

Stable splitting and cohomology of p -local finite groups over the extraspecial p -group of order p^3 and exponent p

NOBUAKI YAGITA

Let p be an odd prime. Let G be a p -local finite group over the extraspecial p -group p_+^{1+2} . In this paper we study the cohomology and the stable splitting of their p -complete classifying space BG .

55P35, 57T25; 55R35, 57T05

1 Introduction

Let us write by E the extraspecial p -group p_+^{1+2} of order p and exponent p for an odd prime p . Let G be a finite group having E as a p -Sylow subgroup, and $BG (= BG_p^\wedge)$ the p -completed classifying space of G . In papers by Tezuka and Yagita [11] and Yagita [13; 14], the cohomology and stable splitting for such groups are studied. In many cases non isomorphic groups have homotopy equivalent p -completed classifying spaces, showing that there are not too many homotopy types of BG , as was first suggested by C B Thomas [12] and D Green [3].

Recently, Ruiz and Viruel [9] classified all p -local finite groups for the p -group E . Their results show that each classifying space BG is homotopic to one of the classifying spaces which were studied in [11] or classifying spaces of three exotic 7-local finite groups. (While descriptions in [11] of $H^*({}^2F_4(2)')_{(3)}$, $H^*(Fi'_{24})_{(7)}$ and $H^*(\mathbb{M})_{(13)}$ contained some errors.)

In Section 2, we recall the results of Ruiz and Viruel. In Section 3, we also recall the cohomology $H^*(BE; \mathbb{Z})/(p, \surd 0)$. In this paper, we simply write

$$H^*(BG) = H^*(BG; \mathbb{Z})/(p, \surd 0)$$

and study them mainly. The cohomology $H^{\text{odd}}(BG; \mathbb{Z}_{(p)})$ and the nilpotents parts in $H^{\text{even}}(BG; \mathbb{Z}_{(p)})$ are given in Section 11. Section 4 is devoted to the explanations of stable splitting of BG according to Dietz, Martino and Priddy. In Section 5, and Section 6, we study cohomology and stable splitting of BG for a finite group G having a 3-Sylow group $(\mathbb{Z}/3)^2$ or $E = 3_+^{1+2}$ respectively. In Section 7 and Section 8, we study cohomology of BG for groups G having a 7-Sylow subgroup $E = 7_+^{1+2}$, and

the three exotic 7-local finite groups. In Section 9, we study their stable splitting. In Section 10 we study the cohomology and stable splitting of the Monster group \mathbb{M} for $p = 13$.

2 p -local finite groups over E

Recall that the extraspecial p -group p_+^{1+2} has a presentation as

$$p_+^{1+2} = \langle a, b, c \mid a^p = b^p = c^p = 1, [a, b] = c, c \in \text{Center} \rangle$$

and denote it simply by E in this paper. We consider p -local finite groups over E , which are generalization of groups whose p -Sylow subgroups are isomorphic to E .

The concept of the p -local finite groups arose in the work of Broto, Levi and Oliver [1] as a generalization of a classical concept of finite groups. The p -local finite group is stated as a triple $\langle S, F, L \rangle$ where S is a p -group, F is a saturated fusion system over a centric linking system L over S (for a detailed definition, see [1]). Given a p -local finite group, we can construct its classifying space $B\langle S, F, L \rangle$ by the realization $|L|_p^\wedge$. Of course if $\langle S, F, L \rangle$ is induced from a finite group G having S as a p -Sylow subgroup, then $B\langle S, F, L \rangle \cong BG$. However note that in general, there exist p -local finite groups which are not induced from finite groups (exotic cases).

Ruiz and Viruel recently determined $\langle p_+^{1+2}, F, L \rangle$ for all odd primes p . We can check the possibility of existence of finite groups only for simple groups and their extensions. Thus they find new exotic 7-local finite groups.

The p -local finite groups $\langle E, F, L \rangle$ are classified by $\text{Out}_F(E)$, number of F^{ec} -radical p -subgroup A (where $A \cong (\mathbb{Z}/p)^2$), and $\text{Aut}_F(A)$ (for details see [9]). When a p -local finite group is induced from a finite group G , then we see easily that $\text{Out}_F(E) \cong W_G(E) (= N_G(E)/E.C_G(E))$ and $\text{Aut}_F(A) \cong W_G(A)$. Moreover A is F^{ec} -radical if and only if $\text{Aut}_F(A) \supset SL_2(\mathbb{F}_p)$ by [9, Lemma 4.1]. When G is a sporadic simple group, F^{ec} -radical follows p -pure.

Theorem 2.1 (Ruiz and Viruel [9]) *If $p \neq 3, 7, 5, 13$, then a p -local finite group $\langle E, F, L \rangle$ is isomorphic to one of the following types.*

- (1) $E: W$ for $W \subset \text{Out}(E)$ and $(|W|, p) = 1$,
- (2) $p^2: SL_2(\mathbb{F}_p).r$ for $r|(p-1)$,
- (3) $SL_3(\mathbb{F}_p): H$ for $H \cong \mathbb{Z}/2, \mathbb{Z}/3$ or S_3 .

When $p = 3, 5, 7$ or 13 , it is either of one of the previous types or of the following types.

- (5) ${}^2F_4(2)'$, J_4 , for $p=3$,
- (6) Th for $p=5$,
- (7) He , $He:2$, Fi'_{24} , Fi_{24} , $O'N$, $O'N:2$, and three exotic 7-local finite groups for $p=7$,
- (8) \mathbb{M} for $p=13$.

For case (1), we know that $H^*(E:W) \cong H^*(E)^W$. Except for these extensions and exotic cases, all $H^{\text{even}}(G; \mathbb{Z})_{(p)}$ are studied by Tezuka and Yagita [11]. In [13], the author studied ways to distinguish $H^{\text{odd}}(G; \mathbb{Z})_{(p)}$ and $H^*(G; \mathbb{Z}/p)$ from $H^{\text{even}}(G; \mathbb{Z})_{(p)}$. The stable splittings for such BG are studied in [14]. However there were some errors in the cohomology of ${}^2F_4(2)'$, Fi'_{24} , \mathbb{M} . In this paper, we study cohomology and stable splitting of BG for $p = 3, 7$ and 13 mainly.

3 Cohomology

In this paper we mainly consider the cohomology $H^*(BG; \mathbb{Z})/(p, \sqrt{0})$ where $\sqrt{0}$ is the ideal generated by nilpotent elements. So we write it simply

$$H^*(BG) = H^*(BG; \mathbb{Z})/(p, \sqrt{0}).$$

Hence we have

$$H^*(B\mathbb{Z}/p) \cong \mathbb{Z}/p[y], \quad H^*(B(\mathbb{Z}/p)^2) \cong \mathbb{Z}/p[y_1, y_2] \text{ with } |y| = |y_i| = 2.$$

Let us write $(\mathbb{Z}/p)^2$ as A and let an A -subgroup of G mean a subgroup isomorphic to $(\mathbb{Z}/p)^2$.

The cohomology of the extraspecial p group $E = p_+^{1+2}$ is well known. In particular recall (Leary [6] and Tezuka–Yagita [11])

$$(3-1) \quad H^*(BE) \cong (\mathbb{Z}/p[y_1, y_2]/(y_1^p y_2 - y_1 y_2^p) \oplus \mathbb{Z}/p\{C\}) \otimes \mathbb{Z}/p[v],$$

where $|y_i| = 2$, $|v| = 2p$, $|C| = 2p - 2$ and $Cy_i = y_i^p$, $C^2 = y_1^{2p-2} + y_2^{2p-2} - y_1^{p-1} y_2^{p-1}$. In this paper we write y_i^{p-1} by Y_i , and v^{p-1} by V , eg $C^2 = Y_1^2 + Y_2^2 - Y_1 Y_2$. The Poincare series of the subalgebra generated by y_i and C are computed

$$\frac{1 - t^{p+1}}{(1-t)(1-t)} + t^{p-1} = \frac{(1 + \dots + t^{p-1}) + t^{p-1}}{(1-t)} = \frac{(1 + \dots + t^{p-1})^2 - t^{2p-2}}{(1-t^{p-1})}.$$

From this Poincare series and (3-1), we get the another expression of $H^*(BE)$

$$(3-2) \quad H^*(BE) \cong \mathbb{Z}/p[C, v] \left\{ y_1^i y_2^j \mid 0 \leq i, j \leq p-1, (i, j) \neq (p-1, p-1) \right\}.$$

The E conjugacy classes of A -subgroups are written by

$$A_i = \langle c, ab^i \rangle \text{ for } 0 \leq i \leq p-1$$

$$A_\infty = \langle c, b \rangle.$$

Letting $H^*(BA_i) \cong \mathbb{Z}/p[y, u]$ and writing $i_{A_i}^*(x) = x|_{A_i}$ for the inclusion $i_{A_i}: A_i \subset E$, the restriction images are given by

$$(3-3) \quad y_1|_{A_i} = y \text{ for } i \in \mathbb{F}_p, y_1|_{A_\infty} = 0, \quad y_2|_{A_i} = iy \text{ for } i \in \mathbb{F}_p, y_2|_{A_\infty} = y,$$

$$C|_{A_i} = y^{p-1}, \quad v|_{A_i} = u^p - y^{p-1}u \text{ for all } i.$$

For an element $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_2(\mathbb{F}_p)$, we can identify $GL_2(\mathbb{F}_p) \cong \text{Out}(E)$ by

$$g(a) = a^\alpha b^\gamma, \quad g(b) = a^\beta b^\delta, \quad g(c) = c^{\det(g)}.$$

Then the action of g on the cohomology is given (see Leary [6] and Tezuka–Yagita [11, page 491]) by

$$(3-4) \quad g^*C = C, \quad g^*y_1 = \alpha y_1 + \beta y_2, \quad g^*y_2 = \gamma y_1 + \delta y_2, \quad g^*v = (\det(g))v.$$

Recall that A is F^{ec} -radical if and only if $SL_2(\mathbb{F}_p) \subset W_G(A)$ (see Ruiz–Viruel [9, Lemma 4.1]).

Theorem 3.1 (Tezuka–Yagita [11, Theorem 4.3], Broto–Levi–Oliver [1]) *Let G have the p -Sylow subgroup E , then we have the isomorphism*

$$H^*(BG) \cong H^*(BE)^{W_G(E)} \cap_{A:F^{\text{ec}}\text{-radical}} i_A^{*-1} H^*(BA)^{W_G(A)}.$$

In [1] and [11], proofs of the above theorem are given only for $H^*(BG; \mathbb{Z}_{(p)})$. A proof for $H^*(BG)$ is explained in Section 11.

4 Stable splitting

Martino–Priddy prove the following theorem of complete stable splitting.

Theorem 4.1 (Martino–Priddy [7]) *Let G be a finite group with a p -Sylow subgroup P . The complete stable splitting of BG is given by*

$$BG \sim \vee \text{rank } A(Q, M) X_M$$

where indecomposable summands X_M range over isomorphism classes of simple $\mathbb{F}_p[\text{Out}(Q)]$ -modules M and over isomorphism classes of subgroups $Q \subset P$.

Remark This theorem also holds for p -local finite groups over P , because all arguments for the proofs are done about the induced maps from some fusion systems of P on stable homotopy types of related classifying spaces.

For the definition of rank $A(Q, M)$ see Martino and Priddy [7]. In particular, when Q is not a subretract (that is not a proper retract of a subgroup) of P (see [7, Definition 2]) and when $W_G(Q) \subset \text{Out}(Q) \cong GL_n(\mathbb{F}_p)$ (see [7, Corollary 4.4 and the proof of Corollary 4.6]), the rank of $A(Q, M)$ is computed by

$$\text{rank } A(Q, M) = \sum \dim_{\mathbb{F}_p}(\bar{W}_G(Q_i)M),$$

where $\bar{W}_G(Q_i) = \sum_{x \in W_G(Q_i)} x$ in $\mathbb{F}_p[GL_n(\mathbb{F}_p)]$ and Q_i ranges over representatives of G -conjugacy classes of subgroups isomorphic to Q .

Recall that $\text{Out}(E) \cong \text{Out}(A) \cong GL_2(\mathbb{F}_p)$. The simple modules of $G = GL_2(\mathbb{F}_p)$ are well known. Let us think of A as the natural two-dimensional representation, and let the determinant representation of G . Then there are $p(p - 1)$ simple $\mathbb{F}_p[G]$ -modules given by $M_{q,k} = S(A)^q \otimes (\det)^k$ for $0 \leq q \leq p - 1, 0 \leq k \leq p - 2$. Harris and Kuhn [4] determined the stable splitting of abelian p -groups. In particular, they showed

Theorem 4.2 (Harris–Kuhn [4]) *Let $\tilde{X}_{q,k} = X_{M_{q,k}}$ (resp. $L(1, k)$) identifying $M_{q,k}$ as an $\mathbb{F}_p[\text{Out}(A)]$ -module (resp. $M_{0,k}$ as an $\mathbb{F}_p[\text{Out}(\mathbb{Z}/p)]$ -module). There is the complete stable splitting*

$$BA \sim \vee_{q,k} (q + 1) \tilde{X}_{q,k} \vee_{q \neq 0} (q + 1) L(1, q),$$

where $0 \leq q \leq p - 1, 0 \leq k \leq p - 2$.

The summand $L(1, p - 1)$ is usually written by $L(1, 0)$.

It is also known $H^+(L(1, q)) \cong \mathbb{Z}/p[y^{p-1}\{y^q\}]$. Since we have the isomorphism

$$H^{2q}(BA) \cong (\mathbb{Z}/p)^{q+1} \cong H^{2q}((q + 1)L(1, q)), \text{ for } 1 \leq q \leq p - 1,$$

we get $H^*(\tilde{X}_{q,k}) \cong 0$ for $* \leq 2(p - 1)$.

Lemma 4.3 *Let H be a finite solvable group with $(p, |H|) = 1$ and M be an $\mathbb{F}_p[H]$ -module. Then we have $\bar{H}(M) = (\sum_{x \in H} x)M \cong M^H \cong H^0(H; M)$.*

Proof First assume $H = \mathbb{Z}/s$ and $x \in \mathbb{Z}/s$ its generator. Then

$$\bar{H}(M) = (1 + x + \dots + x^{s-1})M.$$

Since $(1-x^s) = 0$, we see $\text{Ker}(1-x) \supset \text{Image}(\bar{H})$. The facts that M is a \mathbb{Z}/p -module and $(|H|, p) = 1$ imply $H^*(H; M) = 0$ for $* > 0$. Hence

$$\text{Ker}(1-x)/\text{Image}(1 + \dots + x^{s-1}) \cong H^1(H; M) = 0.$$

Thus we have $\bar{H}(M) = \text{Ker}(1-x) = M^H$.

