Stable splitting and cohomology of p–local finite groups over the extraspecial p–group of order p^3 and exponent p

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Let p be an odd prime. Let G be a p–local finite group over the extraspecial p–group p_+^{1+2}. In this paper we study the cohomology and the stable splitting of their p–complete classifying space BG.

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1 Introduction

Let us write by E the extraspecial p–group p_+^{1+2} of order p and exponent p for an odd prime p. Let G be a finite group having E as a p–Sylow subgroup, and BG $(= BG_p^{\wedge})$ the p–completed classifying space of G. In papers by Tezuka and Yagita [11] and Yagita [13; 14], the cohomology and stable splitting for such groups are studied. In many cases non isomorphic groups have homotopy equivalent p–completed classifying spaces, showing that there are not too many homotopy types of BG, as was first suggested by C B Thomas [12] and D Green [3].

Recently, Ruiz and Viruel [9] classified all p–local finite groups for the p–group E. Their results show that each classifying space BG is homotopic to one of the classifying spaces which were studied in [11] or classifying spaces of three exotic 7–local finite groups. (While descriptions in [11] of $H^*({}^2F_4(2)')_{(3)}$ $H^*(Fi_{24}')_{(7)}$ and $H^*(\mathbb{M})_{(13)}$ contained some errors.)

In Section 2, we recall the results of Ruiz and Viruel. In Section 3, we also recall the cohomology $H^*(BE;\mathbb{Z})/(p,\sqrt{0})$. In this paper, we simply write

$$H^*(BG) = H^*(BG;\mathbb{Z})/(p,\sqrt{0})$$

and study them mainly. The cohomology $H^{\mathrm{odd}}(BG;\mathbb{Z}_{(p)})$ and the nilpotents parts in $H^{\mathrm{even}}(BG;\mathbb{Z}_{(p)})$ are given in Section 11. Section 4 is devoted to the explanations of stable splitting of BG according to Dietz, Martino and Priddy. In Section 5, and Section 6, we study cohomology and stable splitting of BG for a finite group G having a 3–Sylow group $(\mathbb{Z}/3)^2$ or $E = 3_+^{1+2}$ respectively. In Section 7 and Section 8, we study cohomology of BG for groups G having a 7–Sylow subgroup $E = 7_+^{1+2}$, and

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the three exotic 7–local finite groups. In Section 9, we study their stable splitting. In Section 10 we study the cohomology and stable splitting of the Monster group $\mathbb{M}$ for $p=13$.

2 p–local finite groups over E

Recall that the extraspecial p–group p_+^{1+2} has a presentation as

$$p_+^{1+2}=\langle a,b,c|a^p=b^p=c^p=1,[a,b]=c,\ c\in \mathrm{Center}\rangle$$

and denote it simply by E in this paper. We consider p–local finite groups over E, which are generalization of groups whose p–Sylow subgroups are isomorphic to E.

The concept of the p–local finite groups arose in the work of Broto, Levi and Oliver [1] as a generalization of a classical concept of finite groups. The p–local finite group is stated as a triple $\langle S,F,L\rangle$ where S is a p–group, F is a saturated fusion system over a centric linking system L over S (for a detailed definition, see [1]). Given a p–local finite group, we can construct its classifying space $B\langle S,F,L\rangle$ by the realization $|L|_p^{\wedge}$. Of course if $\langle S,F,L\rangle$ is induced from a finite group G having S as a p–Sylow subgroup, then $B\langle S,F,L\rangle\cong BG$. However note that in general, there exist p–local finite groups which are not induced from finite groups (exotic cases).

Ruiz and Viruel recently determined $\langle p_+^{1+2},F,L\rangle$ for all odd primes p. We can check the possibility of existence of finite groups only for simple groups and their extensions. Thus they find new exotic 7–local finite groups.

The p–local finite groups $\langle E,F,L\rangle$ are classified by $\mathrm{Out}_F(E)$, number of F^{ec}–radical p–subgroup A (where $A\cong(\mathbb{Z}/p)^2$), and $\mathrm{Aut}_F(A)$ (for details see [9]). When a p–local finite group is induced from a finite group G, then we see easily that $\mathrm{Out}_F(E)\cong W_G(E)(=N_G(E)/E.C_G(E))$ and $\mathrm{Aut}_F(A)\cong W_G(A)$. Moreover A is F^{ec}–radical if and only if $\mathrm{Aut}_F(A)\supset SL_2(\mathbb{F}_p)$ by [9, Lemma 4.1]. When G is a sporadic simple group, F^{ec}–radical follows p–pure.

Theorem 2.1 (Ruiz and Viruel [9]) *If $p\neq 3,7,5,13$, then a p–local finite group $\langle E,F,L\rangle$ is isomorphic to one of the following types.*

(1) $E{:}W$ *for* $W\subset \mathrm{Out}(E)$ *and* $(|W|,p)=1$,

(2) $p^2{:}SL_2(\mathbb{F}_p).r$ *for* $r|(p-1)$,

(3) $SL_3(\mathbb{F}_p){:}H$ *for* $H\cong\mathbb{Z}/2,\mathbb{Z}/3$ *or* S_3.

When $p=3,5,7$ or 13, it is either of one of the previous types or of the following types.

(5) $^2F_4(2)'$, J_4, *for* p=3,

(6) Th for p=5,

(7) $He, He{:}2, Fi'_{24}, Fi_{24}, O'N, O'N{:}2$, *and three exotic* 7*–local finite groups for* p=7,

(8) $\mathbb{M}$ *for* p=13.

For case (1), we know that $H^*(E{:}W) \cong H^*(E)^W$. Except for these extensions and exotic cases, all $H^{\mathrm{even}}(G;\mathbb{Z})_{(p)}$ are studied by Tezuka and Yagita [11]. In [13], the author studied ways to distinguish $H^{\mathrm{odd}}(G;\mathbb{Z})_{(p)}$ and $H^*(G;\mathbb{Z}/p)$ from $H^{\mathrm{even}}(G;\mathbb{Z})_{(p)}$. The stable splittings for such BG are studied in [14]. However there were some errors in the cohomology of $^2F_4(2)', Fi'_{24}, \mathbb{M}$. In this paper, we study cohomology and stable splitting of BG for $p = 3,7$ and 13 mainly.

3 Cohomology

In this paper we mainly consider the cohomology $H^*(BG;\mathbb{Z})/(p,\sqrt{0})$ where $\sqrt{0}$ is the ideal generated by nilpotent elements. So we write it simply

$$H^*(BG) = H^*(BG;\mathbb{Z})/(p,\sqrt{0}).$$

Hence we have

$$H^*(B\mathbb{Z}/p) \cong \mathbb{Z}/p[y], \quad H^*(B(\mathbb{Z}/p)^2) \cong \mathbb{Z}/p[y_1,y_2] \text{ with } |y| = |y_i| = 2.$$

Let us write $(\mathbb{Z}/p)^2$ as A and let an A–subgroup of G mean a subgroup isomorphic to $(\mathbb{Z}/p)^2$.

The cohomology of the extraspecial p group $E = p_+^{1+2}$ is well known. In particular recall (Leary [6] and Tezuka–Yagita [11])

$$H^*(BE) \cong \big(\mathbb{Z}/p[y_1,y_2]/(y_1^py_2 - y_1y_2^p) \oplus \mathbb{Z}/p\{C\}\big) \otimes \mathbb{Z}/p[v], \tag{3–1}$$

where $|y_i| = 2, |v| = 2p, |C| = 2p-2$ and $Cy_i = y_i^p$, $C^2 = y_1^{2p-2} + y_2^{2p-2} - y_1^{p-1}y_2^{p-1}$. In this paper we write y_i^{p-1} by Y_i, and v^{p-1} by V, eg $C^2 = Y_1^2 + Y_2^2 - Y_1Y_2$. The Poincare series of the subalgebra generated by y_i and C are computed

$$\frac{1-t^{p+1}}{(1-t)(1-t)} + t^{p-1} = \frac{(1+\cdots+t^{p-1}) + t^{p-1}}{(1-t)} = \frac{(1+\cdots+t^{p-1})^2 - t^{2p-2}}{(1-t^{p-1})}.$$

From this Poincare series and (3–1), we get the another expression of $H^*(BE)$

$$H^*(BE) \cong \mathbb{Z}/p[C,v]\left\{y_1^iy_2^j \mid 0 \le i,j \le p-1, (i,j) \ne (p-1,p-1)\right\}. \tag{3–2}$$

The E conjugacy classes of A–subgroups are written by

$$A_i = \langle c, ab^i \rangle \text{ for } 0 \le i \le p-1$$
$$A_\infty = \langle c, b \rangle.$$

Letting $H^*(BA_i) \cong \mathbb{Z}/p[y,u]$ and writing $i^*_{A_i}(x) = x|A_i$ for the inclusion $i_{A_i}\colon A_i \subset E$, the restriction images are given by

$$\text{(3–3)} \quad y_1|A_i = y \text{ for } i \in \mathbb{F}_p,\ y_1|A_\infty = 0, \quad y_2|A_i = iy \text{ for } i \in \mathbb{F}_p,\ y_2|A_\infty = y,$$
$$C|A_i = y^{p-1}, \quad v|A_i = u^p - y^{p-1}u \text{ for all } i.$$

For an element $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in GL_2(\mathbb{F}_p)$, we can identify $GL_2(\mathbb{F}_p) \cong \mathrm{Out}(E)$ by

$$g(a) = a^\alpha b^\gamma, g(b) = a^\beta b^\delta,\ g(c) = c^{\det(g)}.$$

Then the action of g on the cohomology is given (see Leary [6] and Tezuka–Yagita [11, page 491]) by

$$\text{(3–4)} \quad g^*C = C,\ g^*y_1 = \alpha y_1 + \beta y_2,\ g^*y_2 = \gamma y_1 + \delta y_2,\ g^*v = (\det(g))v.$$

Recall that A is F^{ec}–radical if and only if $SL_2(\mathbb{F}_p) \subset W_G(A)$ (see Ruiz–Viruel [9, Lemma 4.1]).

Theorem 3.1 (Tezuka–Yagita [11, Theorem 4.3], Broto–Levi–Oliver [1]) *Let G have the p–Sylow subgroup E, then we have the isomorphism*

$$H^*(BG) \cong H^*(BE)^{W_G(E)} \cap_{A: F^{\mathrm{ec}}-\text{radical}} i_A^{*-1} H^*(BA)^{W_G(A)}.$$

In [1] and [11], proofs of the above theorem are given only for $H^*(BG;\mathbb{Z}_{(p)})$. A proof for $H^*(BG)$ is explained in Section 11.

4 Stable splitting

Martino–Priddy prove the following theorem of complete stable splitting.

Theorem 4.1 (Martino–Priddy [7]) *Let G be a finite group with a p–Sylow subgroup P. The complete stable splitting of BG is given by*

$$BG \sim \vee \operatorname{rank} A(Q,M) X_M$$

where indecomposable summands X_M range over isomorphic classes of simple $\mathbb{F}_p[\mathrm{Out}(Q)]$–modules M and over isomorphism classes of subgroups $Q \subset P$.

Remark This theorem also holds for p–local finite groups over P, because all arguments for the proofs are done about the induced maps from some fusion systems of P on stable homotopy types of related classifying spaces.

For the definition of rank $A(Q,M)$ see Martino and Priddy [7]. In particular, when Q is not a subretract (that is not a proper retract of a subgroup) of P (see [7, Definition 2]) and when $W_G(Q) \subset \mathrm{Out}(Q) \cong GL_n(\mathbb{F}_p)$ (see [7, Corollary 4.4 and the proof of Corollary 4.6]), the rank of $A(Q,M)$ is computed by

$$\mathrm{rank}\, A(Q,M) = \sum \dim_{\mathbb{F}_p}(\bar{W}_G(Q_i)M),$$

where $\bar{W}_G(Q_i) = \sum_{x \in W_G(Q_i)} x$ in $\mathbb{F}_p[GL_n(F_p)]$ and Q_i ranges over representatives of G–conjugacy classes of subgroups isomorphic to Q.

Recall that $\mathrm{Out}(E) \cong \mathrm{Out}(A) \cong GL_2(\mathbb{F}_p)$. The simple modules of $G = GL_2(\mathbb{F}_p)$ are well known. Let us think of A as the natural two-dimensional representation, and det the determinant representation of G. Then there are $p(p-1)$ simple $\mathbb{F}_p[G]$–modules given by $M_{q,k} = S(A)^q \otimes (\det)^k$ for $0 \le q \le p-1, 0 \le k \le p-2$. Harris and Kuhn [4] determined the stable splitting of abelian p–groups. In particular, they showed

Theorem 4.2 (Harris–Kuhn [4]) *Let $\tilde{X}_{q,k} = X_{M_{q,k}}$ (resp. $L(1,k)$) identifying $M_{q,k}$ as an $\mathbb{F}_p[\mathrm{Out}(A)]$–module (resp. $M_{0,k}$ as an $\mathbb{F}_p[\mathrm{Out}(\mathbb{Z}/p)]$–module). There is the complete stable splitting*

$$BA \sim \vee_{q,k}(q+1)\tilde{X}_{q,k} \vee_{q\neq 0} (q+1)L(1,q),$$

where $0 \le q \le p-1$, $0 \le k \le p-2$.

The summand $L(1,p-1)$ is usually written by $L(1,0)$.

It is also known $H^+(L(1,q)) \cong \mathbb{Z}/p[y^{p-1}]\{y^q\}$. Since we have the isomorphism

$$H^{2q}(BA) \cong (\mathbb{Z}/p)^{q+1} \cong H^{2q}((q+1)L(1,q)), \text{ for } 1 \le q \le p-1,$$

we get $H^*(\tilde{X}_{q,k}) \cong 0$ for $* \le 2(p-1)$.

Lemma 4.3 *Let H be a finite solvable group with $(p,|H|)=1$ and M be an $\mathbb{F}_p[H]$–module. Then we have $\bar{H}(M) = (\sum_{x\in H} x)M \cong M^H \cong H^0(H;M)$.*

Proof First assume $H = \mathbb{Z}/s$ and $x \in \mathbb{Z}/s$ its generator. Then

$$\bar{H}(M) = (1 + x + \cdots + x^{s-1})H.$$

Since $(1-x^s)=0$, we see $\mathrm{Ker}(1-x)\supset \mathrm{Image}(\bar{H})$. The facts that M is a $\mathbb{Z}/p$–module and $(|H|,p)=1$ imply $H^*(H;M)=0$ for $*>0$. Hence

$$\mathrm{Ker}(1-x)/\mathrm{Image}(1+\cdots+x^{s-1})\cong H^1(H;M)=0.$$

Thus we have $\bar{H}(M)=\mathrm{Ker}(1-x)=M^H$.

