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The problem session

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This article contains a collection of problems contributed during the course of the conference.

1 The problems presented by Carles Broto

Broto, Levi and Oliver [1; 2] introduced the concept of p–local finite group as an algebraic object modeled on the fusion on the Sylow p–subgroup of a finite group, and attached to it a classifying space which is a p–complete space with properties similar to that of the p–completed classifying space of a finite group. Every finite group G gives rise canonically to a p–local finite group with classifying space homotopy equivalent to $BG^\wedge_p$. But there are also exotic examples; that is, p–local finite groups that cannot be obtained from a finite group in the canonic way and therefore its classifying space is not homotopy equivalent to the p–completed classifying space of any finite group.

Some exotic examples are obtained by Broto and Møller [3] out of p–compact groups (see Dwyer–Wilkerson [4] and Møller [7]). More precisely, to a 1–connected p–compact group X it is attached a family $X(q)$ of p–local finite groups, q a p–adic unit, that approximates X in the sense that the classifying spaces $\{BX(q^{p^m})\}_{m\in\mathbb{N}}$ form a direct system with mapping telescope

$$\operatorname{hocolim}_m BX(q^{p^m}) \simeq BX$$

(up to p–completion). In particular, this shows that every 1–connected p–compact group can be expressed, up to p–completion, as a mapping telescope of p–local finite groups. In view of that, Clarence Wilkerson asked the following question:

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Question 1.1 Is every p–compact group a telescope of finite groups? In more precise words, given a p–compact group X, does there exist a sequence of finite groups and homomorphisms

$$\{G_0 \xrightarrow{f_1} \cdots \xrightarrow{f_i} G_i \xrightarrow{f_{i+1}} \cdots\}$$

and a homotopy equivalence $\mathrm{hocolim}_i\, BG_i \simeq BX$, up to p–completion?

A possibly related question is the following.

Question 1.2 Let $\{BX_m\}_{m\in\mathbb{N}}$ be a direct system of classifying spaces of p–local finite groups and continuous maps $f_m\colon BX_m \to BX_{m+1}$ with $\mathrm{hocolim}_m\, BX_m \simeq BX$, the classifying space of a p–compact group X. Is there a map $\widehat{g}\colon BG_p^\wedge \to BX_m$, for some m, making the diagram

$$\begin{tikzcd} & & & BG_p^\wedge \arrow[dll, "\widehat{g}"'] \arrow[d, "g"] \\ \cdots \arrow[r, "f_m"'] & BX_m \arrow[r, "f_{m+1}"'] & \cdots \arrow[r] & BX \end{tikzcd}$$

homotopy commutative, where $BG_p^\wedge$ is the p–completed classifying space of a finite group (or p–local finite group)?

2 The problem presented by Nguyễn H V Hưng

Let $\mathrm{Sq}^0\colon \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2,\mathbb{F}_2) \to \mathrm{Ext}_{\mathcal{A}}^{s,2t}(\mathbb{F}_2,\mathbb{F}_2)$ be the squaring operation induced on the cohomology of the Steenrod algebra by the Frobenius map $\mathcal{A}_* \to \mathcal{A}_*, x \mapsto x^2$.

Conjecture 2.1 *(Sq^0 is eventually isomorphic on the Ext groups.)*

Let $Im(\mathrm{Sq}^0)^i$ denote the image of $(\mathrm{Sq}^0)^i$ on $Ext_{\mathcal{A}}^s(\mathbb{F}_2,\mathbb{F}_2)$. Then, for arbitrary s, there exists a number $r = r(s)$ depending on s such that

$$(\mathrm{Sq}^0)^{i-r}\colon Im(\mathrm{Sq}^0)^r \to Im(\mathrm{Sq}^0)^i$$

is an isomorphism for every $i \ge r$.

In other words, the conjecture predicts that any finite Sq^0–family in $\mathrm{Ext}_{\mathcal{A}}^s(\mathbb{F}_2,\mathbb{F}_2)$ has at most $r(s)$ nonzero elements.

