Stable splitting and cohomology of $p$–local finite groups over the extraspecial $p$–group of order $p^3$ and exponent $p$

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Let $p$ be an odd prime. Let $G$ be a $p$–local finite group over the extraspecial $p$–group $p^1+2$. In this paper we study the cohomology and the stable splitting of their $p$–complete classifying space $BG$.

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1 Introduction

Let us write by $E$ the extraspecial $p$–group $p^1+2$ of order $p$ and exponent $p$ for an odd prime $p$. Let $G$ be a finite group having $E$ as a $p$–Sylow subgroup, and $BG$ ($= BG_p^1$) the $p$–completed classifying space of $G$. In papers by Tezuka and Yagita [11] and Yagita [13; 14], the cohomology and stable splitting for such groups are studied. In many cases non isomorphic groups have homotopy equivalent $p$–completed classifying spaces, showing that there are not too many homotopy types of $BG$, as was first suggested by C B Thomas [12] and D Green [3].

Recently, Ruiz and Viruel [9] classified all $p$–local finite groups for the $p$–group $E$. Their results show that each classifying space $BG$ is homotopic to one of the classifying spaces which were studied in [11] or classifying spaces of three exotic 7–local finite groups. (While descriptions in [11] of $H^*(F_4(3))$ and $H^*(M(13))$ contained some errors.)

In Section 2, we recall the results of Ruiz and Viruel. In Section 3, we also recall the cohomology $H^*(BE; \mathbb{Z})/(p, \sqrt{0})$. In this paper, we simply write

$$H^*(BG) = H^*(BG; \mathbb{Z})/(p, \sqrt{0})$$

and study them mainly. The cohomology $H^\text{odd}(BG; \mathbb{Z}(p))$ and the nilpotents parts in $H^\text{even}(BG; \mathbb{Z}(p))$ are given in Section 11. Section 4 is devoted to the explanations of stable splitting of $BG$ according to Dietz, Martino and Priddy. In Section 5, and Section 6, we study cohomology and stable splitting of $BG$ for a finite group $G$ having a 3–Sylow group $(\mathbb{Z}/3)^2$ or $E = 3^1+2$ respectively. In Section 7 and Section 8, we study cohomology of $BG$ for groups $G$ having a 7–Sylow subgroup $E = 7^1+2$, and

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the three exotic $7$–local finite groups. In Section 9, we study their stable splitting. In Section 10 we study the cohomology and stable splitting of the Monster group $\mathbb{M}$ for $p = 13$.

## 2 $p$–local finite groups over $E$

Recall that the extraspecial $p$–group $p^+_{1+2}$ has a presentation as
\[ p^+_{1+2} = \langle a, b, c \mid a^p = b^p = c^p = 1, [a, b] = c, c \in \text{Center} \rangle \]
and denote it simply by $E$ in this paper. We consider $p$–local finite groups over $E$, which are generalization of groups whose $p$–Sylow subgroups are isomorphic to $E$.

The concept of the $p$–local finite groups arose in the work of Broto, Levi and Oliver [1] as a generalization of a classical concept of finite groups. The $p$–local finite group is stated as a triple $\langle S, F, L \rangle$ where $S$ is a $p$–group, $F$ is a saturated fusion system over a centric linking system $L$ over $S$ (for a detailed definition, see [1]). Given a $p$–local finite group, we can construct its classifying space $B\langle S, F, L \rangle$ by the realization $|L|^{\wedge}$. Of course if $\langle S, F, L \rangle$ is induced from a finite group $G$ having $S$ as a $p$–Sylow subgroup, then $B\langle S, F, L \rangle \cong BG$. However note that in general, there exist $p$–local finite groups which are not induced from finite groups (exotic cases).

Ruiz and Viruel recently determined $\langle p^+_{1+2}, F, L \rangle$ for all odd primes $p$. We can check the possibility of existence of finite groups only for simple groups and their extensions. Thus they find new exotic $7$–local finite groups.

The $p$–local finite groups $\langle E, F, L \rangle$ are classified by $\text{Out}_F(E)$, number of $F^{\text{ec}}$–radical $p$–subgroup $A$ (where $A \cong (\mathbb{Z}/p)^2$), and $\text{Aut}_F(A)$ (for details see [9]). When a $p$–local finite group is induced from a finite group $G$, then we see easily that $\text{Out}_F(E) \cong W_G(E)(= N_G(E)/E \cdot C_G(E))$ and $\text{Aut}_F(A) \cong W_G(A)$. Moreover $A$ is $F^{\text{ec}}$–radical if and only if $\text{Aut}_F(A) \supset SL_2(\mathbb{F}_p)$ by [9, Lemma 4.1]. When $G$ is a sporadic simple group, $F^{\text{ec}}$–radical follows $p$–pure.

**Theorem 2.1** (Ruiz and Viruel [9]) If $p \neq 3, 7, 5, 13$, then a $p$–local finite group $\langle E, F, L \rangle$ is isomorphic to one of the following types.

1. $E:W$ for $W \subset \text{Out}(E)$ and $|W|, p = 1$,
2. $p^2:SL_2(\mathbb{F}_p), r$ for $r|(p - 1)$,
3. $SL_3(\mathbb{F}_p); H$ for $H \cong \mathbb{Z}/2, \mathbb{Z}/3$ or $S_3$.

When $p = 3, 5, 7$ or $13$, it is either of one of the previous types or of the following types.
Cohomology of $p$–local groups over $p^{1+2}_+$

(5) $^2F_4(2)'$, $J_4$, for $p=3$,
(6) $^2E_6$ for $p=5$,
(7) $He$, $He$: 2, $Fi'_24$, $Fi_24$, $O'N$, $O'N$: 2, and three exotic $7$–local finite groups for $p=7$,
(8) $\mathcal{M}$ for $p=13$.

For case (1), we know that $H^*(E; W) \cong H^*(E; W)$. Except for these extensions and exotic cases, all $H^{\text{even}}(G; \mathbb{Z})(p)$ are studied by Tezuka and Yagita [11]. In [13], the author studied ways to distinguish $H^{\text{odd}}(G; \mathbb{Z})(p)$ and $H^*(G; \mathbb{Z}/p)$ from $H^{\text{even}}(G; \mathbb{Z})(p)$. The stable splittings for such $BG$ are studied in [14]. However there were some errors in the cohomology of $^2F_4(2)'$, $Fi'_24$, $\mathcal{M}$. In this paper, we study cohomology and stable splitting of $BG$ for $p = 3, 7$ and $13$ mainly.

3 Cohomology

In this paper we mainly consider the cohomology $H^*(BG; \mathbb{Z})/(p, \sqrt{0})$ where $\sqrt{0}$ is the ideal generated by nilpotent elements. So we write it simply

$$H^*(BG) = H^*(BG; \mathbb{Z})/(p, \sqrt{0}).$$

Hence we have

$$H^*(BG; \mathbb{Z}/p) \cong \mathbb{Z}/p[y], \quad H^*(BG; \mathbb{Z}/p)^2 \cong \mathbb{Z}/p[y_1, y_2] \text{ with } |y| = |y_i| = 2.$$ 

Let us write $(\mathbb{Z}/p)^2$ as $A$ and let an $A$–subgroup of $G$ mean a subgroup isomorphic to $(\mathbb{Z}/p)^2$.

The cohomology of the extraspecial $p$ group $E = p^{1+2}_+$ is well known. In particular recall (Leary [6] and Tezuka–Yagita [11])

$$(3–1) \quad H^*(BE) \cong (\mathbb{Z}/p[y_1, y_2]/(y_1^p y_2 - y_1 y_2^p) \oplus \mathbb{Z}/p[C]) \otimes \mathbb{Z}/p[v].$$

where $|y_i| = 2$, $|v| = 2p$, $|C| = 2p - 2$ and $Cy_i = y_i^p$, $C^2 = y_1^{2p-2} + y_2^{2p-2} - y_1^{p-1} y_2^{p-1}$. In this paper we write $y_i^{p-1}$ by $Y_i$, and $v^{p-1}$ by $V$, eg $C^2 = Y_1^2 + Y_2^2 - Y_1 Y_2$. The Poincare series of the subalgebra generated by $y_i$ and $C$ are computed

$$1 - t^{p+1} + t^{p-1} - t^2 = \frac{(1 + \cdots + t^{p-1}) + t^{p-1}}{(1-t)(1-t)} = \frac{(1 + \cdots + t^{p-1})^2 - t^{2p-2}}{(1-t)^2}.$$ 

From this Poincare series and (3–1), we get the another expression of $H^*(BE)$

$$(3–2) \quad H^*(BE) \cong \mathbb{Z}/p[C, v] \{y_1^i y_2^j | 0 \leq i, j \leq p-1, (i, j) \neq (p-1, p-1)\}.$$
The $E$ conjugacy classes of $A$–subgroups are written by
\[ A_i = \langle c, ab^i \rangle \text{ for } 0 \leq i \leq p-1 \]
\[ A_\infty = \langle c, b \rangle. \]
Letting $H^*(BA_i) \cong \mathbb{Z}/p[y, u]$ and writing $i_{A_i}^*(x) = x|A_i$ for the inclusion $i_{A_i} : A_i \subset E$, the restriction images are given by
\[(3–3) \quad y_1|A_i = y \text{ for } i \in \mathbb{F}_p, \quad y_2|A_i = i y \text{ for } i \in \mathbb{F}_p, \quad y_1|A_\infty = y, \]
\[ C|A_i = y^{p-1}, \quad v|A_i = u^p - y^{p-1}u \text{ for all } i. \]
For an element $g = (\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}) \in GL_2(\mathbb{F}_p)$, we can identify $GL_2(\mathbb{F}_p) \cong \text{Out}(E)$ by
\[ g(a) = a^\alpha b^\gamma, \quad g(b) = a^\beta b^\delta, \quad g(c) = c^{\text{det}(g)}. \]
Then the action of $g$ on the cohomology is given (see Leary [6] and Tezuka–Yagita [11, page 491]) by
\[(3–4) \quad g^*C = C, \quad g^*y_1 = \alpha y_1 + \beta y_2, \quad g^*y_2 = \gamma y_1 + \delta y_2, \quad g^*v = (\text{det}(g))v. \]
Recall that $A$ is $F^{\text{ec}}$–radical if and only if $SL_2(\mathbb{F}_p) \subset W_G(A)$ (see Ruiz–Viruel [9, Lemma 4.1]).

**Theorem 3.1** (Tezuka–Yagita [11, Theorem 4.3], Broto–Levi–Oliver [1]) Let $G$ have the $p$–Sylow subgroup $E$, then we have the isomorphism
\[ H^*(BG) \cong H^*(BE)^{W_G(E)} \cap A; F^{\text{ec}}–\text{radical } i_{A}^{*–1} H^*(BA)^{W_G(A)}. \]
In [1] and [11], proofs of the above theorem are given only for $H^*(BG; \mathbb{Z}(p))$. A proof for $H^*(BG)$ is explained in Section 11.

### 4 Stable splitting

Martino–Priddy prove the following theorem of complete stable splitting.

**Theorem 4.1** (Martino–Priddy [7]) Let $G$ be a finite group with a $p$–Sylow subgroup $P$. The complete stable splitting of $BG$ is given by
\[ BG \sim \vee \text{rank } A(Q, M) X_M \]
where indecomposable summands $X_M$ range over isomorphic classes of simple $\mathbb{F}_p[\text{Out}(Q)]$–modules $M$ and over isomorphism classes of subgroups $Q \subset P$. 

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Remark  This theorem also holds for $p$–local finite groups over $P$, because all arguments for the proofs are done about the induced maps from some fusion systems of $P$ on stable homotopy types of related classifying spaces.

For the definition of rank $A(Q, M)$ see Martino and Priddy [7]. In particular, when $Q$ is not a subretract (that is not a proper retract of a subgroup) of $P$ (see [7, Definition 2]) and when $W_G(Q) \subset \text{Out}(Q) \cong GL_n(\mathbb{F}_p)$ (see [7, Corollary 4.4 and the proof of Corollary 4.6]), the rank of $A(Q, M)$ is computed by

$$\text{rank } A(Q, M) = \sum \dim_{\mathbb{F}_p}(\tilde{W}_G(Q_i)M),$$

where $\tilde{W}_G(Q_i) = \sum_{x \in W_G(Q_i)} x$ in $\mathbb{F}_p[GL_n(F_p)]$ and $Q_i$ ranges over representatives of $G$–conjugacy classes of subgroups isomorphic to $Q$.

Recall that $\text{Out}(E) \cong \text{Out}(A) \cong GL_2(\mathbb{F}_p)$. The simple modules of $G = GL_2(\mathbb{F}_p)$ are well known. Let us think of $A$ as the natural two-dimensional representation, and det the determinant representation of $G$. Then there are $p(p-1)$ simple $\mathbb{F}_p[G]$–modules given by $M_{q,k} = S(A)^q \otimes (\det)^k$ for $0 \leq q \leq p-1, 0 \leq k \leq p-2$. Harris and Kuhn [4] determined the stable splitting of abelian $p$–groups. In particular, they showed

**Theorem 4.2 (Harris–Kuhn [4])** Let $\bar{X}_{q,k} = X_{M_{q,k}}$ (resp. $L(1,k)$) identifying $M_{q,k}$ as an $\mathbb{F}_p[\text{Out}(A)]$–module (resp. $M_{0,k}$ as an $\mathbb{F}_p[\text{Out}(\mathbb{Z}/p)]$–module). There is the complete stable splitting

$$BA \sim \vee_{q,k} (q+1)\bar{X}_{q,k} \vee q \neq 0 (q+1)L(1,q),$$

where $0 \leq q \leq p-1, 0 \leq k \leq p-2$.

The summand $L(1, p - 1)$ is usually written by $L(1,0)$.

It is also known $H^+ (L(1,q)) \cong \mathbb{Z}/p[y^{p-1}][y^q]$. Since we have the isomorphism

$$H^{2q}(BA) \cong (\mathbb{Z}/p)^{q+1} \cong H^{2q}((q+1)L(1,q)), \text{ for } 1 \leq q \leq p-1,$$

we get $H^*(\bar{X}_{q,k}) \cong 0$ for $* \leq 2(p-1)$.

**Lemma 4.3** Let $H$ be a finite solvable group with $(p, |H|) = 1$ and $M$ be an $\mathbb{F}_p[H]$–module. Then we have $\tilde{H}(M) = (\sum_{x \in H} x)M \cong M^H \cong H^0(H; M)$.

**Proof** First assume $H = \mathbb{Z}/s$ and $x \in \mathbb{Z}/s$ its generator. Then

$$\tilde{H}(M) = (1 + x + \cdots + x^{s-1})H.$$
Thus the lemma is proved.

Thus we have \( Q \in \ker \). In particular

Corollary 4.4 Let \( G \) be a group such that \( \ker(1-x) \supset \ker(1-x) \). The facts that \( M \) is a \( \mathbb{Z}/p \)-module and \( (|H|, p) = 1 \) imply \( H^*(H; M) = 0 \) for \( * > 0 \). Hence

\[
\ker(1-x) / \ker(1-x) = H^1(H; M) = 0.
\]

Thus we have \( H(M) = \ker(1-x) = M^H \).