Suppose that H is a group such that

$$0\to H'\to H\xrightarrow{\pi} H''\to 0$$

and that $\bar{H}'(M')=(M')^{H'}$ (resp. $\bar{H}''(M'')=(M'')^{H''}$) for each $\mathbb{Z}/p[H']$–module M' (resp. $\mathbb{Z}/p[H'']$–module M''). Let σ be a (set theoretical) section of π and denote $\sigma(\bar{H}'')=\sum_{x\in H''}\sigma(x)\in \mathbb{F}_p[H]$. Then

$$\bar{H}(M)=\sigma(\bar{H}'')\bar{H}'(M)=\sigma(\bar{H}'')(M^{H'})=\bar{H}''(M^{H'})=(M^{H'})^{H''}=M^H$$

here the third equation follows from that we can identify $M^{H'}$ as an $\mathbb{F}_p[H'']$–module. Thus the lemma is proved. □

It is known from a result of Suzuki [10, Chapter 3 Theorem 6.17] that any subgroup of $SL_2(\mathbb{F}_{p^n})$, whose order is prime to p is isomorphic to a subgroup of $\mathbb{Z}/s$, $4S_4$, $SL_2(\mathbb{F}_3)$, $SL_2(\mathbb{F}_5)$ or

$$Q_{4n}=\langle x,y|x^n=y^2,y^{-1}xy=x^{-1}\rangle.$$

Corollary 4.4 *Let $H\subset GL_2(\mathbb{F}_p)$ with $(|H|,p)=1$ and H do not have a subgroup isomorphic to $SL_2(\mathbb{F}_3)$ nor $SL_2(\mathbb{F}_5)$. Let $G=A{:}\,H$ and let us write $BG\sim \vee_{q,k}\tilde{n}(H)_{q,k}\tilde{X}_{q,k}\vee_{q'}\tilde{m}(H)_{q'}L(1,q')$. Then*

$$\tilde{n}(H)_{q,k}=\mathrm{rank}_p\ H^0(H;M_{q,k}),$$
$$\tilde{m}(H)_{q'}=\mathrm{rank}_p\ H^{2q'}(BG).$$

In particular $\tilde{n}(H)_{q,0}=\mathrm{rank}_p\ H^{2q}(BG)$.

Proof Since $H^*(\tilde{X}_{q,k})\cong 0$ for $*\le 2(p-1)$, it is immediate that $\tilde{m}(H)_{q'}=\mathrm{rank}_p\ H^{2q'}(G)$. Since $GL_2(\mathbb{F}_p)\cong SL_2(\mathbb{F}_p).\mathbb{F}_p^*$ and $\mathbb{F}_p^*\cong\mathbb{Z}/(p-1)$, each subgroup H in the above satisfies the condition in Lemma 4.3. The first equation is immediate from the lemma. □

Next consider the stable splitting for the extraspecial p–group E. Dietz and Priddy prove the following theorem.

Theorem 4.5 (Dietz–Priddy [2]) *Let $X_{q,k} = X_{M_{q,k}}$ (resp. $L(2,k)$, $L(1,k)$) identifying $M_{q,k}$ as an $\mathbb{F}_p[\mathrm{Out}(E)]$–module (resp. $M_{p-1,k}$ as an $\mathbb{F}_p[\mathrm{Out}(A)]$–module, $\mathbb{F}_p[\mathrm{Out}(\mathbb{Z}/p)]$–module). There is the complete stable splitting*

$$BE \sim \vee_{q,k}(q+1)X_{q,k} \vee_k (p+1)L(2,k) \vee_{q\neq 0} (q+1)L(1,q) \vee L(1,p-1)$$

where $0 \le q \le p-1$, $0 \le k \le p-2$.

Remark Of course $\tilde{X}_{q,k}$ is different from $X_{q,k}$ but $\tilde{X}_{p-1,k} = L(2,k)$.

The number of $L(1,q)$ for $1 \le q < p-1$ is given by the following. Let us consider the decomposition $E/\langle c\rangle \cong \bar{A}_i \oplus \bar{A}_{-i}$ where $\bar{A}_i = \langle ab^i\rangle$ and $\bar{A}_{-0} = \bar{A}_\infty$. We consider the projection $\mathrm{pr}_i\colon E \to \bar{A}_i$. Let $x \in H^1(B\bar{A}_i;\mathbb{Z}/p) = \mathrm{Hom}(\bar{A}_i,\mathbb{Z}/p)$ be the dual of ab^i. Then

$$\mathrm{pr}_i^* x(a) = x(\mathrm{pr}_i(a)) = x(\mathrm{pr}_i(ab^iab^{-i})^{1/2}) = x((ab^i)^{1/2}) = 1/2,$$
$$\mathrm{pr}_i^* x(b) = x(\mathrm{pr}_i(ab^i(ab^{-i})^{-1})^{1/(2i)}) = 1/(2i).$$

Hence for $\beta(x) = y$, we have $\mathrm{pr}_i^*(y) = 1/2y_1 + 1/(2i)y_2$. Therefore the $k+1$ elements $(1/2y_1 + 1/(2i)y_2)^k$, $i = 0,\dots,k$ form a base of $H^{2k}(E/\langle c\rangle;\mathbb{Z}/p) \cong (\mathbb{Z}/p)^{k+1}$ for $k<p-1$. Thus we know the number of $L(1,k)$ is $k+1$ for $0<k<p-1$.

Recall that

$$H^{2q}(BE) \cong \begin{cases} (\mathbb{Z}/p)^{q+1} \cong H^{2q}((q+1)L(1,q)) \text{ for } 0 \le 2 \le p-2 \\ (\mathbb{Z}/p)^{q+2} \cong H^{2p-2}((p+1)L(1,0)) \text{ for } q = p-1. \end{cases}$$

This shows $H^*(X_{q,k}) \cong 0$ for $* \le 2p-2$ since so is $L(2,k)$. The number $n(G)_{q,k}$ of $X_{q,k}$ is only depend on $W_G(E) = H$. Hence we have the following corollary.

Corollary 4.6 *Let G have the p–Sylow subgroup E and $W_G(E) = H$. Let*

$$BG \sim \vee n(G)_{q,k} X_{q,k} \vee m(G,2)_k L(2,k) \vee m(G,1)_k L(1,k).$$

Then $n(G)_{q,k} = \tilde{n}(H)_{q,k}$ and $m(G,1)_k = \mathrm{rank}_p H^{2k}(G)$.

Let $W_G(E) = H$. We also compute the dominant summand by the cohomology $H^*(BE)^H \cong H^*(B(E\colon H))$. Let us write the $\mathbb{Z}/p$–module

$$X_{q,k}(H) = S(A)^q \otimes v^k \cap H^*(B(E\colon H)) \quad with\ S(A)^q = \mathbb{Z}/p\{y_1^q, y_1^{q-1}y_2, \dots, y_2^q\}.$$

Since the module $\mathbb{Z}/p\{v^k\}$ is isomorphic to the H–module $\det^k$, we have the following lemma.

Lemma 4.7 *The number $n_{q,k}(G)$ of $X_{q,k}$ in BG is given by $\mathrm{rank}_p(X_{q,k}(W_G(E)))$.*

Next problem is to seek $m(G,2)_k$. The number $p+1$ for the summand $L(2,k)$ in BE is given as follows. For each E–conjugacy class of A–subgroup $A_i = \langle c, ab^i\rangle, i \in \mathbb{F}_p \cup \infty$, we see

$$W_E(A_i) = N_E(A_i)/A_i = E/A_i \cong \mathbb{Z}/p\{b\} \quad b^*: ab^i \mapsto ab^i c.$$

Let $u = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ in $GL_2(\mathbb{F}_p)$ and $U = \langle u\rangle$ the maximal unipotent subgroup. Then we can identify $W_E(A_i) \cong U$ by $b \mapsto u$. For $y_1^s y_2^l \in M_{q,k}$ (identifying $H^*(BA) \cong S^*(A) = \mathbb{Z}/p[y_1, y_2]$), we can compute

$$\begin{aligned}\bar{W}_E(A) y_1^s y_2^l &= (1+u+\cdots+u^{p-1}) y_1^s y_2^l = \sum_{i=0}^{p-1} (y_1 + i y_2)^s y_2^l \\ &= \textstyle\sum_i \sum_t \binom{s}{t} i^t y_1^{s-t} y_2^t y_2^l \displaystyle = \sum_t \binom{s}{t} \sum_i i^t y_1^{s-t} y_2^{t+l}.\end{aligned}$$

Here $\sum_{i=0}^{p-1} i^t = 0$ for $1 \le t \le p-2$, and $= -1$ for $t = p-1$. Hence we know

$$\dim_p \bar{W}_G(A_i) M_{q,k} = \begin{cases} 0 & \text{for } 1 \le q \le p-2 \\ 1 & \text{for } q = p-1. \end{cases}$$

Thus we know that BE has just one $L(2,k)$ for each E–conjugacy A–subgroup A_i.

Lemma 4.8 *Let A be an F^{ec}–radical subgroup, ie $W_G(A) \supset SL_2(\mathbb{F}_p)$. Then $\bar{W}_G(A)(M_{q,k}) = 0$ for all k and $1 \le q \le p-1$.*

Proof The group $SL_2(\mathbb{F}_p)$ is generated by $u = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $u' = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. We know $\mathrm{Ker}(1-u) \cong \mathbb{Z}/p[y_1^p - y_2^{p-1} y_1, y_2]$ and $\mathrm{Ker}(1-u') \cong \mathbb{Z}/p[y_2^p - y_1^{p-1} y_2, y_1]$. Hence we get $(\mathrm{Ker}(1-u) \cap \mathrm{Ker}(1-u'))^* \cong 0$ for $0 < * \le p-1$. □

Proposition 4.9 *Let G have the p–Sylow subgroup E. The number of $L(2,0)$ in BG is given by*

$$m(G,2)_0 = \sharp_G(A) - \sharp_G(F^{\mathrm{ec}}A)$$

where $\sharp_G(A)$ (resp. $\sharp_G(F^{\mathrm{ec}}A)$) is the number of G–conjugacy classes of A–subgroups (resp. F^{ec}–radical subgroups).

Proof Let us write $K = E{:}W_G(E)$ and $H^*(BE)^{W_G(E)} = H^*(BK)$. From Theorem 3.1, we have

$$H^*(BG) \cong H^*(BK) \cap_{A:\, F^{\mathrm{ec}}-\mathrm{radical}} i_A^{*-1} H^*(BA)^{W_G(A)}. \tag{4–1}$$

Let A be an A–subgroup of K and $x \in W_K(A)$. Recall $A = \langle c, ab^i \rangle$ for some i. Identifying x as an element of $N_G(A) \subset E\colon \mathrm{Out}(E)$ We see $x\langle c \rangle = \langle c \rangle$ from (3–4) and since $\langle c \rangle$ is the center of E. Hence

$$W_K(A) \subset B = U\colon (\mathbb{F}_p^*)^2 \text{ the Borel subgroup.}$$

So we easily see that $\bar{W}_K(y_1^{p-1}) = \lambda y_2^{p-1}$ for some $\lambda \neq 0$ follows from $b^* y_i^{p-1} = y_i^{p-1}$ for $b = \text{diagonal} \in (\mathbb{F}_p)^{*2}$ and the arguments just before Lemma 4.8. We also see $\bar{W}_K(y_1^{p-1-i} y_2^i) = 0$ for $i{>}0$. Hence we have $m(K,2)_0 = \sharp_K(A)$. From the isomorphism (4–1), we have $m(G,2)_0 = \sharp_K(A) - \sharp_G(F^{\mathrm{ec}}A)$.

On the other hand $m(G,2)_0 \leq \sharp_G(A) - \sharp_G(F^{\mathrm{ec}}A)$ from the above lemma. Since $\sharp_K(A) \geq \sharp_G(A)$, we see that $\sharp_K(A) = \sharp_G(A)$ and get the proposition. $\square$

Corollary 4.10 *Let G have the p–Sylow subgroup E. The number of $L(1,0)$ in BG is given by*

$$m(G,1)_{p-1} = \mathrm{rank}_p\, H^{2(p-1)}(G) = \sharp_G(A) - \sharp_G(F^{\mathrm{ec}}A).$$

Proof Since $L(1,0) = L(1,p-1)$ is linked to $L(2,0)$, we know $m(G,1)_{p-1} = m(G,2)_0$. $\square$

Lemma 4.11 *Let $\xi \in \mathbb{F}_p^*$ be a primitive $(p-1)$th root of 1 and $G \supset E{:}\langle \mathrm{diag}(\xi,\xi) \rangle$. If $\xi^{3k} \neq 1$, then BG does not contain the summand $L(2,k)$, ie $m(G,2)_k = 0$.*

Proof It is sufficient to prove the case $G = E{:}\langle \mathrm{diag}(\xi,\xi) \rangle$. Let $G = E{:}\langle \mathrm{diag}(\xi,\xi) \rangle$. Recall $A_i = \langle c, ab^i \rangle$ and

$$\mathrm{diag}(\xi,\xi)\colon ab^i \mapsto (ab^i)^{\xi}, \quad c \mapsto c^{\xi^2}.$$

So the Weyl group is $W_G(A_i) = U\colon \langle \mathrm{diag}(\xi^2,\xi) \rangle$. For $v = \lambda y_1^{p-1} + \cdots \in M_{q,k}$, we have

$$\bar{W}_G(A_i)v = \sum_{i=0}^{p-2} (\xi^{3i})^k \,\mathrm{diag}(\xi^{2i},\xi^i)(1 + \cdots + u^{p-1})v = \sum_{i=0}^{p-2} \xi^{3ik} \lambda y_2^{p-1}.$$

Thus we get the lemma from $\sum_{i=0}^{p-2} \xi^{3ik} = 0$ for $3k \neq 0 \mod (p-1)$ and $= -1$ otherwise. $\square$

5 Cohomology and splitting of $B(\mathbb{Z}/3)^2$

In this section, we study the cohomology and stable splitting of BG for G having a 3–Sylow subgroup $(\mathbb{Z}/3)^2 = A$. In this and next sections, p always means 3. Recall $\mathrm{Out}(A) \cong GL_2(\mathbb{F}_3)$ and $\mathrm{Out}(A)'$ consists the semidihedral group

$$SD_{16} = \langle x, y | x^8 = y^2 = 1, yxy^{-1} = x^3 \rangle.$$

Every 3–local finite group G over A is of type $A{:}W$, $W \subset SD_{16}$. There is the SD_{16}–conjugacy classes of subgroups(here $B \longleftarrow C$ means $B \supset C$)

$$SD_{16} \begin{cases} \longleftarrow Q_8 \longleftarrow \mathbb{Z}/4 \\ \longleftarrow \mathbb{Z}/8 \longleftarrow \mathbb{Z}/4 \longleftarrow \mathbb{Z}/2 \longleftarrow 0 \\ \longleftarrow D_8 \longleftarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \longleftarrow \mathbb{Z}/2 \end{cases}$$