Suppose that H is a group such that

$$0 \rightarrow H' \rightarrow H \xrightarrow{\pi} H'' \rightarrow 0$$

and that $\bar{H}'(M') = (M')^{H'}$ (resp. $\bar{H}''(M'') = (M'')^{H''}$) for each $\mathbb{Z}/p[H']$ -module M' (resp. $\mathbb{Z}/p[H'']$ -module M''). Let σ be a (set theoretical) section of π and denote $\sigma(\bar{H}'') = \sum_{x \in H''} \sigma(x) \in \mathbb{F}_p[H]$. Then

$$\bar{H}(M) = \sigma(\bar{H}'')\bar{H}'(M) = \sigma(\bar{H}'')(M^{H'}) = \bar{H}''(M^{H'}) = (M^{H'})^{H''} = M^H$$

here the third equation follows from that we can identify $M^{H'}$ as an $\mathbb{F}_p[H'']$ -module. Thus the lemma is proved. \square

It is known from a result of Suzuki [10, Chapter 3 Theorem 6.17] that any subgroup of $SL_2(\mathbb{F}_{p^n})$, whose order is prime to p is isomorphic to a subgroup of \mathbb{Z}/s , $4S_4$, $SL_2(\mathbb{F}_3)$, $SL_2(\mathbb{F}_5)$ or

$$Q_{4n} = \langle x, y | x^n = y^2, y^{-1}xy = x^{-1} \rangle.$$

Corollary 4.4 *Let $H \subset GL_2(\mathbb{F}_p)$ with $(|H|, p) = 1$ and H do not have a subgroup isomorphic to $SL_2(\mathbb{F}_3)$ nor $SL_2(\mathbb{F}_5)$. Let $G = A : H$ and let us write $BG \sim \vee_{q,k} \tilde{n}(H)_{q,k} \tilde{X}_{q,k} \vee_{q'} \tilde{m}(H)_{q'} L(1, q')$. Then*

$$\begin{aligned} \tilde{n}(H)_{q,k} &= \text{rank}_p H^0(H; M_{q,k}), \\ \tilde{m}(H)_{q'} &= \text{rank}_p H^{2q'}(BG). \end{aligned}$$

In particular $\tilde{n}(H)_{q,0} = \text{rank}_p H^{2q}(BG)$.

Proof Since $H^*(\tilde{X}_{q,k}) \cong 0$ for $* \leq 2(p-1)$, it is immediate that $\tilde{m}(H)_{q'} = \text{rank}_p H^{2q'}(G)$. Since $GL_2(\mathbb{F}_p) \cong SL_2(\mathbb{F}_p) \cdot \mathbb{F}_p^*$ and $\mathbb{F}_p^* \cong \mathbb{Z}/(p-1)$, each subgroup H in the above satisfies the condition in Lemma 4.3. The first equation is immediate from the lemma. \square

Next consider the stable splitting for the extraspecial p -group E . Dietz and Priddy prove the following theorem.

Theorem 4.5 (Dietz–Priddy [2]) *Let $X_{q,k} = X_{M_{q,k}}$ (resp. $L(2, k)$, $L(1, k)$) identifying $M_{q,k}$ as an $\mathbb{F}_p[\text{Out}(E)]$ -module (resp. $M_{p-1,k}$ as an $\mathbb{F}_p[\text{Out}(A)]$ -module, $\mathbb{F}_p[\text{Out}(\mathbb{Z}/p)]$ -module). There is the complete stable splitting*

$$BE \sim \vee_{q,k} (q+1)X_{q,k} \vee_k (p+1)L(2, k) \vee_{q \neq 0} (q+1)L(1, q) \vee L(1, p-1)$$

where $0 \leq q \leq p-1$, $0 \leq k \leq p-2$.

Remark Of course $\tilde{X}_{q,k}$ is different from $X_{q,k}$ but $\tilde{X}_{p-1,k} = L(2, k)$.

The number of $L(1, q)$ for $1 \leq q < p-1$ is given by the following. Let us consider the decomposition $E/\langle c \rangle \cong \bar{A}_i \oplus \bar{A}_{-i}$ where $\bar{A}_i = \langle ab^i \rangle$ and $\bar{A}_{-0} = \bar{A}_\infty$. We consider the projection $\text{pr}_i: E \rightarrow \bar{A}_i$. Let $x \in H^1(B\bar{A}_i; \mathbb{Z}/p) = \text{Hom}(\bar{A}_i, \mathbb{Z}/p)$ be the dual of ab^i . Then

$$\text{pr}_i^* x(a) = x(\text{pr}_i(a)) = x(\text{pr}_i(ab^i ab^{-i})^{1/2}) = x((ab^i)^{1/2}) = 1/2,$$

$$\text{pr}_i^* x(b) = x(\text{pr}_i(ab^i (ab^{-i})^{-1})^{1/(2i)}) = 1/(2i).$$

Hence for $\beta(x) = y$, we have $\text{pr}_i^*(y) = 1/2y_1 + 1/(2i)y_2$. Therefore the $k+1$ elements $(1/2y_1 + 1/(2i)y_2)^k$, $i = 0, \dots, k$ form a base of $H^{2k}(E/\langle c \rangle; \mathbb{Z}/p) \cong (\mathbb{Z}/p)^{k+1}$ for $k < p-1$. Thus we know the number of $L(1, k)$ is $k+1$ for $0 < k < p-1$.

Recall that

$$H^{2q}(BE) \cong \begin{cases} (\mathbb{Z}/p)^{q+1} \cong H^{2q}((q+1)L(1, q)) & \text{for } 0 \leq 2 \leq p-2 \\ (\mathbb{Z}/p)^{q+2} \cong H^{2p-2}((p+1)L(1, 0)) & \text{for } q = p-1. \end{cases}$$

This shows $H^*(X_{q,k}) \cong 0$ for $* \leq 2p-2$ since so is $L(2, k)$. The number $n(G)_{q,k}$ of $X_{q,k}$ is only depend on $W_G(E) = H$. Hence we have the following corollary.

Corollary 4.6 *Let G have the p -Sylow subgroup E and $W_G(E) = H$. Let*

$$BG \sim \vee n(G)_{q,k} X_{q,k} \vee m(G, 2)_k L(2, k) \vee m(G, 1)_k L(1, k).$$

Then $n(G)_{q,k} = \tilde{n}(H)_{q,k}$ and $m(G, 1)_k = \text{rank}_p H^{2k}(G)$.

Let $W_G(E) = H$. We also compute the dominant summand by the cohomology $H^*(BE)^H \cong H^*(B(E:H))$. Let us write the \mathbb{Z}/p -module

$$X_{q,k}(H) = S(A)^q \otimes v^k \cap H^*(B(E:H)) \quad \text{with } S(A)^q = \mathbb{Z}/p\{y_1^q, y_1^{q-1}y_2, \dots, y_2^q\}.$$

Since the module $\mathbb{Z}/p\{v^k\}$ is isomorphic to the H -module \det^k , we have the following lemma.

Lemma 4.7 The number $n_{q,k}(G)$ of $X_{q,k}$ in BG is given by $\text{rank}_p(X_{q,k}(W_G(E)))$.

Next problem is to seek $m(G, 2)_k$. The number $p + 1$ for the summand $L(2, k)$ in BE is given as follows. For each E -conjugacy class of A -subgroup $A_i = \langle c, ab^i \rangle, i \in \mathbb{F}_p \cup \infty$, we see

$$W_E(A_i) = N_E(A_i)/A_i = E/A_i \cong \mathbb{Z}/p\{b\} \quad b^*: ab^i \mapsto ab^i c.$$

Let $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in $GL_2(\mathbb{F}_p)$ and $U = \langle u \rangle$ the maximal unipotent subgroup. Then we can identify $W_E(A_i) \cong U$ by $b \mapsto u$. For $y_1^s y_2^l \in M_{q,k}$ (identifying $H^*(BA) \cong S^*(A) = \mathbb{Z}/p[y_1, y_2]$), we can compute

$$\begin{aligned} \bar{W}_E(A)y_1^s y_2^l &= (1 + u + \dots + u^{p-1})y_1^s y_2^l = \sum_{i=0}^{p-1} (y_1 + iy_2)^s y_2^l \\ &= \sum_i \sum_t \binom{s}{i} i^t y_1^{s-t} y_2^t y_2^l = \sum_t \binom{s}{t} \sum_i i^t y_1^{s-t} y_2^{t+l}. \end{aligned}$$

Here $\sum_{i=0}^{p-1} i^t = 0$ for $1 \leq t \leq p - 2$, and $= -1$ for $t = p - 1$. Hence we know

$$\dim_p \bar{W}_G(A_i)M_{q,k} = \begin{cases} 0 & \text{for } 1 \leq q \leq p - 2 \\ 1 & \text{for } q = p - 1. \end{cases}$$

Thus we know that BE has just one $L(2, k)$ for each E -conjugacy A -subgroup A_i .

Lemma 4.8 Let A be an F^{ec} -radical subgroup, ie $W_G(A) \supset SL_2(\mathbb{F}_p)$. Then $\bar{W}_G(A)(M_{q,k}) = 0$ for all k and $1 \leq q \leq p - 1$.

Proof The group $SL_2(\mathbb{F}_p)$ is generated by $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $u' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We know $\text{Ker}(1-u) \cong \mathbb{Z}/p[y_1^p - y_2^{p-1} y_1, y_2]$ and $\text{Ker}(1-u') \cong \mathbb{Z}/p[y_2^p - y_1^{p-1} y_2, y_1]$. Hence we get $(\text{Ker}(1-u) \cap \text{Ker}(1-u'))^* \cong 0$ for $0 < * \leq p - 1$. \square

Proposition 4.9 Let G have the p -Sylow subgroup E . The number of $L(2, 0)$ in BG is given by

$$m(G, 2)_0 = \sharp_G(A) - \sharp_G(F^{\text{ec}}A)$$

where $\sharp_G(A)$ (resp. $\sharp_G(F^{\text{ec}}A)$) is the number of G -conjugacy classes of A -subgroups (resp. F^{ec} -radical subgroups).

Proof Let us write $K = E:W_G(E)$ and $H^*(BE)^{W_G(E)} = H^*(BK)$. From Theorem 3.1, we have

$$(4-1) \quad H^*(BG) \cong H^*(BK) \cap_{A: F^{\text{ec}}\text{-radical}} i_A^{*-1} H^*(BA)^{W_G(A)}.$$

Let A be an A -subgroup of K and $x \in W_K(A)$. Recall $A = \langle c, ab^i \rangle$ for some i . Identifying x as an element of $N_G(A) \subset E: \text{Out}(E)$ We see $x\langle c \rangle = \langle c \rangle$ from (3-4) and since $\langle c \rangle$ is the center of E . Hence

$$W_K(A) \subset B = U: (\mathbb{F}_p^*)^2 \text{ the Borel subgroup.}$$

So we easily see that $\bar{W}_K(y_1^{p-1}) = \lambda y_2^{p-1}$ for some $\lambda \neq 0$ follows from $b^* y_i^{p-1} = y_i^{p-1}$ for $b = \text{diagonal} \in (\mathbb{F}_p)^{*2}$ and the arguments just before Lemma 4.8. We also see $\bar{W}_K(y_1^{p-1-i} y_2^i) = 0$ for $i > 0$. Hence we have $m(K, 2)_0 = \sharp_K(A)$. From the isomorphism (4-1), we have $m(G, 2)_0 = \sharp_K(A) - \sharp_G(F^{cc}A)$.

On the other hand $m(G, 2)_0 \leq \sharp_G(A) - \sharp_G(F^{cc}A)$ from the above lemma. Since $\sharp_K(A) \geq \sharp_G(A)$, we see that $\sharp_K(A) = \sharp_G(A)$ and get the proposition. \square

Corollary 4.10 *Let G have the p -Sylow subgroup E . The number of $L(1, 0)$ in BG is given by*

$$m(G, 1)_{p-1} = \text{rank}_p H^{2(p-1)}(G) = \sharp_G(A) - \sharp_G(F^{cc}A).$$

Proof Since $L(1, 0) = L(1, p-1)$ is linked to $L(2, 0)$, we know $m(G, 1)_{p-1} = m(G, 2)_0$. \square

Lemma 4.11 *Let $\xi \in \mathbb{F}_p^*$ be a primitive $(p-1)$ th root of 1 and $G \supset E: \langle \text{diag}(\xi, \xi) \rangle$. If $\xi^{3k} \neq 1$, then BG does not contain the summand $L(2, k)$, ie $m(G, 2)_k = 0$.*

Proof It is sufficient to prove the case $G = E: \langle \text{diag}(\xi, \xi) \rangle$. Let $G = E: \langle \text{diag}(\xi, \xi) \rangle$. Recall $A_i = \langle c, ab^i \rangle$ and

$$\text{diag}(\xi, \xi): ab^i \mapsto (ab^i)^\xi, \quad c \mapsto c^{\xi^2}.$$

So the Weyl group is $W_G(A_i) = U: \langle \text{diag}(\xi^2, \xi) \rangle$. For $v = \lambda y_1^{p-1} + \dots \in M_{q,k}$, we have

$$\bar{W}_G(A_i)v = \sum_{i=0}^{p-2} (\xi^{3i})^k \text{diag}(\xi^{2i}, \xi^i)(1 + \dots + u^{p-1})v = \sum_{i=0}^{p-2} \xi^{3ik} \lambda y_2^{p-1}.$$

Thus we get the lemma from $\sum_{i=0}^{p-2} \xi^{3ik} = 0$ for $3k \neq 0 \pmod{p-1}$ and $= -1$ otherwise. \square

5 Cohomology and splitting of $B(\mathbb{Z}/3)^2$

In this section, we study the cohomology and stable splitting of BG for G having a 3-Sylow subgroup $(\mathbb{Z}/3)^2 = A$. In this and next sections, p always means 3. Recall $\text{Out}(A) \cong GL_2(\mathbb{F}_3)$ and $\text{Out}(A)'$ consists the semidihedral group

$$SD_{16} = \langle x, y \mid x^8 = y^2 = 1, yxy^{-1} = x^3 \rangle.$$

Every 3-local finite group G over A is of type A : W , $W \subset SD_{16}$. There is the SD_{16} -conjugacy classes of subgroups(here $B \leftarrow C$ means $B \supset C$)

$$SD_{16} \begin{cases} \leftarrow Q_8 \leftarrow \mathbb{Z}/4 \\ \leftarrow \mathbb{Z}/8 \leftarrow \mathbb{Z}/4 \leftarrow \mathbb{Z}/2 \leftarrow 0 \\ \leftarrow D_8 \leftarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \leftarrow \mathbb{Z}/2 \end{cases}$$

We can take generators of subgroups in $GL_2(\mathbb{F}_3)$ by the matrices

$$\begin{aligned} \mathbb{Z}/8 &= \langle l \rangle, Q_8 = \langle w, k \rangle, D_8 = \langle w', k \rangle, \mathbb{Z}/4 = \langle w \rangle, \\ \mathbb{Z}/4 &= \langle k \rangle, \mathbb{Z}/2 \oplus \mathbb{Z}/2 = \langle w', m \rangle, \mathbb{Z}/2 = \langle m \rangle, \mathbb{Z}/2 = \langle w' \rangle, \end{aligned}$$

where $l = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$, $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $k = l^2 = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$, $w' = wl = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$ and $m = w^2 = k^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Here we note that k and w are $GL_2(\mathbb{F}_3)$ -conjugate, in fact $uku^{-1} = w$. Hence we note that

$$H^*(B(A: \langle k \rangle)) \cong H^*(B(A: \langle w \rangle)).$$

The cohomology of A is given $H^*(BA) \cong \mathbb{Z}/3[y_1, y_2]$, and the following are immediately

$$H^*(BA)^{\langle m \rangle} \cong \mathbb{Z}/3[y_1^2, y_2^2]\{1, y_1y_2\} \quad H^*(BA)^{\langle w' \rangle} \cong \mathbb{Z}/3[y_1 + y_2, y_2^2].$$

Let us write $Y_i = y_i^2$ and $t = y_1y_2$. The k -action is given $Y_1 \mapsto Y_1 + Y_2 + t$, $Y_2 \mapsto Y_1 + Y_2 - t$, $t \mapsto -Y_1 + Y_2$. So the following are invariant

$$a = -Y_1 + Y_2 + t, a_1 = Y_1(Y_1 + Y_2 + t), a_2 = Y_2(Y_1 + Y_2 - t), b = t(Y_1 - Y_2).$$

Here we note that $a^2 = a_1 + a_2$ and $b^2 = a_1a_2$. We can prove the invariant ring is

$$H^*(BA)^{\langle k \rangle} \cong \mathbb{Z}/3[a_1, a_2]\{1, a, b, ab\}.$$

Next consider the invariant under $Q_8 = \langle w, k \rangle$. The action for w is $a \mapsto -a$, $a_1 \leftrightarrow a_2$, $b \mapsto b$. Hence we get

$$H^*(BA)^{Q_8} \cong \mathbb{Z}/3[a_1 + a_2, a_1a_2]\{1, b\}\{1, (a_1 - a_2)a\}.$$

Let us write $S = \mathbb{Z}/3[a_1 + a_2, a_1a_2]$ and $a' = (a_1 - a_2)a$. The action for l is given $l: Y_1 \mapsto Y_2 \mapsto Y_1 + Y_2 + t \mapsto Y_1 + Y_2 - t \mapsto Y_1$. Hence $l: a \mapsto -a, a_1 \leftrightarrow a_2, b \mapsto -b$. Therefore we get $H^*(BA)^{\langle l \rangle} \cong S\{1, a', ab, (a_1 - a_2)b\}$.