Observation 2.2 We guess that $r(s) = s-2$.

Let us explain the motivation for this conjecture.

Denote by $\mathbb{V}_s$ an s–dimensional vector space over $\mathbb{F}_2$, and let

$$\mathrm{Tr}_s\colon \mathbb{F}_2 \otimes_{\mathrm{GL}_s} PH_d(B\mathbb{V}_s) \to \mathrm{Ext}_{\mathcal{A}}^{s,s+d}(\mathbb{F}_2, \mathbb{F}_2)$$

be the algebraic transfer defined by W Singer as an algebraic version of the geometrical transfer $\mathrm{tr}_s\colon \pi_*^S((B\mathbb{V}_s)_+) \to \pi_*^S(S^0)$. Here $PH_*(B\mathbb{V}_s)$ denotes the primitive part of $H_*(B\mathbb{V}_s)$ consisting of all homology classes that are annihilated by any Steenrod operations of positive degrees.

It has been proved that Tr_s is an isomorphism for $s = 1, 2$ by Singer and for $s = 3$ by Boardman. Among other things, these data together with the fact that $\mathrm{Tr} = \oplus_s \mathrm{Tr}_s$ is an algebra homomorphism show that Tr_s is highly nontrivial. Therefore, the algebraic transfer is expected to be a useful tool in the study of the mysterious cohomology of the Steenrod algebra, $\mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$. Although Singer recognizes that Tr_5 is not isomorphic, his open conjecture predicts that Tr_s is a monomorphism for any s.

According to Boardman and Minami, one has a commutative diagram

$$\begin{tikzcd}
(\mathbb{F}_2 \otimes_{\mathrm{GL}_s} PH_*(B\mathbb{V}_s))_d \arrow[r, "\mathrm{Tr}_s"] \arrow[d, "\mathrm{Sq}^0"'] & \mathrm{Ext}_{\mathcal{A}}^{s,s+d}(\mathbb{F}_2, \mathbb{F}_2) \arrow[d, "\mathrm{Sq}^0"] \\
(\mathbb{F}_2 \otimes_{\mathrm{GL}_s} PH_*(B\mathbb{V}_s))_{2d+s} \arrow[r, "\mathrm{Tr}_s"'] & \mathrm{Ext}_{\mathcal{A}}^{s,2(s+d)}(\mathbb{F}_2, \mathbb{F}_2)
\end{tikzcd},$$

where the left vertical arrow is the Kameko Sq^0 and the right vertical one is the classical squaring operation.

The conjecture comes from the above diagram and the following theorem.

Theorem 2.3 (Hưng [5]) *(Sq^0 is eventually isomorphic on the domain of the transfer.)*

Let $Im(\mathrm{Sq}^0)^i$ denote the image of the Kameko iterated squaring $(\mathrm{Sq}^0)^i$ on the domain of the transfer, $\mathbb{F}_2 \otimes_{\mathrm{GL}_s} PH_*(B\mathbb{V}_s)$. Then

$$(\mathrm{Sq}^0)^{i-s+2}\colon Im(\mathrm{Sq}^0)^{s-2} \to Im(\mathrm{Sq}^0)^i$$

is an isomorphism for every $i \geq s-2$.

3 The problems presented by Nick Kuhn

These three problems are related to pondering the Singer Transfer

$$\tau_{s,t}\colon \mathrm{Hom}_{\mathcal{A}}^{t-s}(H^*((\mathbb{Z}/p)^s;\mathbb{F}_p),\mathbb{F}_p)_{GL_s(\mathbb{Z}/p)} \to \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_p,\mathbb{F}_p)$$

Problem 3.1 Prove or disprove the conjecture that $\mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2,\mathbb{F}_2)=0$ if $\alpha(t)>s$, where, as usual, $\alpha(t)=k$ if $t=2^{i_1}+\cdots+2^{i_k}$ with $i_1<\dots i_k$.