We can take generators of subgroups in $GL_2(\mathbb{F}_3)$ by the matrices

$$\mathbb{Z}/8 = \langle l \rangle, Q_8 = \langle w, k \rangle, D_8 = \langle w', k \rangle, \mathbb{Z}/4 = \langle w \rangle,$$
$$\mathbb{Z}/4 = \langle k \rangle, \mathbb{Z}/2 \oplus \mathbb{Z}/2 = \langle w', m \rangle, \mathbb{Z}/2 = \langle m \rangle, \mathbb{Z}/2 = \langle w' \rangle,$$

where $l = \left(\begin{smallmatrix} 0 & 1 \\ 1 & -1 \end{smallmatrix}\right)$, $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $k = l^2 = \left(\begin{smallmatrix} 1 & -1 \\ -1 & -1 \end{smallmatrix}\right)$, $w' = wl = \left(\begin{smallmatrix} 1 & -1 \\ 0 & -1 \end{smallmatrix}\right)$ and $m = w^2 = k^2 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Here we note that k and w are $GL_2(\mathbb{F}_3)$–conjugate, in fact $uku^{-1} = w$. Hence we note that

$$H^*(B(A{:}\langle k \rangle)) \cong H^*(B(A{:}\langle w \rangle)).$$

The cohomology of A is given $H^*(BA) \cong \mathbb{Z}/3[y_1, y_2]$, and the following are immediately

$$H^*(BA)^{\langle m \rangle} \cong \mathbb{Z}/3[y_1^2, y_2^2]\{1, y_1y_2\} \quad H^*(BA)^{\langle w' \rangle} \cong \mathbb{Z}/3[y_1 + y_2, y_2^2].$$

Let us write $Y_i = y_i^2$ and $t = y_1y_2$. The k–action is given $Y_1 \mapsto Y_1 + Y_2 + t$, $Y_2 \mapsto Y_1 + Y_2 - t$, $t \mapsto -Y_1 + Y_2$. So the following are invariant

$$a = -Y_1 + Y_2 + t,\ a_1 = Y_1(Y_1 + Y_2 + t),\ a_2 = Y_2(Y_1 + Y_2 - t),\ b = t(Y_1 - Y_2).$$

Here we note that $a^2 = a_1 + a_2$ and $b^2 = a_1a_2$. We can prove the invariant ring is

$$H^*(BA)^{\langle k \rangle} \cong \mathbb{Z}/3[a_1, a_2]\{1, a, b, ab\}.$$

Next consider the invariant under $Q_8 = \langle w, k \rangle$. The action for w is $a \mapsto -a$, $a_1 \leftrightarrow a_2$, $b \mapsto b$. Hence we get

$$H^*(BA)^{Q_8} \cong \mathbb{Z}/3[a_1 + a_2, a_1a_2]\{1, b\}\{1, (a_1 - a_2)a\}.$$

Let us write $S = \mathbb{Z}/3[a_1 + a_2, a_1 a_2]$ and $a' = (a_1 - a_2)a$. The action for l is given $l\colon Y_1 \mapsto Y_2 \mapsto Y_1 + Y_2 + t \mapsto Y_1 + Y_2 - t \mapsto Y_1$. Hence $l\colon a \mapsto -a,\ a_1 \leftrightarrow a_2,\ b \mapsto -b$. Therefore we get $H^*(BA)^{\langle l\rangle} \cong S\{1, a', ab, (a_1 - a_2)b\}$.

The action for $w'\colon Y_1 \mapsto Y_1 + Y_2 + t,\ Y_2 \mapsto Y_2$, implies that $w'\colon a \mapsto a,\ a_i \mapsto a_i,\ b \mapsto -b$. Then we can see

$$H^*(BA)^{D_8} = H^*(BA)^{\langle k, w'\rangle} \cong \mathbb{Z}/3[a_1, a_2]\{1, a\} \cong S\{1, a, a_1, a'\}.$$

We also have

$$H^*(BA)^{SD_{16}} \cong H^*(BA)^{Q_8} \cap H^*(BA)^{\mathbb{Z}/8} \cong S\{1, a'\}.$$

Recall the Dickson algebra $DA = \mathbb{Z}/3[\tilde{D}_1, \tilde{D}_2] \cong H^*(BA)^{GL_2(\mathbb{F}_3)}$ where $\tilde{D}_1 = Y_1^3 + Y_1^2 Y_2 + Y_1 Y_2^2 + Y_2^3 = (a_2 - a_1)a = a'$ and $\tilde{D}_2 = (y_1^3 y_2 - y_1 y_2^3)^2 = a_1 a_2$. Using $a^2 = (a_1 + a_2)$ and $\tilde{D}_1^2 = a^6 - a_1 a_2 a^2$, we can write

$$H^*(BA)^{SD_{16}} \cong \mathbb{Z}/3[a^2, \tilde{D}_2]\{1, \tilde{D}_1\} \cong DA\{1, a^2, a^4\}.$$

Theorem 5.1 *Let $G = (\mathbb{Z}/3)^2\colon H$ for $H \subset SD_{16}$. Then BG has the stable splitting given by*

$$\overset{\tilde{X}_{0,0}}{\longleftarrow} SD_{16} \begin{cases} \overset{\tilde{X}_{0,1}}{\longleftarrow} Q_8 \\[2ex] \overset{\tilde{X}_{2,1}}{\longleftarrow} \mathbb{Z}/8 \overset{\tilde{X}_{2,0} \vee \tilde{X}_{0,1} \vee L(1,0)}{\longleftarrow} \mathbb{Z}/4 \overset{2\tilde{X}_{2,0} \vee 2\tilde{X}_{2,1} \vee 2L(1,0)}{\longleftarrow} \mathbb{Z}/2 \overset{2\tilde{X}_{1,0} \vee 2\tilde{X}_{1,1} \vee 2L(1,1)}{\longleftarrow} 0 \\[2ex] \overset{\tilde{X}_{2,0} \vee L(1,0)}{\longleftarrow} D_8 \overset{\tilde{X}_{2,0} \vee \tilde{X}_{2,1} \vee L(1,0)}{\longleftarrow} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \overset{\tilde{X}_{1,0} \vee \tilde{X}_{1,1} \vee L(1,1)}{\longleftarrow} \mathbb{Z}/2 \end{cases}$$

where $\overset{\tilde{X}_1}{\leftarrow} \cdots \overset{\tilde{X}_s}{\leftarrow} H$ means $B((\mathbb{Z}/3)^2{:}H) \sim \tilde{X}_1 \vee \cdots \vee \tilde{X}_s$.

For example

$$B(E\colon SD_{16}) \sim \tilde{X}_{0,0}, \quad B(E\colon Q_8) \sim \tilde{X}_{0,0} \vee \tilde{X}_{0,1}, \quad B(E\colon \mathbb{Z}/8) \sim \tilde{X}_{0,0} \vee \tilde{X}_{2,1}.$$

Main parts of the above splittings are given by the author in [14, (6)] by direct computations of $\bar{W}_G(A)$ (see [14, page 149]). However we get the theorem more easily by using cohomology here. For example, let us consider the case $G = A\colon \langle k\rangle$. The cohomology

$$H^0(BG) \cong \mathbb{Z}/3, \quad H^2(BG) \cong 0, H^4(BG) \cong \mathbb{Z}/3$$

implies that BG contains just one $\tilde{X}_{0,0}, \tilde{X}_{2,0}, L(1,0)$ but does not $\tilde{X}_{1,0}, L(1,1)$. Since $\det(k) = 1$, we also know that $\tilde{X}_{0,1}, \tilde{X}_{2,1}$ are contained. So we can see

$$B(A{:}\mathbb{Z}/4) \sim \tilde{X}_{0,0} \vee \tilde{X}_{0,1} \vee \tilde{X}_{2,0} \vee \tilde{X}_{2,1} \vee L(1,0).$$

Next consider the case $G' = A{:}\langle l\rangle$. The fact $H^4(G) \cong 0$ implies that BG' does not contain $\tilde{X}_{2,0}, L(1,0)$. The determinant $\det(l) = -1$, and $l{:}\ a \mapsto -a$ shows that BG' contains $\tilde{X}_{2,1}$ but does not contain $\tilde{X}_{0,1}$. Hence we know $BG' \sim \tilde{X}_{0,0} \vee \tilde{X}_{2,1}$. Moreover we know $BA{:}SD_{16} \sim \tilde{X}_{0,0}$ since $w{:}\ a \to -a$ but $\det(w) = 1$. Thus we have the graph

$$\overset{\tilde{X}_{0,0}}{\leftarrow} SD_{16} \overset{\tilde{X}_{2,1}}{\longleftarrow} \mathbb{Z}/8 \overset{\tilde{X}_{2,0}\vee\tilde{X}_{0,1}\vee L(1,0)}{\longleftarrow} \mathbb{Z}/4.$$

Similarly we get the other parts of the above graph.

Corollary 5.2 *Let $S = \mathbb{Z}/3[a_1 + a_2, a_1a_2]$. Then we have the isomorphisms*

$$\begin{aligned} H^*(\tilde{X}_{0,0}) &\cong S\{1, \tilde{D}_1\} \\ H^*(\tilde{X}_{0,1}) &\cong S\{b, \tilde{D}_1 b\} \\ H^*(\tilde{X}_{2,1}) &\cong S\{ab, (a_1 - a_2)b\} \\ H^*(\tilde{X}_{2,0} \vee L(1,0)) &\cong S\{a, a_1 - a_2\} \cong DA\{a, a^2, a^3\}. \end{aligned}$$

Here we write down the decomposition of cohomology for a typical case

$$\begin{aligned} H^*(BA)^{\langle k\rangle} &\cong S\{1, a_1 - a_2\}\{1, a\}\{1, b\} \\ &\cong S\{1, a(a_1 - a_2), b, ba(a_1 - a_2), ab, (a_1 - a_2)b, a, (a_1 - a_2)\} \\ &\cong H^*(\tilde{X}_{0,0}) \oplus H^*(\tilde{X}_{0,1}) \oplus H^*(\tilde{X}_{2,1}) \oplus H^*(\tilde{X}_{2,0} \vee L(1,0)). \end{aligned}$$

6 Cohomology and splitting of $B3^{1+2}_+$.

In this section we study the cohomology and stable splitting of BG for G having a 3–Sylow subgroup $E = 3^{1+2}_+$. In the splitting for BE, the summands $X_{q,k}$ are called dominant summands. Moreover the summands $L(2,0) \vee L(1,0)$ is usually written by $M(2)$.

Lemma 6.1 *If $G \supset E{:}\langle \mathrm{diag}(-1,-1)\rangle$ identifying $\mathrm{Out}(E) \cong GL_2(\mathbb{F}_3)$ and G has E as a 3–Sylow subgroup, then*

$$BG \sim (\textit{dominant summands}) \vee (\sharp_G(A) - \sharp_G(F^{\mathrm{ec}}A)(M(2)).$$

Proof From Lemma 4.11, we know $m(G,2)_1 = 0$ ie $L(2,1)$ is not contained. The summand $L(1,1)$ is also not contained, since $H^2(BE)^{\langle \mathrm{diag}(-1,-1)\rangle} \cong 0$. The lemma is almost immediately from Proposition 4.9 and Corollary 4.10. □

Theorem 6.2 *If G has a 3–Sylow subgroup E, then BG is homotopic to the classifying space of one of the following groups. Moreover the stable splitting is given by the graph so that $\overset{X_1}{\leftarrow} \cdots \overset{X_s}{\leftarrow} G$ means $BG \sim X_1 \vee \cdots \vee X_i$ and $EH = E{:}\,H$ for $H \subset SD_{16}$*

$$\overset{X_{0,0}}{\leftarrow} J_4 \begin{cases} \overset{M(2)}{\leftarrow} ESD_{16} \begin{cases} \overset{X_{0,1}}{\leftarrow} EQ_8 \\ \overset{X_{2,1}}{\leftarrow} E\mathbb{Z}/8 \overset{\substack{X_{2,0}\vee X_{0,1}\\ \vee M(2)}}{\longleftarrow} E\mathbb{Z}/4 \overset{\substack{2X_{2,0}\vee 2X_{2,1}\\ \vee 2M(2)}}{\longleftarrow} E\mathbb{Z}/2 \overset{\substack{2X_{1,0}\vee 2X_{1,1}\vee\\ 4L(2,1)\vee 2L(1,1)}}{\longleftarrow} E \\ \overset{X_{2,0}\vee M(2)}{\longleftarrow} ED_8 \overset{X_{2,0}\vee X_{2,1}\vee M(2)}{\longleftarrow} E(\mathbb{Z}/2)^2 \overset{\substack{X_{1,0}\vee X_{1,1}\vee\\ 2L(2,1)\vee L(1,1)}}{\longleftarrow} E\mathbb{Z}/2 \end{cases} \\ \overset{X_{2,0}}{\leftarrow} {}^2F_4(2)' \overset{M(2)}{\leftarrow} M_{24} \overset{X_{2,0}\vee X_{2,1}}{\longleftarrow} M_{12} \overset{M(2)}{\leftarrow} \mathbb{F}_3^2{:}\,GL_2(\mathbb{F}_3) \overset{\substack{X_{1,0}\vee X_{1,1}\vee\\ L(2,1)\vee L(1,1)}}{\longleftarrow} \mathbb{F}_3^2{:}\,SL_2(\mathbb{F}_3) \end{cases}$$

Proof All groups except for $E, E{:}\,\langle w'\rangle$ and $\mathbb{F}_3^2{:}\,SL_2(\mathbb{F}_3)$ contain $E{:}\,\langle\,\mathrm{diag}(-1,-1)\rangle$. Hence we get the theorem from Corollary 4.4, Theorem 5.1 and Lemma 6.1, except for the place for $H^*(BE{:}\,\langle w'\rangle)$ and $H^*(\mathbb{F}_3^2{:}\,SL_2(\mathbb{F}_3))$.

Let $G = E{:}\,\langle w'\rangle$. Note $w'{:}\, y_1 \mapsto y_1 - y_2, y_2 \mapsto -y_2, v \mapsto -v$. Hence $H^2(G) \cong \mathbb{Z}/3\{y_1 + y_2\}$. So BG contains one $L(1,1)$. Next consider the number of $L(2,0)$, $L(2,1)$. The G–conjugacy classes of A–subgroups are $A_0, A_2, A_1 \sim A_\infty$. The Weyl groups are

$$W_G(A_\infty) \cong U, \quad W_G(A_2) \cong U{:}\langle\,\mathrm{diag}(-1,-1)\rangle, \quad W_G(A_0) \cong U{:}\langle\,\mathrm{diag}(-1,1)\rangle,$$

eg $N_G(A_0)/A_0$ is generated by b, w' which is represented by $u, \mathrm{diag}(-1,1)$ respectively. By the arguments similar to the proof of Lemma 4.11, we have that

$$\begin{cases} \dim(\overline{W}_G(A_i)M_{2,0}) = 1 \text{ for all } i \\ \dim(\overline{W}_G(A_i)M_{2,1}) = 1, 1, 0 \text{ for } i = \infty, 2, 0 \text{ respectively.} \end{cases}$$

Thus we show $BG \supset 3L(2,0) \vee 2L(2,1)$ and we get the graph for $G = E{:}\langle w'\rangle$.