Suppose that \( H \) is a group such that

\[
0 \to H' \to H \xrightarrow{\pi} H'' \to 0
\]

and that \( H'(M') = (M')^H' \) (resp. \( H''(M'') = (M'')^H'' \)) for each \( \mathbb{Z}/p[H'] \)-module \( M' \) (resp. \( \mathbb{Z}/p[H''] \)-module \( M'' \)). Let \( \sigma \) be a (set theoretical) section of \( \pi \) and denote \( \sigma(H'') = \sum_{x \in H''} \sigma(x) \in \mathbb{F}_p[H] \). Then

\[
\tilde{H}(M) = \sigma(H'') \tilde{H}(M) = \sigma(H'')(M^H') = \tilde{H}''(M^H') = (M^H)^{H''} = M^H
\]

here the third equation follows from that we can identify \( M^H \) as an \( \mathbb{F}_p[H''] \)-module. Thus the lemma is proved.

It is known from a result of Suzuki [10, Chapter 3 Theorem 6.17] that any subgroup of \( SL_2(\mathbb{F}_{p^n}) \), whose order is prime to \( p \) is isomorphic to a subgroup of \( \mathbb{Z}/s \), \( 4S_4 \), \( SL_2(\mathbb{F}_3) \), \( SL_2(\mathbb{F}_5) \) or

\[
Q_{4n} = \langle x, y | x^n = y^2, y^{-1}xy = x^{-1} \rangle.
\]

Corollary 4.4 Let \( H \subset GL_2(\mathbb{F}_p) \) with \( (|H|, p) = 1 \) and \( H \) do not have a subgroup isomorphic to \( SL_2(\mathbb{F}_3) \) nor \( SL_2(\mathbb{F}_5) \). Let \( G = A : H \) and let us write \( BG \sim \vee_{q,k} \tilde{H}^q(H)_{q,k} X_{q,k} \vee_{q'} \tilde{m}(H)_{q,q'} L(1, q') \). Then

\[
\tilde{n}(H)_{q,k} = \text{rank}_p H^0(H; M_{q,k}),
\]

\[
\tilde{m}(H)_{q,q'} = \text{rank}_p H^{2q'}(BG).
\]

In particular \( \tilde{n}(H)_{q,0} = \text{rank}_p H^{2q}(BG) \).

Proof Since \( H^*(\tilde{X}_{q,k}) \cong 0 \) for \( * \leq 2(p - 1) \), it is immediate that \( \tilde{m}(H)_{q,q'} = \text{rank}_p H^{2q'}(G) \). Since \( GL_2(\mathbb{F}_p) \cong SL_2(\mathbb{F}_p), \mathbb{F}_p^* \) and \( \mathbb{F}_p^* \cong \mathbb{Z}/(p-1) \), each subgroup \( H \) in the above satisfies the condition in Lemma 4.3. The first equation is immediate from the lemma.

Next consider the stable splitting for the extraspecial \( p \)-group \( E \). Dietz and Priddy prove the following theorem.

Theorem 4.5 (Dietz–Priddy [2]) Let $X_{q,k} = X_{M_{q,k}}$ (resp. $L(2,k)$, $L(1,k)$) identifying $M_{q,k}$ as an $\mathbb{F}_p[\text{Out}(E)]$–module (resp. $M_{p-1,k}$ as an $\mathbb{F}_p[\text{Out}(A)]$–module, $\mathbb{F}_p[\text{Out}(\mathbb{Z}/p)]$–module). There is the complete stable splitting

$$BE \sim \vee_{q,k} (q + 1)X_{q,k} \forall (p + 1)L(2,k) \vee_{p \neq 0} (q + 1) L(1,q) \vee L(1,p - 1)$$

where $0 \leq q \leq p - 1$, $0 \leq k \leq p - 2$.

Remark Of course $\tilde{X}_{q,k}$ is different from $X_{q,k}$ but $\tilde{X}_{p-1,k} = L(2,k)$.

The number of $L(1,q)$ for $1 \leq q < p - 1$ is given by the following. Let us consider the decomposition $E/\langle c \rangle \cong \tilde{A}_i \oplus \tilde{A}_{-i}$ where $\tilde{A}_i = \langle ab^i \rangle$ and $\tilde{A}_{-i} = \tilde{A}_\infty$. We consider the projection $pr_i: E \rightarrow \tilde{A}_i$. Let $x \in H^1(B\tilde{A}_i; \mathbb{Z}/p) = \text{Hom}(\tilde{A}_i, \mathbb{Z}/p)$ be the dual of $ab^i$. Then

$$\text{pr}_i^* x(a) = x(pr_i(a)) = x(pr_i(ab^iab^{-i})^{1/2}) = x((ab^i)^{1/2}) = 1/2,$$

$$\text{pr}_i^* x(b) = x(pr_i(ab^i(ab^{-i} - 1)^{1/(2i)})) = 1/(2i).$$

Hence for $\beta(x) = y$, we have $pr_i^*(y) = 1/2y_1 + 1/(2i)y_2$. Therefore the $k + 1$ elements $(1/2y_1 + 1/(2i)y_2)^k$, $i = 0, \ldots, k$ form a base of $H^{2k}(E/\langle c \rangle; \mathbb{Z}/p) \cong (\mathbb{Z}/p)^{k+1}$ for $k < p - 1$. Thus we know the number of $L(1,k)$ is $k + 1$ for $0 < k < p - 1$.

Recall that

$$H^{2q}(BE) \cong \begin{cases} (\mathbb{Z}/p)^{q+1} \cong H^{2q}((q + 1)L(1,q)) & \text{for } 0 \leq 2 \leq p - 2 \\ (\mathbb{Z}/p)^q \cong H^{2p-2}((p + 1)L(1,0)) & \text{for } q = p - 1. \end{cases}$$

This shows $H^*(X_{q,k}) \cong 0$ for $* \leq 2p - 2$ since so is $L(2,k)$. The number $n(G)_{q,k}$ of $X_{q,k}$ is only depend on $W_G(E) = H$. Hence we have the following corollary.

Corollary 4.6 Let $G$ have the $p$–Sylow subgroup $E$ and $W_G(E) = H$. Let

$$BG \sim \vee n(G)_{q,k} X_{q,k} \vee m(G, 2)_k L(2,k) \vee m(G, 1)_k L(1,k).$$

Then $n(G)_{q,k} = \tilde{n}(H)_{q,k}$ and $m(G, 1)_k = \text{rank}_p H^{2k}(G)$.

Let $W_G(E) = H$. We also compute the dominant summand by the cohomology $H^*(BE)^H \cong H^*(B(E; H))$. Let us write the $\mathbb{Z}/p$–module

$$X_{q,k}(H) = S(A)^q \otimes v^k \cap H^*(B(E; H)) \quad \text{with} \quad S(A)^q = \mathbb{Z}/p\{y_1^q, y_1^{q-1}y_2, \ldots, y_2^q\}.$$

Since the module $\mathbb{Z}/p\{v^k\}$ is isomorphic to the $H$–module det$^k$, we have the following lemma.
Lemma 4.7  The number $n_{q,k}(G)\text{ of } X_{q,k}$ in $BG$ is given by rank$_p(X_{q,k}(W_G(E)))$.

Next problem is to seek $m(G, 2)_k$. The number $p + 1$ for the summand $L(2, k)$ in $BE$ is given as follows. For each $E$–conjugacy class of $A$–subgroup $A_i = \langle c, ab^i \rangle, i \in \mathbb{F}_p \cup \infty$, we see

$$W_E(A_i) = N_E(A_i)/A_i = E/A_i \cong \mathbb{Z}/p[b] \quad b^*: ab^i \mapsto ab^ic.$$ Let $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in $GL_2(\mathbb{F}_p)$ and $U = \{u\}$ the maximal unipotent subgroup. Then we can identify $W_E(A_i) \cong U$ by $b \mapsto u$. For $y_1^i y_2^j \in M_{q,k}$ (identifying $H^*(BA) \cong S^*(A) = \mathbb{Z}/p[y_1, y_2]$), we can compute

$$\bar{W}_E(A) y_1^i y_2^j = (1 + u + \cdots + u^{p-1}) y_1^i y_2^j = \sum_{i=0}^{p-1} (y_1 + iy_2)^i y_2^j \quad = \sum_i \sum_t (\binom{t}{i}) i^t y_1^{t-i} y_2^j = \sum_t (\binom{t}{i}) \sum_i i^t y_1^{t-i} y_2^{j+t}.$$ Here $\sum_{t=0}^{p-1} i^t = 0$ for $1 \leq t \leq p - 2$, and $= -1$ for $t = p - 1$. Hence we know

$$\text{dim}_p \bar{W}_G(A_i) M_{q,k} = \begin{cases} 0 & \text{for } 1 \leq q \leq p - 2 \\ 1 & \text{for } q = p - 1. \end{cases}$$ Thus we know that $BE$ has just one $L(2, k)$ for each $E$–conjugacy $A$–subgroup $A_i$.

Lemma 4.8  Let $A$ be an $F^{ec}$–radical subgroup, i.e. $W_G(A) \supset SL_2(\mathbb{F}_p)$. Then $\bar{W}_G(A)(M_{q,k}) = 0$ for all $k$ and $1 \leq q \leq p - 1$.

Proof  The group $SL_2(\mathbb{F}_p)$ is generated by $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $u' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We know Ker$(1-u) \cong \mathbb{Z}/p[y_1^p - y_1, y_2]$ and Ker$(1-u') \cong \mathbb{Z}/p[y_2^p - y_2, y_1]$. Hence we get Ker$(1-u) \cap$ Ker$(1-u') \neq 0$ for $0 \leq q \leq p - 1$.

Proposition 4.9  Let $G$ have the $p$–Sylow subgroup $E$. The number of $L(2, 0)$ in $BG$ is given by

$$m(G, 2)_0 = \#_G(A) - \#_G(F^{ec}A)$$

where $\#_G(A)$ (resp. $\#_G(F^{ec}A)$) is the number of $G$–conjugacy classes of $A$–subgroups (resp. $F^{ec}$–radical subgroups).

Proof  Let us write $K = E: W_G(E)$ and $H^*(BE)^W_G(E) = H^*(BK)$. From Theorem 3.1, we have

$$H^*(BG) \cong H^*(BE) \cap A; F^{ec}$–radical $i^{*}_{A}H^*(BA)^W_G(A).$$

Thus we get the lemma from $P$.

Let $A$ be an $A$–subgroup of $K$ and $x \in W_K(A)$. Recall $A = \langle c, ab^i \rangle$ for some $i$. Identifying $x$ as an element of $N_G(A) \subset E$: Out$(E)$ We see $x\langle c \rangle = \langle c \rangle$ from (3–4) and since $\langle c \rangle$ is the center of $E$. Hence

$$W_K(A) \subset B = U:\left(\mathbb{F}_p^*\right)^2$$
the Borel subgroup.

So we easily see that $\bar{W}_K(y_1^{p-1}) = \lambda y_2^{p-1}$ for some $\lambda \neq 0$ follows from $b^* y_i^{p-1} = y_i^{p-1}$ for $b = \text{diagonal} \in \left(\mathbb{F}_p\right)^{\times 2}$ and the arguments just before Lemma 4.8. We also see $\bar{W}_K(y_1^{p-1-i} y_2^i) = 0$ for $i > 0$. Hence we have $m(K, 0) = \#_K(A)$. From the isomorphism (4–1), we have $m(G, 2)_0 = \#_K(A) - \#_G(F^{\text{or}}A)$.

On the other hand $m(G, 2)_0 \leq \#_G(A) - \#_G(F^{\text{or}}A)$ from the above lemma. Since $\#_K(A) \geq \#_G(A)$, we see that $\#_K(A) = \#_G(A)$ and get the proposition. \hfill \Box

**Corollary 4.10** Let $G$ have the $p$–Sylow subgroup $E$. The number of $L(1, 0)$ in $BG$ is given by

$$m(G, 1)_{p-1} = \text{rank}_p H^{2(p-1)}(G) = \#_G(A) - \#_G(F^{\text{or}}A).$$

**Proof** Since $L(1, 0) = L(1, p - 1)$ is linked to $L(2, 0)$, we know $m(G, 1)_{p-1} = m(G, 2)_0$. \hfill \Box

**Lemma 4.11** Let $\xi \in \mathbb{F}_p^*$ be a primitive $(p - 1)$th root of 1 and $G \supset E: \langle \text{diag}(\xi, \xi) \rangle$. If $\xi^{3k} \neq 1$, then $BG$ does not contain the summand $L(2, k)$, ie $m(G, 2)_k = 0$.

**Proof** It is sufficient to prove the case $G = E: \langle \text{diag}(\xi, \xi) \rangle$. For $G = E: \langle \text{diag}(\xi, \xi) \rangle$. Recall $A_i = \langle c, ab^i \rangle$ and

$$\text{diag}(\xi, \xi): ab^i \mapsto (ab^i)^\xi, \; c \mapsto c^\xi.$$ 

So the Weyl group is $W_G(A_i) = U: \langle \text{diag}(\xi^2, \xi) \rangle$. For $v = \lambda y_1^{p-1} \cdots \in M_{q, k}$, we have

$$\bar{W}_G(A_i)v = \sum_{i=0}^{p-2} (\xi^{3i})^k \text{diag}(\xi^2, \xi^i)(1 + \cdots + u^{p-1})v = \sum_{i=0}^{p-2} \xi^{3ik} \lambda y_2^{p-1}. $$

Thus we get the lemma from $\sum_{i=0}^{p-2} \xi^{3ik} = 0$ for $3k \neq 0 \mod (p - 1)$ and $= -1$ otherwise. \hfill \Box
5 Cohomology and splitting of $B(\mathbb{Z}/3)^2$

In this section, we study the cohomology and stable splitting of $BG$ for $G$ having a 3–Sylow subgroup $(\mathbb{Z}/3)^2 = A$. In this and next sections, $p$ always means 3. Recall $\text{Out}(A) \cong G_2(\mathbb{F}_3)$ and $\text{Out}(A)'$ consists the semidihedral group

$$SD_{16} = \langle x, y | x^8 = y^2 = 1, yxy^{-1} = x^3 \rangle.$$

Every 3–local finite group $G$ over $A$ is of type $A:W$, $W \subset SD_{16}$. There is the $SD_{16}$–conjugacy classes of subgroups (here $B \leftarrow C$ means $B \supset C$)

$$SD_{16} \rightarrow Q_8 \leftarrow \mathbb{Z}/4 \leftarrow \mathbb{Z}/8 \leftarrow \mathbb{Z}/4 \leftarrow \mathbb{Z}/2 \leftarrow 0 \leftarrow D_8 \leftarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \leftarrow \mathbb{Z}/2.$$