Note that Reg Wood's verification of the Peterson conjecture shows that the domain of $\tau_{s,t}$ is 0 if $\alpha(t)>s$, and I am conjecturing that the range of $\tau_{s,t}$ vanishes in these same places. Bob Bruner has checked that the conjecture is true in the range of known Ext calculations.

Problem 3.2 Can $\tau_{*,*}$ be viewed as the edge homomorphism of some spectral sequence?

This problem is not so well posed, but the point would be to find some sensible way to measure the failure of $\tau_{*,*}$ to be an isomorphism of bigraded algebras.

Problem 3.3 Fix s. Is there a uniform calculation of $\mathrm{Ext}_{\mathcal{A}}^{s,*}(\mathbb{F}_p,\mathbb{F}_p)$ for all large enough p?

For example, $\mathrm{Ext}_{\mathcal{A}}^{1,*}(\mathbb{F}_p,\mathbb{F}_p)$ can be viewed as 'independent of p' for $p>2$. The heuristic here goes as follows: $\mathrm{Ext}_{\mathcal{A}}^{s,*+s}(\mathbb{F}_p,\mathbb{F}_p)$ is similar to the representation theoretic object $\mathrm{Hom}_{\mathcal{A}}^{*}(H^*((\mathbb{Z}/p)^s;\mathbb{F}_p),\mathbb{F}_p)_{GL_s(\mathbb{Z}/p)}$, and thus is perhaps 'independent of p' for large p analogous to similar situations studied in the work of Anderson, Jantzen, and Soergel.

4 The problems presented by John H Palmieri

Let p be a prime. Let $P_* \cong \mathbb{F}_p[\xi_1,\xi_2,\xi_3,\dots]$ be the polynomial part of the dual Steenrod algebra, and write P for its dual; thus when $p=2$, P is the mod 2 Steenrod algebra, and when p is odd, P is a quotient Hopf algebra of the mod p Steenrod algebra. Consider $\mathrm{Ext}_{\mathrm{P}}^{*,*}(\mathbb{F}_{\mathrm{p}},\mathbb{F}_{\mathrm{p}})$.

The pth power map on P_* induces an algebra endomorphism $\mathcal{P}^0$ on Ext, which with respect to the grading acts like this:

$$\mathcal{P}^0\colon \mathrm{Ext}_{\mathrm{P}}^{\mathrm{s,t}}(\mathbb{F}_{\mathrm{p}},\mathbb{F}_{\mathrm{p}}) \to \mathrm{Ext}_{\mathrm{P}}^{\mathrm{s,pt}}(\mathbb{F}_{\mathrm{p}},\mathbb{F}_{\mathrm{p}}).$$

Conjecture 4.1 *Fix $s>0$. For all $z \in (\mathcal{P}^0)^{-1}\mathrm{Ext}_{\mathrm{P}}^{\mathrm{s,t}}(\mathbb{F}_{\mathrm{p}},\mathbb{F}_{\mathrm{p}})$, z is nilpotent. Equivalently, for all $z \in \mathrm{Ext}_{\mathrm{P}}^{\mathrm{s,t}}(\mathbb{F}_{\mathrm{p}},\mathbb{F}_{\mathrm{p}})$, there exists an $n \geq 0$ so that $(\mathcal{P}^0)^n z$ is nilpotent.*

The following is proved in [8].

Theorem 4.2 *Conjecture 4.1 is true when $p=2$.*

Hand and computer calculations when $p=2$ suggest the following.

Conjecture 4.3 *In Conjecture 4.1, one may take n to be 1. That is, for all $z \in \mathrm{Ext}_{\mathrm{P}}^{\mathrm{s,t}}(\mathbb{F}_{\mathrm{p}},\mathbb{F}_{\mathrm{p}})$, $\mathcal{P}^0 z$ is nilpotent.*

5 The problems presented by Stewart Priddy

Let $G_n = GL_n(\mathbb{F}_p)$. Quillen has shown $\operatorname{colim}_n H_i(G_n;\mathbb{F}_p)=0$ for $i>0$. The classes of $H^*(G_n;\mathbb{F}_p)$ are unstable characteristic classes for representations over $\mathbb{F}_p$.