For the place $G = \mathbb{F}_3^2{:}\,SL_2(\mathbb{F}_3)$, we see $W_G(A_\infty) \cong SL_2(\mathbb{F}_3)$. We also have

$$\begin{cases} \dim(\overline{W}_G(A_i)M_{2,0}) = 0, 1, 1 \text{ for } i = \infty, 2, 0 \text{ respectively} \\ \dim(\overline{W}_G(A_i)M_{2,1}) = 0, 1, 0 \text{ for } i = \infty, 2, 0 \text{ respectively.} \end{cases}$$

Thus we can see the graph for the place $H^*(\mathbb{F}_3^2{:}\,SL_2(\mathbb{F}_3))$. □

Remark From Tezuka–Yagita [11], Yagita [13] and Theorem 2.1, we have the following homotopy equivalences (localized at 3).

$$BJ_4 \cong BRu, \quad BM_{24} \cong BHe, \quad BM_{12} \cong BGL_3(\mathbb{F}_3)$$

$$B(E\colon SD_{16}) \cong BG_2(2) \cong BG_2(4), \quad B(E\colon D_8) \cong BHJ \cong BU_3(3).$$

We write down the cohomologies explicitly (see also Tezuka–Yagita [11] and Yagita [14]). First we compute $H^*(B(E\colon H))$. The following cohomologies are easily computed

$$H^*(BE)^{\langle m\rangle} \cong \mathbb{Z}/3[C,v]\{1, y_1y_2, Y_1, Y_2\}, \quad H^*(BE)^{\langle w\rangle} \cong \mathbb{Z}/3[C,v]\{1, Y_1+Y_2\}.$$
$$H^*(BE)^{\langle k\rangle} \cong \mathbb{Z}/3[C,v]\{1,a\} \text{ where } a = -Y_1+Y_2+y_1y_2,\ C^2 = a^2.$$

Recall that $V = v^{p-1}$ and C multiplicatively generate $H^*(BE)^{\mathrm{Out}(E)}$. Let us write

$$CA = \mathbb{Z}/p[C,V] \cong H^*(BE)^{\mathrm{Out}(E)}.$$

Then we have

$$H^*(BE)^{\langle w'\rangle} \cong CA\{1, y_1', Y_1', Y_2, Y_2y_1', y_2v, y_1'y_2v, Y_1'y_2v\} \text{ with } y_1' = y_1+y_2$$
$$H^*(BE)^{\langle w',m\rangle} \cong CA\{1, a, a', Y_2\} \text{ where } a' = (t+Y_2)v = y_1'y_2v.$$

We can compute

$$H^*(BE)^{Q_8} \cong H^*(BE)^{\langle k\rangle} \cap H^*(BE)^{\langle w\rangle} \cong \mathbb{Z}/3[C,v] \cong CA\{1,v\},$$
$$H^*(BE)^{D_8} \cong CA\{1,a\}, \quad H^*(BE)^{\langle l\rangle} \cong CA\{1,av\}.$$

Hence we have $H^*(BE)^{SD_{16}} \cong CA$.

Let $D_1 = C^p + V$ and $D_2 = CV$. Then it is known that

$$D_1|A_i = \tilde{D}_1,\ D_2|A_i = \tilde{D}_2 \text{ for all } i \in \mathbb{F}_p \cup \infty.$$

So we also write $DA \cong \mathbb{Z}/p[D_1, D_2]$. Since $CD_1 - D_2 = C^{p+1}$, we can write $CA \cong DA\{1, C, C^2, \ldots, C^p\}$.

Now return to the case $p = 3$ and we get (see [11])

$$H^*(BJ_4) \cong H^*(BE)^{SD_{16}} \cap i_0^{*-1} H^*(BA_0)^{GL_2(\mathbb{F}_3)} \cong DA.$$

Proposition 6.3 *There are isomorphisms for $|a''| = 4$,*

$$H^*({}^2F_4(2)') \cong DA\{1, (D_1 - C^3)a''\}, \quad H^*(M_{24}) \cong DA \oplus CA\{a''\}.$$

Proof Let $G = M_{24}$. Then G has just two G–conjugacy classes of A–subgroups

$$\{A_0, A_2\}, \quad \{A_1, A_\infty\}.$$

It is known that one is F^{ec}–radical and the other is not. Suppose that A_0 is F^{ec}–radical. Then $W_G(A_0) \cong GL_2(\mathbb{F}_3)$. Let $a'' = a + C$. Then

$$a''|A_0 = (-Y_1 + Y_2 + y_1 y_2 + C)|A_0 = 0, \quad a''|A_\infty = -Y.$$

By Theorem 3.1

$$H^*(BM_{24}) \cong H^*(BE)^{D_8} \cap i_{A_0}^{*-1} H^*(BA_0)^{W_G(A_0)},$$

we get the isomorphism for M_{24}. When A_∞ is a F^{ec}–radical, we take $a'' = a - c$. Then we get the same result.

For $G =^2 F_4(2)'$, the both conjugacy classes are F^{ec}–subgroups and $W_G(A_\infty) \cong GL_2(\mathbb{F}_3)$. Hence (for case $a'' = a + C$)

$$H^*(B^2F_4(2)') \cong H^*(BM_{24}) \cap i_{A_\infty}^{*-1} H^*(BA_\infty)^{GL_2(\mathbb{F}_3)}.$$

We know

$$(D_1 - C^3)a''|A_0 = 0, \quad (D_1 - C^3)a''|A_\infty = -VY = -\tilde{D}_2.$$

Thus we get the cohomology of ${}^2F_4(2)'$. □

Remark In [11; 14], we take

$$(\mathbb{Z}/2)^2 = \langle \operatorname{diag}(\pm 1, \pm 1) \rangle, \quad D_8 = \langle \operatorname{diag}(\pm 1, \pm 1), w \rangle.$$

For this case, the M_{24}–conjugacy classes of A–subgroups are $A_0 \sim A_\infty$, $A_1 \sim A_2$, and we can take $a'' = C - Y_1 - Y_2$. The expressions of $H^*(M_{12})$, $H^*(A{:}GL_2(\mathbb{F}_3))$ become more simple (see [11; 14]), in fact,

$$H^*(B^2F_4(2)') \cong DA\{1, (Y_1 + Y_2)V\}.$$

Remark [11, Corollary 6.3] and [14, Corollary 3.7] were not correct. This followed from an error in [11, Theorem 6.1]. This theorem is only correct with adding the assumption that there are exactly two G conjugacy classes of A–subgroups such that one is p–pure and the other is not. This assumption is always satisfied for sporadic simple groups but not for ${}^2F_4(2)'$.

Corollary 6.4 *There are isomorphisms of cohomologies*

$$H^*(X_{2,0}) \cong DA\{D_2\}, \quad H^*(X_{2,1}) \cong CA\{av\} \text{ where } (av)^2 = CD_2$$
$$H^*(X_{0,1}) \cong CA\{v\}, \quad H^*(M(2)) \cong DA\{C, C^2, C^3\} \text{ where } C^4 = CD_1 - D_2.$$

Here we write down typical examples. First recall

$$CA \cong DA\{1, C, C^2, C^3\} \cong H^*(X_{0,0}) \oplus H^*(M(2))$$
$$CA\{C\} \cong DA\{C, C^2, C^3,\ D_2\} \cong H^*(M(2)) \oplus H^*(X_{2,0}).$$

Thus the decomposition for $H^*(BE)^{D_8}$ gives the isomorphisms

$$CA\{1, a''\} \cong CA\{1, C\} \cong H^*(X_{0,0}) \oplus H^*(M(2)) \oplus H^*(X_{2,0}) \oplus H^*(M(2)).$$

Similarly the decomposition for $H^*(BE)^{\langle k\rangle}$ gives the isomorphism

$$CA\{1, a, v, av\} \cong H^*(BE)^{D_8} \oplus H^*(X_{0,1}) \oplus H^*(X_{2,1}).$$

We recall here Lemma 4.7 and the module

$$X_{q,k}(\langle k\rangle) = S(V)^q \otimes v^k \cap H^*(B(E{:}\langle k\rangle)).$$

Then it is easily seen that

$$X_{0,0}(\langle k\rangle) = \{1\}, X_{2,0}(\langle k\rangle) = \{a\}, X_{0,1}(\langle k\rangle) = \{v\}, X_{2,1}(\langle k\rangle) = \{av\}.$$

Hence we also see $B(E{:}\langle k\rangle)$ has the dominant summands $X_{0,0} \vee X_{2,0} \vee X_{0,1} \vee X_{2,1}$. Moreover it has non dominant summands $2M(2)$ since $H^4(B(E{:}\langle k\rangle)) \cong \mathbb{Z}/3\{C, a\}$. Thus we can give an another proof of Theorem 6.2 from Lemma 4.7 and the cohomologies $H^*(BG)$.

7 Cohomology for $B7_+^{1+2}$ I.

In this section, we assume $p = 7$ and $E = 7_+^{1+2}$. We are interested in groups $O'N, O'N{:}2, He, He{:}2, Fi'_{24}, Fi_{24}$ and three exotic 7–local groups. Denote them by RV_1, RV_2, RV_3 according the numbering in [9]. We have the diagram from Ruiz and Viruel

$$\begin{cases} \xleftarrow{3SD_{32}} \overset{SL_2(\mathbb{F}_7):2}{RV_3} \xleftarrow{3SD_{16}} \overset{SL_2(\mathbb{F}_7):2, SL_2(\mathbb{F}_7):2}{RV_2} \xleftarrow{3SD_{16}} \overset{SL_2(\mathbb{F}_7):2}{O'N{:}2} \xleftarrow{3D_8} \overset{SL_2(\mathbb{F}_7):2, Sl_2(\mathbb{F}_7):2}{O'N} \\ \\ \xleftarrow{6^2:2} \overset{SL_2(\mathbb{F}_7):2, GL_2(\mathbb{F}_7)}{RV_1} \xleftarrow{6^2:2} \overset{SL_2(\mathbb{F}_7):2}{Fi_{24}} \xleftarrow{6S_3} \overset{SL_2(\mathbb{F}_7):2, SL_2(\mathbb{F}_7):2}{Fi'_{24}} \xleftarrow{6S_3} \overset{SL_2(\mathbb{F}_7):2}{He{:}2} \xleftarrow{3S_3} \overset{SL_2(\mathbb{F}_7)}{He} \end{cases}$$

Here $\xleftarrow{H} \overset{W_1,\ldots,W_2}{G}$ means $W_G(E) \cong H, W_i = W_G(A_i)$ for G–conjugacy classes of $F^{\mathrm{ec}}A-$ subgroups A_i.

In this section, we study the cohomology of $O'N, RV_2, RV_3$. First we study the cohomology of $G = O'N$. The multiplicative generators of $H^*(BE)^{3D_8}$ are still studied in [11, Lemma 7.10]. We will study more detailed cohomology structures here.

Lemma 7.1 *There is the CA–module isomorphism*

$$H^*(BE)^{3D_8} \cong CA\{1,a,a^2,a^3/V,a^4/V,a^5/V,b,ab/V,a^2b/V,d,ad,a^2d\},$$

where $a=(y_1^2+y_2^2)v^2$, $b=y_1^2y_2^2v^4$ and $d=(y_1y_2^3-y_1^3y_2)v$.

Proof The group $3D_8\subset GL_2(\mathbb{F}_7)$ is generated by $\mathrm{diag}(-1,1)$, $(2,2)$ and $w=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$. If $y_1^iy_2^jv^k$ is invariant under $\mathrm{diag}(-1,1), \mathrm{diag}(1,-1)$ and $\mathrm{diag}(2,2)$, then $i=j=k \bmod(2)$ and $i+j+2k=0 \bmod(3)$. When $i,j\le 6, k\le 5$ but $(i,j)\ne(6,6)$, the invariant monomials have the following terms, $y_1^2v^2$, $y_1^4v^4$, y_1^6, $y_1^2y_2^2v^4$, $y_1^4y_2^4v^2$, $y_1y_2v^5$, $y_1^3y_2^3v^3$, $y_1^5y_2^5v$, $y_1^2y_2^4$, $y_1^2y_2^6v^2$, $y_1^4y_2^6v^4$, $y_1y_2^3v$, $y_1y_2^5v^3$, $y_1^3y_2^5v^5$ and terms obtained by exchanging y_1 and y_2. Recall that w: $y_1\mapsto y_2, y_2\mapsto -y_1$ and $v\to v$. From the expression of (3–2), we have

$$H^*(BE)^{3D_8}\cong CA\{1,a,a^2,a',b,b',c,c',c'',d,ad,bd\}$$

where $a=(y_1^2+y_2^2)v^2$, $a'=y_1^6+y_2^6$, $b=y_1^2y_2^2v^4$, $b'=y_1^4y_2^4v^2$, $c=(y_1^2y_2^4+y_1^4y_2^2)$, $c'=(y_1^2y_2^6+y_1^6y_2^2)v^2$, $c''=(y_1^4y_2^6+y_1^6y_2^4)v^4$, $d=(y_1y_2^3-y_1^3y_2)v$, $ad=(y_1y_2^5-y_1^5y_2)v^3$ and $bd=(y_1^3y_2^5-y_1^5y_2^3)v^5$. Here $a^2d=bd$ from $(y_1^6-y_2^6)y_1y_2=0$ in $H^*(BE)$. It is easily seen that $b'V=b^2$, $cV=ab$, $c'V=(a^2-2b)b$ and $c''V=ab^2$. Moreover we get

$$\begin{aligned}
a^3/V &= (y_1^2+y_2^2)^3=(y_1^6+y_2^6)+3y_1^2y_2^2(y_1^2+y_2^2)=a'+3ab\\
a^4/V &= (y_1^2+y_2^2)^4v^2=((y_1^8+y_2^8)+4y_1^2y_2^2(y_1^4+y_2^4)+6y_1y_2^4)v^2\\
&= aC+4c'+6b'\\
a^5/V &= ((y_1^{10}+y_2^{10})+5y_1^2y_2^2(y_1^6+y_2^6)+10y_1^4y_2^4(y_1^2+y_2^2))v^4\\
&= c'C+10bC+10c''.
\end{aligned}$$

Hence, we can take generators $a^4/V, a^5/V, ab/V, a^2b/V$ for b',c'',c,c' respectively, and get the lemma. □

Note that the computations shows

$$\begin{aligned}
a^6 &= (y_1^2+y_2^2)^6v^{12}=(y_1^{12}-y_1^{10}y_2^2+y_1^8y_2^4-y_1^6y_2^6+y_1^4y_2^8-y_1^2y_2^{10}+y_2^{12})V^2\\
&= (y_1^{12}-y_1^6y_2^6+y_2^{12})V^2=C^2V^2=D_2^2,
\end{aligned}$$

where we use the fact $y_1^7y_2-y_1y_2^7=0$.

Lemma 7.2 $H^*(BE)^{3SD_{16}}\cong CA\{1,a,a^2,a^3/V,a^4/V,a^5/V\}$.

Proof Take the matrix $k'=\begin{pmatrix}-1&1\\-1&-1\end{pmatrix}$ such that $\langle 3D_8,k'\rangle\cong 3SD_{16}$. Then we have

$$k'^*\colon a=(y_1^2+y_2^2)v^2\mapsto((-y_1+y_2)^2+(-y_1-y_2)^2)(2v)^2=a,$$
$$b=y_1^2y_2^2v^4\mapsto(y_1^2-y_2^2)^2(2v)^4=2(a^2-4b)=2a^2-b.$$

(If we take $\tilde{b}=b-a^2$, then $k'^*\colon\tilde{b}\mapsto-\tilde{b}$.) Similarly we can compute $k'\colon d\mapsto -d$. Then the lemma is almost immediate from the preceding lemma. □

Lemma 7.3 $H^*(BE)^{3SD_{32}}\cong CA\{1,a^2,a^4/V\}$.