The action for w' : $Y_1 \mapsto Y_1 + Y_2 + t, Y_2 \mapsto Y_2$, implies that $w': a \mapsto a, a_i \mapsto a_i, b \mapsto -b$. Then we can see

$$H^*(BA)^{D_8} = H^*(BA)^{\langle k, w' \rangle} \cong \mathbb{Z}/3[a_1, a_2]\{1, a\} \cong S\{1, a, a_1, a'\}.$$

We also have

$$H^*(BA)^{SD_{16}} \cong H^*(BA)^{Q_8} \cap H^*(BA)^{\mathbb{Z}/8} \cong S\{1, a'\}.$$

Recall the Dickson algebra $DA = \mathbb{Z}/3[\tilde{D}_1, \tilde{D}_2] \cong H^*(BA)^{GL_2(\mathbb{F}_3)}$ where $\tilde{D}_1 = Y_1^3 + Y_1^2Y_2 + Y_1Y_2^2 + Y_2^3 = (a_2 - a_1)a = a'$ and $\tilde{D}_2 = (y_1^3y_2 - y_1y_2^3)^2 = a_1a_2$. Using $a^2 = (a_1 + a_2)$ and $\tilde{D}_1^2 = a^6 - a_1a_2a^2$, we can write

$$H^*(BA)^{SD_{16}} \cong \mathbb{Z}/3[a^2, \tilde{D}_2]\{1, \tilde{D}_1\} \cong DA\{1, a^2, a^4\}.$$

Theorem 5.1 *Let $G = (\mathbb{Z}/3)^2:H$ for $H \subset SD_{16}$. Then BG has the stable splitting given by*

$$\begin{array}{l} \tilde{X}_{0,0} \leftarrow SD_{16} \left\{ \begin{array}{l} \tilde{X}_{0,1} \leftarrow Q_8 \\ \tilde{X}_{2,1} \leftarrow \mathbb{Z}/8 \quad \tilde{X}_{2,0} \vee \tilde{X}_{0,1} \vee L(1,0) \leftarrow \mathbb{Z}/4 \quad 2\tilde{X}_{2,0} \vee 2\tilde{X}_{2,1} \vee 2L(1,0) \leftarrow \mathbb{Z}/2 \quad 2\tilde{X}_{1,0} \vee 2\tilde{X}_{1,1} \vee 2L(1,1) \leftarrow 0 \\ \tilde{X}_{2,0} \vee L(1,0) \leftarrow D_8 \quad \tilde{X}_{2,0} \vee \tilde{X}_{2,1} \vee L(1,0) \leftarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \quad \tilde{X}_{1,0} \vee \tilde{X}_{1,1} \vee L(1,1) \leftarrow \mathbb{Z}/2 \end{array} \right. \end{array}$$

where $\tilde{X}_1 \leftarrow \dots \leftarrow \tilde{X}_s H$ means $B((\mathbb{Z}/3)^2:H) \sim \tilde{X}_1 \vee \dots \vee \tilde{X}_s$.

For example

$$B(E:SD_{16}) \sim \tilde{X}_{0,0}, \quad B(E:Q_8) \sim \tilde{X}_{0,0} \vee \tilde{X}_{0,1}, \quad B(E:\mathbb{Z}/8) \sim \tilde{X}_{0,0} \vee \tilde{X}_{2,1}.$$

Main parts of the above splittings are given by the author in [14, (6)] by direct computations of $\bar{W}_G(A)$ (see [14, page 149]). However we get the theorem more easily by using cohomology here. For example, let us consider the case $G = A:\langle k \rangle$. The cohomology

$$H^0(BG) \cong \mathbb{Z}/3, \quad H^2(BG) \cong 0, \quad H^4(BG) \cong \mathbb{Z}/3$$

implies that BG contains just one $\tilde{X}_{0,0}, \tilde{X}_{2,0}, L(1,0)$ but does not $\tilde{X}_{1,0}, L(1,1)$. Since $\det(k) = 1$, we also know that $\tilde{X}_{0,1}, \tilde{X}_{2,1}$ are contained. So we can see

$$B(A:\mathbb{Z}/4) \sim \tilde{X}_{0,0} \vee \tilde{X}_{0,1} \vee \tilde{X}_{2,0} \vee \tilde{X}_{2,1} \vee L(1,0).$$

Next consider the case $G' = A: \langle l \rangle$. The fact $H^4(G) \cong 0$ implies that BG' does not contain $\tilde{X}_{2,0}, L(1, 0)$. The determinant $\det(l) = -1$, and $l: a \mapsto -a$ shows that BG' contains $\tilde{X}_{2,1}$ but does not contain $\tilde{X}_{0,1}$. Hence we know $BG' \sim \tilde{X}_{0,0} \vee \tilde{X}_{2,1}$. Moreover we know $BA: SD_{16} \sim \tilde{X}_{0,0}$ since $w: a \rightarrow -a$ but $\det(w) = 1$. Thus we have the graph

$$\begin{array}{ccccccc} \tilde{X}_{0,0} & & \tilde{X}_{2,1} & & \tilde{X}_{2,0} \vee \tilde{X}_{0,1} \vee L(1,0) & & \\ \leftarrow & SD_{16} & \leftarrow & \mathbb{Z}/8 & \leftarrow & \mathbb{Z}/4 & \end{array}$$

Similarly we get the other parts of the above graph.

Corollary 5.2 *Let $S = \mathbb{Z}/3[a_1 + a_2, a_1 a_2]$. Then we have the isomorphisms*

$$\begin{aligned} H^*(\tilde{X}_{0,0}) &\cong S\{1, \tilde{D}_1\} \\ H^*(\tilde{X}_{0,1}) &\cong S\{b, \tilde{D}_1 b\} \\ H^*(\tilde{X}_{2,1}) &\cong S\{ab, (a_1 - a_2)b\} \\ H^*(\tilde{X}_{2,0} \vee L(1, 0)) &\cong S\{a, a_1 - a_2\} \cong DA\{a, a^2, a^3\}. \end{aligned}$$

Here we write down the decomposition of cohomology for a typical case

$$\begin{aligned} H^*(BA)^{(k)} &\cong S\{1, a_1 - a_2\}\{1, a\}\{1, b\} \\ &\cong S\{1, a(a_1 - a_2), b, ba(a_1 - a_2), ab, (a_1 - a_2)b, a, (a_1 - a_2)\} \\ &\cong H^*(\tilde{X}_{0,0}) \oplus H^*(\tilde{X}_{0,1}) \oplus H^*(\tilde{X}_{2,1}) \oplus H^*(\tilde{X}_{2,0} \vee L(1, 0)). \end{aligned}$$

6 Cohomology and splitting of $B3_+^{1+2}$.

In this section we study the cohomology and stable splitting of BG for G having a 3-Sylow subgroup $E = 3_+^{1+2}$. In the splitting for BE , the summands $X_{q,k}$ are called dominant summands. Moreover the summands $L(2, 0) \vee L(1, 0)$ is usually written by $M(2)$.

Lemma 6.1 *If $G \supset E: \langle \text{diag}(-1, -1) \rangle$ identifying $\text{Out}(E) \cong GL_2(\mathbb{F}_3)$ and G has E as a 3-Sylow subgroup, then*

$$BG \sim (\text{dominant summands}) \vee (\#_G(A) - \#_G(F^{\text{ec}}A))(M(2)).$$

Proof From Lemma 4.11, we know $m(G, 2)_1 = 0$ ie $L(2, 1)$ is not contained. The summand $L(1, 1)$ is also not contained, since $H^2(BE)^{\langle \text{diag}(-1, -1) \rangle} \cong 0$. The lemma is almost immediately from Proposition 4.9 and Corollary 4.10. \square

Theorem 6.2 *If G has a 3-Sylow subgroup E , then BG is homotopic to the classifying space of one of the following groups. Moreover the stable splitting is given by the graph so that $\overset{X_1}{\leftarrow} \dots \overset{X_s}{\leftarrow} G$ means $BG \sim X_1 \vee \dots \vee X_i$ and $EH = E: H$ for $H \subset SD_{16}$*

$$\overset{X_{0,0}}{\leftarrow} J_4 \left\{ \begin{array}{l} \overset{M(2)}{\leftarrow} ESD_{16} \left\{ \begin{array}{l} \overset{X_{0,1}}{\leftarrow} EQ_8 \\ \overset{X_{2,1}}{\leftarrow} EZ/8 \quad \overset{X_{2,0} \vee X_{0,1}}{\vee \leftarrow} M(2) \quad \overset{2X_{2,0} \vee 2X_{2,1}}{\vee \leftarrow} M(2) \quad \overset{2X_{1,0} \vee 2X_{1,1} \vee}{\vee \leftarrow} 4L(2,1) \vee 2L(1,1) \quad E \\ \overset{X_{2,0} \vee M(2)}{\leftarrow} ED_8 \quad \overset{X_{2,0} \vee X_{2,1} \vee M(2)}{\vee \leftarrow} M(2) \quad \overset{X_{1,0} \vee X_{1,1} \vee}{\vee \leftarrow} 2L(2,1) \vee L(1,1) \quad EZ/2 \end{array} \right. \\ \overset{X_{2,0}}{\leftarrow} {}_2F_4(2) \quad \overset{M(2)}{\leftarrow} M_{24} \quad \overset{X_{2,0} \vee X_{2,1}}{\vee \leftarrow} M_{12} \quad \overset{M(2)}{\leftarrow} \mathbb{F}_3^2: GL_2(\mathbb{F}_3) \quad \overset{X_{1,0} \vee X_{1,1} \vee}{\vee \leftarrow} L(2,1) \vee L(1,1) \quad \mathbb{F}_3^2: SL_2(\mathbb{F}_3) \end{array} \right.$$

Proof All groups except for $E, E: \langle w' \rangle$ and $\mathbb{F}_3^2: SL_2(\mathbb{F}_3)$ contain $E: \langle \text{diag}(-1, -1) \rangle$. Hence we get the theorem from Corollary 4.4, Theorem 5.1 and Lemma 6.1, except for the place for $H^*(BE: \langle w' \rangle)$ and $H^*(\mathbb{F}_3^2: SL_2(\mathbb{F}_3))$.

Let $G = E: \langle w' \rangle$. Note $w': y_1 \mapsto y_1 - y_2, y_2 \mapsto -y_2, v \mapsto -v$. Hence $H^2(G) \cong \mathbb{Z}/3\{y_1 + y_2\}$. So BG contains one $L(1, 1)$. Next consider the number of $L(2, 0), L(2, 1)$. The G -conjugacy classes of A -subgroups are $A_0, A_2, A_1 \sim A_\infty$. The Weyl groups are

$$W_G(A_\infty) \cong U, \quad W_G(A_2) \cong U: \langle \text{diag}(-1, -1) \rangle, \quad W_G(A_0) \cong U: \langle \text{diag}(-1, 1) \rangle,$$

eg $N_G(A_0)/A_0$ is generated by b, w' which is represented by $u, \text{diag}(-1, 1)$ respectively. By the arguments similar to the proof of Lemma 4.11, we have that

$$\begin{cases} \dim(\bar{W}_G(A_i)M_{2,0}) = 1 \text{ for all } i \\ \dim(\bar{W}_G(A_i)M_{2,1}) = 1, 1, 0 \text{ for } i = \infty, 2, 0 \text{ respectively.} \end{cases}$$

Thus we show $BG \supset 3L(2, 0) \vee 2L(2, 1)$ and we get the graph for $G = E: \langle w' \rangle$.

For the place $G = \mathbb{F}_3^2: SL_2(\mathbb{F}_3)$, we see $W_G(A_\infty) \cong SL_2(\mathbb{F}_3)$. We also have

$$\begin{cases} \dim(\bar{W}_G(A_i)M_{2,0}) = 0, 1, 1 \text{ for } i = \infty, 2, 0 \text{ respectively} \\ \dim(\bar{W}_G(A_i)M_{2,1}) = 0, 1, 0 \text{ for } i = \infty, 2, 0 \text{ respectively.} \end{cases}$$

Thus we can see the graph for the place $H^*(\mathbb{F}_3^2: SL_2(\mathbb{F}_3))$. □

Remark From Tezuka–Yagita [11], Yagita [13] and Theorem 2.1, we have the following homotopy equivalences (localized at 3).

$$BJ_4 \cong BRu, \quad BM_{24} \cong BHe, \quad BM_{12} \cong BGL_3(\mathbb{F}_3)$$

$$B(E:SD_{16}) \cong BG_2(2) \cong BG_2(4), \quad B(E:D_8) \cong BHJ \cong BU_3(3).$$

We write down the cohomologies explicitly (see also Tezuka–Yagita [11] and Yagita [14]). First we compute $H^*(B(E:H))$. The following cohomologies are easily computed

$$\begin{aligned} H^*(BE)^{(m)} &\cong \mathbb{Z}/3[C, v]\{1, y_1 y_2, Y_1, Y_2\}, & H^*(BE)^{(w)} &\cong \mathbb{Z}/3[C, v]\{1, Y_1 + Y_2\}. \\ H^*(BE)^{(k)} &\cong \mathbb{Z}/3[C, v]\{1, a\} \text{ where } a = -Y_1 + Y_2 + y_1 y_2, & C^2 &= a^2. \end{aligned}$$

Recall that $V = v^{p-1}$ and C multiplicatively generate $H^*(BE)^{\text{Out}(E)}$. Let us write

$$CA = \mathbb{Z}/p[C, V] \cong H^*(BE)^{\text{Out}(E)}.$$

Then we have

$$\begin{aligned} H^*(BE)^{(w')} &\cong CA\{1, y'_1, Y'_1, Y_2, Y_2 y'_1, y_2 v, y'_1 y_2 v, Y'_1 y_2 v\} \text{ with } y'_1 = y_1 + y_2 \\ H^*(BE)^{(w', m)} &\cong CA\{1, a, a', Y_2\} \text{ where } a' = (t + Y_2)v = y'_1 y_2 v. \end{aligned}$$

We can compute

$$\begin{aligned} H^*(BE)^{Q_8} &\cong H^*(BE)^{(k)} \cap H^*(BE)^{(w)} \cong \mathbb{Z}/3[C, v] \cong CA\{1, v\}, \\ H^*(BE)^{D_8} &\cong CA\{1, a\}, \quad H^*(BE)^{(l)} \cong CA\{1, av\}. \end{aligned}$$

Hence we have $H^*(BE)^{SD_{16}} \cong CA$.

