Degeneration of Heegaard genus, a survey

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We survey known (and unknown) results about the behavior of Heegaard genus of 3-manifolds constructed via various gluings. The constructions we consider are (1) gluing together two 3-manifolds with incompressible boundary, (2) gluing together the boundary components of surface $\times I$, and (3) gluing a handlebody to the boundary of a 3-manifold. We detail those cases in which it is known when the Heegaard genus is less than what is expected after gluing.

57N10, 57M99; 57M27

1 Introduction

In this paper we survey known (and unknown) results about the Heegaard genus of 3-manifolds constructed via some gluing map. In particular we will be concerned here with compact orientable 3-manifolds M constructed in one of three ways. For the first construction, let X and Y be compact, orientable, irreducible 3-manifolds each with a single boundary component homeomorphic to a closed orientable surface F, and suppose that ∂X and ∂Y are essential in X and Y.

Construction 1 Glue ∂X to ∂Y via a map φ. We write $M = X \cup_F Y$ or $M = X \cup_\varphi Y$.

For the second construction we begin with $F \times I$.

Construction 2 Glue $F \times \{0\}$ to $F \times \{1\}$ via a map φ. We write $M = F \times_\varphi S^1$.

Finally, for the last construction considered here let $\mathcal{H}(F)$ denote the handlebody whose boundary is homeomorphic to F.

Construction 3 Glue $\partial\mathcal{H}(F)$ to ∂X via a map φ. We write $M = X \cup \mathcal{H}(F)$ or $M = X \cup_\varphi \mathcal{H}(F)$.

The *Heegaard genus* of M, denoted $g(M)$, is the minimal value g such that M admits a Heegaard splitting of genus g. The genus of the surface F is denoted $g(F)$.

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The following definition can be made for any compact, orientable 3-manifold M. For simplicity, we assume here that M is closed.

Definition 1.1 For F a separating surface in M, let X and Y denote the components of M cut along F. Let $V_X \cup_{H_X} W_X$ and $V_Y \cup_{H_Y} W_Y$ denote Heegaard splittings of X and Y, respectively, such that $F \subset \partial V_X, \partial W_Y$. Then there exists a product neighborhood $F \times I$ of F such that V_X equals $F \times [0, \frac{1}{2}]$ with 1-handles attached along $F \times \{0\}$ and W_Y equals $F \times [\frac{1}{2}, 1]$ with 1-handles attached along $F \times \{1\}$. Form a homeomorphism of M by deforming $F \times I$ to $F \times \{\frac{1}{2}\}$ so that the disks of attachment of the 1-handles in V_X and in W_Y end up disjoint on $F \times \{\frac{1}{2}\} = F$. This yields compression bodies $V = V_Y \cup \{$1-handles in $V_X\}$, and $W = W_X \cup \{$1-handles in $W_Y\}$, giving a Heegaard splitting $V \cup_H W$. Such a splitting is called an *amalgamation along* F.

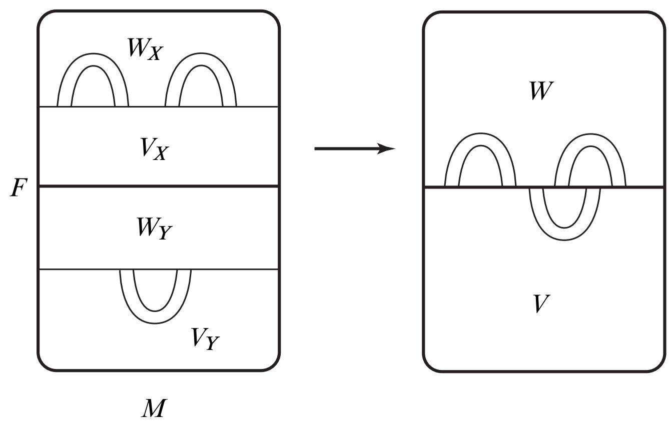

Figure 1: A schematic for the construction of an amalgamation along F

By amalgamating minimal genus Heegaard splittings along F (or along two copies of F if $M = F \times_\varphi S^1$) one obtains the following inequalities in each of the three constructions:

(1) $g(M) \le g(X) + g(Y) - g(F)$

(2) $g(M) \le 2g(F) + 1$

(3) $g(M) \le g(X)$

Our discussion will be concerned with when each of the inequalities is either strict ("degeneration is possible") or is in fact an equality ("no degeneration").

If a Heegaard splitting of a 3-manifold is of minimal genus, then it is unstabilized. One way that degeneration of Heegaard genus can occur in the above situations is that the amalgamation of minimal genus splittings results in a stabilized splitting. This leads to the following natural question:

Question 1.2 When is the amalgamation of unstabilized Heegaard splittings unstabilized?

The paper is organized as follows. In Sections 2 – 4 we discuss the issue of degeneration of Heegaard genus under the three types of gluing mentioned above, concluding each section with a discussion of Question 1.2. These sections are organized by the genus of F as the results tend to be based more on genus than on which of the three constructions we are considering. The results on Heegaard genus degeneration from these sections are summarized in a table at the end of Section 4. In Section 5 we review known results bounding how much degeneration of Heegaard genus can occur. For basic definitions and notions related to Heegaard splittings, see Scharlemann [26].