Proof Take the matrix $l'=\begin{pmatrix}-1&3\\-3&-1\end{pmatrix}$ so that $l'^2=k'$ and $\langle 3SD_8,l'\rangle\cong 3SD_{32}$. We see that

$$l'^*\colon a=(y_1^2+y_2^2)v^2\mapsto((-y_1+3y_2)^2+(-3y_1-y_2)^2)(3v)^2=-a,$$

which shows the lemma. □

Theorem 7.4 *There is the isomorphism with $C'=C-a^3/V$*

$$H^*(BO'N)\cong DA\{1,a,a^2,b,ab,a^2b\}\oplus CA\{d,ad,a^2d,C',C'a,C'a^2\}$$

Proof Let $G=O'N$. The orbits of $N_G(E)$–action of A–subgroups in E are given by $\{A_0,A_\infty\}$,$\{A_1,A_6\}$ and $\{A_2,A_3,A_4,A_5\}$. From Ruiz and Viruel [9], A_0, A_∞, A_1 and A_6 are F^{ec}–radical subgroups. Hence we know that

$$H^*(O'N)\cong H^*(BE)^{3D_8}\cap i_{A_0}^{*-1}H^*(BA_0)^{SL_2(\mathbb{F}_7)\colon 2}\cap i_{A_1}^{*-1}H^*(BA_1)^{SL_2(\mathbb{F}_7)\colon 2}.$$

For element $x=d$ or $x=C'$, the restrictions are $x|A_0=x|A_1=0$. Hence we see that $CA\{x\}$ are contained in $H^*(BG)$. We can take $C',C'a,C'a^2$ instead of a^3/V, a^4/V and a^5/V as the CA–module generators since $a^3/V=(C-C')$. Moreover we know $CA\{C',C'a,C'a^2\}\subset H^*(BG)$.

It is known that $\mathbb{Z}/p[y,u]^{SL_p(\mathbb{F}_p)}\cong\mathbb{Z}/p[\tilde{D}_1,\tilde{D}_2']$ where $\tilde{D}_2'=y_1u^p-y_1^pu$ and $(\tilde{D}_2')^{p-1}=\tilde{D}_2$. Hence we know $\mathbb{Z}/7[y,u]^{SL_2(\mathbb{F}_7)\colon 2}\cong\mathbb{Z}/7[\tilde{D}_1,(\tilde{D}_2)^2]$.

Since $y_1v|A=\tilde{D}_2'$ we see $a|A_0=(\tilde{D}_2')^2, a|A_1=2(\tilde{D}_2')^2$. Hence a,a^2 are in $H^*(BG)$. The fact $b|A_0=0$ and $b|A_1=(\tilde{D}_2')^4$, implies that $b\in H^*(BG)$. Hence all a^ib^j are also in $H^*(BG)$. □

Next we consider the group $G=O'N\colon 2$. Its Weyl group $W_G(E)$ is isomorphic to $3SD_{16}$. So we have $H^*(B(O'N{:}2))\cong H^*(BO'N)\cap H^*(BE)^{3SD_{16}}$.

Corollary 7.5 $H^*(B(O'N{:}2))\cong(DA\{1,a,a^2\}\oplus CA\{C',C'a,C'a^2\})$.

Corollary 7.6 $H^*(BRV_2) \cong DA\{1, a, a^2, a^3, a^4, a^5\}$.

Proof Let $G = RV_2$. Since A_2 is also F^{ec}–radical and $W_G(A_2) = SL_2(\mathbb{F}_7){:}\,2$. Hence we have

$$H^*(BG) \cong H^*(BE)^{3SD_{16}} \cap i_{A_2}^{*-1} H^*(BA_2)^{SL_2(\mathbb{F}_7):\,2}.$$

Hence we have the corollary of the theorem. □

Since $H^*(BRV_3) \cong H^*(BE)^{3SD_{32}} \cap H^*(BRV_2)$, we have the following corollary.

Corollary 7.7 $H^*(BRV_3) \cong DA\{1, a^2, a^4\}$.

Corollary 7.7 can also be proved in the following way.

Proof Let $G = RV_3$. Since there is just one G–conjugacy class of A–subgroups, by Quillen's theorem [8], we know

$$H^*(BRV_3) \subset H^*(BA_0)^{SL_2(\mathbb{F}_7):\,2} \cong DA\{1, (\tilde{D}_2')^2, (\tilde{D}_2')^4\} \text{ with } (\tilde{D}_2')^6 = \tilde{D}_2.$$

Note that $a^2|A_0 = (\tilde{D}_2')^4$, $a^4|A_0 = (\tilde{D}_2')^2\tilde{D}_2$ and $D_2|A_0 = \tilde{D}_2$. The fact $k'^*{:}\, a \mapsto -a$ implies that $DA\{a^2, a^4\} \subset H^*(BG)$ but $DA\{a, a^3, a^5\} \cap H^*(BG) = 0$. □

Corollary 7.6 can also be proved in the following way.

Proof Let $G = RV_2$. Since there is just two G–conjugacy classes of A–subgroups, by Quillen's theorem [8], we know

$$H^*(BRV_2) \subset H^*(BA_0)^{SL_2(\mathbb{F}_7):\,2} \times H^*(BA_2)^{SL_2(\mathbb{F}_7):\,2}$$

Since $a \in H^*(BRV_2)$, the map $i_0^*{:}\, H^*(BRV_2) \to H^*(BA_0)^{SL_2(\mathbb{F}_7):\,2}$ is epimorphism. Take $b' = b^2 - 2a^2 b$ so that $b'|A_0 = b'|A_1 = 0$. Hence

$$\operatorname{Ker} i_{A_0}^* \supset DA\{b', b'a, C'V\}.$$

Moreover $b'|A_2 = (\bar{D}_2')^2\bar{D}_2, b'a|A_2 = (\bar{D}_2')^4\bar{D}_2, c'V|A_2 = (\bar{D}_2)$. Since $(\bar{D}_2')^2$ itself is not in the image of $i_{A_2}^*$, we get the isomorphism

$$H^*(BRV_2) \cong DA\{1, a, a^2\} \oplus DA\{c'V, b', b'a\}. \qquad \square$$

8 Cohomology for $B7_+^{1+2}$ II

In this section, we study cohomology of He, Fi_{24}, RV_1.

First we consider the group $G = He$. The multiplicative generators of $H^*(He)$ are still computed by Leary [5]. We will study more detailed cohomology structures here. The Weyl group is $W_G(He) \cong 3S_3$.

Lemma 8.1 *The invariant $H^*(BE)^{3S_3}$ is isomorphic to*

$$CA\otimes\{\mathbb{Z}/7\{1,\bar b,\bar b^2\}\{1,\bar a,\bar b^3/V\}\oplus\mathbb{Z}/7\{\bar d\}\{1,\bar a,\bar b,\bar b^2/V,\bar b^3/V\}\oplus\mathbb{Z}/7\{\bar a^2\}),$$

where $\bar a=(y_1^3+y_2^3)$, $\bar b=y_1y_2v^2$ and $\bar d=(y_1^3-y_2^3)v^3$.

Proof The group $3S_3\subset GL_2(\mathbb{F}_7)$ is generated by $T'=\{\mathrm{diag}(\lambda,\mu)|\lambda^3=\mu^3=1)$ and $w'=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. If $y_1^iy_2^jv^k$ is invariant under T', then $i=j=-k \bmod(3)$. When $i,j\le 6, k\le 5$ but $(i,j)\ne(6,6)$, the invariant monomials have the following terms

$$\{1,\bar c=\bar b^3/V=y_1^3y_2^3\}\{1,v^3\}\{1,\bar b=y_1y_2v^2,\bar b'=y_1^2y_2^2v\}$$
$$\{1,v^3\}\{y_1^3,y_1^6,y_1y_2^4v^2,y_1^2y_2^5v,y_1^3y_2^6\},$$

and terms obtained by exchanging y_1 and y_2. Recall that w': $y_1\mapsto y_2, y_2\mapsto y_1$ and $v\to -v$. The following elements are invariant

$$\begin{array}{lll}\overline{ab}=(y_1y_2^4+y_1^4y_2)v^2, & \bar a\bar b^2=(y_1^2y_2^5+y_1^5y_2^2)v^4, & \overline{ac}=(y_1^3y_2^6+y_1^6y_2^3),\\ \overline{bd}=(y_1y_2^4-y_1^4y_2)v^5, & \bar b^2\bar d/V=(y_1^2y_2^5-y_2^5y_2^2)v, & \overline{cd}=(y_1^3y_2^6-y_1^6y_2^3)v^3\\ \overline{ad}=(y_1^6-y_2^6)v^3, & \bar a\bar b^3/V=y_1^3y_2^6+y_1^6y_2^3. & \end{array}$$

Thus we get the lemma from (3–2). □

Lemma 8.2 $H^*(BE)^{6S_3}\cong CA\otimes(\mathbb{Z}/7\{1,\bar b,\bar b^2\}\{1,\bar b^3/V\}\oplus\mathbb{Z}/7\{\bar d\bar a,\bar a^2\})$.

Proof We can think $6S_3=\langle S_3,\mathrm{diag}(-1,-1)\rangle$. The action $\mathrm{diag}(-1,-1)$ are given by $\bar a\mapsto -\bar a$,$\bar b\mapsto\bar b$, and $\bar d\mapsto -\bar d$. From Lemma 8.1, we have the lemma. □

Lemma 8.3 $H^*(BE)^{6^2:2}\cong CA\{1,\bar b^2,\bar c'',\bar b^4/V\}$ *where* $\bar c''=\bar a^2-2\bar b^3/V-2C$.

Proof We can think $6^2{:}2=\langle 3S_6,\mathrm{diag}(3,1)\rangle$. The action $\mathrm{diag}(3,1)$ are given by $\bar a^2\mapsto\bar a^2-4\bar c$, $\bar b\mapsto -\bar b$, $\bar c\mapsto -\bar c$, $\overline{da}\mapsto -\overline{da}$. For example $\bar b=y_1y_2v^2\mapsto (3y_1)y_2(3v)^2=-\bar b$. Moreover we have $\bar c''=Y_1+Y_2-2C\mapsto\bar c''$. Thus we have the lemma. □

Theorem 8.4 *Let $\overline{c}' = C + \overline{a}^3/V$. Then there is the isomorphism*

$$H^*(BHe) \cong DA\{1, \overline{b}, \overline{b}^2, \overline{d}, \overline{d}\overline{b}, \overline{d}\overline{b}^2\} \oplus CA\{\{\overline{a}, \overline{c}'\}\{1, \overline{b}, \overline{b}^2, \overline{d}\}, \overline{a}^2, \overline{a}^2\overline{c}'\}.$$

Proof Let $G = He$. The orbits of $N_G(E)$–action of A–subgroups in E are given by

$$\{A_0, A_\infty\}, \quad \{A_1, A_2, A_4\}, \quad \{A_3, A_5, A_6\}.$$

Since A_6 is the F^{ec}–radical (see Leary [6]), we have

$$H^*(BHe) \cong H^*(BE)^{3S_3} \cap i_{A_6}^{*-1} H^*(BA_6)^{SL_2(\mathbb{F}_7)}.$$

For element $x = \overline{a}$ or $x = C + \overline{c} = C + y_1^3 y_2^3$, the restrictions are $x|A_6 = 0$, eg $\overline{a}|A_6 = (y^3 + (-y)^3) = 0$. Hence we see that $CA\{x\}$ are contained in $H^*(BG)$.

Since $\overline{b} = y_1 y_2 v^2$, we see $\overline{b}|A_0 = -y^2 v^2 = -(\tilde{D}_2')^2$. Similarly $\overline{d}|A_6 = 2(\tilde{D}_2')^3$. Thus we can compute $H^*(BHe)$. □

Corollary 8.5 $H^*(B(He{:}\,2)) \cong DA\{1, \overline{b}, \overline{b}^2\} \oplus CA\{\overline{c}', \overline{c}'\overline{b}, \overline{c}'\overline{b}^2, \overline{a}^2, \overline{a}\overline{d}\}$.

Theorem 8.6 *There is the isomorphism*

$$H^*(BFi_{24}') \cong DA\{1, \overline{b}, \overline{b}^2, \overline{a}^2 V, \overline{c}'\overline{b}V, \overline{c}'\overline{b}^2 V\} \oplus CA\{\overline{c}'', \overline{a}\overline{d}\} \text{ where } \overline{c}'' = \overline{a}^2 - 2\overline{c}'.$$

Proof Let $G = Fi_{24}'$. Since A_1 is also F^{ec}–radical and $W_G(A_1) = SL_2(\mathbb{F}_7){:}\,2$. Hence we have

$$H^*(BG) \cong H^*(B(He{:}\,2)) \cap i_{A_1}^{*-1} H^*(BA_1)^{SL_2(\mathbb{F}_7){:}\,2}.$$

For the elements $x = \overline{a}\overline{d}, \overline{c}''(= Y_1 + Y_2 - 2C)$, we see $x|A_1 = x|A_6 = 0$. Hence these elements are in $H^*(BG)$. Note that $\overline{b}|A_1 = (\tilde{D}_2')^2$ and $\overline{b} \in H^*(BG)$. We also know $\overline{a}^2 V|A_1 = \tilde{D}_2$. □

Since $H^*(BFi_{24}) \cong H^*(BFi_{24}') \cap H^*(BE)^{6^2:2}$ and $\overline{b}^4 = 1/2(\overline{a}^2 - 2C - \overline{c}'')V$, we have the following corollary.

Corollary 8.7 $H^*(BFi_{24}) \cong (DA\{1, \overline{b}^2, \overline{b}^4\} \oplus CA\{\overline{c}''\})$.

For $G = RV_1$, The subgroup A_0 is also F^{ec}–radical, we see

$$H^*(BRV_1) \cong H^*(BFi_{24}) \cap i_0^{-1*} H^*(BA_0)^{GL_2(\mathbb{F}_7)}$$

Hence we have the following corollary.

Corollary 8.8 $H^*(BRV_1) \cong DA\{1, \overline{b}^2, \overline{b}^4, D_2''\}$ *with* $\overline{b}^6 = D_2^2 + D_2'' D_2$.

Proof Let $D_2'' = \bar{c}''V = \bar{c}''(D_1 - C^6\bar{c}'')$. Then we have

$$\bar{b}^6 = Y_1Y_2V^2 = (Y_1+Y_2-C)CV^2 = (C+(Y_1+Y_2-2C)CV^2 = D_2^2 + (\bar{c}''V)D_2.$$

Thus the corollary is proved. □

9 Stable splitting for $B7_+^{1+2}$

Let G be groups considered in the preceding two sections, eg $O'N, O'N{:}2, \ldots, RV_1$. First consider the dominant summands $X_{q,k}$. From Corollary 4.6, the dominant summands are only related to $H = W_G(E)$. Recall the notation $X_{q,k}(H)$ in Lemma 4.7. The module $X_{q,k}(H)$ is still given in the preceding sections.