Let $D_1 = C^p + V$ and $D_2 = CV$. Then it is known that

$$D_1|A_i = \tilde{D}_1, \quad D_2|A_i = \tilde{D}_2 \text{ for all } i \in \mathbb{F}_p \cup \infty.$$

So we also write $DA \cong \mathbb{Z}/p[D_1, D_2]$. Since $CD_1 - D_2 = C^{p+1}$, we can write $CA \cong DA\{1, C, C^2, \dots, C^p\}$.

Now return to the case $p = 3$ and we get (see [11])

$$H^*(BJ_4) \cong H^*(BE)^{SD_{16}} \cap i_0^{*-1} H^*(BA_0)^{GL_2(\mathbb{F}_3)} \cong DA.$$

Proposition 6.3 *There are isomorphisms for $|a''| = 4$,*

$$H^*({}^2F_4(2)') \cong DA\{1, (D_1 - C^3)a''\}, \quad H^*(M_{24}) \cong DA \oplus CA\{a''\}.$$

Proof Let $G = M_{24}$. Then G has just two G -conjugacy classes of A -subgroups

$$\{A_0, A_2\}, \quad \{A_1, A_\infty\}.$$

It is known that one is F^{ec} -radical and the other is not. Suppose that A_0 is F^{ec} -radical. Then $W_G(A_0) \cong GL_2(\mathbb{F}_3)$. Let $a'' = a + C$. Then

$$a''|_{A_0} = (-Y_1 + Y_2 + y_1y_2 + C)|_{A_0} = 0, \quad a''|_{A_\infty} = -Y.$$

By Theorem 3.1

$$H^*(BM_{24}) \cong H^*(BE)^{D_8} \cap i_{A_0}^{*-1} H^*(BA_0)^{W_G(A_0)},$$

we get the isomorphism for M_{24} . When A_∞ is a F^{ec} -radical, we take $a'' = a - c$. Then we get the same result.

For $G = {}^2F_4(2)'$, the both conjugacy classes are F^{ec} -subgroups and $W_G(A_\infty) \cong GL_2(\mathbb{F}_3)$. Hence (for case $a'' = a + C$)

$$H^*(B^2F_4(2)') \cong H^*(BM_{24}) \cap i_{A_\infty}^{*-1} H^*(BA_\infty)^{GL_2(\mathbb{F}_3)}.$$

We know

$$(D_1 - C^3)a''|_{A_0} = 0, \quad (D_1 - C^3)a''|_{A_\infty} = -VY = -\tilde{D}_2.$$

Thus we get the cohomology of ${}^2F_4(2)'$. □

Remark In [11; 14], we take

$$(\mathbb{Z}/2)^2 = \langle \text{diag}(\pm 1, \pm 1) \rangle, \quad D_8 = \langle \text{diag}(\pm 1, \pm 1), w \rangle.$$

For this case, the M_{24} -conjugacy classes of A -subgroups are $A_0 \sim A_\infty$, $A_1 \sim A_2$, and we can take $a'' = C - Y_1 - Y_2$. The expressions of $H^*(M_{12})$, $H^*(A:GL_2(\mathbb{F}_3))$ become more simple (see [11; 14]), in fact,

$$H^*(B^2F_4(2)') \cong DA\{1, (Y_1 + Y_2)V\}.$$

Remark [11, Corollary 6.3] and [14, Corollary 3.7] were not correct. This followed from an error in [11, Theorem 6.1]. This theorem is only correct with adding the assumption that there are exactly two G conjugacy classes of A -subgroups such that one is p -pure and the other is not. This assumption is always satisfied for sporadic simple groups but not for ${}^2F_4(2)'$.

Corollary 6.4 *There are isomorphisms of cohomologies*

$$H^*(X_{2,0}) \cong DA\{D_2\}, \quad H^*(X_{2,1}) \cong CA\{av\} \text{ where } (av)^2 = CD_2$$

$$H^*(X_{0,1}) \cong CA\{v\}, \quad H^*(M(2)) \cong DA\{C, C^2, C^3\} \text{ where } C^4 = CD_1 - D_2.$$

Here we write down typical examples. First recall

$$CA \cong DA\{1, C, C^2, C^3\} \cong H^*(X_{0,0}) \oplus H^*(M(2))$$

$$CA\{C\} \cong DA\{C, C^2, C^3, D_2\} \cong H^*(M(2)) \oplus H^*(X_{2,0}).$$

Thus the decomposition for $H^*(BE)^{D_8}$ gives the isomorphisms

$$CA\{1, a''\} \cong CA\{1, C\} \cong H^*(X_{0,0}) \oplus H^*(M(2)) \oplus H^*(X_{2,0}) \oplus H^*(M(2)).$$

Similarly the decomposition for $H^*(BE)^{\langle k \rangle}$ gives the isomorphism

$$CA\{1, a, v, av\} \cong H^*(BE)^{D_8} \oplus H^*(X_{0,1}) \oplus H^*(X_{2,1}).$$

We recall here Lemma 4.7 and the module

$$X_{q,k}(\langle k \rangle) = S(V)^q \otimes v^k \cap H^*(B(E: \langle k \rangle)).$$

Then it is easily seen that

$$X_{0,0}(\langle k \rangle) = \{1\}, X_{2,0}(\langle k \rangle) = \{a\}, X_{0,1}(\langle k \rangle) = \{v\}, X_{2,1}(\langle k \rangle) = \{av\}.$$

Hence we also see $B(E: \langle k \rangle)$ has the dominant summands $X_{0,0} \vee X_{2,0} \vee X_{0,1} \vee X_{2,1}$. Moreover it has non dominant summands $2M(2)$ since $H^4(B(E: \langle k \rangle)) \cong \mathbb{Z}/3\{C, a\}$. Thus we can give an another proof of Theorem 6.2 from Lemma 4.7 and the cohomologies $H^*(BG)$.

7 Cohomology for $B 7_+^{1+2}$ I.

In this section, we assume $p = 7$ and $E = 7_+^{1+2}$. We are interested in groups $O'N, O'N:2, He, He:2, Fi'_{24}, Fi_{24}$ and three exotic 7-local groups. Denote them by RV_1, RV_2, RV_3 according the numbering in [9]. We have the diagram from Ruiz and Viruel

$$\left\{ \begin{array}{ccccccc} \xleftarrow{3SD_{32}} & \xleftarrow{SL_2(\mathbb{F}_7):2} & \xleftarrow{3SD_{16}} & \xleftarrow{SL_2(\mathbb{F}_7):2, SL_2(\mathbb{F}_7):2} & \xleftarrow{3SD_{16}} & \xleftarrow{SL_2(\mathbb{F}_7):2} & \xleftarrow{3D_8} & \xleftarrow{SL_2(\mathbb{F}_7):2, SL_2(\mathbb{F}_7):2} \\ & RV_3 & & RV_2 & & O'N:2 & & O'N \\ \xleftarrow{6^2:2} & \xleftarrow{SL_2(\mathbb{F}_7):2, GL_2(\mathbb{F}_7)} & \xleftarrow{6^2:2} & \xleftarrow{SL_2(\mathbb{F}_7):2} & \xleftarrow{6S_3} & \xleftarrow{SL_2(\mathbb{F}_7):2, SL_2(\mathbb{F}_7):2} & \xleftarrow{6S_3} & \xleftarrow{SL_2(\mathbb{F}_7):2} & \xleftarrow{3S_3} & \xleftarrow{SL_2(\mathbb{F}_7)} \\ & RV_1 & & Fi_{24} & & Fi'_{24} & & He:2 & & He \end{array} \right.$$

Here $\xleftarrow{H} \frac{W_1, \dots, W_2}{G}$ means $W_G(E) \cong H, W_i = W_G(A_i)$ for G -conjugacy classes of $F^{ec}A$ -subgroups A_i .

In this section, we study the cohomology of $O'N, RV_2, RV_3$. First we study the cohomology of $G = O'N$. The multiplicative generators of $H^*(BE)^{3D_8}$ are still studied in [11, Lemma 7.10]. We will study more detailed cohomology structures here.

Lemma 7.1 *There is the CA -module isomorphism*

$$H^*(BE)^{3D_8} \cong CA\{1, a, a^2, a^3/V, a^4/V, a^5/V, b, ab/V, a^2b/V, d, ad, a^2d\},$$

where $a = (y_1^2 + y_2^2)v^2, b = y_1^2y_2^2v^4$ and $d = (y_1y_2^3 - y_1^3y_2)v$.

Proof The group $3D_8 \subset GL_2(\mathbb{F}_7)$ is generated by $\text{diag}(-1, 1), (2, 2)$ and $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. If $y_1^i y_2^j v^k$ is invariant under $\text{diag}(-1, 1), \text{diag}(1, -1)$ and $\text{diag}(2, 2)$, then $i = j = k \pmod{2}$ and $i + j + 2k = 0 \pmod{3}$. When $i, j \leq 6, k \leq 5$ but $(i, j) \neq (6, 6)$, the invariant monomials have the following terms, $y_1^2v^2, y_1^4v^4, y_1^6, y_1^2y_2^2v^4, y_1^4y_2^4v^2, y_1y_2v^5, y_1^3y_2^3v^3, y_1^5y_2^5v, y_1^2y_2^4, y_1^2y_2^6v^2, y_1^4y_2^6v^4, y_1y_2^3v, y_1y_2^5v^3, y_1^3y_2^5v^5$ and terms obtained by exchanging y_1 and y_2 . Recall that $w: y_1 \mapsto y_2, y_2 \mapsto -y_1$ and $v \rightarrow v$. From the expression of (3-2), we have

$$H^*(BE)^{3D_8} \cong CA\{1, a, a^2, a', b, b', c, c', c'', d, ad, bd\}$$

where $a = (y_1^2 + y_2^2)v^2, a' = y_1^6 + y_2^6, b = y_1^2y_2^2v^4, b' = y_1^4y_2^4v^2, c = (y_1^2y_2^4 + y_1^4y_2^2), c' = (y_1^2y_2^6 + y_1^6y_2^2)v^2, c'' = (y_1^4y_2^6 + y_1^6y_2^4)v^4, d = (y_1y_2^3 - y_1^3y_2)v, ad = (y_1y_2^5 - y_1^5y_2)v^3$ and $bd = (y_1^3y_2^5 - y_1^5y_2^3)v^5$. Here $a^2d = bd$ from $(y_1^6 - y_2^6)y_1y_2 = 0$ in $H^*(BE)$. It is easily seen that $b'V = b^2, cV = ab, c'V = (a^2 - 2b)b$ and $c''V = ab^2$. Moreover we get

$$\begin{aligned} a^3/V &= (y_1^2 + y_2^2)^3 = (y_1^6 + y_2^6) + 3y_1^2y_2^2(y_1^2 + y_2^2) = a' + 3ab \\ a^4/V &= (y_1^2 + y_2^2)^4v^2 = ((y_1^8 + y_2^8) + 4y_1^2y_2^2(y_1^4 + y_2^4) + 6y_1y_2^4)v^2 \\ &= aC + 4c' + 6b' \\ a^5/V &= ((y_1^{10} + y_2^{10}) + 5y_1^2y_2^2(y_1^6 + y_2^6) + 10y_1^4y_2^4(y_1^2 + y_2^2))v^4 \\ &= c'C + 10bC + 10c''. \end{aligned}$$

Hence, we can take generators $a^4/V, a^5/V, ab/V, a^2b/V$ for b', c'', c, c' respectively, and get the lemma. \square

Note that the computations shows

$$\begin{aligned} a^6 &= (y_1^2 + y_2^2)^6v^{12} = (y_1^{12} - y_1^{10}y_2^2 + y_1^8y_2^4 - y_1^6y_2^6 + y_1^4y_2^8 - y_1^2y_2^{10} + y_2^{12})V^2 \\ &= (y_1^{12} - y_1^6y_2^6 + y_2^{12})V^2 = C^2V^2 = D_2^2, \end{aligned}$$

where we use the fact $y_1^7y_2 - y_1y_2^7 = 0$.

Lemma 7.2 $H^*(BE)^{3SD_{16}} \cong CA\{1, a, a^2, a^3/V, a^4/V, a^5/V\}$.

Proof Take the matrix $k' = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$ such that $\langle 3D_8, k' \rangle \cong 3SD_{16}$. Then we have

$$\begin{aligned} k'^*: a &= (y_1^2 + y_2^2)v^2 \mapsto ((-y_1 + y_2)^2 + (-y_1 - y_2)^2)(2v)^2 = a, \\ b &= y_1^2 y_2^2 v^4 \mapsto (y_1^2 - y_2^2)^2 (2v)^4 = 2(a^2 - 4b) = 2a^2 - b. \end{aligned}$$

(If we take $\tilde{b} = b - a^2$, then $k'^*: \tilde{b} \mapsto -\tilde{b}$.) Similarly we can compute $k': d \mapsto -d$. Then the lemma is almost immediate from the preceding lemma. \square

Lemma 7.3 $H^*(BE)^{3SD_{32}} \cong CA\{1, a^2, a^4/V\}$.

Proof Take the matrix $l' = \begin{pmatrix} -1 & 3 \\ -3 & -1 \end{pmatrix}$ so that $l'^2 = k'$ and $\langle 3SD_8, l' \rangle \cong 3SD_{32}$. We see that

$$l'^*: a = (y_1^2 + y_2^2)v^2 \mapsto ((-y_1 + 3y_2)^2 + (-3y_1 - y_2)^2)(3v)^2 = -a,$$

which shows the lemma. \square

Theorem 7.4 *There is the isomorphism with $C' = C - a^3/V$*

$$H^*(BO'N) \cong DA\{1, a, a^2, b, ab, a^2b\} \oplus CA\{d, ad, a^2d, C', C'a, C'a^2\}$$

Proof Let $G = O'N$. The orbits of $N_G(E)$ -action of A -subgroups in E are given by $\{A_0, A_\infty\}, \{A_1, A_6\}$ and $\{A_2, A_3, A_4, A_5\}$. From Ruiz and Viruel [9], A_0, A_∞, A_1 and A_6 are F^{ec} -radical subgroups. Hence we know that

$$H^*(O'N) \cong H^*(BE)^{3D_8} \cap i_{A_0}^{*-1} H^*(BA_0)^{SL_2(\mathbb{F}_7):2} \cap i_{A_1}^{*-1} H^*(BA_1)^{SL_2(\mathbb{F}_7):2}.$$

For element $x = d$ or $x = C'$, the restrictions are $x|_{A_0} = x|_{A_1} = 0$. Hence we see that $CA\{x\}$ are contained in $H^*(BG)$. We can take $C', C'a, C'a^2$ instead of $a^3/V, a^4/V$ and a^5/V as the CA -module generators since $a^3/V = (C - C')$. Moreover we know $CA\{C', C'a, C'a^2\} \subset H^*(BG)$.

It is known that $\mathbb{Z}/p[y, u]^{SL_p(\mathbb{F}_p)} \cong \mathbb{Z}/p[\tilde{D}'_1, \tilde{D}'_2]$ where $\tilde{D}'_2 = y_1 u^p - y_1^p u$ and $(\tilde{D}'_2)^{p-1} = \tilde{D}'_2$. Hence we know $\mathbb{Z}/7[y, u]^{SL_2(\mathbb{F}_7):2} \cong \mathbb{Z}/7[\tilde{D}'_1, (\tilde{D}'_2)^2]$.