We can take generators of subgroups in $GL_2(\mathbb{F}_3)$ by the matrices

$$\mathbb{Z}/8 = \langle l \rangle, Q_8 = \langle w, k \rangle, D_8 = \langle w', k \rangle, \mathbb{Z}/4 = \langle w \rangle, \mathbb{Z}/4 = \langle k \rangle, \mathbb{Z}/2 \oplus \mathbb{Z}/2 = \langle w', m \rangle, \mathbb{Z}/2 = \langle w \rangle, \mathbb{Z}/2 = \langle w \rangle,$$

where $l = \left( \begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array} \right)$, $w = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$, $k = l^2 = \left( \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right)$, $w' = w = \left( \begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array} \right)$ and $m = w^2 = k^2 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$. Here we note that $k$ and $w$ are $GL_2(\mathbb{F}_3)$–conjugate, in fact $uku^{-1} = w$. Hence we note that

$$H^* (B(A; \langle k \rangle)) \cong H^* (B(A; \langle w \rangle)).$$

The cohomology of $A$ is given $H^* (BA) \cong \mathbb{Z}/3[y_1, y_2]$, and the following are immediately

$$H^* (BA)^{(m)} \cong \mathbb{Z}/3[y_1^2, y_2^2][1, y_1 y_2] \quad H^* (BA)^{(w)} \cong \mathbb{Z}/3[y_1 + y_2, y_2^2].$$

Let us write $Y_i = y_i^2$ and $t = y_1 y_2$. The $k$–action is given $Y_1 \mapsto Y_1 + Y_2 + t$, $Y_2 \mapsto Y_1 + Y_2 - t$, $t \mapsto -Y_1 + Y_2$. So the following are invariant

$$a = -Y_1 + Y_2 + t, \quad a_1 = Y_1(Y_1 + Y_2 + t), \quad a_2 = Y_2(Y_1 + Y_2 - t), \quad b = t(Y_1 - Y_2).$$

Here we note that $a^2 = a_1 + a_2$ and $b^2 = a_1 a_2$. We can prove the invariant ring is

$$H^* (BA)^{(k)} \cong \mathbb{Z}/3[a_1, a_2][1, a, b, ab].$$

Next consider the invariant under $Q_8 = \langle w, k \rangle$. The action for $w$ is $a \mapsto -a$, $a_1 \mapsto a_2$, $b \mapsto b$. Hence we get

$$H^* (BA)^{Q_8} \cong \mathbb{Z}/3[1 + a_2, a_1 a_2][1, b][1, (a_1 - a_2)a].$$
Let us write \( S = \mathbb{Z}/3[a_1 + a_2, a_1 a_2] \) and \( a' = (a_1 - a_2)a \). The action for \( l \) is given \( l: Y_1 \mapsto Y_2 \mapsto Y_1 + Y_2 + t \mapsto Y_1 + Y_2 - t \mapsto Y_1 \). Hence \( l: a \mapsto -a, a_1 \mapsto a_2, b \mapsto -b \).

Therefore we get \( H^*(BA)^{(l)} \cong S\{1, a', ab, (a_1 - a_2)b\} \).

The action for \( w' : Y_1 \mapsto Y_1 + Y_2 + t, Y_2 \mapsto Y_2 \), implies that \( w' : a \mapsto a, a_1 \mapsto a_1, b \mapsto -b \). Then we can see

\[
H^*(BA)^{D_8} = H^*(BA)^{(k, w')} \cong \mathbb{Z}/3[a_1, a_2][1, a] \cong S\{1, a, a_1, a'\}.
\]

We also have

\[
H^*(BA)^{SD_{16}} \cong H^*(BA)^{Q_8} \cap H^*(BA)^{\mathbb{Z}/8} \cong S\{1, a'\}.
\]

Recall the Dickson algebra \( DA = \mathbb{Z}/3[\tilde{D}_1, \tilde{D}_2] \cong H^*(BA)^{GL_2(\mathbb{F}_3)} \) where \( \tilde{D}_1 = Y_1^3 + Y_2^2 Y_1 + Y_1^2 Y_2 + Y_3^2 = (a_2 - a_1)a = a' \) and \( \tilde{D}_2 = (y_3^2 y_2 - y_1 y_3^2)^2 = a_1 a_2 \).

Using \( a^2 = (a_1 + a_2) \) and \( \tilde{D}_1 = a^6 - a_1 a_2 a^2 \), we can write

\[
H^*(BA)^{SD_{16}} \cong \mathbb{Z}/3[a^2, \tilde{D}_2][1, \tilde{D}_1] \cong DA\{1, a^2, a'^4\}.
\]

**Theorem 5.1** Let \( G = (\mathbb{Z}/3)^2 \): \( H \) for \( H \subset SD_{16} \). Then \( BG \) has the stable splitting given by

\[
\begin{align*}
\hat{X}_0,0 & \leftarrow SD_{16} \\
\hat{X}_0,1 & \leftarrow Q_8 \\
\hat{X}_2,0 & \leftarrow 8 \\
\hat{X}_2,0 \vee \hat{X}_0,1 & \leftarrow L(1,0) \\
\hat{X}_2,0 \vee 2 \hat{X}_0,1 \vee 2 L(1,0) & \leftarrow 4 \\
2 \hat{X}_0,1 \vee 2 \hat{X}_1,1 \vee 2 L(1,1) & \leftarrow 0 \\
\hat{X}_2,0 \vee L(1,0) & \leftarrow P_8 \\
\hat{X}_2,0 \vee \hat{X}_0,1 \vee L(1,0) & \leftarrow 2 \oplus 2 \\
\hat{X}_1,0 \vee \hat{X}_0,1 \vee L(1,1) & \leftarrow 2 \vee 2 \\
\hat{X}_1 & \leftarrow H.
\end{align*}
\]

where \( \leftarrow \cdot \cdots \leftarrow H \) means \( B((\mathbb{Z}/3)^2; H) \sim \hat{X}_1 \vee \cdots \vee \hat{X}_s \).

For example

\[
B(E; SD_{16}) \sim \hat{X}_0,0, \quad B(E; Q_8) \sim \hat{X}_0,0 \vee \hat{X}_0,1, \quad B(E; \mathbb{Z}/8) \sim \hat{X}_0,0 \vee \hat{X}_2,1.
\]

Main parts of the above splittings are given by the author in [14, (6)] by direct computations of \( \tilde{W}_G(A) \) (see [14, page 149]). However we get the theorem more easily by using cohomology here. For example, let us consider the case \( G = A: \langle k \rangle \). The cohomology

\[
H^0(BG) \cong \mathbb{Z}/3, \quad H^2(BG) \cong 0, \quad H^4(BG) \cong \mathbb{Z}/3
\]

implies that \( BG \) contains just one \( \hat{X}_0,0, \hat{X}_2,0, L(1,0) \) but does not \( \hat{X}_1,0, L(1,1) \). Since \( \det(k) = 1 \), we also know that \( \hat{X}_0,1, \hat{X}_2,1 \) are contained. So we can see

\[
B(A; \mathbb{Z}/4) \sim \hat{X}_0,0 \vee \hat{X}_0,1 \vee \hat{X}_2,0 \vee \hat{X}_2,1 \vee L(1,0).
\]
Next consider the case $G' = A: \langle l \rangle$. The fact $H^4(G) \cong 0$ implies that $BG'$ does not contain $\tilde{x}_{0,0}, L(1,0)$. The determinant $\text{det}(l) = -1$, and $l: a \mapsto -a$ shows that $BG'$ contains $\tilde{x}_{2,1}$ but does not contain $\tilde{x}_{0,1}$. Hence we know $BG' \sim \tilde{x}_{0,0} \vee \tilde{x}_{2,1}$. Moreover we know $BA: SD_{16} \sim \tilde{x}_{0,0}$ since $w: a \mapsto -a$ but $\text{det}(w) = 1$. Thus we have the graph
\[
\tilde{x}_{0,0} \leftarrow SD_{16} \overset{\mathbb{Z}/8}{\longrightarrow} \tilde{x}_{2,0} \vee \tilde{x}_{2,1} \quad \mathbb{Z}/4.
\]
Similarly we get the other parts of the above graph.

**Corollary 5.2** Let $S = \mathbb{Z}/3[a_1 + a_2, a_1a_2]$. Then we have the isomorphisms
\[
H^*(\tilde{x}_{0,0}) \cong S\{1, \tilde{D}_1\}
\]
\[
H^*(\tilde{x}_{0,1}) \cong S\{b, \tilde{D}_1b\}
\]
\[
H^*(\tilde{x}_{2,1}) \cong S\{ab, (a_1 - a_2)b\}
\]
\[
H^*(\tilde{x}_{2,0} \vee L(1,0)) \cong S\{a, a_1 - a_2\} \cong DA\{a, a^2, a^3\}.
\]

Here we write down the decomposition of cohomology for a typical case
\[
H^*(BA)^{(k)} \cong S\{1, a_1 - a_2\}\{1, a\}\{1, b\}
\]
\[
\cong S\{1, a(a_1 - a_2), b, ba(a_1 - a_2), ab, (a_1 - a_2)ba, (a_1 - a_2)\}
\]
\[
\cong H^*(\tilde{x}_{0,0}) \oplus H^*(\tilde{x}_{0,1}) \oplus H^*(\tilde{x}_{2,1}) \oplus H^*(\tilde{x}_{2,0} \vee L(1,0)).
\]

### 6 Cohomology and splitting of $B3^1+2$.

In this section we study the cohomology and stable splitting of $BG$ for $G$ having a $3$–Sylow subgroup $E = 3^1+2$. In the splitting for $BE$, the summands $X_{q,k}$ are called dominant summands. Moreover the summands $L(2,0) \vee L(1,0)$ is usually written by $M(2)$.

**Lemma 6.1** If $G \supset E: \langle \text{diag}(-1,-1) \rangle$ identifying $\text{Out}(E) \cong GL_2(\mathbb{F}_3)$ and $G$ has $E$ as a $3$–Sylow subgroup, then
\[
BG \sim (\text{dominant summands}) \vee (\phi_G(A) - \phi_G(F^{\text{ec}} A)(M(2))).
\]

**Proof** From Lemma 4.11, we know $m(G,2)_1 = 0$ ie $L(2,1)$ is not contained. The summand $L(1,1)$ is also not contained, since $H^2(BE)^{\langle \text{diag}(-1,-1) \rangle} \cong 0$. The lemma is almost immediately from Proposition 4.9 and Corollary 4.10. \(\square\)
Thus we can see the graph for the place \( H \). Thus we show all groups except for \( H \) means \( BG \) is homotopic to the classifying space of one of the following groups. Moreover the stable splitting is given by the graph so that \( X_1 \times \cdots \times X_i \) means \( EH = E; H \) for \( H \in SD_{16} \)

\[
\begin{align*}
X_{0,0} & \leftarrow j_4 \\
X_{0,1} & \leftarrow EQ_8 \\
X_{2,1} & \leftarrow EZ/8 & X_{2,0} \vee X_{0,1} & \leftarrow M(2) & X_{2,0} \vee X_{2,1} \vee M(2) & \leftarrow EZ/2 \\
X_{2,0} M(2) & \leftarrow ED_8 & X_{2,0} \vee X_{2,1} \vee M(2) & \leftarrow 4L(2,1) \vee 2L(1,1) & X_{1,0} \vee X_{1,1} \vee \leftarrow EZ/2 \\
2F_4(2)^{\vee} & \leftarrow M(2) & X_{2,0} \vee X_{1,0} \vee X_{1,1} \vee & \leftarrow GL_2(F_3) & X_{1,0} \vee X_{1,1} \vee X_{1,2} \vee & \leftarrow 2L(1,1) \vee L(1,1) \\
M(2) & \leftarrow 2M_4 & M_{12} & \leftarrow \mathbb{F}^2_3: GL_2(F_3) \\
M(2) & \leftarrow \mathbb{F}^2_3: SL_2(F_3) &
\end{align*}
\]

**Proof** All groups except for \( E; E: \langle w' \rangle \) and \( \mathbb{F}^2_3: SL_2(F_3) \) contain \( E: \langle \text{diag}(-1, -1) \rangle \). Hence we get the theorem from Corollary 4.4, Theorem 5.1 and Lemma 6.1, except for the place for \( H^*(BE: \langle w' \rangle) \) and \( H^*(\mathbb{F}^2_3: SL_2(F_3)) \).

Let \( G = E: \langle w' \rangle \). Note \( w': y_1 \mapsto y_1 - y_2, y_2 \mapsto -y_2, v \mapsto -v \). Hence \( H^2(G) \cong \mathbb{Z}/3 \{y_1 + y_2\} \). So \( BG \) contains one \( L(1, 1) \). Next consider the number of \( L(2, 0) \) and \( L(2, 1) \). The \( G \)–conjugacy classes of \( A \)–subgroups are \( A_0, A_2, A_1 \sim A_\infty \). The Weyl groups are

\[
W_G(A_\infty) \cong U, \quad W_G(A_2) \cong U: \langle \text{diag}(-1, -1) \rangle, \quad W_G(A_0) \cong U: \langle \text{diag}(-1, 1) \rangle,
\]

eg \( N_G(A_0) / A_0 \) is generated by \( b, w' \) which is represented by \( u, \text{diag}(-1, 1) \) respectively. By the arguments similar to the proof of Lemma 4.11, we have that

\[
\begin{align*}
\dim(\overline{W}_G(A_i), M_{2,0}) = 1 & \text{ for all } i \\
\dim(\overline{W}_G(A_i), M_{2,1}) = 1, 1, 0 & \text{ for } i = \infty, 2, 0 \text{ respectively}.
\end{align*}
\]

Thus we show \( BG \supset 3L(2,0) \vee 2L(2,1) \) and we get the graph for \( G = E: \langle w' \rangle \).

For the place \( G = \mathbb{F}^2_3: SL_2(F_3) \), we see \( W_G(A_\infty) \cong SL_2(F_3) \). We also have

\[
\begin{align*}
\dim(\overline{W}_G(A_i), M_{2,0}) = 0, 1, 1 & \text{ for } i = \infty, 2, 0 \text{ respectively} \\
\dim(\overline{W}_G(A_i), M_{2,1}) = 0, 1, 0 & \text{ for } i = \infty, 2, 0 \text{ respectively}.
\end{align*}
\]

Thus we can see the graph for the place \( H^*(\mathbb{F}^2_3: SL_2(F_3)) \).