From Lemma 7.1, Lemma 7.2, Lemma 7.3, Lemma 8.1, Lemma 8.2 and Lemma 8.3 we have

$$H = 3D_8;\ X_{6,0} = \{a^3/V, a^2b/V\}, X_{4,4} = \{a^2, b\}, X_{2,2} = \{a\},$$
$$X_{4,1} = \{d\}, X_{6,3} = \{ad\}$$
$$H = 3SD_{16};\ X_{6,0} = \{a^3/V\}, X_{4,4} = \{a^2\}, X_{2,2} = \{a\}$$
$$H = 3SD_{32};\ X_{4,4} = \{a^2\}$$
$$H = 3S_3;\ X_{6,0} = \{\bar{b}^3/V, \bar{a}^2\}, X_{4,4} = \{\bar{b}^2\}, X_{2,2} = \{\bar{b}\},$$
$$X_{6,3} = \{\bar{a}\bar{d}\}, X_{3,0} = \{\bar{a}\}, X_{5,2} = \{\bar{a}\bar{b}\}, X_{3,3} = \{\bar{d}\}, X_{5,5} = \{\bar{d}\bar{b}\}$$
$$H = 6S_3;\ X_{6,0} = \{\bar{b}^3/V, \bar{a}^2\}, X_{2,2} = \{\bar{b}\}, X_{4,4} = \{\bar{b}^2\}, X_{6,3} = \{\bar{a}\bar{d}\}$$
$$H = 6^2{:}2;\ X_{6,0} = \{\bar{a}^2 - 2\bar{b}^3/V\}, X_{4,4} = \{\bar{b}^2\}.$$

For example, ignoring nondominant summands, we have the following diagram

$$\xleftarrow{X_{0,0}\vee X_{4,4}} B(E{:}3SD_{32}) \xleftarrow{X_{6,0}\vee X_{2,2}} B(E{:}3SD_{16}) \xleftarrow{X_{6,0}\vee X_{4,4}\vee X_{4,1}\vee X_{6,3}} B(E{:}3D_8).$$

From Corollary 4.4, the number $m(G,1)_k$ is given by $\mathrm{rank}_p\, H^{2k}(BG)$ for $k\langle p-1$ and $\mathrm{rank}_p\, H^{2p-2}(G)$ for $k=0$. For example when $G = E{:}3S_3$,

$$m(G,1)_0 = 3, m(G,1)_3 = 1, \quad m(G,1)_k = 0 \text{ for } k \neq 0, \neq 3.$$

Lemma 9.1 *Let G be one of the $O'N, O'N, \dots, Fi'_{24}, RV_1$. Then the number $m(G,1)_k$ for $L(1,k)$ is given by*

$$m(G,1)_0 = \begin{cases} 2 \text{ for } G = He, He{:}2 \\ 1 \text{ for } G = O'N, O'N{:}2, Fi_{24}, Fi'_{24} \end{cases}$$
$$m(G,1)_3 = \begin{cases} 1 \text{ for } G = He, \\ m(G,1)_k = 0 \text{ otherwise.} \end{cases}$$

Now we consider the number $m(G,2)_k$ of the non dominant summand $L(2,k)$.

Lemma 9.2 *The classifying spaces BG for $G = O'N, O'N{:}2$ have the non dominant summands $M(2) \vee L(2,2) \vee L(2,4)$.*

Proof We only consider the case $G = O'N$, and the case $O'N{:}2$ is almost the same. The non F^{ec}–radical groups are $\{A_2, A_3, A_4, A_5\}$ (recall the proof of Theorem 7.4). The group $W_G(E) = 3D_8 \cong \langle \operatorname{diag}(2,2), \operatorname{diag}(1,-1), w\rangle$. Hence the normalizer group is
$$N_G(A_2) = E{:}\langle \operatorname{diag}(2,2), \operatorname{diag}(-1,-1)\rangle.$$
Here note that $w, \operatorname{diag}(1,-1)$ are not in the normalizer, eg $w{:}\, \langle c, ab^2\rangle \to \langle c, a^2b^{-2}\rangle = \langle c, ab^6\rangle$. Since $\operatorname{diag}(2,2){:}\, ab^2 \mapsto (ab^2)^2$, $c \mapsto c^4$ and $\operatorname{diag}(-1,-1){:}\, ab^2 \mapsto (ab^2)^{-1}$, $c \mapsto c$, the Weyl groups are
$$W_G(A_2) \cong U{:}\langle \operatorname{diag}(4,2), \operatorname{diag}(1,-1)\rangle.$$
Let $W_1 = U{:}\ \operatorname{diag}\langle 4,2\rangle$. For $v = \lambda y_1^{p-1} \in M_{p-1,k}$, we have $\overline{W}_1 v = \lambda y_2^{p-1}$ since $2^3 = 1$, from the argument in the proof of Lemma 4.11. Moreover
$$\overline{\langle \operatorname{diag}(1,-1)\rangle}\, y_2^{p-1} = (1+(-1)^k)\, y_2^{p-1},$$
implies that the BG contains $L(2,k)$ if and only if k even. □

Lemma 9.3 *The classifying space BHe (resp. $B(He{:}2), Fi'_{24}, Fi_{24}$) contains the non dominant summands*

$$2M(2) \vee L(2,2) \vee L(2,4) \vee L(2,3) \vee L(1,3)$$
$$(\text{resp. } 2M(2) \vee L(2,2) \vee L(2,4),\quad M(2),\quad M(2)).$$

Proof First consider the case $G = He$. The non F^{ec}–radical group are
$$\{A_0, A_\infty\}, \quad \{A_1, A_2, A_4\}.$$

The group $W_G(E) \cong 3S_3 = \langle \mathrm{diag}(2,1), w' \rangle$. So we see $N_G(A_0) = E{:}\langle \mathrm{diag}(2,1) \rangle$, and this implies $W_G(A_0) \cong U{:}\langle \mathrm{diag}(2,2) \rangle$. The fact $4^k = 0 \bmod (7)$ implies $k = 3 \bmod (6)$. Hence BG contains the summand

$$M(2) \vee L(2,3) \vee L(1,3)$$

which is induced from BA_0.

Next consider the summands induced from BA_1. The normalizer and Weyl group are $N_G(A_1) = E{:}\langle w' \rangle$ and $W_G(A_1) = U{:}\langle \mathrm{diag}(-1,1) \rangle$ since $w'{:}ab \mapsto ab, c \mapsto -c$. So we get

$$M(2) \vee L(2,2) \vee L(2,4)$$

which is induced from BA_1.

For $G = He{:}2$, we see $\mathrm{diag}(-1,-1) \in W_G(E)$, this implies that $\mathrm{diag}(-1,-1) \in N_G(A)$ and $\mathrm{diag}(1,-1) \in W_G(A_0)$. This means that the non dominant summand induced from BA_0 is $M(2)$ but is not $L(2,3)$. We also know $U{:}\ \mathrm{diag}(1,-1) \in W_G(A_1)$ but the summand induced from BA_1 are not changed.

For groups Fi_{24}', Fi_{24}, the non F^{ec}–radical groups make just one G–conjugacy class $\{A_0, A_\infty\}$. So BG dose not contain the summands induced from BA_1. $\square$

Theorem 9.4 *When $p = 7$, we have the following stable decompositions of BG so that $\overset{X_1}{\leftarrow} \cdots \overset{X_s}{\leftarrow} G$ means that $BG \sim X_1 \vee \cdots \vee X_s$*

$$\overset{X_{0,0}}{\longleftarrow} \begin{cases} \overset{X_{4,4}}{\longleftarrow} RV_3 \overset{X_{6,0}\vee X_{2,2}}{\longleftarrow} RV_2 \overset{M(2)\vee L(2,2)\vee L(2,4)}{\longleftarrow} O'N{:}2 \overset{X_{6,0}\vee X_{4,4}\vee X_{4,1}\vee X_{6,3}}{\longleftarrow} O'N \\[2ex] \overset{X_{6,0}\vee X_{4,4}}{\longleftarrow} RV_1 \overset{M(2)}{\longleftarrow} Fi_{24} \overset{X_{6,0}\vee X_{6,3}\vee X_{2,2}}{\longleftarrow} Fi_{24}' \overset{M(2)\vee L(2,2)\vee L(2,4)}{\longleftarrow} He{:}2 \\[2ex] \qquad\qquad \overset{X_{3,0}\vee X_{5,2}\vee X_{3,3}\vee X_{5,5}\vee L(2,3)\vee L(1,3)}{\longleftarrow} He. \end{cases}$$

We write down the cohomology of stable summands. At first we see that $H^*(X_{0,0}) \cong H^*(BRV_3) \cap H^*(BRV_1) \cong DA$. Here note that elements $a^2 - (y_1 y_2)^2 v^4$ in Section 7 and $\bar{b}^2 = y_1^2 y_2^2 v^4$ in Section 8 are not equivalent under the action in $GL_7(\mathbb{F}_7)$ because $y_1^2 + y_2^2$ is indecomposable in $\mathbb{Z}/7[y_1, y_2]$.

From the cohomologies, $H^*(BRV_3)$ and $H^*(BRV_2)$, then $H^*(X_{4,4}) \cong DA\{a^2, a^4\}$ and $H^*(X_{6,0} \vee X_{2,2}) \cong DA\{a, a^3, a^5\}$.

On the other hand, we know $H^*(X_{6,0})$ from the cohomology $H^*(BRV_1)$. Thus we get the following lemma.

Lemma 9.5 *There are isomorphisms of cohomologies*

$$H^*(X_{0,0}) \cong DA,\ H^*(X_{4,4}) \cong DA\{a^2, a^4\}$$
$$H^*(X_{6,0}) \cong DA\{D_2\} \cong DA\{a^3\},\ H^*(X_{2,2}) \cong DA\{a, a^5\}.$$

Let us write $M\{a\} = DA\{1, C, \dots, C^{p-1}\}\{a\}$. From the facts that $D_2 = CV$, $D_1 = C^p + V$ and $D_2 = C(D_1 - C^p) = CD_1 - C^{p+1}$, we have two decompositions

$$CA\{a\} \cong DA\{1, C, \dots, C^p\}\{a\} \cong DA\{a\} \oplus M\{Ca\} \cong M\{a\} \oplus DA\{Va\}.$$

From the cohomology of $H^*(Fi_{24})$, we know the following lemma.

Lemma 9.6 $H^*(M(2)) \cong M\{C\}$.

Comparing the cohomology $H^*(B(He{:}\,2)) \cong H^*(BFi_{24}') \oplus M\{\bar{a}^2, \bar{c}'\bar{b}, \bar{c}'\bar{b}^2\}$, we have the isomorphisms

$$H^*(M(2)) \cong M\{\bar{a}^2\},\ H^*(L(2,2) \vee L(2,4)) \cong M\{\bar{c}'\bar{b}, \bar{c}'\bar{b}^2\}.$$

From $H^*(BFi_{24}') \cong H^*(BFi_{24}) \oplus DA\{\bar{a}^2V, \bar{c}'\bar{b}V, \bar{c}'\bar{b}^2V\} \oplus CA\{\bar{a}\bar{d}\}$, we also know that

$$H^*(X_{6,3}) \cong CA\{\bar{a}\bar{d}\},\ H^*(X_{6,0} \vee X_{2,2}) \cong DA\{\bar{a}^2V, \bar{c}'\bar{b}V, \bar{c}'\bar{b}^2V\}.$$

We still get $H^*(BFi_{24}) \cong H^*(BRV_1) \oplus M\{\bar{c}''\}$ and $H^*(M(2)) \cong M\{\bar{c}''\}$.

Next consider the cohomology of groups studied in Section 7 eg $O'N$. There is the isomorphism

$$H^*(BO'N) \cong H^*(BO'N{:}\,2) \oplus DA\{b, b^2, ab^2\} \oplus CA\{d, da, da^2\}.$$

Indeed, we have

$$H^*(X_{6,0} \vee X_{4,4}) \cong DA\{b, b^2, ab^2\} \cong DA\{a^2, a^3, a^4\}$$
$$H^*(X_{6,3}) \cong CA\{da\}$$
$$H^*(X_{4,1}) \cong CA\{d, da^2\}.$$

We also have the isomorphism $H^*(BO'N{:}2) \cong H^*(BRV_2) \oplus M\{C', C'a, C'a^2\}$ and $H^*(M(2) \vee L(2,2) \vee L(2,4)) \cong M\{C', C'a, C'a^2\}$.

Recall that

$$H^*(BE)^{3SD_{32}} \cong CA\{1, a^2, a^4/V\} \cong DA\{1, a^2, a^4\} \oplus M\{C, a^2C, a^4/V\},$$

in fact $H^*(M(2) \vee L(2,2) \vee L(2,4)) \cong M\{C, a^2C, a^4/V\}$.

10 The cohomology of $\mathbb{M}$ for $p=13$

In this section, we consider the case $p=13$ and $G=\mathbb{M}$ the Fisher–Griess Monster group. It is know that $W_G(E)\cong 3\times 4S_4$. The G–conjugacy classes of A–subgroups are divided two classes ; one is F^{ec}–radical and the other is not. The class of F^{ec}–radical groups contains 6 E–conjugacy classes (see Ruiz–Viruel [9]). (The description of [11, (4.1)] was not correct, and the description of $H^*(B\mathbb{M})$ in [11, Theorem 6.6] was not correct.) The Weyl group $W_G(A)\cong SL_2(\mathbb{F}_{13}).4$ for each F^{ec}–radical subgroup A.

Since $S_4\cong PGL_2(\mathbb{F}_3)$ [S], we have the presentation of

$$S_4=\langle x,y,z|x^3=y^3=z^2=(xy)^2=1, zxz^{-1}=y\rangle.$$

(Take $x=u$, $y=u'$ in Lemma 4.8, and $z=w$ in Section 5.) By arguments in the proof of Suzuki [10, Chapter 3 (6.24)], we can take elements x,y,z in $GL_2(\mathbb{F}_{13})$ by

$$x=\begin{pmatrix}3&0\\0&9\end{pmatrix},\quad y=\begin{pmatrix}5&-4\\-2&7\end{pmatrix},\quad z=\begin{pmatrix}2&2\\1&-2\end{pmatrix}, \tag{10–1}$$

so that we have

$$x^3=y^3=1,\ zxz^{-1}=y,\ (xy)^2=-1,\ z^2=\mathrm{diag}(6,6).$$

Hence we can identify

$$3\times 4S_4\cong\langle x,y,z\rangle\subset GL_2(\mathbb{F}_{13}). \tag{10–2}$$

It is almost immediate that $H^*(BE)^{\langle x\rangle}$ (resp. $H^*(BE)^{\langle -1\rangle}$) is multiplicatively generated by y_1y_2, y_1^3, y_2^3 (resp. y_1y_2, y_1^2, y_2^2) as a $\mathbb{Z}/(13)[C,v]$–algebra. Hence we can write

$$H^*(BE)^{\langle x,-1\rangle}\cong\mathbb{Z}/(13)[C,v]\{\{1,y_1y_2,\dots,(y_1y_2)^5\}\{(y_1y_2)^6,y_1^6,y_2^6\},\\ y_1^{12},y_2^{12},y_1^{12}y_2^6,y_1^6y_2^{12}\}. \tag{10–3}$$

For the invariant $H^*(BE)^{\langle y,-1\rangle}$, we get the similar result exchanging y_i to $(z^{-1})^*y_i$ since $zxz^{-1}=y$. Indeed $(z^{-1})^*\colon H^*(BE)^{\langle x,-1\rangle}\cong H^*(BE)^{\langle y,-1\rangle}$.