Since $y_1 v|_A = \tilde{D}'_2$ we see $a|_{A_0} = (\tilde{D}'_2)^2, a|_{A_1} = 2(\tilde{D}'_2)^2$. Hence a, a^2 are in $H^*(BG)$. The fact $b|_{A_0} = 0$ and $b|_{A_1} = (\tilde{D}'_2)^4$, implies that $b \in H^*(BG)$. Hence all $a^i b^j$ are also in $H^*(BG)$. \square

Next we consider the group $G = O'N:2$. Its Weyl group $W_G(E)$ is isomorphic to $3SD_{16}$. So we have $H^*(B(O'N:2)) \cong H^*(BO'N) \cap H^*(BE)^{3SD_{16}}$.

Corollary 7.5 $H^*(B(O'N:2)) \cong (DA\{1, a, a^2\} \oplus CA\{C', C'a, C'a^2\})$.

Corollary 7.6 $H^*(BRV_2) \cong DA\{1, a, a^2, a^3, a^4, a^5\}$.

Proof Let $G = RV_2$. Since A_2 is also F^{cc} -radical and $W_G(A_2) = SL_2(\mathbb{F}_7):2$. Hence we have

$$H^*(BG) \cong H^*(BE)^{3SD_{16}} \cap i_{A_2}^{*-1} H^*(BA_2)^{SL_2(\mathbb{F}_7):2}.$$

Hence we have the corollary of the theorem. \square

Since $H^*(BRV_3) \cong H^*(BE)^{3SD_{32}} \cap H^*(BRV_2)$, we have the following corollary.

Corollary 7.7 $H^*(BRV_3) \cong DA\{1, a^2, a^4\}$.

Corollary 7.7 can also be proved in the following way.

Proof Let $G = RV_3$. Since there is just one G -conjugacy class of A -subgroups, by Quillen's theorem [8], we know

$$H^*(BRV_3) \subset H^*(BA_0)^{SL_2(\mathbb{F}_7):2} \cong DA\{1, (\tilde{D}'_2)^2, (\tilde{D}'_2)^4\} \text{ with } (\tilde{D}'_2)^6 = \tilde{D}_2.$$

Note that $a^2|_{A_0} = (\tilde{D}'_2)^4$, $a^4|_{A_0} = (\tilde{D}'_2)^2 \tilde{D}_2$ and $D_2|_{A_0} = \tilde{D}_2$. The fact $k'^*: a \mapsto -a$ implies that $DA\{a^2, a^4\} \subset H^*(BG)$ but $DA\{a, a^3, a^5\} \cap H^*(BG) = 0$. \square

Corollary 7.6 can also be proved in the following way.

Proof Let $G = RV_2$. Since there is just two G -conjugacy classes of A -subgroups, by Quillen's theorem [8], we know

$$H^*(BRV_2) \subset H^*(BA_0)^{SL_2(\mathbb{F}_7):2} \times H^*(BA_2)^{SL_2(\mathbb{F}_7):2}$$

Since $a \in H^*(BRV_2)$, the map $i_0^*: H^*(BRV_2) \rightarrow H^*(BA_0)^{SL_2(\mathbb{F}_7):2}$ is epimorphism. Take $b' = b^2 - 2a^2b$ so that $b'|_{A_0} = b'|_{A_1} = 0$. Hence

$$\text{Ker } i_{A_0}^* \supset DA\{b', b'a, c'V\}.$$

Moreover $b'|_{A_2} = (\bar{D}'_2)^2 \bar{D}_2$, $b'a|_{A_2} = (\bar{D}'_2)^4 \bar{D}_2$, $c'V|_{A_2} = (\bar{D}_2)$. Since $(\bar{D}'_2)^2$ itself is not in the image of $i_{A_2}^*$, we get the isomorphism

$$H^*(BRV_2) \cong DA\{1, a, a^2\} \oplus DA\{c'V, b', b'a\}. \quad \square$$

8 Cohomology for $B\mathbb{7}^{1+2}_+$ II

In this section, we study cohomology of He, Fi_{24}, RV_1 .

First we consider the group $G = He$. The multiplicative generators of $H^*(He)$ are still computed by Leary [5]. We will study more detailed cohomology structures here. The Weyl group is $W_G(He) \cong 3S_3$.

Lemma 8.1 *The invariant $H^*(BE)^{3S_3}$ is isomorphic to*

$$CA \otimes \{\mathbb{Z}/7\{1, \bar{b}, \bar{b}^2\}\{1, \bar{a}, \bar{b}^3/V\} \oplus \mathbb{Z}/7\{\bar{d}\}\{1, \bar{a}, \bar{b}, \bar{b}^2/V, \bar{b}^3/V\} \oplus \mathbb{Z}/7\{\bar{a}^2\}\},$$

where $\bar{a} = (y_1^3 + y_2^3)$, $\bar{b} = y_1 y_2 v^2$ and $\bar{d} = (y_1^3 - y_2^3)v^3$.

Proof The group $3S_3 \subset GL_2(\mathbb{F}_7)$ is generated by $T' = \{\text{diag}(\lambda, \mu) \mid \lambda^3 = \mu^3 = 1\}$ and $w' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. If $y_1^i y_2^j v^k$ is invariant under T' , then $i = j = -k \pmod{3}$. When $i, j \leq 6, k \leq 5$ but $(i, j) \neq (6, 6)$, the invariant monomials have the following terms

$$\{1, \bar{c} = \bar{b}^3/V = y_1^3 y_2^3\}\{1, v^3\}\{1, \bar{b} = y_1 y_2 v^2, \bar{b}' = y_1^2 y_2^2 v\} \\ \{1, v^3\}\{y_1^3, y_1^6, y_1 y_2^4 v^2, y_1^2 y_2^5 v, y_1^3 y_2^6\},$$

and terms obtained by exchanging y_1 and y_2 . Recall that $w': y_1 \mapsto y_2, y_2 \mapsto y_1$ and $v \rightarrow -v$. The following elements are invariant

$$\bar{a}\bar{b} = (y_1 y_2^4 + y_1^4 y_2)v^2, \quad \bar{a}\bar{b}^2 = (y_1^2 y_2^5 + y_1^5 y_2^2)v^4, \quad \bar{a}\bar{c} = (y_1^3 y_2^6 + y_1^6 y_2^3), \\ \bar{b}\bar{d} = (y_1 y_2^4 - y_1^4 y_2)v^5, \quad \bar{b}^2\bar{d}/V = (y_1^2 y_2^5 - y_2^5 y_1^2)v, \quad \bar{c}\bar{d} = (y_1^3 y_2^6 - y_1^6 y_2^3)v^3 \\ \bar{a}\bar{d} = (y_1^6 - y_2^6)v^3, \quad \bar{a}\bar{b}^3/V = y_1^3 y_2^6 + y_1^6 y_2^3.$$

Thus we get the lemma from (3–2). □

Lemma 8.2 $H^*(BE)^{6S_3} \cong CA \otimes (\mathbb{Z}/7\{1, \bar{b}, \bar{b}^2\}\{1, \bar{b}^3/V\} \oplus \mathbb{Z}/7\{\bar{d}\bar{a}, \bar{a}^2\})$.

Proof We can think $6S_3 = \langle S_3, \text{diag}(-1, -1) \rangle$. The action $\text{diag}(-1, -1)$ are given by $\bar{a} \mapsto -\bar{a}, \bar{b} \mapsto \bar{b}$, and $\bar{d} \mapsto -\bar{d}$. From Lemma 8.1, we have the lemma. □

Lemma 8.3 $H^*(BE)^{6^2:2} \cong CA\{1, \bar{b}^2, \bar{c}'', \bar{b}^4/V\}$ where $\bar{c}'' = \bar{a}^2 - 2\bar{b}^3/V - 2C$.

Proof We can think $6^2:2 = \langle 3S_6, \text{diag}(3, 1) \rangle$. The action $\text{diag}(3, 1)$ are given by $\bar{a}^2 \mapsto \bar{a}^2 - 4\bar{c}, \bar{b} \mapsto -\bar{b}, \bar{c} \mapsto -\bar{c}, \bar{d}\bar{a} \mapsto -\bar{d}\bar{a}$. For example $\bar{b} = y_1 y_2 v^2 \mapsto (3y_1)y_2(3v)^2 = -\bar{b}$. Moreover we have $\bar{c}'' = Y_1 + Y_2 - 2C \mapsto \bar{c}''$. Thus we have the lemma. □

Theorem 8.4 Let $\bar{c}' = C + \bar{a}^3/V$. Then there is the isomorphism

$$H^*(BHe) \cong DA\{1, \bar{b}, \bar{b}^2, \bar{d}, \bar{d}\bar{b}, \bar{d}\bar{b}^2\} \oplus CA\{\{\bar{a}, \bar{c}'\}\{1, \bar{b}, \bar{b}^2, \bar{d}\}, \bar{a}^2, \bar{a}^2\bar{c}'\}.$$

Proof Let $G = He$. The orbits of $N_G(E)$ -action of A -subgroups in E are given by

$$\{A_0, A_\infty\}, \quad \{A_1, A_2, A_4\}, \quad \{A_3, A_5, A_6\}.$$

Since A_6 is the F^{ec} -radical (see Leary [6]), we have

$$H^*(BHe) \cong H^*(BE)^{3S_3} \cap i_{A_6}^{*-1} H^*(BA_6)^{SL_2(\mathbb{F}_7)}.$$

For element $x = \bar{a}$ or $x = C + \bar{c} = C + y_1^3 y_2^3$, the restrictions are $x|_{A_6} = 0$, eg $\bar{a}|_{A_6} = (y^3 + (-y)^3) = 0$. Hence we see that $CA\{x\}$ are contained in $H^*(BG)$.

Since $\bar{b} = y_1 y_2 v^2$, we see $\bar{b}|_{A_0} = -y^2 v^2 = -(\tilde{D}'_2)^2$. Similarly $\bar{d}|_{A_6} = 2(\tilde{D}'_2)^3$. Thus we can compute $H^*(BHe)$. \square

Corollary 8.5 $H^*(B(He:2)) \cong DA\{1, \bar{b}, \bar{b}^2\} \oplus CA\{\bar{c}', \bar{c}'\bar{b}, \bar{c}'\bar{b}^2, \bar{a}^2, \bar{a}\bar{d}\}.$

Theorem 8.6 There is the isomorphism

$$H^*(BFi'_{24}) \cong DA\{1, \bar{b}, \bar{b}^2, \bar{a}^2 V, \bar{c}'\bar{b}V, \bar{c}'\bar{b}^2 V\} \oplus CA\{\bar{c}'', \bar{a}\bar{d}\} \text{ where } \bar{c}'' = \bar{a}^2 - 2\bar{c}'.$$

Proof Let $G = F_{24}$. Since A_1 is also F^{ec} -radical and $W_G(A_1) = SL_2(\mathbb{F}_7):2$. Hence we have

$$H^*(BG) \cong H^*(B(He:2)) \cap i_{A_1}^{*-1} H^*(BA_1)^{SL_2(\mathbb{F}_7):2}.$$

For the elements $x = \bar{a}\bar{d}, \bar{c}'' (= Y_1 + Y_2 - 2C)$, we see $x|_{A_1} = x|_{A_6} = 0$. Hence these elements are in $H^*(BG)$. Note that $\bar{b}|_{A_1} = (\tilde{D}'_2)^2$ and $\bar{b} \in H^*(BG)$. We also know $\bar{a}^2 V|_{A_1} = \tilde{D}_2$. \square

Since $H^*(BFi_{24}) \cong H^*(BFi'_{24}) \cap H^*(BE)^{6^2:2}$ and $\bar{b}^4 = 1/2(\bar{a}^2 - 2C - \bar{c}'')V$, we have the following corollary.

Corollary 8.7 $H^*(BFi_{24}) \cong (DA\{1, \bar{b}^2, \bar{b}^4\} \oplus CA\{\bar{c}''\})$.

For $G = RV_1$, The subgroup A_0 is also F^{ec} -radical, we see

$$H^*(BRV_1) \cong H^*(BFi_{24}) \cap i_0^{-1} H^*(BA_0)^{GL_2(\mathbb{F}_7)}$$

Hence we have the following corollary.

Corollary 8.8 $H^*(BRV_1) \cong DA\{1, \bar{b}^2, \bar{b}^4, D''_2\}$ with $\bar{b}^6 = D_2^2 + D''_2 D_2$.

Proof Let $D_2'' = \bar{c}''V = \bar{c}''(D_1 - C^6\bar{c}'')$. Then we have

$$\bar{b}^6 = Y_1Y_2V^2 = (Y_1 + Y_2 - C)CV^2 = (C + (Y_1 + Y_2 - 2C)C)V^2 = D_2^2 + (\bar{c}''V)D_2.$$

Thus the corollary is proved. □

9 Stable splitting for $B\mathbb{7}_+^{1+2}$

Let G be groups considered in the preceding two sections, eg $O'N, O'N:2, \dots, RV_1$. First consider the dominant summands $X_{q,k}$. From Corollary 4.6, the dominant summands are only related to $H = W_G(E)$. Recall the notation $X_{q,k}(H)$ in Lemma 4.7. The module $X_{q,k}(H)$ is still given in the preceding sections.

From Lemma 7.1, Lemma 7.2, Lemma 7.3, Lemma 8.1, Lemma 8.2 and Lemma 8.3 we have

$$\begin{aligned} H = 3D_8; X_{6,0} &= \{a^3/V, a^2b/V\}, X_{4,4} = \{a^2, b\}, X_{2,2} = \{a\}, \\ &X_{4,1} = \{d\}, X_{6,3} = \{ad\} \\ H = 3SD_{16}; X_{6,0} &= \{a^3/V\}, X_{4,4} = \{a^2\}, X_{2,2} = \{a\} \\ H = 3SD_{32}; X_{4,4} &= \{a^2\} \\ H = 3S_3; X_{6,0} &= \{\bar{b}^3/V, \bar{a}^2\}, X_{4,4} = \{\bar{b}^2\}, X_{2,2} = \{\bar{b}\}, \\ &X_{6,3} = \{\bar{a}\bar{d}\}, X_{3,0} = \{\bar{a}\}, X_{5,2} = \{\bar{a}\bar{b}\}, X_{3,3} = \{\bar{d}\}, X_{5,5} = \{\bar{d}\bar{b}\} \\ H = 6S_3; X_{6,0} &= \{\bar{b}^3/V, \bar{a}^2\}, X_{2,2} = \{\bar{b}\}, X_{4,4} = \{\bar{b}^2\}, X_{6,3} = \{\bar{a}\bar{d}\} \\ H = 6^2:2; X_{6,0} &= \{\bar{a}^2 - 2\bar{b}^3/V\}, X_{4,4} = \{\bar{b}^2\}. \end{aligned}$$

For example, ignoring nondominant summands, we have the following diagram

$$\begin{array}{ccccc} & X_{0,0} \vee X_{4,4} & & X_{6,0} \vee X_{2,2} & & X_{6,0} \vee X_{4,4} \vee X_{4,1} \vee X_{6,3} \\ & \longleftarrow & B(E:3SD_{32}) & \longleftarrow & B(E:3SD_{16}) & \longleftarrow & B(E:3D_8). \end{array}$$

From Corollary 4.4, the number $m(G, 1)_k$ is given by $\text{rank}_p H^{2k}(BG)$ for $k(p-1)$ and $\text{rank}_p H^{2p-2}(G)$ for $k=0$. For example when $G = E:3S_3$,

$$m(G, 1)_0 = 3, m(G, 1)_3 = 1, \quad m(G, 1)_k = 0 \text{ for } k \neq 0, \neq 3.$$

Lemma 9.1 *Let G be one of the $O'N, O'N, \dots, Fi'_{24}, RV_1$. Then the number $m(G, 1)_k$ for $L(1, k)$ is given by*

$$m(G, 1)_0 = \begin{cases} 2 & \text{for } G = He, He:2 \\ 1 & \text{for } G = O'N, O'N:2, Fi_{24}, Fi'_{24} \end{cases}$$

$$m(G, 1)_3 = \begin{cases} 1 & \text{for } G = He, \\ m(G, 1)_k = 0 & \text{otherwise.} \end{cases}$$

Now we consider the number $m(G, 2)_k$ of the non dominant summand $L(2, k)$.