**Remark** From Tezuka–Yagita [11], Yagita [13] and Theorem 2.1, we have the following homotopy equivalences (localized at 3).

\[
BJ_4 \cong BRu, \quad BM_{24} \cong BHe, \quad BM_{12} \cong BGL_3(F_3)
\]
We write down the cohomologies explicitly (see also Tezuka–Yagita [11] and Yagita [14]). First we compute $H^*(B(E; H))$. The following cohomologies are easily computed

\[ H^*(BE)^{(m)} \cong \mathbb{Z}/3[C, v][1, y_1y_2, Y_1, Y_2], \quad H^*(BE)^{(w)} \cong \mathbb{Z}/3[C, v][1, Y_1 + Y_2]. \]

\[ H^*(BE)^{(k)} \cong \mathbb{Z}/3[C, v][1, a] \text{ where } a = -Y_1 + Y_2 + y_1y_2, \quad C^2 = a^2. \]

Recall that $V = v^{p-1}$ and $C$ multiplicatively generate $H^*(BE)^{\text{Out}(E)}$. Let us write

\[ CA = \mathbb{Z}/p[C, V] \cong H^*(BE)^{\text{Out}(E)}. \]

Then we have

\[ H^*(BE)^{(w')} \cong CA[1, y'_1, Y'_1, Y'_2, Y_2, y_2, y'_2, v, y'_1y_2v, Y'_1y'_2v] \text{ with } y'_1 = y_1 + y_2 \]

\[ H^*(BE)^{(w', m)} \cong CA[1, a', Y_2] \text{ where } a' = (t + Y_2)v = y'_1y_2v. \]

We can compute

\[ H^*(BE)^{D_8} \cong H^*(BE)^{(k)} \cap H^*(BE)^{(w)} \cong \mathbb{Z}/3[C, v] \cong CA[1, v], \]

\[ H^*(BE)^{D_8} \cong CA[1, a], \quad H^*(BE)^{(l)} \cong CA[1, av]. \]

Hence we have $H^*(BE)^{SD_{16}} \cong CA$.

Let $D_1 = C^p + V$ and $D_2 = CV$. Then it is known that

\[ D_1|A_i = \tilde{D}_1, \quad D_2|A_i = \tilde{D}_2 \text{ for all } i \in \mathbb{F}_p \cup \infty. \]

So we also write $DA \cong \mathbb{Z}/p[D_1, D_2]$. Since $CD_1 - D_2 = C^{p+1}$, we can write $CA \cong DA[1, C, C^2, \ldots, C^p]$.

Now return to the case $p = 3$ and we get (see [11])

\[ H^*(BJ_4) \cong H^*(BE)^{SD_{16}} \cap i^{*}_0^{-1} H^*(BA_0)^{GL_2(F_3)} \cong DA. \]

**Proposition 6.3** There are isomorphisms for $|a''| = 4$,

\[ H^*(\tilde{F}_4(2)) \cong DA[1, (D_1 - C^3)a''], \quad H^*(M_{24}) \cong DA \oplus CA[a'']. \]

**Proof** Let $G = M_{24}$. Then $G$ has just two $G$–conjugacy classes of $A$–subgroups

\[ \{A_0, A_2\}, \quad \{A_1, A_3\}. \]
It is known that one is $F^{ec}$--radical and the other is not. Suppose that $A_0$ is $F^{ec}$--radical. Then $W_G(A_0) \cong GL_2(F_3)$. Let $a'' = a + C$. Then

$$a''|A_0 = (-Y_1 + Y_2 + y_1y_2 + C)|A_0 = 0, \quad a''|A_\infty = -Y.$$ 

By Theorem 3.1

$$H^*(BM_{24}) \cong H^*(BE)^{D_8} \cap i_{A_0}^* H^*(BA_0)W_G(A_0),$$

we get the isomorphism for $M_{24}$. When $A_\infty$ is a $F^{ec}$--radical, we take $a'' = a - C$. Then we get the same result.

For $G =^2 F_4(2)'$, the both conjugacy classes are $F^{ec}$--subgroups and $W_G(A_\infty) \cong GL_2(F_3)$. Hence (for case $a'' = a + C$)

$$H^*(B^2 F_4(2)') \cong H^*(BM_{24}) \cap i_{A_\infty}^* H^*(BA_\infty)GL_2(F_3).$$

We know

$$(D_1 - C^3)a''|A_0 = 0, \quad (D_1 - C^3)a''|A_\infty = -VV = -\tilde{D}_2.$$ 

Thus we get the cohomology of $^2 F_4(2)'$. 

\[ \square \]

**Remark** In [11; 14], we take

$$\langle \mathbb{Z}/2 \rangle^2 = \langle \text{diag}(\pm 1, \pm 1) \rangle, \quad D_8 = \langle \text{diag}(\pm 1, \pm 1), w \rangle.$$ 

For this case, the $M_{24}$--conjugacy classes of $A$--subgroups are $A_0 \sim A_\infty$, $A_1 \sim A_2$, and we can take $a'' = C - Y_1 - Y_2$. The expressions of $H^*(M_{12})$, $H^*(A:GL_2(F_3))$ become more simple (see [11; 14]), in fact,

$$H^*(B^2 F_4(2)') \cong DA\{1, (Y_1 + Y_2)V\}.$$ 

**Remark** [11, Corollary 6.3] and [14, Corollary 3.7] were not correct. This followed from an error in [11, Theorem 6.1]. This theorem is only correct with adding the assumption that there are exactly two $G$ conjugacy classes of $A$--subgroups such that one is $p$--pure and the other is not. This assumption is always satisfied for sporadic simple groups but not for $^2 F_4(2)'$.

**Corollary 6.4** There are isomorphisms of cohomologies

$$H^*(X_{2,0}) \cong DA\{D_2\}, \quad H^*(X_{2,1}) \cong CA\{av\} \text{ where } (av)^2 = CD_2,$$

$$H^*(X_{0,1}) \cong CA\{v\}, \quad H^*(M(2)) \cong DA\{C, C^2, C^3\} \text{ where } C^4 = CD_1 - D_2.$$
Here we write down typical examples. First recall

\[ CA \cong DA\{1, C, C^2, C^3\} \cong H^*(X_{0,0}) \oplus H^*(M(2)) \]
\[ CA\{C\} \cong DA\{C, C^2, C^3, D_2\} \cong H^*(M(2)) \oplus H^*(X_{2,0}). \]

Thus the decomposition for \( H^*(BE)^D \) gives the isomorphisms

\[ CA\{1, a''\} \cong CA\{1, C\} \cong H^*(X_{0,0}) \oplus H^*(M(2)) \oplus H^*(X_{2,0}) \oplus H^*(M(2)). \]

Similarly the decomposition for \( H^*(BE)^{\langle k \rangle} \) gives the isomorphism

\[ CA\{1, a, v, av\} \cong H^*(BE)^D \oplus H^*(X_{0,1}) \oplus H^*(X_{2,1}). \]

We recall here Lemma 4.7 and the module

\[ X_{q,k}(\langle k \rangle) = S(V)^q \otimes v^k \cap H^*(B(E: \langle k \rangle)). \]

Then it is easily seen that

\[ X_{0,0}(\langle k \rangle) = \{1\}, X_{2,0}(\langle k \rangle) = \{a\}, X_{0,1}(\langle k \rangle) = \{v\}, X_{2,1}(\langle k \rangle) = \{av\}. \]

Hence we also see \( B(E: \langle k \rangle) \) has the dominant summands \( X_{0,0} \lor X_{2,0} \lor X_{0,1} \lor X_{2,1} \). Moreover it has no dominant summands \( 2M(2) \) since \( H^4(B(E: \langle k \rangle)) \cong \mathbb{Z}/3\{C, a\} \). Thus we can give another proof of Theorem 6.2 from Lemma 4.7 and the cohomologies \( H^*(BG) \).

7 Cohomology for \( B\, 7^{1+2}_+ \). I.

In this section, we assume \( p = 7 \) and \( E = 7^{1+2}_+ \). We are interested in groups \( O'N, O'N:2, He, He:2, F_{i_2}^1, F_{i_2}^2 \) and three exotic 7-local groups. Denote them by \( RV_1, RV_2, RV_3 \) according the numbering in [9]. We have the diagram from Ruiz and Viruel

\[
\begin{align*}
3SD_2 & \quad SL_2(\ell^7):2 \quad 3SD_16 \quad SL_2(\ell^7):2, SL_2(\ell^7):2 \quad 3SD_16 \quad SL_2(\ell^7):2, SL_2(\ell^7):2 \quad 3SD_16 \quad SL_2(\ell^7):2, SL_2(\ell^7):2 \quad 3SD_16 \quad SL_2(\ell^7):2, SL_2(\ell^7):2 \\
RV_3 & \quad RV_2 \quad RV_2 \quad RV_2 \quad RV_2 \quad RV_2 \quad RV_2 \quad RV_2 \quad RV_2 \\
O'N & \quad O'N:2 \quad O'N:2 \quad O'N:2 \quad O'N:2 \quad O'N:2 \quad O'N:2 \quad O'N:2 \quad O'N:2 \\
F_{i_2}^1 & \quad F_{i_2}^1 \quad F_{i_2}^1 \quad F_{i_2}^1 \quad F_{i_2}^1 \quad F_{i_2}^1 \quad F_{i_2}^1 \quad F_{i_2}^1 \quad F_{i_2}^1 \\
6_2 & \quad 6_2 \quad 6_2 \quad 6_2 \quad 6_2 \quad 6_2 \quad 6_2 \quad 6_2 \quad 6_2 \\
RV_1 & \quad RV_1 \quad RV_1 \quad RV_1 \quad RV_1 \quad RV_1 \quad RV_1 \quad RV_1 \quad RV_1 \\
F_{i_2} & \quad F_{i_2} \quad F_{i_2} \quad F_{i_2} \quad F_{i_2} \quad F_{i_2} \quad F_{i_2} \quad F_{i_2} \quad F_{i_2} \\
He & \quad He:2 \quad He:2 \quad He:2 \quad He:2 \quad He:2 \quad He:2 \quad He:2 \quad He:2 \\
H & \quad W_{1,...,W_2} \quad W_{1,...,W_2} \quad W_{1,...,W_2} \quad W_{1,...,W_2} \quad W_{1,...,W_2} \quad W_{1,...,W_2} \quad W_{1,...,W_2} \quad W_{1,...,W_2}
\end{align*}
\]

Here \( H \xleftarrow{W_1,...,W_2} G \) means \( W_G(E) \cong H, W_i = W_G(A_i) \) for \( G \)-conjugacy classes of \( F^{\text{rec}} A \)- subgroups \( A_i \).

In this section, we study the cohomology of \( O'N, RV_2, RV_3 \). First we study the cohomology of \( G = O'N \). The multiplicative generators of \( H^*(BE)^3D_2 \) are still studied in [11, Lemma 7.10]. We will study more detailed cohomology structures here.
Lemma 7.1 There is the CA–module isomorphism

\[ H^*(BE)^{3D_8} \cong CA\{1, a, a^2, a^3/V, a^4/V, a^5/V, b, ab/V, a^2b/V, d, ad, a^2d\}, \]

where \( a = (y_1^2 + y_2^2)v^2, b = y_1^3y_2^2v^4 \) and \( d = (y_1y_2^3 - y_1^3y_2)v \).

Proof The group \( 3D_8 \subset GL_2(\mathbb{F}_7) \) is generated by \( \text{diag}(-1, 1), (2, 2) \) and \( w = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \).

If \( y_1^iv_2^jv^k \) is invariant under \( \text{diag}(-1, 1), \text{diag}(1, -1) \) and \( \text{diag}(2, 2) \), then \( i = j = k \mod(2) \) and \( i + j + 2k = 0 \mod(3) \).

When \( i, j, k \leq 6, k \leq 5 \) but \( i, j \neq (6, 6) \), the invariant monomials have the following terms, \( y_1^2v^2, y_1^4v^4, y_1^6v_1^2y_2v^4, y_1^4y_2^4v^2, y_1y_2v^5, y_1^3y_2^3v^3, y_1^5y_2^5v, y_1^3y_2^5v, y_1^2y_2^2v^2, y_1^4y_2^6v^4, y_1y_2^3v, y_1y_2^5v^3, y_1^3y_2^5v^5 \) and terms obtained by exchanging \( y_1 \) and \( y_2 \). Recall that \( w: y_1 \mapsto y_2, y_2 \mapsto -y_1 \) and \( v \mapsto v \).

From the expression of (3–2), we have

\[ H^*(BE)^{3D_8} \cong CA\{1, a, a^2, a^4, b, b^2, e, c, c^", d, ad, bd\} \]

where \( a = (y_1^2 + y_2^2)v^2, a^2 = y_1^6 + y_2^6, b = y_1^7y_2^2v^4, b^2 = y_1^3y_2^6v^2, c = (y_1^7y_2^4 + y_1^4y_2^2), \)

\( c^" = (y_1^7y_2^4 + y_1^4y_2^2)v^2, \)

\( a^4/V = (y_1^2 + y_2^2)^4v^2 = (y_1^6 + y_2^6 + 3y_1^2y_2^2(y_1^2 + y_2^2) = a^4 + 3ab \)

\( a^5/V = (y_1^{10} + y_2^{10}) + 5y_1^3y_2^2(y_1^4 + y_2^4) + 10y_1^{14}y_2^4(y_1^2 + y_2^2)v^4 \)

\( = c^"C + 10bC + 10c^". \)

Hence, we can take generators \( a^4/V, a^5/V, ab/V, a^2b/V \) for \( b^", c^", c, c^" \) respectively, and get the lemma. \( \square \)

Note that the computations shows

\[ a^6 = (y_1^2 + y_2^2)^6v^2 = (y_1^{12} - y_1y_2^2 + y_1^8y_2^4 - y_1^6y_2^6 + y_1^4y_2^8 - y_1^2y_2^{10} + y_2^{12})V^2 \]

\( = (y_1^{12} - y_1y_2^2 + y_2^{12})V^2 = C^2V^2 = D_2^2, \)

where we use the fact \( y_1^7y_2 - y_1y_2^7 = 0 \).

Lemma 7.2 \( H^*(BE)^{3SD_16} \cong CA\{1, a, a^2, a^3/V, a^4/V, a^5/V}\).
Proof Take the matrix \(k' = \left( \begin{array}{cc} 1 & 1 \\ -1 & 0 \end{array} \right) \) such that \(\langle 3D_8, k' \rangle \cong 3SD_{16} \). Then we have
\[
k'^*: a = (y_1^2 + y_2^2)v^2 \mapsto ((-y_1 + y_2)^2 + (-y_1 - y_2)^2)(2v)^2 = a,
\]
\[
b = y_1^2y_2^2v^4 \mapsto (y_1^2 - y_2^2)^2(2v)^4 = 2(a^2 - 4b) = 2a^2 - b.
\]
(If we take \(\tilde{b} = b - a^2\), then \(k'^*: \tilde{b} \mapsto -\tilde{b}\) ) Similarly we can compute \(k': d \mapsto -d\).
Then the lemma is almost immediate from the preceding lemma. \(\Box\)

Lemma 7.3 \(H^*(BE)^3SD_{32} \cong CA\{1, a^2, a^4/V\}\).