To seek invariants, we recall the relation between the A–subgroups and elements in $H^2(BE;\mathbb{Z}/p)$. For $0\neq y=\alpha y_1+\beta y_2\in H^2(BE;\mathbb{Z}/p)$, let $A_y=A_{-(\alpha/\beta)}$ so that $y|A_y=0$. This induces the $1-1$ correspondence,

$$(H^2(BE;\mathbb{Z}/p)-\{0\})/F_p^*\leftrightarrow\{A_i|i\in F_p\cup\{\infty\}\},\quad y\leftrightarrow A_y.$$

Considering the map $g^{-1}A_i\xrightarrow{g}A_i\subset E\xrightarrow{\beta^{-1}y}\mathbb{Z}/p$, we easily see $A_{g^*y}=g^{-1}A_y$.

For example, the order 3 element x induces the maps

$$x^*\colon y_1 - y_2 \mapsto 3y_1 - 9y_2 \mapsto 9y_1 - 3y_2 \mapsto y_1 - y_2$$
$$x^{-1}\colon A_{y_1-y_2} = \langle c, ab\rangle \to \langle c, a^9b^3\rangle \to \langle c, a^3b^9\rangle \to \langle c, ab\rangle.$$

In particular A_1, A_9, A_3 are in the same x–orbit of A–subgroups. Similarly the $\langle x\rangle$–conjugacy classes of A is given

$$\{A_0\}, \{A_\infty\}, \{A_1, A_3, A_9\}, \{A_2, A_5, A_6\}, \{A_4, A_{10}, A_{12}\}, \{A_7, A_8, A_{11}\}.$$

The $\langle y\rangle$–conjugacy classes are just $\{zA_i\}$ for $\langle x\rangle$–conjugacy classes $\{A_i\}$.

$$\{A_7 = zA_0\}, \{A_{12}\}, \{A_3, A_1, A_5\}, \{A_6, A_9, A_2\}, \{A_{11}, A_8, A_\infty\}, \{A_0, A_{10}, A_4\}.$$

Hence we have the $\langle x, y\rangle$–conjugacy classes

$$C_1 = \{A_1, A_2, A_3, A_5, A_6, A_9\}, C_2 = \{A_0, A_4, A_{10}, A_{12}\}, C_3 = \{A_\infty, A_7, A_8, A_{11}\}.$$

At last we note $\langle x, y, z\rangle$–conjugacy classes are two classes $C_1, C_2 \cup C_3$.

Let us write the $\langle x\rangle$–invariant

$$\text{(10–4)}\qquad \begin{aligned} u_6 &= \Pi_{A_i\in C_1}(y_2 - iy_1) = (y_2 - y_1)(y_2 - 2y_1)\cdots(y_2 - 9y_1) \\ &= y_2^6 - 9y_1^3y_2^3 + 8y_1^6. \end{aligned}$$

Then u_6 is also invariant under y^* because the $\langle x, y\rangle$–conjugacy class C_1 divides two $\langle y\rangle$–conjugacy classes

$$C_1 = \{A_1, A_3, A_5\} \cup \{A_2, A_6, A_9\}$$

and the element u_6 is rewritten as

$$u_6 = \lambda(\Pi_{i=0}^2 y^{i*}(y_2 - y_1)).(\Pi_{i=0}^2 y^{i*}(y_2 - 2y_1)) \text{ for } \lambda \neq 0 \in \mathbb{Z}/(13).$$

We also note that $u_6|A_i = 0$ if and only if $i \in C_1$. Similarly the following elements are $\langle x, y\rangle$–invariant,

$$\text{(10–5)}\qquad \begin{aligned} u_8 &= \Pi_{A_i\in C_2\cup C_3}(y_2 - iy_1) = y_1y_2(y_2^6 + 9y_1^3y_2^3 + 8y_1^6) \\ u_{12} &= \Pi_{A_i\in C_2}(y_2 - iy_1)^3 = (y_2^4 + y_1^3y_2)^3 \\ &= \lambda(\Pi_{i=0}^2 x^{i*}y_2)(\Pi_{i=0}^2 x^{i*}(y_2 - 4y_1))^3 \\ &= \lambda'(\Pi_{i=0}^2 y^{i*}(y_2 - 12y_1))(\Pi_{i=0}^2 y^{i*}y_2)^3 v \\ u'_{12} &= \Pi_{A_i\in C_3}(y_2 - iy_1)^3 = (y_1y_2^3 + 8y_1^4)^3. \end{aligned}$$

Of course $(u_{12}u'_{12})^{1/3} = u_8$ and $u_6u_8 = 0$. Moreover direct computation shows $u_6^2 = u_{12} + 5u'_{12}$.

Lemma 10.1 $H^*(BE)^{\langle x,y\rangle} \cong \mathbb{Z}/(13)[C,v]\{1,u_6,u_6^2,u_6^3,u_8,u_8^2,u_{12}\}$.

Proof Recall (10–3) to compute

$$H^*(BE)^{\langle x,y\rangle} \cong H^*(BE)^{\langle x,-1\rangle} \cap H^*(BE)^{\langle y,-1\rangle}.$$

Since $(z^{-1})^*(y_1y_2)^i \neq (y_1y_2)^i$ for $1 \le i \le p-2$, from (10–3) we know invariants of the lowest positive degree are of the form

$$u = \gamma y_2^6 + \alpha y_2^3 y_1^3 + \beta y_1^6.$$

Then $u' = u - \gamma u_6$ is also invariant with $u'|A_\infty = 0$. Hence $u'|A_i = 0$ for all $A_i \in C_3$. Thus we know $u' = \lambda y_1^2 (u'_{12})^{1/3}$. But this is not $\langle y\rangle$–invariant for $\lambda \neq 0$, because $(u')^3 = \lambda^3 y_1^6 u'_{12}$ is invariant, while y_1^6 is not $\langle y\rangle$–invariant. Thus we know $u' = 0$.

Any 16–dimensional invariant is form of

$$u = y_1y_2(\gamma y_2^6 + \alpha y_2^3 y_1^3 + \beta y_1^6).$$

Since $u|A_0 = u|A_\infty = 0$, we know $u|A_i = 0$ for all $A_i \in C_2 \cup C_3$. Hence we know

$$u = \gamma u_{12}^{1/3} (u'_{12})^{1/3} = \gamma u_8.$$

By the similar arguments, we can prove the lemma for degree ≤ 24.

For $24 <$degree< 48, we only need consider the elements $u' = 0 \bmod (y_1y_2)$. For example, $H^{18}(BE;\mathbb{Z}/13)^{\langle x,-1\rangle}$ is generated by

$$\{(y_1y_2)^9, (y_1y_2)^3C, y_1^6C, y_2^6C, y_1^6y_2^{12}, y_1^{12}y_2^6\}.$$

But we can take off $y_1^6C = y_1^{18}$, $y_2^6C = y_2^{18}$ by $\lambda u_6^3 + \mu C u_6$ so that $u' = 0 \bmod (y_1y_2)$.

Hence we can take u' so that u_8 divides u' from the arguments similar to the case of degree$=16$. Let us write $u' = u'' u_8$. Then we can write

$$u'' = y_1^k y_2^k (\lambda_1 y_1^6 + \lambda_2 y_1^3 y_2^3) + \lambda_3 (y_1y_2)^{k-3} C,$$

taking off $\lambda y_1^k y_2^k u_6$ if necessary since $u_6u_8 = 0$. (Of course, for $k<3$, $\lambda_3 = 0$.) Since $u_8|A_i \neq 0$ and $u_6|A_i = 0$ for $i \in C_1$, we have

$$(u'' - y^* u'')|A_i = 0 \text{ for } i \in C_1.$$

Since $y^*y_1 = 5y_1 - 4y_2$ and $y^*y_2 = -2y_1 + 7y_2$, we have

$$\begin{aligned}(u'' - y^*u'')|A_i = \lambda_1 (i^k - &(5-4i)^{6+k}(-2+7i)^k) \\ &+ \lambda_2 (i^{k+3} - (5-4i)^{k+3}(-2+7i)^{k+3}) \\ &+ \lambda_3 (i^{k-3} - (5-4i)^{k-3}(-2+7i)^{k-3}).\end{aligned}$$

We will prove that we can take all $\lambda_i = 0$. Let us write $U = u'' - y^*u''$. We then have the following cases.

(1) The case $k = 0$, ie degree=14. If we take $i = 1$,

$$U|A_1 = \lambda_1(1-1) + \lambda_2(1 - 1^3 5^3) = 0.$$

So we have $\lambda_2 = 0$. We also see $\lambda_1 = 0$ since $U|A_3 = \lambda_1(1 - (5-12)^6) = 2\lambda_1 = 0$.

(2) The case $k = 1$. Since $y_1 y_2 u_6 - u_8 = -18 y_1^4 y_2^4$, we can assume $\lambda_2 = 0$ taking off λu_8^2 if necessary. We have also $\lambda_1 = 0$ from $U|A_1 = \lambda_1(1^1 - 1^7 5^1) = 0$.

(3) The case $k = 2$. We get the the result $U|A_1 = 2\lambda_1 + 4\lambda_2$, $U|A_3 = 5\lambda_1 + 5\lambda_2$.

(4) The case $k = 3$. First considering Cu_8, we may take $\lambda_3 = 0$. The result is given by $U|A_1 = 6\lambda_1 + 2\lambda_2$ and $U|A_2 = 7\lambda_1 + 9\lambda_2$.

(5) The case $k = 4$. The result follows from

$$U|A_1 = 6\lambda_2 + 9\lambda_3, U|A_3 = 6\lambda_1 + 6\lambda_2 + 6\lambda_3, U|A_5 = 2\lambda_1 - 4\lambda_2 + 6\lambda_3.$$

Hence the lemma is proved. □

Next consider the invariant under $\langle x, y, \mathrm{diag}(6,6)\rangle$. The action for $\mathrm{diag}(6,6)$ is given by $y_1^i y_2^j v^k \mapsto 6^{i+j+2k} y_1^i y_2^j v^k$. Hence the invariant property implies $i + j + 2k = 0 \bmod(12)$. Thus $H^*(BE)^{\langle x,y,\mathrm{diag}(6,6)\rangle}$ is generated as a CA–algebra by

$$\{1, u_6 v^3, u_8 v^2, u_{12}, u'_{12}, v^6\}.$$

Lemma 10.2 *The invariant $H^*(BE)^{3\times 4S_4} \cong H^*(BE)^{\langle x,y,z\rangle}$ is isomorphic to*

$$CA\{1, u_6v^3, (u_6v^3)^2, (u_6v^3)^3, u_8v^8, (u_8v^8)^2/V, (u_{12} - 5u'_{12})\}.$$

Proof We only need compute z^*–action. Since

$$3 \times 4S_4 \cong \langle x, y, \mathrm{diag}(6,6)\rangle{:}\langle z\rangle,$$

the z^*–action on $H^*(BE)^{\langle x,y,\mathrm{diag}(6,6)\rangle}$ is an involution. Let $u_6v^3 = u_6(y_1,y_2)v^3$. First note $u_6|A_\infty = u_6(0,y) = y^6$. On the other hand, its z^*–action is

$$\begin{aligned} z^*u_6v^3|A_\infty &= u_6(2y_1 + 2y_2, y_1 - 2y_2)(-6v)^3|A_\infty = u_6(2y, -2y)(-6v)^3 \\ &= ((-2)^6 - 9(-2)^3(2)^3 + 8(2)^6)(-6)^3 y^6 v^3 \\ &= (1+9+8)8y^6v^3 = y^6v^3. \end{aligned}$$

Hence we know u_6v^3 is invariant, while u_6v^9 is not.

Similarly we know

$$u_8 v^2|A_1 = u_8(y,y)v^2 = 5y^8 v^2, \quad z^* u_8 v^2|A_1 = -5y^8 v^2.$$

Hence $u_8 v^8$ and $u_8^2 v^4$ are invariant but $u_8 v^2$ is not.

For the action u_{12}, we have

$$\begin{array}{rclrclrclrcl} u_{12}|A_0 &=& 0, & u_{12}|A_\infty &=& y^{12}, & u'_{12}|A_0 &=& 5y^{12}, & u'_{12}|A_\infty &=& 0, \\ z^*u_{12}|A_0 &=& y^{12}, & z^*u_{12}|A_\infty &=& 0, & z^*u'_{12}|A_0 &=& 0, & z^*u'_{12}|A_\infty &=& 5y^{12}. \end{array}$$

Thus we get $z^*u_{12} = (1/5)u'_{12}$, $k^*u'_{12} = 5u_{12}$. Hence we know $u_{12} + (1/5)u'_{12}$ and $(u_4^3 - (1/5)u'_{12})v^6 = (u_6 v^3)^2$ are invariants. Thus we can prove the lemma. □

Theorem 10.3 *For $p = 13$, the cohomology $H^*(B\mathbb{M})$ is isomorphic to*

$$DA\{1, u_8v^8, (u_8v^8)^2\} \oplus CA\{u_6v^3, (u_6v^3)^2, (u_6v^3)^3, (u_{12} - 5u'_{12} - 3C)\}.$$

Proof Direct computation shows

$$u_{12} - 5u'_{12} = y_2^{12} - 2y_2^9 y_1^3 + 3y_2^3 y_1^9 + y_1^{12},$$

and hence $u_{12} - 5u'_{12} - 3C|A_1 = 0$, indeed, the restriction is zero for each $A_i \in C_1$. The isomorphism

$$H^*(B\mathbb{M}) \cong H^*(BE)^{3\times 4S_4} \cap i_{A_1}^{-*}(H^*(BA_1)^{SL_4(\mathbb{F}_{13}).4},$$

completes the proof. □

The stable splitting is given by the following theorem.

Theorem 10.4 *We have the stable splitting*

$$B\mathbb{M} \sim X_{0,0} \vee X_{12,0} \vee X_{12,6} \vee X_{6,3} \vee X_{8,8} \vee M(2),$$
$$B(E{:}\,3\times 4S_4) \sim B\mathbb{M} \vee M(2) \vee L(2,4) \vee L(2,8).$$

Proof Let $H = E{:}\,3 \times 4S_4$. Recall that

$$X_{q,k}(H) = (S(A)^q \otimes v^k) \cap H^*(BH) \quad 0 \le q \le 12, 0 \le k \le 11.$$

We already know

$$X_{*,*}(H) = \mathbb{Z}/(13)\{1, u_8v^8, u_6v^3, u_6^2v^6, u_{12} - 5u'_{12}\}.$$

Hence BH has the dominant summands in the theorem.