Lemma 9.2 *The classifying spaces BG for $G = O'N, O'N:2$ have the non dominant summands $M(2) \vee L(2, 2) \vee L(2, 4)$.*

Proof We only consider the case $G = O'N$, and the case $O'N:2$ is almost the same. The non F^{ec} -radical groups are $\{A_2, A_3, A_4, A_5\}$ (recall the proof of Theorem 7.4). The group $W_G(E) = 3D_8 \cong \langle \text{diag}(2, 2), \text{diag}(1, -1), w \rangle$. Hence the normalizer group is

$$N_G(A_2) = E: \langle \text{diag}(2, 2), \text{diag}(-1, -1) \rangle.$$

Here note that $w, \text{diag}(1, -1)$ are not in the normalizer, eg $w: \langle c, ab^2 \rangle \rightarrow \langle c, a^2b^{-2} \rangle = \langle c, ab^6 \rangle$. Since $\text{diag}(2, 2): ab^2 \mapsto (ab^2)^2, c \mapsto c^4$ and $\text{diag}(-1, -1): ab^2 \mapsto (ab^2)^{-1}, c \mapsto c$, the Weyl groups are

$$W_G(A_2) \cong U: \langle \text{diag}(4, 2), \text{diag}(1, -1) \rangle.$$

Let $W_1 = U: \text{diag} \langle 4, 2 \rangle$. For $v = \lambda y_1^{p-1} \in M_{p-1, k}$, we have $\bar{W}_1 v = \lambda y_2^{p-1}$ since $2^3 = 1$, from the argument in the proof of Lemma 4.11. Moreover

$$\overline{\langle \text{diag}(1, -1) \rangle} y_2^{p-1} = (1 + (-1)^k) y_2^{p-1},$$

implies that the BG contains $L(2, k)$ if and only if k even. □

Lemma 9.3 *The classifying space BHe (resp. $B(He:2), Fi'_{24}, Fi_{24}$) contains the non dominant summands*

$$2M(2) \vee L(2, 2) \vee L(2, 4) \vee L(2, 3) \vee L(1, 3)$$

$$\text{(resp. } 2M(2) \vee L(2, 2) \vee L(2, 4), M(2), M(2)).$$

Proof First consider the case $G = He$. The non F^{ec} -radical group are

$$\{A_0, A_\infty\}, \quad \{A_1, A_2, A_4\}.$$

The group $W_G(E) \cong 3S_3 = \langle \text{diag}(2, 1), w' \rangle$. So we see $N_G(A_0) = E: \langle \text{diag}(2, 1) \rangle$, and this implies $W_G(A_0) \cong U: \langle \text{diag}(2, 2) \rangle$. The fact $4^k = 0 \pmod{7}$ implies $k = 3 \pmod{6}$. Hence BG contains the summand

$$M(2) \vee L(2, 3) \vee L(1, 3)$$

which is induced from BA_0 .

Next consider the summands induced from BA_1 . The normalizer and Weyl group are $N_G(A_1) = E: \langle w' \rangle$ and $W_G(A_1) = U: \langle \text{diag}(-1, 1) \rangle$ since $w': ab \mapsto ab, c \mapsto -c$. So we get

$$M(2) \vee L(2, 2) \vee L(2, 4)$$

which is induced from BA_1 .

For $G = He:2$, we see $\text{diag}(-1, -1) \in W_G(E)$, this implies that $\text{diag}(-1, -1) \in N_G(A)$ and $\text{diag}(1, -1) \in W_G(A_0)$. This means that the non dominant summand induced from BA_0 is $M(2)$ but is not $L(2, 3)$. We also know $U: \text{diag}(1, -1) \in W_G(A_1)$ but the summand induced from BA_1 are not changed.

For groups Fi'_{24}, Fi_{24} , the non F^{ec} -radical groups make just one G -conjugacy class $\{A_0, A_\infty\}$. So BG dose not contain the summands induced from BA_1 . \square

Theorem 9.4 When $p = 7$, we have the following stable decompositions of BG so that $\overset{X_1}{\leftarrow} \cdots \overset{X_s}{\leftarrow} G$ means that $BG \sim X_1 \vee \cdots \vee X_s$

$$\overset{X_{0,0}}{\leftarrow} \left\{ \begin{array}{l} \overset{X_{4,4}}{\leftarrow} RV_3 \xleftarrow{X_{6,0} \vee X_{2,2}} \overset{RV_2}{\leftarrow} \xleftarrow{M(2) \vee L(2,2) \vee L(2,4)} \overset{O'N:2}{\leftarrow} \xleftarrow{\overset{X_{6,0} \vee X_{4,4}}{\vee X_{4,1} \vee X_{6,3}}} \overset{O'N}{\leftarrow} \\ \overset{X_{6,0} \vee X_{4,4}}{\leftarrow} \overset{RV_1}{\leftarrow} \xleftarrow{M(2)} \overset{Fi_{24}}{\leftarrow} \xleftarrow{X_{6,0} \vee X_{6,3} \vee X_{2,2}} \overset{Fi'_{24}}{\leftarrow} \xleftarrow{M(2) \vee L(2,2) \vee L(2,4)} \overset{He:2}{\leftarrow} \\ \xleftarrow{X_{3,0} \vee X_{5,2} \vee X_{3,3} \vee X_{5,5} \vee L(2,3) \vee L(1,3)} \overset{He}{\leftarrow} \end{array} \right.$$

We write down the cohomology of stable summands. At first we see that $H^*(X_{0,0}) \cong H^*(BRV_3) \cap H^*(BRV_1) \cong DA$. Here note that elements $a^2 - (y_1 y_2)^2 v^4$ in Section 7 and $\bar{b}^2 = y_1^2 y_2^2 v^4$ in Section 8 are not equivalent under the action in $GL_7(\mathbb{F}_7)$ because $y_1^2 + y_2^2$ is indecomposable in $\mathbb{Z}/7[y_1, y_2]$.

From the cohomologies, $H^*(BRV_3)$ and $H^*(BRV_2)$, then $H^*(X_{4,4}) \cong DA\{a^2, a^4\}$ and $H^*(X_{6,0} \vee X_{2,2}) \cong DA\{a, a^3, a^5\}$.

On the other hand, we know $H^*(X_{6,0})$ from the cohomology $H^*(BRV_1)$. Thus we get the following lemma.

Lemma 9.5 *There are isomorphisms of cohomologies*

$$\begin{aligned} H^*(X_{0,0}) &\cong DA, H^*(X_{4,4}) \cong DA\{a^2, a^4\} \\ H^*(X_{6,0}) &\cong DA\{D_2\} \cong DA\{a^3\}, H^*(X_{2,2}) \cong DA\{a, a^5\}. \end{aligned}$$

Let us write $M\{a\} = DA\{1, C, \dots, C^{p-1}\}\{a\}$. From the facts that $D_2 = CV$, $D_1 = C^p + V$ and $D_2 = C(D_1 - C^p) = CD_1 - C^{p+1}$, we have two decompositions

$$CA\{a\} \cong DA\{1, C, \dots, C^p\}\{a\} \cong DA\{a\} \oplus M\{Ca\} \cong M\{a\} \oplus DA\{Va\}.$$

From the cohomology of $H^*(Fi_{24})$, we know the following lemma.

Lemma 9.6 $H^*(M(2)) \cong M\{C\}$.

Comparing the cohomology $H^*(B(He:2)) \cong H^*(BFi'_{24}) \oplus M\{\bar{a}^2, \bar{c}'\bar{b}, \bar{c}'\bar{b}^2\}$, we have the isomorphisms

$$H^*(M(2)) \cong M\{\bar{a}^2\}, H^*(L(2,2) \vee L(2,4)) \cong M\{\bar{c}'\bar{b}, \bar{c}'\bar{b}^2\}.$$

From $H^*(BFi'_{24}) \cong H^*(BFi_{24}) \oplus DA\{\bar{a}^2V, \bar{c}'\bar{b}V, \bar{c}'\bar{b}^2V\} \oplus CA\{\bar{a}\bar{d}\}$, we also know that

$$H^*(X_{6,3}) \cong CA\{\bar{a}\bar{d}\}, H^*(X_{6,0} \vee X_{2,2}) \cong DA\{\bar{a}^2V, \bar{c}'\bar{b}V, \bar{c}'\bar{b}^2V\}.$$

We still get $H^*(BFi_{24}) \cong H^*(BRV_1) \oplus M\{\bar{c}''\}$ and $H^*(M(2)) \cong M\{\bar{c}''\}$.

Next consider the cohomology of groups studied in Section 7 eg $O'N$. There is the isomorphism

$$H^*(BO'N) \cong H^*(BO'N:2) \oplus DA\{b, b^2, ab^2\} \oplus CA\{d, da, da^2\}.$$

Indeed, we have

$$\begin{aligned} H^*(X_{6,0} \vee X_{4,4}) &\cong DA\{b, b^2, ab^2\} \cong DA\{a^2, a^3, a^4\} \\ H^*(X_{6,3}) &\cong CA\{da\} \\ H^*(X_{4,1}) &\cong CA\{d, da^2\}. \end{aligned}$$

We also have the isomorphism $H^*(BO'N:2) \cong H^*(BRV_2) \oplus M\{C', C'a, C'a^2\}$ and $H^*(M(2) \vee L(2,2) \vee L(2,4)) \cong M\{C', C'a, C'a^2\}$.

Recall that

$$H^*(BE)^{3SD_{32}} \cong CA\{1, a^2, a^4/V\} \cong DA\{1, a^2, a^4\} \oplus M\{C, a^2C, a^4/V\},$$

in fact $H^*(M(2) \vee L(2,2) \vee L(2,4)) \cong M\{C, a^2C, a^4/V\}$.

10 The cohomology of \mathbb{M} for $p = 13$

In this section, we consider the case $p = 13$ and $G = \mathbb{M}$ the Fisher–Griess Monster group. It is known that $W_G(E) \cong 3 \times 4S_4$. The G -conjugacy classes of A -subgroups are divided into two classes; one is F^{ec} -radical and the other is not. The class of F^{ec} -radical groups contains 6 E -conjugacy classes (see Ruiz–Viruel [9]). (The description of [11, (4.1)] was not correct, and the description of $H^*(B\mathbb{M})$ in [11, Theorem 6.6] was not correct.) The Weyl group $W_G(A) \cong SL_2(\mathbb{F}_{13})$ for each F^{ec} -radical subgroup A .

Since $S_4 \cong PGL_2(\mathbb{F}_3)$ [S], we have the presentation of

$$S_4 = \langle x, y, z \mid x^3 = y^3 = z^2 = (xy)^2 = 1, zxz^{-1} = y \rangle.$$

(Take $x = u, y = u'$ in Lemma 4.8, and $z = w$ in Section 5.) By arguments in the proof of Suzuki [10, Chapter 3 (6.24)], we can take elements x, y, z in $GL_2(\mathbb{F}_{13})$ by

$$(10-1) \quad x = \begin{pmatrix} 3 & 0 \\ 0 & 9 \end{pmatrix}, \quad y = \begin{pmatrix} 5 & -4 \\ -2 & 7 \end{pmatrix}, \quad z = \begin{pmatrix} 2 & 2 \\ 1 & -2 \end{pmatrix},$$

so that we have

$$x^3 = y^3 = 1, \quad zxz^{-1} = y, \quad (xy)^2 = -1, \quad z^2 = \text{diag}(6, 6).$$

Hence we can identify

$$(10-2) \quad 3 \times 4S_4 \cong \langle x, y, z \rangle \subset GL_2(\mathbb{F}_{13}).$$

It is almost immediate that $H^*(BE)^{\langle x \rangle}$ (resp. $H^*(BE)^{\langle -1 \rangle}$) is multiplicatively generated by $y_1 y_2, y_1^3, y_2^3$ (resp. $y_1 y_2, y_1^2, y_2^2$) as a $\mathbb{Z}/(13)[C, v]$ -algebra. Hence we can write

$$(10-3) \quad H^*(BE)^{\langle x, -1 \rangle} \cong \mathbb{Z}/(13)[C, v] \{ \{1, y_1 y_2, \dots, (y_1 y_2)^5\} \{ (y_1 y_2)^6, y_1^6, y_2^6 \}, \\ y_1^{12}, y_2^{12}, y_1^{12} y_2^6, y_1^6 y_2^{12} \}.$$

For the invariant $H^*(BE)^{\langle y, -1 \rangle}$, we get the similar result exchanging y_i to $(z^{-1})^* y_i$ since $z x z^{-1} = y$. Indeed $(z^{-1})^*: H^*(BE)^{\langle x, -1 \rangle} \cong H^*(BE)^{\langle y, -1 \rangle}$.

To seek invariants, we recall the relation between the A -subgroups and elements in $H^2(BE; \mathbb{Z}/p)$. For $0 \neq y = \alpha y_1 + \beta y_2 \in H^2(BE; \mathbb{Z}/p)$, let $A_y = A_{-(\alpha/\beta)}$ so that $y|_{A_y} = 0$. This induces the 1–1 correspondence,

$$(H^2(BE; \mathbb{Z}/p) - \{0\})/F_p^* \leftrightarrow \{A_i \mid i \in F_p \cup \{\infty\}\}, \quad y \leftrightarrow A_y.$$

Considering the map $g^{-1} A_i \xrightarrow{g} A_i \subset E \xrightarrow{\beta^{-1} y} \mathbb{Z}/p$, we easily see $A_{g^* y} = g^{-1} A_y$.

For example, the order 3 element x induces the maps

$$\begin{aligned} x^*: y_1 - y_2 &\mapsto 3y_1 - 9y_2 \mapsto 9y_1 - 3y_2 \mapsto y_1 - y_2 \\ x^{-1}: A_{y_1-y_2} = \langle c, ab \rangle &\rightarrow \langle c, a^9b^3 \rangle \rightarrow \langle c, a^3b^9 \rangle \rightarrow \langle c, ab \rangle. \end{aligned}$$

In particular A_1, A_9, A_3 are in the same x -orbit of A -subgroups. Similarly the $\langle x \rangle$ -conjugacy classes of A is given

$$\{A_0\}, \{A_\infty\}, \{A_1, A_3, A_9\}, \{A_2, A_5, A_6\}, \{A_4, A_{10}, A_{12}\}, \{A_7, A_8, A_{11}\}.$$

The $\langle y \rangle$ -conjugacy classes are just $\{zA_i\}$ for $\langle x \rangle$ -conjugacy classes $\{A_i\}$.

$$\{A_7 = zA_0\}, \{A_{12}\}, \{A_3, A_1, A_5\}, \{A_6, A_9, A_2\}, \{A_{11}, A_8, A_\infty\}, \{A_0, A_{10}, A_4\}.$$

Hence we have the $\langle x, y \rangle$ -conjugacy classes

$$C_1 = \{A_1, A_2, A_3, A_5, A_6, A_9\}, C_2 = \{A_0, A_4, A_{10}, A_{12}\}, C_3 = \{A_\infty, A_7, A_8, A_{11}\}.$$

At last we note $\langle x, y, z \rangle$ -conjugacy classes are two classes $C_1, C_2 \cup C_3$.