Proof Take the matrix \(l' = \left( \begin{array}{cc} 1 & r \\ -1 & 0 \end{array} \right) \) so that \(l'^2 = k'\) and \(\langle 3SD_8, l' \rangle \cong 3SD_{32}\). We see that
\[
l'^*: a = (y_1^2 + y_2^2)v^2 \mapsto ((-y_1 + 3y_2)^2 + (-y_1 - y_2)^2)(3v)^2 = -a,
\]
which shows the lemma. \(\Box\)

Theorem 7.4 There is the isomorphism with \(C' = C - a^3/V\)
\[
H^*(BO'N) \cong DA\{1, a, a^2, b, ab, a^2b\} \oplus CA\{d, ad, a^2d, C', C'a, C'a^2\}
\]

Proof Let \(G = O'N\). The orbits of \(N_G(E)\)--action of \(A\)--subgroups in \(E\) are given by \(\{A_0, A_\infty\}, \{A_1, A_6\}\) and \(\{A_2, A_3, A_4, A_5\}\). From Ruiz and Viruel [9], \(A_0, A_\infty, A_1\) and \(A_6\) are \(F\)--radical subgroups. Hence we know that
\[
H^*(O'N) \cong H^*(BE)^3D_8 \cap i_{A_0}^{-1}H^*(BA_0)^{SL_2(\mathbb{F}_2)} \cap i_{A_1}^{-1}H^*(BA_1)^{SL_2(\mathbb{F}_2)^2}.
\]
For element \(x = d\) or \(x = C'\), the restrictions are \(x|A_0 = x|A_1 = 0\). Hence we see that \(CA\{x\}\) are contained in \(H^*(BG)\). We can take \(C', C'a, C'a^2\) instead of \(a^3/V, a^4/V\) and \(a^5/V\) as the \(CA\)--module generators since \(a^3/V = (C - C')\). Moreover we know \(CA\{C', C'a, C'a^2\} \subset H^*(BG)\).

It is known that \(\mathbb{Z}/p[y, u]^{SL_p(\mathbb{F}_2)} \cong \mathbb{Z}/p[\tilde{D}_1, \tilde{D}_2]\) where \(\tilde{D}_2' = y_1u^p - y_1^p\) and \((\tilde{D}_2')^{p-1} = \tilde{D}_2\). Hence we know \(\mathbb{Z}/\mathbb{Z}[y, u]^{SL_2(\mathbb{F}_2)^2} \cong \mathbb{Z}/\mathbb{Z}[\tilde{D}_1, (\tilde{D}_2)^2]\).

Since \(y_1v|A = \tilde{D}_2\) we see \(a|A_0 = (\tilde{D}_2)^2, a|A_1 = 2(\tilde{D}_2)^2\). Hence \(a, a^2\) are in \(H^*(BG)\). The fact \(b|A_0 = 0\) and \(b|A_1 = (\tilde{D}_2)^4\), implies that \(b \in H^*(BG)\). Hence all \(a^k b^j\) are also in \(H^*(BG)\). \(\Box\)

Next we consider the group \(G = O'N:2\). Its Weyl group \(W_G(E)\) is isomorphic to \(3SD_{16}\). So we have \(H^*(B(O'N:2)) \cong H^*(BO'N) \cap H^*(BE)^3SD_{16}\).

Corollary 7.5 \(H^*(B(O'N:2)) \cong (DA\{1, a, a^2\} \oplus CA\{C', C'a, C'a^2\})\).
Corollary 7.6 \( H^*(BRV_2) \cong DA\{1, a, a^2, a^3, a^4, a^5\} \).

**Proof** Let \( G = RV_2 \). Since \( A_2 \) is also \( F^{cc} \)-radical and \( W_G(A_2) = SL_2(\mathbb{F}_7) : 2 \). Hence we have
\[
H^*(BG) \cong H^*(BE)^{3SD_{16}} \cap i_{A_2}^{* -1} H^*(BA_2)^{SL_2(\mathbb{F}_7) : 2}.
\]
Hence we have the corollary of the theorem. \( \square \)

Since \( H^*(BRV_3) \cong H^*(BE)^{3SD_{32}} \cap H^*(BRV_2) \), we have the following corollary.

**Corollary 7.7** \( H^*(BRV_3) \cong DA\{1, a^2, a^4\} \).

Corollary 7.7 can also be proved in the following way.

**Proof** Let \( G = RV_3 \). Since there is just one \( G \)-conjugacy class of \( A \)-subgroups, by Quillen’s theorem [8], we know
\[
H^*(BRV_3) \subset H^*(BA_0)^{SL_2(\mathbb{F}_7) : 2} \cong DA\{1, (\tilde{D}_2)^2, (\tilde{D}_2)^4\} \text{ with } (\tilde{D}_2)^6 = \tilde{D}_2.
\]
Note that \( a^2 | A_0 = (\tilde{D}_2)^4, a^4 | A_0 = (\tilde{D}_2)^2 \tilde{D}_2 \) and \( D_2 | A_0 = \tilde{D}_2 \). The fact \( k^* : a \mapsto -a \) implies that \( DA\{a^2, a^4\} \subset H^*(BG) \) but \( DA\{a, a^3, a^5\} \cap H^*(BG) = 0 \). \( \square \)

Corollary 7.6 can also be proved in the following way.

**Proof** Let \( G = RV_2 \). Since there is just two \( G \)-conjugacy classes of \( A \)-subgroups, by Quillen’s theorem [8], we know
\[
H^*(BRV_2) \subset H^*(BA_0)^{SL_2(\mathbb{F}_7) : 2} \times H^*(BA_2)^{SL_2(\mathbb{F}_7) : 2}
\]
Since \( a \in H^*(BRV_2) \), the map \( i_0^* : H^*(BRV_2) \to H^*(BA_0)^{SL_2(\mathbb{F}_7) : 2} \) is epimorphism. Take \( b' = b^2 - 2a^2b \) so that \( b'| A_0 = b'| A_1 = 0 \). Hence
\[
\text{Ker } i_0^* \supset DA\{b', b'a, C'V\}.
\]
Moreover \( b'| A_2 = (\tilde{D}_2)^2 \tilde{D}_2, b'a| A_2 = (\tilde{D}_2)^4 \tilde{D}_2, c'V| A_2 = (\tilde{D}_2) \). Since \( (\tilde{D}_2)^2 \) itself is not in the image of \( i_{A_2}^* \), we get the isomorphism
\[
H^*(BRV_2) \cong DA\{1, a, a^2\} \oplus DA\{c'V, b', b'a\}.
\] \( \square \)
8  Cohomology for $B^7 + 2$ II

In this section, we study cohomology of $He, Fi_{24}, RV_1$.

First we consider the group $G = He$. The multiplicative generators of $H^*(He)$ are still computed by Leary [5]. We will study more detailed cohomology structures here. The Weyl group is $W_G(He) \cong S_3$.

**Lemma 8.1** The invariant $H^*(BE)^{S_3}$ is isomorphic to

$$CA \otimes \mathbb{Z}/7\{1, \overline{a}, \overline{b}^2, y \} \oplus \mathbb{Z}/7\{\overline{d}\} \{1, \overline{a}, \overline{b}, \overline{b}^2, v / V \} \oplus \mathbb{Z}/7\{\overline{a}^2\}.$$

where $\overline{a} = (y_1^3 + y_2^3)$, $\overline{b} = y_1 y_2 v^2$ and $\overline{d} = (y_1^3 - y_2^3)v^3$.

**Proof** The group $3S_3 \subset GL_2(\mathbb{F}_7)$ is generated by $T' = \{\text{diag}(\lambda, \mu) | \lambda^3 = \mu^3 = 1\}$ and $w' = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. If $y_1^i y_2^j v^k$ is invariant under $T'$, then $i = j = -k \mod(3)$. When $j \leq 6, k \leq 5$ but $(i, j) \neq (6, 6)$, the invariant monomials have the following terms

$$\{1, \overline{c} = \overline{b}^3 / V = y_1^3 y_2^3, 1, v \} \{1, \overline{b} = y_1 y_2 v^2, \overline{d} = y_1^3 y_2^3 v \}$$

and terms obtained by exchanging $y_1$ and $y_2$. Recall that $w'$: $y_1 \mapsto y_2, y_2 \mapsto y_1$ and $v \mapsto -v$. The following elements are invariant

$$\begin{align*}
\overline{a} \overline{b} &= (y_1^3 y_2^4 + y_1^4 y_2) v^2, \\
\overline{a} \overline{b}^2 &= (y_1^3 y_2^5 + y_1^4 y_2^3) v^4, \\
\overline{a} \overline{c} &= (y_1^3 y_2^6 + y_1^6 y_2^3), \\
\overline{b} \overline{d} &= (y_1^3 y_2^4 - y_1^4 y_2) v^5, \\
\overline{b} \overline{d}^2 / V &= (y_1^3 y_2^5 - y_1^4 y_2^3) v, \\
\overline{c} \overline{d} &= (y_1^3 y_2^6 - y_1^6 y_2^3) v^3 \\
\overline{a} \overline{d} &= (y_1^3 y_2 - y_1^6 y_2^3), \\
\overline{a} \overline{d}^2 / V &= y_1^3 y_2^6 + y_1^6 y_2^3.
\end{align*}$$

Thus we get the lemma from (3–2). □

**Lemma 8.2** $H^*(BE)^{S_3} \cong CA \otimes (\mathbb{Z}/7\{1, \overline{b}, \overline{b}^2, y \} \oplus \mathbb{Z}/7\{\overline{d} \overline{a}, \overline{a}^2\})$.

**Proof** We can think $6S_3 = \langle S_3, \text{diag}(-1, -1) \rangle$. The action $\text{diag}(-1, -1)$ are given by $\overline{a} \mapsto -\overline{a}, \overline{b} \mapsto \overline{b}$, and $\overline{d} \mapsto -\overline{d}$. From Lemma 8.1, we have the lemma. □

**Lemma 8.3** $H^*(BE)^{6^2 \cdot 2} \cong CA\{1, \overline{b}, \overline{c}^2, \overline{c}^2 / V \}$ where $\overline{c}^2 = \overline{a}^2 - 2\overline{b}^3 / V - 2C$.

**Proof** We can think $6^2 \cdot 2 = \langle 3S_6, \text{diag}(3, 1) \rangle$. The action $\text{diag}(3, 1)$ are given by $\overline{a}^2 \mapsto \overline{a}^2 - 4\overline{c}, \overline{b} \mapsto -\overline{b}, \overline{c} \mapsto -\overline{c}, \overline{d} \overline{a} \mapsto -\overline{d} \overline{a}$. For example $\overline{b} = y_1 y_2 v^2 \mapsto (3y_1) y_2 (3v) = -\overline{b}$. Moreover we have $\overline{c}^2 = Y_1 + Y_2 - 2C \mapsto \overline{c}^2$. Thus we have the lemma. □
Theorem 8.4  Let $c' = C + a^3/V$. Then there is the isomorphism
\[ H^*(BH\epsilon) \cong DA\{1, \bar{b}, \bar{b}^2, \bar{d}, \bar{d}b, \bar{d}b^2\} \oplus CA\{\bar{a}, c', \{1, \bar{b}, \bar{b}^2, \bar{d}\}, \bar{a}^2, \bar{a}^2c'. \]

Proof  Let $G = H\epsilon$. The orbits of $N_G(E)$–action of $A$–subgroups in $E$ are given by
\[ \{A_0, A_\infty\}, \{A_1, A_2, A_4\}, \{A_3, A_5, A_6\}. \]
Since $A_6$ is the $F^{ec}$–radical (see Leary [6]), we have
\[ H^*(BH\epsilon) \cong H^*(BE)^3S_3 \cap i_{A_6}^{*} H^*(BA_6)^{SL_2(\mathbb{F}_7)}. \]
For element $x = \bar{a}$ or $x = C + c' = C + y_1^3y_2^3$, the restrictions are $x|A_6 = 0$, eg $\bar{d}|A_6 = (y^3 + (-y)^3) = 0$. Hence we see that $CA\{x\}$ are contained in $H^*(BG)$.
Since $\bar{b} = y_1y_2v^2$, we see $\bar{b}|A_0 = -y^2v^2 = -(\bar{D}_2)^2$. Similarly $\bar{d}|A_6 = 2(\bar{D}_2)^3$. Thus we can compute $H^*(BH\epsilon)$.

Corollary 8.5  $H^*(B(H\epsilon; 2)) \cong DA\{1, \bar{b}, \bar{b}^2\} \oplus CA\{c', \bar{c}', \bar{c}'b^2, \bar{a}^2, \bar{a}d\}$.

Theorem 8.6  There is the isomorphism
\[ H^*(BFi'_{24}) \cong DA\{1, \bar{b}, \bar{b}^2, \bar{a}^2V, \bar{c}' \bar{b}V, \bar{c}' \bar{b}^2V\} \oplus CA\{\bar{c}'', \bar{a}d\} \text{ where } \bar{c}'' = \bar{a}^2 - 2c'. \]

Proof  Let $G = Fi'_{24}$. Since $A_1$ is also $F^{ec}$–radical and $W_G(A_1) = SL_2(\mathbb{F}_7); 2$. Hence we have
\[ H^*(BG) \cong H^*(B(H\epsilon; 2)) \cap i_{A_1}^{*} H^*(BA_1)^{SL_2(\mathbb{F}_7); 2}. \]
For the elements $x = \bar{a}d, \bar{c}''(= Y_1 + Y_2 - 2C)$, we see $x|A_1 = x|A_6 = 0$. Hence these elements are in $H^*(BG)$. Note that $\bar{b}|A_1 = (\bar{D}_2)^2$ and $\bar{b} \in H^*(BG)$. We also know $\bar{a}^2V|A_1 = \bar{D}_2$.

Since $H^*(BFi_{24}) \cong H^*(BFi'_{24}) \cap H^*(BE)^{6^2}; 2$ and $\bar{b}^4 = 1/2(\bar{a}^2 - 2C - c'')V$, we have the following corollary.

Corollary 8.7  $H^*(BFi_{24}) \cong (DA\{1, \bar{b}^2, \bar{b}^4\} \oplus CA\{c''\})$.

For $G = RV_1$, The subgroup $A_0$ is also $F^{ec}$–radical, we see
\[ H^*(BRV_1) \cong H^*(BFi_{24}) \cap i_0^{*} H^*(BA_0)^{GL_2(\mathbb{F}_7)} \]
Hence we have the following corollary.

Corollary 8.8  $H^*(BRV_1) \cong DA\{1, \bar{b}^2, \bar{b}^4, D_2^\epsilon\}$ with $\bar{b}^6 = D_2^2 + D_2^\epsilon D_2$. 

4.7. The module $X$. Thus the corollary is proved.

**Proof** Let $D'' = \bar{c}''V = \bar{c}''(D_1 - C^6\bar{c}'')$. Then we have

$$\bar{b}^6 = Y_1Y_2V^2 = (Y_1 + Y_2 - C)CV^2 = (C + (Y_1 + Y_2 - 2C)CV^2 = D_2^2 + (\bar{c}''V)D_2.$$  

Thus the corollary is proved. \(\square\)

9  Stable splitting for $B 7^{1+2}$

Let $G$ be groups considered in the preceding two sections, eg $O'N, O'N: 2, \ldots, RV_1$. First consider the dominant summands $X_{q,k}$. From Corollary 4.6, the dominant summands are only related to $H = WG(E)$. Recall the notation $X_{q,k}(H)$ in Lemma 4.7. The module $X_{q,k}(H)$ is still given in the preceding sections.