Now let $p=2$; Maazen has shown $H^i(G_{2n};\mathbb{F}_2)=0$ for $0<i<n$. What about $i=n$? A maximal elementary p–subgroup of G_{2n} is

$$A_{n,n} = \begin{pmatrix} I_n & * \\ 0 & I_n \end{pmatrix}.$$

$A_{n,n} \approx (\mathbb{Z}/2)^{n^2}$ and

$$H^*(A_{n,n};\mathbb{F}_2) = \mathbb{F}[x_{i,j} \mid 1 \leq i,j \leq n]$$

where $x_{i,j}$ are the obvious 1–dimensional generators. The normalizer $N_{G_{2n}}(A_{n,n}) = G_n \times G_n$; thus the restriction homomorphism has the form

$$\text{res: } H^*(G_{2n};\mathbb{F}_2) \longrightarrow H^*(A_{n,n};\mathbb{F}_2)^{G_n \times G_n}$$

We consider the image in dimension n. Let

$$\det_n = \det x_{i,j}$$

Then $\det_n \in H^n(A_{n,n};\mathbb{F}_2)^{G_n \times G_n}$

Question 5.1 Is $\det_n \in \operatorname{im}(\text{res})$?

This is true for $n=1,2$ (see Milgram and Priddy [6]). There is also an analogous question for p odd but one must take into account the action of the diagonal matrices.

Question 5.2 Is $H^n(G_{2n};\mathbb{F}_2)=\mathbb{F}_2$ generated by $\det_n$?

6 The problem presented by Nobuaki Yagita

Let us write by E the extraspecial p–group p_+^{1+2} of order p^3 and exponent p for an odd prime p. In Theorem 6.2 of my paper in the present proceedings (Stable splitting and cohomology of p–local finite groups over the extraspecial p–group of order p^3 and exponent p), we have a graph which shows stable splitting of BG for all (p-local) finite groups having a Sylow p–subgroup E for $p = 3$. The problem is to write down the similar graph for $p = 7$. A partial result is given in Theorem 9.4 in the same paper.

References

[1] **C Broto**, **R Levi**, **B Oliver**, *The homotopy theory of fusion systems*, J. Amer. Math. Soc. 16 (2003) 779–856 MR1992826

[2] **C Broto**, **R Levi**, **B Oliver**, *The theory of p–local groups: a survey*, from: "Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K–theory", Contemp. Math. 346, Amer. Math. Soc., Providence, RI (2004) 51–84 MR2066496

[3] **C Broto**, **J Møller**, *Homotopy finite Chevalley versions of p–compact groups*, preprint

[4] **W G Dwyer**, **C W Wilkerson**, *Homotopy fixed-point methods for Lie groups and finite loop spaces*, Ann. of Math. (2) 139 (1994) 395–442 MR1274096

[5] **N H V Hưng**, *The cohomology of the Steenrod algebra and representations of the general linear groups*, Trans. Amer. Math. Soc. 357 (2005) 4065–4089 MR2159700

[6] **R J Milgram**, **S B Priddy**, *Invariant theory and $H^*(\mathrm{GL}_n(\mathbb{F}_p);\mathbb{F}_p)$*, from: "Proceedings of the Northwestern conference on cohomology of groups (Evanston, Ill., 1985)", volume 44 (1987) 291–302 MR885113

[7] **J M Møller**, *Homotopy Lie groups*, Bull. Amer. Math. Soc. (N.S.) 32 (1995) 413–428 MR1322786

[8] **J H Palmieri**, *Some quotient Hopf algebras of the dual Steenrod algebra*, Trans. Amer. Math. Soc. 358 (2006) 671–685 MR2177035

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