Let us write the $\langle x \rangle$ -invariant

$$\begin{aligned} (10-4) \quad u_6 &= \prod_{A_i \in C_1} (y_2 - iy_1) = (y_2 - y_1)(y_2 - 2y_1) \cdots (y_2 - 9y_1) \\ &= y_2^6 - 9y_1^3y_2^3 + 8y_1^6. \end{aligned}$$

Then u_6 is also invariant under y^* because the $\langle x, y \rangle$ -conjugacy class C_1 divides two $\langle y \rangle$ -conjugacy classes

$$C_1 = \{A_1, A_3, A_5\} \cup \{A_2, A_6, A_9\}$$

and the element u_6 is rewritten as

$$u_6 = \lambda(\prod_{i=0}^2 y^{i*}(y_2 - y_1)) \cdot (\prod_{i=0}^2 y^{i*}(y_2 - 2y_1)) \text{ for } \lambda \neq 0 \in \mathbb{Z}/(13).$$

We also note that $u_6|_{A_i} = 0$ if and only if $i \in C_1$. Similarly the following elements are $\langle x, y \rangle$ -invariant,

$$\begin{aligned} (10-5) \quad u_8 &= \prod_{A_i \in C_2 \cup C_3} (y_2 - iy_1) = y_1y_2(y_2^6 + 9y_1^3y_2^3 + 8y_1^6) \\ u_{12} &= \prod_{A_i \in C_2} (y_2 - iy_1)^3 = (y_2^4 + y_1^3y_2)^3 \\ &= \lambda(\prod_{i=0}^2 x^{i*}y_2)(\prod_{i=0}^2 x^{i*}(y_2 - 4y_1))^3 \\ &= \lambda'(\prod_{i=0}^2 y^{i*}(y_2 - 12y_1))(\prod_{i=0}^2 y^{i*}y_2)^3 v \\ u'_{12} &= \prod_{A_i \in C_3} (y_2 - iy_1)^3 = (y_1y_2^3 + 8y_1^4)^3. \end{aligned}$$

Of course $(u_{12}u'_{12})^{1/3} = u_8$ and $u_6u_8 = 0$. Moreover direct computation shows $u_6^2 = u_{12} + 5u'_{12}$.

Lemma 10.1 $H^*(BE)^{\langle x,y \rangle} \cong \mathbb{Z}/(13)[C, v]\{1, u_6, u_6^2, u_6^3, u_8, u_8^2, u_{12}\}$.

Proof Recall (10–3) to compute

$$H^*(BE)^{\langle x,y \rangle} \cong H^*(BE)^{\langle x,-1 \rangle} \cap H^*(BE)^{\langle y,-1 \rangle}.$$

Since $(z^{-1})^*(y_1 y_2)^i \neq (y_1 y_2)^i$ for $1 \leq i \leq p - 2$, from (10–3) we know invariants of the lowest positive degree are of the form

$$u = \gamma y_2^6 + \alpha y_2^3 y_1^3 + \beta y_1^6.$$

Then $u' = u - \gamma u_6$ is also invariant with $u'|_{A_\infty} = 0$. Hence $u'|_{A_i} = 0$ for all $A_i \in C_3$. Thus we know $u' = \lambda y_1^2 (u'_{12})^{1/3}$. But this is not $\langle y \rangle$ -invariant for $\lambda \neq 0$, because $(u')^3 = \lambda^3 y_1^6 u'_{12}$ is invariant, while y_1^6 is not $\langle y \rangle$ -invariant. Thus we know $u' = 0$.

Any 16-dimensional invariant is form of

$$u = y_1 y_2 (\gamma y_2^6 + \alpha y_2^3 y_1^3 + \beta y_1^6).$$

Since $u|_{A_0} = u|_{A_\infty} = 0$, we know $u|_{A_i} = 0$ for all $A_i \in C_2 \cup C_3$. Hence we know

$$u = \gamma u_{12}^{1/3} (u'_{12})^{1/3} = \gamma u_8.$$

By the similar arguments, we can prove the lemma for degree ≤ 24 .

For $24 < \text{degree} < 48$, we only need consider the elements $u' = 0 \pmod{(y_1 y_2)}$. For example, $H^{18}(BE; \mathbb{Z}/13)^{\langle x,-1 \rangle}$ is generated by

$$\{(y_1 y_2)^9, (y_1 y_2)^3 C, y_1^6 C, y_2^6 C, y_1^6 y_2^{12}, y_1^{12} y_2^6\}.$$

But we can take off $y_1^6 C = y_1^{18}$, $y_2^6 C = y_2^{18}$ by $\lambda u_6^3 + \mu C u_6$ so that $u' = 0 \pmod{(y_1 y_2)}$.

Hence we can take u' so that u_8 divides u' from the arguments similar to the case of degree=16. Let us write $u' = u'' u_8$. Then we can write

$$u'' = y_1^k y_2^k (\lambda_1 y_1^6 + \lambda_2 y_1^3 y_2^3) + \lambda_3 (y_1 y_2)^{k-3} C,$$

taking off $\lambda y_1^k y_2^k u_6$ if necessary since $u_6 u_8 = 0$. (Of course, for $k < 3$, $\lambda_3 = 0$.) Since $u_8|_{A_i} \neq 0$ and $u_6|_{A_i} = 0$ for $i \in C_1$, we have

$$(u'' - y^* u'')|_{A_i} = 0 \text{ for } i \in C_1.$$

Since $y^* y_1 = 5y_1 - 4y_2$ and $y^* y_2 = -2y_1 + 7y_2$, we have

$$\begin{aligned} (u'' - y^* u'')|_{A_i} &= \lambda_1 (i^k - (5 - 4i)^{6+k} (-2 + 7i)^k) \\ &\quad + \lambda_2 (i^{k+3} - (5 - 4i)^{k+3} (-2 + 7i)^{k+3}) \\ &\quad + \lambda_3 (i^{k-3} - (5 - 4i)^{k-3} (-2 + 7i)^{k-3}). \end{aligned}$$

We will prove that we can take all $\lambda_i = 0$. Let us write $U = u'' - y^*u''$. We then have the following cases.

- (1) The case $k = 0$, ie degree=14. If we take $i = 1$,

$$U|_{A_1} = \lambda_1(1 - 1) + \lambda_2(1 - 1^3 5^3) = 0.$$

So we have $\lambda_2 = 0$. We also see $\lambda_1 = 0$ since $U|_{A_3} = \lambda_1(1 - (5 - 12)^6) = 2\lambda_1 = 0$.

- (2) The case $k = 1$. Since $y_1 y_2 u_6 - u_8 = -18y_1^4 y_2^4$, we can assume $\lambda_2 = 0$ taking off λu_8^2 if necessary. We have also $\lambda_1 = 0$ from $U|_{A_1} = \lambda_1(1^1 - 1^7 5^1) = 0$.
- (3) The case $k = 2$. We get the the result $U|_{A_1} = 2\lambda_1 + 4\lambda_2$, $U|_{A_3} = 5\lambda_1 + 5\lambda_2$.
- (4) The case $k = 3$. First considering Cu_8 , we may take $\lambda_3 = 0$. The result is given by $U|_{A_1} = 6\lambda_1 + 2\lambda_2$ and $U|_{A_2} = 7\lambda_1 + 9\lambda_2$.
- (5) The case $k = 4$. The result follows from

$$U|_{A_1} = 6\lambda_2 + 9\lambda_3, U|_{A_3} = 6\lambda_1 + 6\lambda_2 + 6\lambda_3, U|_{A_5} = 2\lambda_1 - 4\lambda_2 + 6\lambda_3.$$

Hence the lemma is proved. □

Next consider the invariant under $\langle x, y, \text{diag}(6, 6) \rangle$. The action for $\text{diag}(6, 6)$ is given by $y_1^i y_2^j v^k \mapsto 6^{i+j+2k} y_1^i y_2^j v^k$. Hence the invariant property implies $i + j + 2k = 0 \pmod{12}$. Thus $H^*(BE)^{\langle x, y, \text{diag}(6, 6) \rangle}$ is generated as a CA -algebra by

$$\{1, u_6 v^3, u_8 v^2, u_{12}, u'_{12}, v^6\}.$$

Lemma 10.2 *The invariant $H^*(BE)^{3 \times 4S_4} \cong H^*(BE)^{\langle x, y, z \rangle}$ is isomorphic to*

$$CA\{1, u_6 v^3, (u_6 v^3)^2, (u_6 v^3)^3, u_8 v^8, (u_8 v^8)^2 / V, (u_{12} - 5u'_{12})\}.$$

Proof We only need compute z^* -action. Since

$$3 \times 4S_4 \cong \langle x, y, \text{diag}(6, 6) \rangle : \langle z \rangle,$$

the z^* -action on $H^*(BE)^{\langle x, y, \text{diag}(6, 6) \rangle}$ is an involution. Let $u_6 v^3 = u_6(y_1, y_2)v^3$. First note $u_6|_{A_\infty} = u_6(0, y) = y^6$. On the other hand, its z^* -action is

$$\begin{aligned} z^* u_6 v^3 |_{A_\infty} &= u_6(2y_1 + 2y_2, y_1 - 2y_2)(-6v)^3 |_{A_\infty} = u_6(2y, -2y)(-6v)^3 \\ &= ((-2)^6 - 9(-2)^3(2)^3 + 8(2)^6)(-6)^3 y^6 v^3 \\ &= (1 + 9 + 8)8y^6 v^3 = y^6 v^3. \end{aligned}$$

Hence we know $u_6 v^3$ is invariant, while $u_6 v^9$ is not.

Similarly we know

$$u_8 v^2|_{A_1} = u_8(y, y)v^2 = 5y^8 v^2, \quad z^* u_8 v^2|_{A_1} = -5y^8 v^2.$$

Hence $u_8 v^8$ and $u_8^2 v^4$ are invariant but $u_8 v^2$ is not.

For the action u_{12} , we have

$$\begin{aligned} u_{12}|_{A_0} &= 0, & u_{12}|_{A_\infty} &= y^{12}, & u'_{12}|_{A_0} &= 5y^{12}, & u'_{12}|_{A_\infty} &= 0, \\ z^* u_{12}|_{A_0} &= y^{12}, & z^* u_{12}|_{A_\infty} &= 0, & z^* u'_{12}|_{A_0} &= 0, & z^* u'_{12}|_{A_\infty} &= 5y^{12}. \end{aligned}$$

Thus we get $z^* u_{12} = (1/5)u'_{12}$, $k^* u'_{12} = 5u_{12}$. Hence we know $u_{12} + (1/5)u'_{12}$ and $(u_4^3 - (1/5)u'_{12})v^6 = (u_6 v^3)^2$ are invariants. Thus we can prove the lemma. \square

Theorem 10.3 For $p = 13$, the cohomology $H^*(B\mathbb{M})$ is isomorphic to

$$DA\{1, u_8 v^8, (u_8 v^8)^2\} \oplus CA\{u_6 v^3, (u_6 v^3)^2, (u_6 v^3)^3, (u_{12} - 5u'_{12} - 3C)\}.$$

Proof Direct computation shows

$$u_{12} - 5u'_{12} = y_2^{12} - 2y_2^9 y_1^3 + 3y_2^3 y_1^9 + y_1^{12},$$

and hence $u_{12} - 5u'_{12} - 3C|_{A_1} = 0$, indeed, the restriction is zero for each $A_i \in C_1$. The isomorphism

$$H^*(B\mathbb{M}) \cong H^*(BE)^{3 \times 4S_4} \cap i_{A_1}^{-*}(H^*(BA_1)^{SL_4(\mathbb{F}_{13})})^4,$$

completes the proof. \square

The stable splitting is given by the following theorem.

Theorem 10.4 We have the stable splitting

$$\begin{aligned} B\mathbb{M} &\sim X_{0,0} \vee X_{12,0} \vee X_{12,6} \vee X_{6,3} \vee X_{8,8} \vee M(2), \\ B(E: 3 \times 4S_4) &\sim B\mathbb{M} \vee M(2) \vee L(2, 4) \vee L(2, 8). \end{aligned}$$

Proof Let $H = E: 3 \times 4S_4$. Recall that

$$X_{q,k}(H) = (S(A)^q \otimes v^k) \cap H^*(BH) \quad 0 \leq q \leq 12, 0 \leq k \leq 11.$$

We already know

$$X_{*,*}(H) = \mathbb{Z}/(13)\{1, u_8 v^8, u_6 v^3, u_6^2 v^6, u_{12} - 5u'_{12}\}.$$

Hence BH has the dominant summands in the theorem.

The normalizer groups of A_0, A_1 are given

$$N_H(A_0) = E: \langle x, \text{diag}(6, 6) \rangle, N_H(A_1) = E: \langle \text{diag}(6, 6) \rangle.$$

Hence the Weyl groups are

$$W_H(A_0) = U: \langle \text{diag}(1, 3), \text{diag}(6^2, 6) \rangle, W_H(A_1) = U: \langle \text{diag}(6^2, 6) \rangle.$$

From the arguments of Lemma 4.11, the non-dominant summands induced from BA_1 are $M(2) \vee L(2, 4) \vee L(2, 8)$. We also know the non-dominant summands from BA_0 are $M(2)$. This follows from

$$\overline{\langle \text{diag}(1, 3) \rangle} y_2^{p-1} = \sum_{i=0}^2 (3^i)^k y_2^{p-1} \quad \text{for } y_2^{p-1} \in M_{p-1,k}$$

and this is nonzero mod(13) if and only if $k \equiv 0 \pmod{3}$. □

Remark It is known $H^*(Th) \cong DA$ for $p = 5$ in [11]. Hence all cohomology $H^*(BG)$ for groups G in Theorem 2.1 (4)–(7) are explicitly known. For (1)–(3), see also Tezuka–Yagita [11].

11 Nilpotent parts of $H^*(BG; \mathbb{Z}_{(p)})$

It is known that $p^2 H^*(BE; \mathbb{Z}) = 0$ (see Tezuka–Yagita [11] and Leary [6]) and

$$pH^{>0}(BE; \mathbb{Z}) \cong \mathbb{Z}/p\{pv, pv^2, \dots\}.$$

In particular $H^{\text{odd}}(BE; \mathbb{Z})$ is all just p -torsion. There is a decomposition

$$H^{\text{even}}(BE; \mathbb{Z})/p \cong H^*(BE) \oplus N \text{ with } N = \mathbb{Z}/p[V]\{b_1, \dots, b_{p-3}\}$$

where $b_i = \text{Cor}_{A_0}^E(u^{i+1})$, $|b_i| = 2i + 2$. (Note for $p = 3, N = 0$.) The restriction images $b_i|_{A_j} = 0$ for all $j \in \mathbb{F}_p \cup \infty$. For $g \in GL_2(\mathbb{F}_p)$, the induced action is given by $g^*(b_i) = \det(g)^{i+1} b_i$ by the definition of b_i .