From Lemma 7.1, Lemma 7.2, Lemma 7.3, Lemma 8.1, Lemma 8.2 and Lemma 8.3 we have

$$H = 3D_8; X_{6,0} = \{a^3 / V, a^2b / V\}, X_{4,4} = \{a^2, b\}, X_{2,2} = \{a\},$$

$$X_{4,1} = \{d\}, X_{6,3} = \{ad\}$$

$$H = 3SD_{16}; X_{6,0} = \{a^3 / V\}, X_{4,4} = \{a^2\}, X_{2,2} = \{a\}$$

$$H = 3SD_{32}; X_{4,4} = \{a^2\}$$

$$H = 3S_3; X_{6,0} = \{\bar{b}^3 / V, \bar{a}^2\}, X_{4,4} = \{\bar{b}^2\}, X_{2,2} = \{\bar{b}\},$$

$$X_{6,3} = \{\bar{a}d\}, X_{3,0} = \{\bar{a}\}, X_{5,2} = \{\bar{a}b\}, X_{3,3} = \{\bar{d}\}, X_{5,5} = \{\bar{a}b\}$$

$$H = 6S_3; X_{6,0} = \{\bar{b}^3 / V, \bar{a}^2\}, X_{2,2} = \{\bar{b}\}, X_{4,4} = \{\bar{b}^2\}, X_{6,3} = \{\bar{a}d\}$$

$$H = 6^{2:2}; X_{6,0} = \{\bar{a}^2 - 2\bar{b}^3 / V\}, X_{4,4} = \{\bar{b}^2\}.$$  

For example, ignoring nondominant summands, we have the following diagram

$$X_{0,0} \vee X_{4,4} \quad \begin{array} {c} \leftarrow \quad B(E: 3SD_{32}) \quad \begin{array} {c} \leftarrow \quad B(E: 3SD_{16}) \quad \begin{array} {c} \leftarrow \quad B(E: 3SD_{32}) \quad \begin{array} {c} \leftarrow \quad B(E: 3D_8). \end{array} \end{array} \end{array} \end{array}$$  

From Corollary 4.4, the number $m(G, 1)_k$ is given by rank$_p H^{2k}(BG)$ for $k \mid p - 1$ and rank$_p H^{2^p-2}(G)$ for $k = 0$. For example when $G = E: 3S_3$, $m(G, 1)_0 = 3, m(G, 1)_3 = 1, m(G, 1)_k = 0$ for $k \neq 0, \neq 3$.  

Lemma 9.1 Let $G$ be one of the $O'N, O'N, \ldots, Fi'_{24}, RV_1$. Then the number $m(G, 1)_k$ for $L(1, k)$ is given by

\[
m(G, 1)_0 = \begin{cases} 
2 & \text{for } G = He, He:2 \\
1 & \text{for } G = O'N, O'N:2, Fi_{24}, Fi'_{24}
\end{cases}
\]

\[
m(G, 1)_3 = \begin{cases} 
1 & \text{for } G = He, \\
m(G, 1)_k = 0 & \text{otherwise.}
\end{cases}
\]

Now we consider the number $m(G, 2)_k$ of the non dominant summand $L(2, k)$.

Lemma 9.2 The classifying spaces $BG$ for $G = O'N, O'N:2$ have the non dominant summands $M(2) \vee L(2, 2) \vee L(2, 4)$.

Proof We only consider the case $G = O’N$, and the case $O’N:2$ is almost the same. The non $F_{\text{cc}}$–radical groups are $\{A_2, A_3, A_4, A_5\}$ (recall the proof of Theorem 7.4). The group $W_G(E) = 3D_8 \cong \langle \text{diag}(2, 2), \text{diag}(1, -1), w \rangle$. Hence the normalizer group is

\[N_G(A_2) = E; \langle \text{diag}(2, 2), \text{diag}(1, -1) \rangle.\]

Here note that $w, \text{diag}(1, -1)$ are not in the normalizer, eg $w: \langle c, ab^2 \rangle \rightarrow \langle c, a^2b^{-2} \rangle = \langle c, ab^6 \rangle$. Since $\text{diag}(2, 2): ab^2 \mapsto (ab^2)^2, c \mapsto c^4$ and $\text{diag}(1, -1): ab^2 \mapsto (ab^2)^{-1}, c \mapsto c$, the Weyl groups are

\[W_G(A_2) \cong U; \langle \text{diag}(4, 2), \text{diag}(1, -1) \rangle.\]

Let $W_1 = U; \text{diag}(4, 2)$. For $v = \lambda_1 y_1^{p-1} \in M_{p-1,k}$, we have $\overline{W}_1 v = \lambda_1 y_2^{p-1}$ since $2^3 = 1$, from the argument in the proof of Lemma 4.11. Moreover

\[
\langle \text{diag}(1, -1) \rangle y_2^{p-1} = (1 + (-1)^k) y_2^{p-1},
\]

implies that the $BG$ contains $L(2, k)$ if and only if $k$ even. \hfill \Box

Lemma 9.3 The classifying space $BHe$ (resp. $B(He; 2), Fi'_{24}, Fi_{24}$) contains the non dominant summands

\[2M(2) \vee L(2, 2) \vee L(2, 4) \vee L(2, 3) \vee L(1, 3)\]

(resp. $2M(2) \vee L(2, 2) \vee L(2, 4), M(2), M(2)$).

Proof First consider the case $G = He$. The non $F_{\text{cc}}$–radical group are

\[\{A_0, A_\infty\}, \{A_1, A_2, A_4\}\]

The group \( W_G(E) \cong 3S_3 = \langle \text{diag}(2, 1), w' \rangle \). So we see \( N_G(A_0) = E: \langle \text{diag}(2, 1) \rangle \), and this implies \( W_G(A_0) \cong U: \langle \text{diag}(2, 2) \rangle \). The fact \( 4^k = 0 \mod(7) \) implies \( k = 3 \mod(6) \). Hence \( BG \) contains the summand

\[
M(2) \vee L(2, 3) \vee L(1, 3)
\]

which is induced from \( BA_0 \).

Next consider the summands induced from \( BA_1 \). The normalizer and Weyl group are \( N_G(A_1) = E: \langle w' \rangle \) and \( W_G(A_1) = U: \langle \text{diag}(-1, 1) \rangle \) since \( w':ab \leftrightarrow ab, c \mapsto -c \). So we get

\[
M(2) \vee L(2, 2) \vee L(2, 4)
\]

which is induced from \( BA_1 \).

For \( G = He:2 \), we see \( \text{diag}(-1, -1) \in W_G(E) \), this implies that \( \text{diag}(-1, -1) \in N_G(A) \) and \( \text{diag}(1, -1) \in W_G(A_0) \). This means that the non dominant summand induced from \( BA_0 \) is \( M(2) \) but is not \( L(2, 3) \). We also know \( U: \langle \text{diag}(1, -1) \rangle \in W_G(A_1) \) but the summand induced from \( BA_1 \) are not changed.

For groups \( FI_{24}, FI_{24} \), the non \( F^\text{ec} \)-radical groups make just one \( G \)-conjugacy class \( \{A_0, A_\infty\} \). So \( BG \) dose not contain the summands induced from \( BA_1 \).

**Theorem 9.4** When \( p = 7 \), we have the following stable decompositions of \( BG \) so that \( \langle X_1, \ldots, X_s \rangle \rangle G \) means that \( BG \sim X_1 \vee \cdots \vee X_s \)

\[
\begin{align*}
X_{0,0} & \xleftarrow{RV_3} X_{0,0} \times X_{2,2} \\
& \xleftarrow{RV_2} M(2) \vee L(2,2) \vee L(2,4), \\
& \xleftarrow{O'N:2} X_{6,0} \times X_{4,4} \\
& \xleftarrow{O'N} X_{6,0} \times X_{4,4} \\
& \xleftarrow{RV_1} M(22) \\
& \xleftarrow{FI_{24}} X_{6,0} \times X_{6,2} \times X_{2,2} \\
& \xleftarrow{FI_{24}} M(2) \vee L(2,2) \vee L(2,4) \\
& \xleftarrow{He:2} X_{3,0} \times X_{5,2} \times X_{3,1} \times X_{5,1} \vee L(2,3) \vee L(1,3) \\
& \xleftarrow{He}.
\end{align*}
\]

We write down the cohomology of stable summands. At first we see that \( H^*(X_{0,0}) \cong H^*(BRV_3) \cap H^*(BRV_1) \cong DA \). Here note that elements \( a^2 - (y_1 y_2)^2 u^4 \) in Section 7 and \( b^2 = y_1^2 y_2^2 y^4 \) in Section 8 are not equivalent under the action in \( GL_7(\mathbb{F}_7) \) because \( y_1^2 + y_2^2 \) is indecomposable in \( \mathbb{Z}/7[y_1, y_2] \).

From the cohomologies, \( H^*(BRV_3) \) and \( H^*(BRV_2) \), then \( H^*(X_{4,4}) \cong DA\{a^2, a^4\} \) and \( H^*(X_{6,0} \vee X_{2,2}) \cong DA\{a, a^3, a^5\} \).

On the other hand, we know \( H^*(X_{6,0}) \) from the cohomology \( H^*(BRV_1) \). Thus we get the following lemma.
Lemma 9.5  There are isomorphisms of cohomologies
\[ H^*(X_{0,0}) \cong DA, H^*(X_{4,4}) \cong DA\{a^2, a^4\} \]
\[ H^*(X_{6,0}) \cong DA\{D_2\} \cong DA\{a^3\}, H^*(X_{2,2}) \cong DA\{a, a^5\}. \]

Let us write \( M\{a\} = DA\{1, C, \ldots, C^{p-1}\}\{a\} \). From the facts that \( D_2 = CV, D_1 = C + V \) and \( D_2 = C(D_1 - C^P) = CD_1 - C^P + 1 \), we have two decompositions
\[ CA\{a\} \cong DA\{1, C, \ldots, C^P\}\{a\} \cong DA\{a\} \oplus M\{Ca\} \cong M\{a\} \oplus DA\{Va\}. \]

From the cohomology of \( H^*(Fi_24) \), we know the following lemma.

Lemma 9.6  \( H^*(M(2)) \cong M\{C\}. \)

Comparing the cohomology \( H^*(B(He; 2)) \cong H^*(BFi_{24}) \oplus M\{a^2, \overline{c'}d, \overline{c'}b^2\} \), we have the isomorphisms
\[ H^*(M(2)) \cong M\{a^2\}, H^*(L(2, 2) \lor L(2, 4)) \cong M\{\overline{c'}d, \overline{c'}b^2\}. \]

From \( H^*(BFi_{24}) \cong H^*(BFi_{24}) \oplus DA\{\overline{a}^2, \overline{c'}d, \overline{c'}b^2\} \), we also know that
\[ H^*(X_{6,3}) \cong CA\{\overline{a}\}, H^*(X_{6,0} \lor X_{2,2}) \cong DA\{\overline{a}^2, \overline{c'}d, \overline{c'}b^2\}. \]

We still get \( H^*(BFi_{24}) \cong H^*(BRV_1) \oplus M\{\overline{c''}\} \) and \( H^*(M(2)) \cong M\{\overline{c''}\} \).

Next consider the cohomology of groups studied in Section 7 eg \( O'N \). There is the isomorphism
\[ H^*(BO'N) \cong H^*(BO'N; 2) \oplus DA\{b, b^2, ab^2\} \oplus CA\{d, da, da^2\}. \]

Indeed, we have
\[ H^*(X_{6,0} \lor X_{4,4}) \cong DA\{b, b^2, ab^2\} \cong DA\{a^2, a^3, a^4\} \]
\[ H^*(X_{6,3}) \cong CA\{da\} \]
\[ H^*(X_{4,3}) \cong CA\{d, da^2\}. \]

We also have the isomorphism \( H^*(BO'N; 2) \cong H^*(BRV_2) \oplus M\{C', C'a, C'a^2\} \) and
\[ H^*(M(2) \lor L(2, 2) \lor L(2, 4)) \cong M\{C', C'a, C'a^2\}. \]

Recall that
\[ H^*(BE)^3SD_{32} \cong CA\{1, a^2, a^4 / V\} \cong DA\{1, a^2, a^4\} \oplus M\{C, a^2C, a^4 / V\}, \]
in fact \( H^*(M(2) \lor L(2, 2) \lor L(2, 4)) \cong M\{C, a^2C, a^4 / V\}. \)
10 The cohomology of \( \mathbb{M} \) for \( p = 13 \)

In this section, we consider the case \( p = 13 \) and \( G = \mathbb{M} \) the Fisher–Griess Monster group. It is known that \( W_G(E) \cong 3 \times 4S_4 \). The \( G \)–conjugacy classes of \( A \)–subgroups are divided into two classes; one is \( F^c \)–radical and the other is not. The class of \( F^c \)–radical groups contains 6 \( E \)–conjugacy classes (see Ruiz–Viruel [9]). (The description of [11, (4.1)] was not correct, and the description of \( H^*(B\mathbb{M}) \) in [11, Theorem 6.6] was not correct.) The Weyl group \( W_G(A) \cong SL_2(\mathbb{F}_{13}) \). for each \( F^c \)–radical subgroup \( A \).

Since \( S_4 \cong PGL_2(\mathbb{F}_3) \) [S], we have the presentation of
\[
S_4 = \langle x, y, z \mid x^3 = y^3 = z^2 = (xy)^2 = 1, zxz^{-1} = y \rangle.
\]
(Take \( x = u, y = u' \) in Lemma 4.8, and \( z = w \) in Section 5.) By arguments in the proof of Suzuki [10, Chapter 3 (6.24)], we can take elements \( x, y, z \) in \( GL_2(\mathbb{F}_{13}) \) by
\[
(10–1) \quad x = \begin{pmatrix} 3 & 0 \\ 0 & 9 \end{pmatrix}, \quad y = \begin{pmatrix} 5 & -4 \\ -2 & 7 \end{pmatrix}, \quad z = \begin{pmatrix} 2 & 2 \\ 1 & -2 \end{pmatrix},
\]
so that we have
\[
x^3 = y^3 = 1, \quad zxz^{-1} = y, \quad (xy)^2 = -1, \quad z^2 = \text{diag}(6, 6).
\]

Hence we can identify
\[
(10–2) \quad 3 \times 4S_4 \cong \langle x, y, z \rangle \subset GL_2(\mathbb{F}_{13}).
\]

It is almost immediate that \( H^*(BE)^{(x)} \) (resp. \( H^*(BE)^{(x)} \)) is multiplicatively generated by \( y_1y_2, y_1^3, y_2^3 \) (resp. \( y_1y_2, y_1^2, y_2^2 \)) as a \( \mathbb{Z}/(13)[C, v] \)–algebra. Hence we can write
\[
(10–3) \quad H^*(BE)^{(x)} \cong \mathbb{Z}/(13)[C, v]\{1, y_1y_2, \ldots, (y_1y_2)^5\}(y_1^6, y_2^6, y_1^6, y_2^6).
\]

For the invariant \( H^*(BE)^{(y)} \), we get the similar result exchanging \( y_i \) to \( (z^{-1})^*y_i \) since \( zxz^{-1} = y \). Indeed \( (z^{-1})^*: H^*(BE)^{(x)} \cong H^*(BE)^{(y)} \).

