S A Mitchell**, **S B Priddy**, *Stable splittings derived from the Steinberg module*, Topology 22 (1983) 285–298 MR710102

[17] **M F Mothebe**, *Generators of the polynomial algebra $F_2[x_1, \dots, x_n]$ as a module over the Steenrod algebra*, Comm. Algebra 30 (2002) 2213–2228 MR1904635

[18] **F P Peterson**, *Generators of* $\mathbf{H}^*(\mathbb{R}P^\infty \wedge \mathbb{R}P^\infty)$ *as a module over the Steenrod algebra*, Abstracts Amer. Math. Soc. 833-55-89 (1987)

[19] **B E Sagan**, *The symmetric group*, second edition, Graduate Texts in Mathematics 203, Springer, New York (2001) MR1824028

[20] **W M Singer**, *On the action of Steenrod squares on polynomial algebras*, Proc. Amer. Math. Soc. 111 (1991) 577–583 MR1045150

[21] **R P Stanley**, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics 62, Cambridge University Press, Cambridge (1999) MR1676282

[22] **N E Steenrod**, **D B A Epstein**, *Cohomology operations*, Annals of Mathematics Studies 50, Princeton University Press, Princeton, N.J. (1962) MR0145525

[23] **G Walker**, **R M W Wood**, *Linking first occurrence polynomials over $\mathbb{F}_2$ by Steenrod operations*, J. Algebra 246 (2001) 739–760 MR1872123

[24] **R M W Wood**, *Steenrod squares of polynomials and the Peterson conjecture*, Math. Proc. Cambridge Philos. Soc. 105 (1989) 307–309 MR974986

[25] **R M W Wood**, *Differential operators and the Steenrod algebra*, Proc. London Math. Soc. (3) 75 (1997) 194–220 MR1444319

[26] **R M W Wood**, *Problems in the Steenrod algebra*, Bull. London Math. Soc. 30 (1998) 449–517 MR1643834

[27] **R M W Wood**, *Hit problems and the Steenrod algebra*, from: "Proceedings of the summer school 'Interactions between algebraic topology and invariant theory', a satellite conference of the third European congress of mathematics, Ioannina University, Greece" (2000) 65–103

[28] **R M W Wood**, *Invariants of linear groups as modules over the Steenrod algebra*, from: "Ingo2003, Invariant theory and its interactions with related fields, Göttingen" (2003)

[29] **R M W Wood**, *The Peterson conjecture for algebras of invariants*, from: "Invariant theory in all characteristics", CRM Proc. Lecture Notes 35, Amer. Math. Soc., Providence, RI (2004) 275–280 MR2066475

School of Mathematics, University of Manchester
Manchester, M13 9PL, United Kingdom

Grant.Walker@manchester.ac.uk, Reg.Wood@manchester.ac.uk

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