Note that

$$2 = |y_i| < |b_j| = 2(j + 1) < |C| = 2p - 2 < |v| = 2p.$$

So $g^*(y_i)$ is given by (3–4) also in $H^*(BE; \mathbb{Z})$ and $g^*(v) = \det(g)v \pmod{p}$. Hence we can identify that

$$H^*(BE)^H = (H^{\text{even}}(BE; \mathbb{Z})/(p, N))^H \subset H^{\text{even}}(BE; \mathbb{Z}/p)^H.$$

Let us write the reduction map by $q: H^*(BE; \mathbb{Z}) \rightarrow H^*(BE; \mathbb{Z}/p)$.

Lemma 11.1 *Let $H \subset GL_2(\mathbb{F}_p)$ and $(|H|, p) = 1$. If $x \in H^*(BE)^H$, then there is $x' \in H^*(BE; \mathbb{Z})^H$ such that $q(x') = x$.*

Proof Let $x \in H^*(BE)^H$ and $G = E : H$. Then we can think $x \in H^*(BE; \mathbb{Z}/p)^H \cong H^*(BG; \mathbb{Z}/p)$ and $\beta(x) = 0$. By the exact sequence

$$H^{\text{even}}(BG; \mathbb{Z}_{(p)}) \xrightarrow{q} H^{\text{even}}(BG; \mathbb{Z}/p) \xrightarrow{\delta} H^{\text{odd}}(BG; \mathbb{Z}_{(p)}),$$

we easily see that $x \in \text{Image}(q)$ since $q\delta(x) = \beta(x) = 0$ and $q|H^{\text{odd}}(BG; \mathbb{Z}_{(p)})$ is injective. Since $H^*(BG; R) \cong H^*(BE; R)^H$ for $R = \mathbb{Z}_{(p)}$ or \mathbb{Z}/p , we get the lemma. \square

Proof of Theorem 3.1 From Tezuka–Yagita [11, Theorem 4.3] and Broto–Levi–Oliver [1], we have the isomorphism

$$H^*(BG; \mathbb{Z})_{(p)} \cong H^*(BE; \mathbb{Z})^{W_G(E)} \cap_{A: F^{\text{ec-radical}}} i_A^{*-1} H^*(BA; \mathbb{Z})^{W_G(A)}.$$

The theorem is immediate from the above lemma and the fact that $H^{\text{even}>0}(BA; \mathbb{Z}) \cong H^{*>0}(BA)$. \square

Let us write $N(G) = H^*(BG; \mathbb{Z}) \cap N$. Then

$$H^{\text{even}}(BG; \mathbb{Z})/p \cong H^*(BG) \oplus N(G).$$

The nilpotent parts $N(G)$ depends only on the group $\text{Det}(G) = \{\det(g) | g \in W_G(E)\} \subset \mathbb{F}_p^*$, in fact, $N(G) = N^{W_G(E)} = N^{\text{Det}(G)}$.

Lemma 11.2 *If $\text{Det}(G) \cong \mathbb{F}_p^*$ (eg $G = O'N, He, \dots, RV_3$ for $p = 7$, or $G = \mathbb{M}$ for $p = 13$), then*

$$N(G) \cong \mathbb{Z}/p[V]\{b_i v^{p-2-i} | 1 \leq i \leq p-3\}.$$

Lemma 11.3 *Let G have a 7–Sylow subgroup E . Then, we have*

$$N(G) = \begin{cases} \mathbb{Z}/7[V]\{b_1 v^4, b_2 v^3, b_3 v^2, b_4 v\} & \text{if } \text{Det}(G) = \mathbb{F}_7^* \\ \mathbb{Z}/7[v^3]\{b_1 v, b_2, b_3 v^2, b_4 v\} & \text{if } \text{Det}(G) \cong \mathbb{Z}/3 \\ \mathbb{Z}/7[v^2]\{b_1, b_2 v, b_3, b_4 v\} & \text{if } \text{Det}(G) \cong \mathbb{Z}/2 \\ \mathbb{Z}/7[v]\{b_1, b_2, b_3, b_4\} & \text{if } \text{Det}(G) \cong \{1\}. \end{cases}$$

Now we consider the odd dimensional elements. Recall that

$$H^{\text{odd}}(BA; \mathbb{Z}) \cong \mathbb{Z}/p[y_1, y_2]\{\alpha\},$$

where $\alpha = \beta(x_1 x_2) \in H^*(BA; \mathbb{Z}/p) \cong \mathbb{Z}/p[y_1, y_2] \otimes \Lambda(x_1, x_2)$ with $\beta(x_i) = y_i$. Of course $g^*(\alpha) = \det(g)\alpha$ for $g \in \text{Out}(A)$. For example $H^{\text{odd}}(B(A:Q_8)) \cong H^*(B(A:Q_8))\{\alpha\}$ since $\text{Det}(A:Q_8) = \{1\}$.

Recall the Milnor operation $Q_{i+1} = [P^{p^n} Q_i - Q_i P^{p^n}]$, $Q_0 = \beta$. It is known that

$$Q_1(\alpha) = y_1^p y_2 - y_1 y_2^p = \tilde{D}'_2 \text{ with } (\tilde{D}'_2)^{p-1} = \tilde{D}_2.$$

The submodule of $H^*(X; \mathbb{Z}/p)$ generated by (just) p -torsion additive generators can be identified with $Q_0 H^*(X; \mathbb{Z}/p)$. Since $Q_i Q_0 = -Q_0 Q_i$, we can extend the map [13, page 377]

$$Q_i: Q_0 H^*(X; \mathbb{Z}/p) \xrightarrow{Q_i} Q_0 H^*(X; \mathbb{Z}/p) \subset H^*(X; \mathbb{Z}/p).$$

Since all elements in $H^{\text{odd}}(BA; \mathbb{Z})$ are (just) p -torsion, we can define the map

$$Q_1: H^{\text{odd}}(BA; \mathbb{Z}) \rightarrow H^{\text{even}}(BA; \mathbb{Z}) = H^{\text{even}}(BA).$$

Moreover this map is injective.

Lemma 11.4 (Yagita [13]) *Let G have the p -Sylow subgroup $A = (\mathbb{Z}/p)^2$. Then*

$$Q_1: H^{\text{odd}}(BG; \mathbb{Z}/p) \cong (H^{\text{even}}(BG) \cap J(G)),$$

with $J(G) = \text{Ideal}(y_1^p y_2 - y_1 y_2^p) \subset H^{\text{even}}(BA)$.

Corollary 11.5 *For $p = 3$, there are isomorphisms*

$$H^{\text{odd}}(BA; \mathbb{Z})^{Z/8} \cong S\{b, a'b, a, (a_1 - a_2)\}\{\alpha\}$$

$$H^{\text{odd}}(BA; \mathbb{Z})^{D_8} \cong S\{1, a, a_1, a'\}\{b\alpha\}$$

$$H^{\text{odd}}(BA; \mathbb{Z})^{SD_{16}} \cong S\{1, a'\}\{b\alpha\}.$$

Proof We only prove the case $G = A: \mathbb{Z}/8$ since the proof of the other cases are similar. Note in §5 the element $Q_1(\alpha)$ is written by b and $b^2 = a_1 a_2$. Recall $S = \mathbb{Z}/3[a_1 + a_2, a_1 a_2]$. Hence we get

$$\begin{aligned} H^*(BA)^{(l)} \cap J(G) &\cong S\{1, a', ab, (a_1 - a_2)b\} \cap \text{Ideal}(b) \\ &= S\{b^2, b^2 a', ab, (a_1 - a_2)b\} \\ &= S\{b, ba', a, (a_1 - a_2)\}\{Q_1(\alpha)\}. \end{aligned}$$

The corollary follows. □

By Lewis, we can write [6; 11]

$$H^{\text{odd}}(BE; \mathbb{Z}) \cong \mathbb{Z}/p[y_1, y_2]/(y_1\alpha_2 - y_2\alpha_1, y_1^p\alpha_2 - y_2^p\alpha_1)\{\alpha_1, \alpha_2\},$$

where $|\alpha_i| = 3$. It is also known that $Q_1(\alpha_i) = y_i v$ and $Q_1: H^{\text{odd}}(BE; \mathbb{Z}_{(p)}) \rightarrow H^{\text{even}}(BE) \subset H^{\text{even}}(BE; \mathbb{Z})/p$ is injective [13]. Using this we can prove the following lemma.

Lemma 11.6 (Yagita [13]) *Let G have the p -Sylow subgroup E . Then*

$$Q_1: H^{\text{odd}}(BG) \cong (H^{\text{even}}(BG) \cap J(G))$$

with $J(G) = \text{Ideal}(y_i v) \subset H^{\text{even}}(BE)$.

From the above lemma we easily compute the odd dimensional elements. Note that

$$D_2 = CV \notin J(E) \text{ but } D_2^2 = C^2V^2 = (Y_1^2 + Y_2^2 - Y_1Y_2)V^2 \in J(E).$$

Let us write $\alpha = (Y_1y_1^{p-2}\alpha_1 + Y_2y_2^{p-2}\alpha_2 - Y_1y_2^{p-2}\alpha_2)Vv^{p-2}$ so that $Q_1(\alpha) = D_2^2$.

Corollary 11.7 $H^{\text{odd}}(B^2F_4(2)'; \mathbb{Z}_{(3)}) \cong DA\{\alpha, \alpha'\}$ with $\alpha' = (y_1\alpha_1 + y_2\alpha_2)v$.

Proof Recall that $H^*(B^2F_4(2)') \cong DA\{1, (Y_1 + Y_2)V\}$ from the remark of Proposition 6.3. The result is easily obtained from $Q_1(\alpha) = D_2^2, Q_1(\alpha') = (Y_1 + Y_2)V$. \square

Corollary 11.8 *There are isomorphisms*

$$\begin{aligned} H^{\text{odd}}(BRV_3; \mathbb{Z}_{(7)}) &\cong DA\{a, a^3, a^5\}\{\alpha'\} \\ H^{\text{odd}}(BRV_2; \mathbb{Z}_{(7)}) &\cong DA\{1, a, \dots, a^5\}\{\alpha'\}, \end{aligned}$$

with $\alpha' = (y_1\alpha_1 + y_2\alpha_2)v$.

Proof We can easily compute

$$Q_1(\alpha') = Q_1((y_1\alpha_1 + y_2\alpha_2)v) = (y_1Q_1(\alpha_1) + y_2Q_1(\alpha_2))v = (y_1^2 + y_2^2)v^2 = a.$$

Recall that $H^*(BRV_3) \cong DA\{1, a^2, a^4\}$. We get

$$H^*(BRV_3) \cap \text{Ideal}(y_i v) = DA\{D_2^2, a^2, a^4\} = DA\{a^5, a, a^3\}(Q_1\alpha'),$$

and the corollary follows. \square

Corollary 11.9 $H^{\text{odd}}(BRV_1; \mathbb{Z}_{(7)}) \cong DA\{\bar{b}, \bar{b}^3, \bar{b}^5\}\{\alpha''\} \oplus DA\{\alpha\}$ where $\alpha'' = y_1v\alpha_2$.

Proof Recall Corollary 8.8. We have $C\bar{c}'' = C(Y_1 + Y_2 - 2C) = -Y_1^2 - Y_2^2 + 2Y_1Y_2$. Hence we can see $Q_1(\alpha) = -D_2\bar{c}''V$. \square

Corollary 11.10 *The cohomology $H^{\text{odd}}(B\mathbb{M}; \mathbb{Z}_{(13)})$ is isomorphic to*

$$DA\{\alpha, \alpha_8, (u_8v^8)\alpha_8\} \oplus CA\{\alpha_6, (u_6v^3)\alpha_6, (u_6v^3)^2\alpha_6, \alpha_{12}\},$$

where

$$\alpha_8 = y_2(y_2^6 + 9y_2^3y_1^3 + 8y_1^6)v^7\alpha_1$$

$$\alpha_6 = (y_2^5\alpha_2 - 9y_2^2y_1^3\alpha_2 + 8y_1^5y\alpha_1)v^2$$

$$\alpha_{12} = C(y_2^{11}\alpha_2 - 2y_2^8y_1^3\alpha_2 + 3y_2^2y_1^9\alpha_2 + y_1^{11}\alpha_1)v^{11} - 3\alpha/V.$$

Proof It is almost immediate that

$$Q_1(\alpha_8) = u_8v^8, \quad Q_1(\alpha_6) = u_6v^3, \quad Q_1(\alpha_{12}) = (u_{12} - 5u'_{12} - 3C)CV.$$

From Theorem 10.3, we get the corollary. \square

References

- [1] **C Broto, R Levi, B Oliver**, *The homotopy theory of fusion systems*, J. Amer. Math. Soc. 16 (2003) 779–856 MR1992826
- [2] **J Dietz, S Priddy**, *The stable homotopy type of rank two p -groups*, from: “Homotopy theory and its applications (Cocoyoc, 1993)”, Contemp. Math. 188, Amer. Math. Soc., Providence, RI (1995) 93–103 MR1349132
- [3] **D J Green**, *On the cohomology of the sporadic simple group J_4* , Math. Proc. Cambridge Philos. Soc. 113 (1993) 253–266 MR1198410
- [4] **J C Harris, N J Kuhn**, *Stable decompositions of classifying spaces of finite abelian p -groups*, Math. Proc. Cambridge Philos. Soc. 103 (1988) 427–449 MR932667
- [5] **I Leary**, *The integral cohomology rings of certain finite groups*, PhD thesis, Cambridge University (1990)
- [6] **I Leary**, *The integral cohomology rings of some p -groups*, Math. Proc. Cambridge Philos. Soc. 110 (1991) 25–32 MR1104598
- [7] **J Martino, S Priddy**, *The complete stable splitting for the classifying space of a finite group*, Topology 31 (1992) 143–156 MR1153242
- [8] **D Quillen**, *The spectrum of an equivariant cohomology ring. I, II*, Ann. of Math. (2) 94 (1971) 549–572; *ibid.* (2) 94 (1971), 573–602 MR0298694
- [9] **A Ruiz, A Viruel**, *The classification of p -local finite groups over the extraspecial group of order p^3 and exponent p* , Math. Z. 248 (2004) 45–65 MR2092721

- [10] **M Suzuki**, *Group theory. I*, Grundlehren der Mathematischen Wissenschaften 247, Springer, Berlin (1982) MR648772 Translated from the Japanese by the author
- [11] **M Tezuka, N Yagita**, *On odd prime components of cohomologies of sporadic simple groups and the rings of universal stable elements*, J. Algebra 183 (1996) 483–513 MR1399037
- [12] **C B Thomas**, *Characteristic classes and 2–modular representations for some sporadic simple groups. II*, from: “Algebraic topology Poznań 1989”, Lecture Notes in Math. 1474, Springer, Berlin (1991) 371–381 MR1133913
- [13] **N Yagita**, *On odd degree parts of cohomology of sporadic simple groups whose Sylow p –subgroup is the extra-special p –group of order p^3* , J. Algebra 201 (1998) 373–391 MR1612382
- [14] **N Yagita**, *Classifying spaces of sporadic simple groups for odd primes*, J. London Math. Soc. (2) 60 (1999) 143–154 MR1721821

*Department of Mathematics, Faculty of Education, Ibaraki University
Mito, Ibaraki, Japan*

yagita@mx.ibaraki.ac.jp

Received: 28 December 2004 Revised: 17 July 2005