To seek invariants, we recall the relation between the \( A \)–subgroups and elements in \( H^2(BE; \mathbb{Z}/p) \). For \( 0 \neq y = \alpha y_1 + \beta y_2 \in H^2(BE; \mathbb{Z}/p) \), let \( A_y = A_{-((\alpha, \beta))} \) so that \( y|A_y = 0 \). This induces the 1–1 correspondence,
\[
(H^2(BE; \mathbb{Z}/p) - \{0\})/F^*_p \leftrightarrow \{A_i \mid i \in F_p \cup \{\infty\}\}, \quad y \leftrightarrow A_y.
\]

Considering the map \( g^{-1}A_i \xrightarrow{g} A_i \subset E^\delta \rightarrow \mathbb{Z}/p \), we easily see \( A_y^* \yields g^{-1} A_y \).
For example, the order 3 element $x$ induces the maps
\[ x^*: y_1 - y_2 \mapsto 3y_1 - 9y_2 \mapsto 9y_1 - 3y_2 \mapsto y_1 - y_2 \]
\[ x^{-1}: A_1 - y_2 = \langle c, ab \rangle \mapsto \langle c, a^9b^3 \rangle \mapsto \langle c, a^3b^9 \rangle \mapsto \langle c, ab \rangle. \]

In particular $A_1, A_9, A_3$ are in the same $x$–orbit of $A$–subgroups. Similarly the $\langle x \rangle$–conjugacy classes of $A$ is given
\[ \{A_0\}, \{A_\infty\}, \{A_1, A_3, A_9\}, \{A_2, A_5, A_6\}, \{A_4, A_{10}, A_{12}\}, \{A_7, A_8, A_{11}\}. \]

The $\langle y \rangle$–conjugacy classes are just $\{zA_i\}$ for $\langle x \rangle$–conjugacy classes $\{A_i\}$.
\[ \{A_7 = zA_0\}, \{A_{12}\}, \{A_3, A_1, A_5\}, \{A_6, A_9, A_2\}, \{A_{11}, A_8, A_\infty\}, \{A_0, A_{10}, A_4\}. \]

Hence we have the $\langle x, y \rangle$–conjugacy classes
\[ C_1 = \{A_1, A_2, A_3, A_5, A_6, A_9\}, C_2 = \{A_0, A_4, A_{10}, A_{12}\}, C_3 = \{A_\infty, A_7, A_8, A_{11}\}. \]

At last we note $\langle x, y, z \rangle$–conjugacy classes are two classes $C_1, C_2 \cup C_3$.

Let us write the $\langle x \rangle$–invariant
\[(10–4) \quad u_6 = \prod_{i \in C_1}(y_2 - iy_1) = (y_2 - y_1)(y_2 - 2y_1) \cdots (y_2 - 9y_1)
= y_2^6 - 9y_1^3y_2^3 + 8y_1^6. \]

Then $u_6$ is also invariant under $y^*$ because the $\langle x, y \rangle$–conjugacy class $C_1$ divides two $\langle y \rangle$–conjugacy classes
\[ C_1 = \{A_1, A_3, A_5\} \cup \{A_2, A_6, A_9\} \]

and the element $u_6$ is rewritten as
\[ u_6 = \lambda(\prod_{i=0}^2 y_i^2(y_2 - y_1)).(\prod_{i=0}^2 y_i^2(y_2 - 2y_1)) \text{ for } \lambda \neq 0 \in \mathbb{Z}/(13). \]

We also note that $u_6|A_i = 0$ if and only if $i \in C_1$. Similarly the following elements are $\langle x, y \rangle$–invariant,
\[(10–5) \quad u_8 = \prod_{i \in C_2 \cup C_3}(y_2 - iy_1) = y_1y_2(y_2^6 + 9y_1^3y_2^3 + 8y_1^6)
\quad u_{12} = \prod_{i \in C_2}(y_2 - iy_1)^3 = (y_2^4 + y_1^3y_2)^3
\quad = \lambda(\prod_{i=0}^2 y_i^2)(\prod_{i=0}^2 y_i^2)(y_2 - 4y_1)^3
\quad = \lambda'(\prod_{i=0}^2 y_i^2(y_2 - 12y_1))(\prod_{i=0}^2 y_i^2y_2)^3 v
\quad u'_{12} = \prod_{i \in C_3}(y_2 - iy_1)^3 = (y_1y_2^3 + 8y_1^4)^3. \]

Of course $(u_{12}u'_{12})^{1/3} = u_8$ and $u_6u_8 = 0$. Moreover direct computation shows $u_6^2 = u_{12} + 5u'_{12}$. 

Lemma 10.1 \( H^*(BE)^{(x,y)} \cong \mathbb{Z}/(13)[C, u]\{1, u_6, u_6^2, u_6^3, u_8, u_8^2, u_{12}\}. \)

Proof Recall (10–3) to compute
\[
H^*(BE)^{(x,y)} \cong H^*(BE)^{(x,-1)} \cap H^*(BE)^{(y,-1)}.
\]
Since \((z^{-1})^*(y_1 y_2)^l \neq (y_1 y_2)^l\) for \(1 \leq i \leq p - 2\), from (10–3) we know invariants of the lowest positive degree are of the form
\[
u = \gamma y_2^6 + \alpha y_2^3 y_2^3 + \beta y_2^6.
\]
Then \(u' = u - \gamma u_6\) is also invariant with \(u'|A_\infty = 0\). Hence \(u'|A_i = 0\) for all \(A_i \in C_3\).
Thus we know \(u' = \lambda y_2^3 y_2^3\). But this is not \((y)\)-invariant for \(\lambda \neq 0\), because \((u')^3 = \lambda y_2^3 y_2^3\) is invariant, while \(y_2^3\) is not \((y)\)-invariant. Thus we know \(u' = 0\).

Any 16–dimensional invariant is form of
\[
u = y_2 y_2 (\gamma y_2^6 + \alpha y_2^3 y_2^3 + \beta y_2^6).
\]
Since \(u|A_0 = u|A_\infty = 0\), we know \(u|A_i = 0\) for all \(A_i \in C_2 \cup C_3\). Hence we know
\[
u = \gamma u_2^{1/3} (u_2^{1/3}) = \gamma u_8.
\]
By the similar arguments, we can prove the lemma for degree \(\leq 24\).

For \(24 < \text{degree} < 48\), we only need consider the elements \(u' = 0 \mod(y_1 y_2)\). For example, \(H^{18}(BE; \mathbb{Z}/13)^{(x,-1)}\) is generated by
\[
\{(y_1 y_2)^9, (y_1 y_2)^3 C, y_1^6 C, y_2^6 C, y_1 y_2^{12}, y_1^{12} y_2^{12}\}.
\]
But we can take off \(y_2^6 C = y_1^1, y_2^6 C = y_2^1\) by \(\lambda u_2^3 + \mu C u_6\) so that \(u' = 0 \mod(y_1 y_2)\).

Hence we can take \(u'\) so that \(u_8\) divides \(u'\) from the arguments similar to the case of degree=16. Let us write \(u' = u''u_8\). Then we can write
\[
u'' = y_2 x^k (\lambda_1 y_1^6 + \lambda_2 y_2^3 y_2^3 + \lambda_3 (y_1 y_2)^k - C),
\]
taking off \(\lambda y_1^k y_2^2 u_6\) if necessary since \(u_6 u_8 = 0\). (Of course, for \(k < 3, \lambda_3 = 0\).) Since \(u_8|A_i \neq 0\) and \(u_6|A_i = 0\) for \(i \in C_1\), we have
\[
(u'' - \gamma y^* u'')|A_i = 0 \text{ for } i \in C_1.
\]
Since \(y^* y_1 = 5 y_1 - 4 y_2\) and \(y^* y_2 = -2 y_1 + 7 y_2\), we have
\[
(u'' - \gamma y^* u'')|A_i = \lambda_1 (i^k - (5 - 4i)^6 + k (-2 + 7i)^k)
+ \lambda_2 (i^{k+3} - (5 - 4i)^{k+3} (-2 + 7i)^{k+3})
+ \lambda_3 (i^{k-3} - (5 - 4i)^{k-3} (-2 + 7i)^{k-3}).
\]
We will prove that we can take all $\lambda_i = 0$. Let us write $U = u'' - y^*u''$. We then have the following cases.

1. The case $k = 0$, ie degree=14. If we take $i = 1$,

$$U|A_1 = \lambda_1 (1 - 1) + \lambda_2 (1 - 1^3 5^3) = 0.$$ 

So we have $\lambda_2 = 0$. We also see $\lambda_1 = 0$ since $U|A_3 = \lambda_1 (1 - (5 - 12)^6) = 2\lambda_1 = 0$.

2. The case $k = 1$. Since $y_1 y_2 u_6 - u_8 = -18y_1^4 y_2^4$, we can assume $\lambda_2 = 0$ taking off $\lambda u_6^2$ if necessary. We have also $\lambda_1 = 0$ from $U|A_1 = \lambda_1 (1^1 - 1^7 1^1) = 0$.

3. The case $k = 2$. We get the the result $U|A_1 = 2\lambda_1 + 4\lambda_2$, $U|A_3 = 5\lambda_1 + 5\lambda_2$.

4. The case $k = 3$. First considering $C\gamma_8$, we may take $\lambda_3 = 0$. The result is given by $U|A_1 = 6\lambda_1 + 2\lambda_2$ and $U|A_2 = 7\lambda_1 + 9\lambda_2$.

5. The case $k = 4$. The result follows from

$$U|A_1 = 6\lambda_2 + 9\lambda_3, U|A_3 = 6\lambda_1 + 6\lambda_2 + 6\lambda_3, U|A_5 = 2\lambda_1 - 4\lambda_2 + 6\lambda_3.$$

Hence the lemma is proved.

Next consider the invariant under $\langle x, y, \text{diag}(6, 6) \rangle$. The action for $\text{diag}(6, 6)$ is given by $y_1^i y_2^j v^k \mapsto 6^i + j + 2k y_1^i y_2^j v^k$. Hence the invariant property implies $i + j + 2k = 0 \mod(12)$. Thus $H^*(BE)^{\langle x, y, \text{diag}(6, 6) \rangle}$ is generated as a $CA$–algebra by

$$\{1, u_6 v^3, u_8 v^2, u_{12}, u'_{12}, v^6\}.$$

**Lemma 10.2** The invariant $H^*(BE)^{3 \times 4S_4} \cong H^*(BE)^{\langle x, y, v \rangle}$ is isomorphic to

$$CA\{1, u_6 v^3, (u_6 v^3)^2, (u_6 v^3)^3, u_8 v^8, (u_8 v^8)^2 / V, (u_{12} - 5u'_{12})\}.$$

**Proof** We only need compute $z^*$–action. Since

$$3 \times 4S_4 \cong \langle x, y, \text{diag}(6, 6) \rangle : (z),$$

the $z^*$–action on $H^*(BE)^{\langle x, y, \text{diag}(6, 6) \rangle}$ is an involution. Let $u_6 v^3 = u_6(y_1, y_2)v^3$.

First note $u_6|A_\infty = u_6(0, y) = y^6$. On the other hand, its $z^*$–action is

$$z^* u_6 v^3|A_\infty = u_6(2y_1 + 2y_2, y_1 - 2y_2)(-6v)^3|A_\infty = u_6(2y, -2y)(-6v)^3$$

$$= ((-2)^6 - 9(-2)^3 (2)^3 + 8(2)^6)(-6)^3 y^6 v^3$$

$$= (1 + 9 + 8)y^6 v^3 = y^6 v^3.$$

Hence we know $u_6 v^3$ is invariant, while $u_6 v^9$ is not.
Similarly we know
\[ u_8v^2|A_1 = u_8(y, y)v^2 = 5y^8v^2, \quad z^*u_8v^2|A_1 = -5y^8v^2. \]
Hence \( u_8v^8 \) and \( u_8^2v^4 \) are invariant but \( u_8v^2 \) is not.

For the action \( u_{12} \), we have
\[
\begin{align*}
&u_{12}|A_0 = 0, \quad u_{12}|A_{\infty} = y^{12}, \quad u_{12}'|A_0 = 5y^{12}, \quad u_{12}'|A_{\infty} = 0, \\
&z^*u_{12}|A_0 = y^{12}, \quad z^*u_{12}|A_{\infty} = 0, \quad z^*u_{12}'|A_0 = 0, \quad z^*u_{12}'|A_{\infty} = 5y^{12}.
\end{align*}
\]
Thus we get \( z^*u_{12} = (1/5)u_{12}' \), \( k^*u_{12}' = 5u_{12} \). Hence we know \( u_{12} + (1/5)u_{12}' \) and \( (u_3^2 - (1/5)u_{12}')v^6 = (u_6v^3)^2 \) are invariants. Thus we can prove the lemma. \( \square \)

**Theorem 10.3** For \( p = 13 \), the cohomology \( H^*(B\mathbb{M}) \) is isomorphic to
\[
DA\{1, u_8v^8, (u_8v^8)^2\} \oplus CA\{u_6v^3, (u_6v^3)^2, (u_6v^3)^3, (u_{12} - 5u_{12}' - 3C)\}.
\]

**Proof** Direct computation shows
\[
u_{12} - 5u_{12}' = y_{12}^2 - 2y_2^9y_1^3 + 3y_2^3y_1^9 + y_{12}^{12},
\]
and hence \( u_{12} - 5u_{12}' - 3C|A_1 = 0 \), indeed, the restriction is zero for each \( A_i \in C_1 \). The isomorphism
\[
H^*(B\mathbb{M}) \cong H^*(BE)^{3 \times 4S_4} \cap i_{A_1}^-(H^*(BA_1)^{SL_4(\mathbb{C}_{13})}).^4,
\]
completes the proof. \( \square \)

The stable splitting is given by the following theorem.