The normalizer groups of A_0, A_1 are given

$$N_H(A_0) = E{:}\langle x, \mathrm{diag}(6,6)\rangle, N_H(A_1) = E{:}\langle \mathrm{diag}(6,6)\rangle.$$

Hence the Weyl groups are

$$W_H(A_0) = U{:}\langle \mathrm{diag}(1,3), \mathrm{diag}(6^2,6)\rangle, W_H(A_1) = U{:}\langle \mathrm{diag}(6^2,6)\rangle.$$

From the arguments of Lemma 4.11, the non-dominant summands induced from BA_1 are $M(2) \vee L(2,4) \vee L(2,8)$. We also know the non-dominant summands from BA_0 are $M(2)$. This follows from

$$\overline{\langle \mathrm{diag}(1,3)\rangle} y_2^{p-1} = \sum_{i=0}^{2} (3^i)^k y_2^{p-1} \quad for\ y_2^{p-1} \in M_{p-1,k}$$

and this is nonzero mod(13) if and only if $k = 0$ mod(3). □

Remark It is known $H^*(Th) \cong DA$ for $p = 5$ in [11]. Hence all cohomology $H^*(BG)$ for groups G in Theorem 2.1 (4)–(7) are explicitly known. For (1)–(3), see also Tezuka–Yagita [11].

11 Nilpotent parts of $H^*(BG;\mathbb{Z}_{(p)})$

It is known that $p^2H^*(BE;\mathbb{Z}) = 0$ (see Tezuka–Yagita [11] and Leary [6]) and

$$pH^{*>0}(BE;\mathbb{Z}) \cong \mathbb{Z}/p\{pv, pv^2, \ldots\}.$$

In particular $H^{\mathrm{odd}}(BE;\mathbb{Z})$ is all just p–torsion. There is a decomposition

$$H^{\mathrm{even}}(BE;\mathbb{Z})/p \cong H^*(BE) \oplus N \text{ with } N = \mathbb{Z}/p[V]\{b_1, \ldots, b_{p-3}\}$$

where $b_i = Cor^E_{A_0}(u^{i+1}), |b_i| = 2i+2$. (Note for $p = 3, N = 0$.) The restriction images $b_i|A_j = 0$ for all $j \in \mathbb{F}_p \cup \infty$. For $g \in GL_2(\mathbb{F}_p)$, the induced action is given by $g^*(b_i) = \det(g)^{i+1}b_i$ by the definition of b_i.

Note that

$$2 = |y_i| < |b_j| = 2(j+1) < |C| = 2p-2 < |v| = 2p.$$

So $g^*(y_i)$ is given by (3–4) also in $H^*(BE;\mathbb{Z})$ and $g^*(v) = \det(g)v$ mod(p). Hence we can identify that

$$H^*(BE)^H = (H^{\mathrm{even}}(BE;\mathbb{Z})/(p,N))^H \subset H^{\mathrm{even}}(BE;\mathbb{Z}/p)^H.$$

Let us write the reduction map by $q{:}\ H^*(BE;\mathbb{Z}) \to H^*(BE;\mathbb{Z}/p)$.

Lemma 11.1 *Let $H \subset GL_2(\mathbb{F}_p)$ and $(|H|, p) = 1$. If $x \in H^*(BE)^H$, then there is $x' \in H^*(BE;\mathbb{Z})^H$ such that $q(x') = x$.*

Proof Let $x \in H^*(BE)^H$ and $G = E\colon H$. Then we can think $x \in H^*(BE;\mathbb{Z}/p)^H \cong H^*(BG;\mathbb{Z}/p)$ and $\beta(x) = 0$. By the exact sequence

$$H^{\mathrm{even}}(BG;\mathbb{Z}_{(p)}) \xrightarrow{q} H^{\mathrm{even}}(BG;\mathbb{Z}/p)) \xrightarrow{\delta} H^{\mathrm{odd}}(BG;\mathbb{Z}_{(p)}),$$

we easily see that $x \in Image(q)$ since $q\delta(x) = \beta(x) = 0$ and $q|H^{\mathrm{odd}}(BG;\mathbb{Z}_{(p)})$ is injective. Since $H^*(BG;R) \cong H^*(BE;R)^H$ for $R = \mathbb{Z}_{(p)}$ or $\mathbb{Z}/p$, we get the lemma. □

Proof of Theorem 3.1 From Tezuka–Yagita [11, Theorem 4.3] and Broto–Levi–Oliver [1], we have the isomorphism

$$H^*(BG;\mathbb{Z})_{(p)} \cong H^*(BE;\mathbb{Z})^{W_G(E)} \cap_{A\colon F^{\mathrm{ec}}-\mathrm{radical}} i_A^{*-1} H^*(BA;\mathbb{Z})^{W_G(A)}.$$

The theorem is immediate from the above lemma and the fact that $H^{\mathrm{even}>0}(BA;\mathbb{Z}) \cong H^{*>0}(BA)$. □

Let us write $N(G) = H^*(BG;\mathbb{Z}) \cap N$. Then

$$H^{\mathrm{even}}(BG;\mathbb{Z})/p \cong H^*(BG) \oplus N(G).$$

The nilpotent parts $N(G)$ depends only on the group $\mathrm{Det}(G) = \{\det(g)|g \in W_G(E)\} \subset \mathbb{F}_p^*$, in fact, $N(G) = N^{W_G(E)} = N^{\mathrm{Det}(G)}$.

Lemma 11.2 *If $\mathrm{Det}(G) \cong \mathbb{F}_p^*$ (eg $G = O'N, He, \ldots, RV_3$ for $p = 7$, or $G = \mathbb{M}$ for $p = 13$), then*

$$N(G) \cong \mathbb{Z}/p[V]\{b_i v^{p-2-i} | 1 \le i \le p-3\}.$$

Lemma 11.3 *Let G have a 7–Sylow subgroup E. Then, we have*

$$N(G) = \begin{cases} \mathbb{Z}/7[V]\{b_1v^4, b_2v^3, b_3v^2, b_4v\} \text{ if } \mathrm{Det}(G) = \mathbb{F}_7^* \\ \mathbb{Z}/7[v^3]\{b_1v, b_2, b_3v^2, b_4v\} \text{ if } \mathrm{Det}(G) \cong \mathbb{Z}/3 \\ \mathbb{Z}/7[v^2]\{b_1, b_2v, b_3, b_4v\} \text{ if } \mathrm{Det}(G) \cong \mathbb{Z}/2 \\ \mathbb{Z}/7[v]\{b_1, b_2, b_3, b_4\} \text{ if } \mathrm{Det}(G) \cong \{1\}. \end{cases}$$

Now we consider the odd dimensional elements. Recall that

$$H^{\mathrm{odd}}(BA;\mathbb{Z}) \cong \mathbb{Z}/p[y_1, y_2]\{\alpha\},$$

where $\alpha = \beta(x_1x_2) \in H^*(BA;\mathbb{Z}/p) \cong \mathbb{Z}/p[y_1,y_2] \otimes \Lambda(x_1,x_2)$ with $\beta(x_i) = y_i$. Of course $g^*(\alpha) = \det(g)\alpha$ for $g \in \mathrm{Out}(A)$. For example $H^{\mathrm{odd}}(B(A{:}Q_8)) \cong H^*(B(A{:}Q_8))\{\alpha\}$ since $\mathrm{Det}(A{:}\,Q_8) = \{1\}$.

Recall the Milnor operation $Q_{i+1} = [P^{p^n}Q_i - Q_iP^{p^n}]$, $Q_0 = \beta$. It is known that

$$Q_1(\alpha) = y_1^py_2 - y_1y_2^p = \tilde{D}_2' \text{ with } (\tilde{D}_2')^{p-1} = \tilde{D}_2.$$

The submodule of $H^*(X;\mathbb{Z}_{(p)})$ generated by (just) p–torsion additive generators can be identified with $Q_0H^*(X;\mathbb{Z}/p)$. Since $Q_iQ_0 = -Q_0Q_i$, we can extend the map [13, page 377]

$$Q_i{:}\; Q_0H^*(X;\mathbb{Z}/p) \xrightarrow{Q_i} Q_0H^*(X;\mathbb{Z}/p) \subset H^*(X;\mathbb{Z}_{(p)}).$$

Since all elements in $H^{\mathrm{odd}}(BA;\mathbb{Z})$ are (just) p–torsion, we can define the map

$$Q_1{:}\; H^{\mathrm{odd}}(BA;\mathbb{Z}) \to H^{\mathrm{even}}(BA;\mathbb{Z}) = H^{\mathrm{even}}(BA).$$

Moreover this map is injective.

Lemma 11.4 (Yagita [13]) *Let G have the p–Sylow subgroup $A = (\mathbb{Z}/p)^2$. Then*

$$Q_1:\; H^{\mathrm{odd}}(BG;\mathbb{Z}_{(p)}) \cong (H^{\mathrm{even}}(BG) \cap J(G)),$$

with $J(G) = \mathrm{Ideal}(y_1^py_2 - y_1y_2^p) \subset H^{\mathrm{even}}(BA)$.

Corollary 11.5 *For $p = 3$, there are isomorphisms*

$$H^{\mathrm{odd}}(BA;\mathbb{Z})^{Z/8} \cong S\{b, a'b, a, (a_1 - a_2)\}\{\alpha\}$$
$$H^{\mathrm{odd}}(BA;\mathbb{Z})^{D_8} \cong S\{1, a, a_1, a'\}\{b\alpha\}$$
$$H^{\mathrm{odd}}(BA;\mathbb{Z})^{SD_{16}} \cong S\{1, a'\}\{b\alpha\}.$$

Proof We only prove the case $G = A{:}\,\mathbb{Z}/8$ since the proof of the other cases are similar. Note in §5 the element $Q_1(\alpha)$ is written by b and $b^2 = a_1a_2$. Recall $S = \mathbb{Z}/3[a_1 + a_2, a_1a_2]$. Hence we get

$$\begin{aligned} H^*(BA)^{\langle l\rangle} \cap J(G) &\cong S\{1, a', ab, (a_1 - a_2)b\} \cap \mathrm{Ideal}(b) \\ &= S\{b^2, b^2a', ab, (a_1 - a_2)b\} \\ &= S\{b, ba', a, (a_1 - a_2)\}\{Q_1(\alpha)\}. \end{aligned}$$

The corollary follows. □

By Lewis, we can write [6; 11]

$$H^{\mathrm{odd}}(BE;\mathbb{Z})\cong \mathbb{Z}/p[y_1,y_2]/(y_1\alpha_2-y_2\alpha_1, y_1^p\alpha_2-y_2^p\alpha_1)\{\alpha_1,\alpha_2\},$$

where $|\alpha_i|=3$. It is also known that $Q_1(\alpha_i)=y_iv$ and $Q_1\colon H^{\mathrm{odd}}(BE;\mathbb{Z}_{(p)})\to H^{\mathrm{even}}(BE)\subset H^{\mathrm{even}}(BE;\mathbb{Z})/p$ is injective [13]. Using this we can prove the following lemma.

Lemma 11.6 (Yagita [13]) *Let G have the p–Sylow subgroup E. Then*

$$Q_1\colon\ H^{\mathrm{odd}}(BG)\cong (H^{\mathrm{even}}(BG)\cap J(G))$$

with $J(G)=\mathrm{Ideal}(y_iv)\subset H^{\mathrm{even}}(BE)$.

From the above lemma we easily compute the odd dimensional elements. Note that

$$D_2=CV\notin J(E) \text{ but } D_2^2=C^2V^2=(Y_1^2+Y_2^2-Y_1Y_2)V^2\in J(E).$$

Let us write $\alpha=(Y_1y_1^{p-2}\alpha_1+Y_2y_2^{p-2}\alpha_2-Y_1y_2^{p-2}\alpha_2)Vv^{p-2}$ so that $Q_1(\alpha)=D_2^2$.

Corollary 11.7 $H^{\mathrm{odd}}(B^2F_4(2)';\mathbb{Z}_{(3)})\cong DA\{\alpha,\alpha'\}$ *with $\alpha'=(y_1\alpha_1+y_2\alpha_2)v$.*

Proof Recall that $H^*(B^2F_4(2)')\cong DA\{1,(Y_1+Y_2)V\}$ from the remark of Proposition 6.3. The result is easily obtained from $Q_1(\alpha)=D_2^2$, $Q_1(\alpha')=(Y_1+Y_2)V$. □

Corollary 11.8 *There are isomorphisms*

$$H^{\mathrm{odd}}(BRV_3;\mathbb{Z}_{(7)})\cong DA\{a,a^3,a^5\}\{\alpha'\}$$
$$H^{\mathrm{odd}}(BRV_2;\mathbb{Z}_{(7)})\cong DA\{1,a,\dots,a^5\}\{\alpha'\},$$

with $\alpha'=(y_1\alpha_1+y_2\alpha_2)v$.

Proof We can easily compute

$$Q_1(\alpha')=Q_1((y_1\alpha_1+y_2\alpha_2)v)=(y_1Q_1(\alpha_1)+y_2Q_1(\alpha_2))v=(y_1^2+y_2^2)v^2=a.$$

Recall that $H^*(BRV_3)\cong DA\{1,a^2,a^4\}$. We get

$$H^*(BRV_3)\cap\mathrm{Ideal}(y_iv)=DA\{D_2^2,a^2,a^4\}=DA\{a^5,a,a^3\}(Q_1\alpha'),$$

and the corollary follows. □

Corollary 11.9 $H^{\mathrm{odd}}(BRV_1;\mathbb{Z}_{(7)})\cong DA\{\bar b,\bar b^3,\bar b^5\}\{\alpha''\}\oplus DA\{\alpha\}$ *where $\alpha''=y_1v\alpha_2$.*

Proof Recall Corollary 8.8.We have $C\overline{c}'' = C(Y_1+Y_2-2C) = -Y_1^2 - Y_2^2 + 2Y_1Y_2$. Hence we can see $Q_1(\alpha) = -D_2\overline{c}''V$. □

Corollary 11.10 *The cohomology $H^{\mathrm{odd}}(B\mathbb{M};\mathbb{Z}_{(13)})$ is isomorphic to*

$$DA\{\alpha,\alpha_8,(u_8v^8)\alpha_8\} \oplus CA\{\alpha_6,(u_6v^3)\alpha_6,(u_6v^3)^2\alpha_6,\alpha_{12}\},$$

where

$$\begin{aligned}
\alpha_8 &= y_2(y_2^6+9y_2^3y_1^3+8y_1^6)v^7\alpha_1\\
\alpha_6 &= (y_2^5\alpha_2 - 9y_2^2y_1^3\alpha_2 + 8y_1^5y\alpha_1)v^2\\
\alpha_{12} &= C(y_2^{11}\alpha_2 - 2y_2^8y_1^3\alpha_2 + 3y_2^2y_1^9\alpha_2 + y_1^{11}\alpha_1)v^{11} - 3\alpha/V.
\end{aligned}$$

Proof It is almost immediate that

$$Q_1(\alpha_8) = u_8v^8,\quad Q_1(\alpha_6) = u_6v^3,\quad Q_1(\alpha_{12}) = (u_{12} - 5u'_{12} - 3C)CV.$$

From Theorem 10.3, we get the corollary. □

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