**Theorem 10.4** We have the stable splitting
\[
\begin{align*}
B\mathbb{M} &\sim X_{0,0} \vee X_{12,0} \vee X_{12,6} \vee X_{6,3} \vee X_{8,8} \vee M(2), \\
B(E: 3 \times 4S_4) &\sim B\mathbb{M} \vee M(2) \vee L(2, 4) \vee L(2, 8).
\end{align*}
\]

**Proof** Let \( H = E: 3 \times 4S_4 \). Recall that
\[
X_{q,k}(H) = (S(A)^q \otimes v^k) \cap H^*(BH) \quad 0 \leq q \leq 12, 0 \leq k \leq 11.
\]
We already know
\[
X_{*,*}(H) = \mathbb{Z}/(13)\{1, u_8v^8, u_6v^3, u_6^2v^6, u_{12} - 5u_{12}'\}.
\]
Hence \( BH \) has the dominant summands in the theorem.
The normalizer groups of \( A_0, A_1 \) are given
\[
N_H(A_0) = E: \langle x, \text{diag}(6, 6) \rangle, \quad N_H(A_1) = E: \langle \text{diag}(6, 6) \rangle.
\]
Hence the Weyl groups are
\[
W_H(A_0) = U: \langle \text{diag}(1, 3), \text{diag}(6^2, 6) \rangle, \quad W_H(A_1) = U: \langle \text{diag}(6^2, 6) \rangle.
\]
From the arguments of Lemma 4.11, the non-dominant summands induced from \( BA_1 \) are \( M(2) \vee L(2, 4) \vee L(2, 8) \). We also know the non-dominant summands from \( BA_0 \) are \( M(2) \). This follows from
\[
\left( \text{diag}(1, 3) \right) y_2^{p-1} = \sum_{i=0}^{2} (3^i y_2^{p-1}) \quad \text{for } y_2^{p-1} \in M_{p-1,k}
\]
and this is nonzero mod(13) if and only if \( k = 0 \mod(3) \).

**Remark** It is known \( H^*(TH) \cong DA \) for \( p = 5 \) in [11]. Hence all cohomology \( H^*(BG) \) for groups \( G \) in Theorem 2.1 (4)–(7) are explicitly known. For (1)–(3), see also Tezuka–Yagita [11].

## 11 Nilpotent parts of \( H^*(BG; \mathbb{Z}(p)) \)

It is known that \( p^2 H^*(BE; \mathbb{Z}) = 0 \) (see Tezuka–Yagita [11] and Leary [6]) and
\[
pH^{*>0}(BE; \mathbb{Z}) \cong \mathbb{Z}/p^2\{pv, pv^2, \ldots\}.
\]
In particular \( H^\text{odd}(BE; \mathbb{Z}) \) is all just \( p \)-torsion. There is a decomposition
\[
H^\text{even}(BE; \mathbb{Z})/p \cong H^*(BE) \oplus N \quad \text{with } N = \mathbb{Z}/p[V]_i[b_1, \ldots, b_{p-3}]
\]
where \( b_i = Cor_{A_0}^E(u^{i+1}), |b_i| = 2i + 2 \). (Note for \( p = 3, N = 0 \).) The restriction images \( b_i|A_j = 0 \) for all \( j \in \mathbb{F}_p \cup \infty \). For \( g \in GL_2(\mathbb{F}_p) \), the induced action is given by \( g^*(b_i) = \text{det}(g)^i b_i \) by the definition of \( b_i \).

Note that
\[
2 = |y_i| < |b_j| = 2(j + 1) < |C| = 2p - 2 < |v| = 2p.
\]
So \( g^*(y_i) \) is given by (3–4) also in \( H^*(BE; \mathbb{Z}) \) and \( g^*(v) = \text{det}(g)v \mod(p) \). Hence we can identify that
\[
H^*(BE)^H = (H^\text{even}(BE; \mathbb{Z})/(p, N))^H \subset H^\text{even}(BE; \mathbb{Z}/p)^H.
\]
Let us write the reduction map by \( q: H^*(BE; \mathbb{Z}) \to H^*(BE; \mathbb{Z}/p) \).

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Lemma 11.1  Let $H \subset GL_2(\mathbb{F}_p)$ and $(|H|, p) = 1$. If $x \in H^*(BE)^H$, then there is $x' \in H^*(BE; \mathbb{Z})^H$ such that $q(x') = x$.

Proof  Let $x \in H^*(BE)^H$ and $G = E; H$. Then we can think $x \in H^*(BE; \mathbb{Z}/p)^H \cong H^*(BG; \mathbb{Z}/p)$ and $\beta(x) = 0$. By the exact sequence

$$H^\text{even}(BG; \mathbb{Z}(p)) \to H^\text{even}(BG; \mathbb{Z}/p) \to H^\text{odd}(BG; \mathbb{Z}(p)).$$

we easily see that $x \in \text{Image}(q)$ since $q\delta(x) = \beta(x) = 0$ and $q|H^\text{odd}(BG; \mathbb{Z}(p))$ is injective. Since $H^*(BG; R) \cong H^*(BE; R)^H$ for $R = \mathbb{Z}(p)$ or $\mathbb{Z}/p$, we get the lemma. \[\square\]

Proof of Theorem 3.1  From Tezuka–Yagita [11, Theorem 4.3] and Broto–Levi–Oliver [11], we have the isomorphism

$$H^*(BG; \mathbb{Z})(p) \cong H^*(BE; \mathbb{Z})^{W_G(E)} \cap A; F^{\infty-\text{radical}} i^*^{-1} H^*(BA; \mathbb{Z})^{W_G(A)}.$$

The theorem is immediate from the above lemma and the fact that $H^\text{even}>0(BA; \mathbb{Z}) \cong H^{*>0}(BA)$. \[\square\]

Let us write $N(G) = H^*(BG; \mathbb{Z}) \cap N$. Then

$$H^\text{even}(BG; \mathbb{Z})/p \cong H^*(BG) \oplus N(G).$$

The nilpotent parts $N(G)$ depends only on the group $\text{Det}(G) = \{\det(g) | g \in W_G(E)\} \subset \mathbb{F}_p^*$, in fact, $N(G) = N^{W_G(E)} \cong N^{\text{Det}(G)}$.

Lemma 11.2  If $\text{Det}(G) \cong \mathbb{F}_p^*$ (e.g. $G = O'N, He, \ldots, RV_3$ for $p = 7$, or $G = \mathbb{M}$ for $p = 13$), then

$$N(G) \cong \mathbb{Z}/p[V]; b_1 v^{p-2-i} | 1 \leq i \leq p - 3.$$  

Lemma 11.3  Let $G$ have a 7–Sylow subgroup $E$. Then, we have

$$N(G) = \begin{cases} 
\mathbb{Z}/7[V]; b_1 v^4, b_2 v^3, b_3 v^2, b_4 v) & \text{if } \text{Det}(G) = \mathbb{F}_7^* \\
\mathbb{Z}/7[v^3]; b_1 v, b_2, b_3 v^2, b_4 v) & \text{if } \text{Det}(G) \cong \mathbb{Z}/3 \\
\mathbb{Z}/7[v^2]; b_1, b_2 v, b_3, b_4 v) & \text{if } \text{Det}(G) \cong \mathbb{Z}/2 \\
\mathbb{Z}/7[v]; b_1, b_2, b_3, b_4) & \text{if } \text{Det}(G) \cong \{1\}.
\end{cases}$$

Now we consider the odd dimensional elements. Recall that

$$H^\text{odd}(BA; \mathbb{Z}) \cong \mathbb{Z}/p[y_1, y_2][\alpha].$$
where \( \alpha = \beta(x_1x_2) \in H^*(BA;\mathbb{Z}/p) \cong \mathbb{Z}/p[y_1, y_2] \otimes \Lambda(x_1, x_2) \) with \( \beta(x_i) = y_i \).

Of course \( g^*(\alpha) = \text{det}(g)\alpha \) for \( g \in \text{Out}(A) \). For example \( H^\text{odd}(B(A;Q_8)) \cong H^*(B(A;Q_8))\{\alpha\} \) since \( \text{Det}(A;Q_8) = \{1\} \).

Recall the Milnor operation \( Q_{i+1} = [Pp^n Q_i - Q_i Pp^n], Q_0 = \beta \). It is known that
\[
Q_1(\alpha) = y_1^p y_2 - y_1 y_2^p = \tilde{D}_2 \text{ with } (\tilde{D}_2)^{p-1} = \tilde{D}_2.
\]

The submodule of \( H^* (X;\mathbb{Z}/p) \) generated by (just) \( p \)-torsion additive generators can be identified with \( Q_0 H^* (X;\mathbb{Z}/p) \). Since \( Q_i Q_0 = -Q_0 Q_i \), we can extend the map [13, page 377]

\[
Q_i : Q_0 H^* (X;\mathbb{Z}/p) \to Q_0 H^* (X;\mathbb{Z}/p) \subset H^* (X;\mathbb{Z} G).
\]

Since all elements in \( H^\text{odd}(BA;\mathbb{Z}) \) are (just) \( p \)-torsion, we can define the map
\[
Q_1 : H^\text{odd}(BA;\mathbb{Z}) \to H^\text{even}(BA;\mathbb{Z}) = H^\text{even}(BA).
\]

Moreover this map is injective.

**Lemma 11.4** (Yagita [13]) Let \( G \) have the \( p \)-Sylow subgroup \( A = (\mathbb{Z}/p)^2 \). Then
\[
Q_1 : H^\text{odd}(BG;\mathbb{Z}(p)) \cong (H^\text{even}(BG) \cap J(G)),
\]
with \( J(G) = \text{Ideal}(y_1^p y_2 - y_1 y_2^p) \subset H^\text{even}(BA) \).

**Corollary 11.5** For \( p = 3 \), there are isomorphisms
\[
H^\text{odd}(BA;\mathbb{Z})^{Z/8} \cong S\{b, a'b, a, (a_1 - a_2)\}\{\alpha\}
\]
\[
H^\text{odd}(BA;\mathbb{Z})^{D_8} \cong S\{1, a, a', b\}\{ba\}
\]
\[
H^\text{odd}(BA;\mathbb{Z})^{S^{D_16}_d} \cong S\{1, a'\}\{ba\}.
\]

**Proof** We only prove the case \( G = A;\mathbb{Z}/8 \) since the proof of the other cases are similar. Note in §5 the element \( Q_1(\alpha) \) is written by \( b \) and \( b^2 = a_1 a_2 \). Recall \( S = \mathbb{Z}/3[a_1 + a_2, a_1 a_2] \). Hence we get
\[
H^* (BA)^{D_8} \cap J(G) \cong S\{1, a', ab, (a_1 - a_2)b\} \cap \text{Ideal}(b)
\]
\[
= S\{b^2, b'a', ab, (a_1 - a_2)b\}
\]
\[
= S\{b, ba', a, (a_1 - a_2)\}\{Q_1(\alpha)\}.
\]

The corollary follows. \( \square \)
By Lewis, we can write \[6; 11\]

\[H^{\text{odd}}(BE; \mathbb{Z}) \cong \mathbb{Z} / p[y_1, y_2]/(y_1\alpha_2 - y_2\alpha_1, y_1^p\alpha_2 - y_2^p\alpha_1)\{\alpha_1, \alpha_2}\],

where \(|\alpha_i| = 3\). It is also known that \(Q_1(\alpha) = y_i v\) and \(Q_1; H^{\text{odd}}(BE; \mathbb{Z}(p)) \to H^{\text{even}}(BE) \subset H^{\text{even}}(BE; \mathbb{Z})/p\) is injective [13]. Using this we can prove the following lemma.

**Lemma 11.6** (Yagita [13]) Let \(G\) have the \(p\)-Sylow subgroup \(E\). Then

\[Q_1; H^{\text{odd}}(BG) \cong (H^{\text{even}}(BG) \cap J(G))\]

with \(J(G) = \text{Ideal}(y_i v) \subset H^{\text{even}}(BE)\).

From the above lemma we easily compute the odd dimensional elements. Note that

\[D_2 = CV \not\in J(E)\]

but \(D_2^2 = C^2 V^2 = (Y_1^2 + Y_2^2 - Y_1 Y_2) V^2 \in J(E)\).

Let us write \(\alpha = (Y_1 y_1^{p-2} \alpha_1 + Y_2 y_2^{p-2} \alpha_2 - Y_1 y_2^{p-2} \alpha_2) V v^{p-2}\) so that \(Q_1(\alpha) = D_2^2\).

**Corollary 11.7** \(H^{\text{odd}}(B^2 F_4(2)^*; \mathbb{Z}(3)) \cong DA\{a, a'\}\) with \(a' = (y_1 \alpha_1 + y_2 \alpha_2) v\).

**Proof** Recall that \(H^*(B^2 F_4(2)^*) \cong DA\{1, (Y_1 + Y_2) V\}\) from the remark of Proposition 6.3. The result is easily obtained from \(Q_1(\alpha) = D_2^2, Q_1(\alpha') = (Y_1 + Y_2) V\). \(\square\)

**Corollary 11.8** There are isomorphisms

\[H^{\text{odd}}(BRV_3; \mathbb{Z}(7)) \cong DA\{a, a^3, a^5\}\{a'\}\]

\[H^{\text{odd}}(BRV_2; \mathbb{Z}(7)) \cong DA\{1, a, \ldots, a^5\}\{a'\},\]

with \(a' = (y_1 \alpha_1 + y_2 \alpha_2) v\).

**Proof** We can easily compute

\(Q_1(\alpha') = Q_1((y_1 \alpha_1 + y_2 \alpha_2) v) = (y_1 Q_1(\alpha_1) + y_2 Q_1(\alpha_2)) v = (y_1^2 + y_2^2) v^2 = a\).

Recall that \(H^*(BRV_3) \cong DA\{1, a^2, a^4\}\). We get

\[H^*(BRV_3) \cap \text{Ideal}(y_i v) = DA\{D_2^2, a^2, a^4\} = DA\{a^5, a, a^3\}(Q_1 \alpha'),\]

and the corollary follows. \(\square\)

**Corollary 11.9** \(H^{\text{odd}}(BRV_1; \mathbb{Z}(7)) \cong DA\{b, b^3, b^5\}\{a''\} \oplus DA\{a\}\) where \(a'' = y_1 v \alpha_2\).
Cohomology of \( p \)-local groups over \( p^{1+2} \)

**Proof** Recall Corollary 8.8. We have \( C\mathfrak{G}'' = C(Y_1 + Y_2 - 2C) = -Y_1^2 - Y_2^2 + 2Y_1Y_2 \). Hence we can see \( Q_1(\alpha) = -D_2\mathfrak{G}''V \).

**Corollary 11.10** The cohomology \( H^{\text{odd}}(B\mathfrak{G}; \mathbb{Z}_{(13)}) \) is isomorphic to

\[
DA\{\alpha, \alpha_8, (u_8v^8)\alpha_8\} \oplus CA\{\alpha_6, (u_6v^3)\alpha_6, (u_6v^3)^2\alpha_6, \alpha_{12}\},
\]

where

\[
\alpha_8 = y_2(y_2^6 + 9y_2^3y_1^3 + 8y_1^6)v^7\alpha_1,
\]

\[
\alpha_6 = (y_2^5\alpha_2 - 9y_2^3y_1^3\alpha_2 + 8y_1^5,\alpha_1)v^2,
\]

\[
\alpha_{12} = C(y_2^{11}\alpha_2 - 2y_2^8y_1^3\alpha_2 + 3y_2^2y_1^9\alpha_2 + y_1^{11}\alpha_1)v^{11} - 3\alpha/V.
\]

**Proof** It is almost immediate that

\[
Q_1(\alpha_8) = u_8v^8, \quad Q_1(\alpha_6) = u_6v^3, \quad Q_1(\alpha_{12}) = (u_{12} - 5u_{12} - 3C)V.
\]

From Theorem 10.3, we get the corollary.

**References**


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