An interesting property of Heegaard genus is that it provides an upper bound on the rank of the fundamental group of a 3-manifold M, since a genus g Heegaard splitting of M can be used to construct a presentation of $\pi_1(M)$ with g generators. There are several results about the degeneration of rank of the fundamental group of manifolds formed via some gluing map, many analogous to those stated in this paper about Heegaard genus. While the analogy between Heegaard genus and rank is interesting and worth mentioning, we do not attempt to include a detailed discussion of it here.

2 Sphere gluings

2.1 No degeneration

We begin by showing that Heegaard genus does not degenerate in any of the three constructions when F is a sphere. The case that $M = X \cup_{S^2} Y$ is implied by the following classic result of Haken.

Theorem 2.1 (Haken's Lemma [7]) *Let $V \cup_H W$ be a Heegaard splitting of M and suppose that M contains an essential 2-sphere. Then there is an essential 2-sphere F such that $H \cap F$ is a single simple closed curve essential on H.*

If the sphere F is separating so that M is the connected sum of two 3-manifolds X and Y, then Haken's Lemma implies that a Heegaard splitting of M is obtained from the "connected sum" of Heegaard splittings of X and Y. In particular:

Corollary 2.2 *If $M = X\#Y$, then $g(M) = g(X) + g(Y)$.*

Proof Given Heegaard splittings of X and Y let B_X and B_Y be open embedded 3-balls in X and Y, respectively, each intersecting the respective Heegaard surface in an open equatorial disk. Form the connected sum of X and Y by gluing $X - B_X$ to $Y - B_Y$ so that the component of the Heegaard surface in $X - B_X$ meets the component of the Heegaard surface in $Y - B_Y$ (note that there are two ways to do this, yielding possibly non-isotopic splittings). This yields a Heegaard splitting of M, implying that $g(M) \le g(X) + g(Y)$.

For the reverse inequality, assume that X and Y are irreducible. Consider a Heegaard splitting $V \cup_H W$ of M of minimal genus. Since M contains an essential sphere, by Haken's Lemma it contains one meeting H in an essential simple closed curve. Since X and Y are irreducible, M contains a unique essential sphere, hence H intersects the connect sum sphere S^2 in a simple closed curve. Cutting along this sphere and filling the resulting boundary components with 3-balls containing equatorial disks, we obtain Heegaard splittings of X and Y. If one of these splittings is not of minimal genus, then as above it could have been used to form a Heegaard splitting of M of smaller genus, a contradiction. Thus the splittings of X and Y must be of minimal genus. This implies $g(M) = g(X) + g(Y)$.

To remove the assumption that X and Y are irreducible, use Milnor's result on the uniqueness of prime decomposition for 3-manifolds [17] and proceed by induction. □

If $M = S^2 \times_\varphi S^1$ is orientable then there is only one possibility for the map φ, which implies $M = S^2 \times S^1$. The Heegaard genus of this manifold is one.

Finally, if M has a sphere boundary and we glue a genus zero handle-body (a 3-ball) to it then the Heegaard genus does not change. Thus in all three constructions, Heegaard genus does not degenerate when F is a sphere.

2.2 Stabilization and connected sum

In the case that F is a sphere Question 1.2 was originally asked by C McA Gordon, who conjectured that the connected sum of unstabilized Heegaard splittings is never stabilized [11]. A proof of this conjecture has been announced independently by the first author and Qiu.

Theorem 2.3 [1; 21] *Let H_X and H_Y be unstabilized Heegaard surfaces in X and Y, respectively. Then $H_X\#H_Y$ is an unstabilized Heegaard splitting surface in $X\#Y$.*

The splitting surface $H_X\#H_Y$ is defined as in the proof of Corollary 2.2.

3 Torus Gluing

3.1 Degeneration is possible

Unlike the sphere case, there are examples where Heegaard genus degenerates when F is a torus. The following result of Schultens and Weidmann shows that the amount of degeneration can be arbitrarily large.

Theorem 3.1 [28] *Let n be a positive integer. Then there exist manifolds $M_n = X_n \cup_{T^2} Y_n$ such that*

$$g(M_n) \leq g(X_n) + g(Y_n) - n.$$

They in fact construct examples of unstabilized Heegaard splittings of X_n and Y_n such that the resulting amalgamated Heegaard splitting of M_n can be destabilized n times.

If $M = T^2 \times_\varphi S^1$ is a torus bundle, degeneration of Heegaard genus is also possible. Taking two genus 2 Heegaard splittings of $T^2 \times I$ and amalgamating gives a Heegaard splitting of genus 3 of M, implying that $g(M) \leq 3$. Cooper and Scharlemann have characterized precisely which solvmanifolds have $g(M) = 2$.

Theorem 3.2 [6] *Let $M = T^2 \times_\varphi S^1$ be a solvmanifold, and suppose the monodromy φ can be expressed as*

$$\begin{pmatrix} \pm m & -1 \\ 1 & 0 \end{pmatrix}.$$

If $m \geq 3$, then $g(M) = 2$.

Moreover, they show that there are precisely two genus 2 splittings if $m = 3$, and only one genus 2 splitting if $m \geq 4$. It should be noted that manifolds which are torus bundles but not solvmanifolds, namely flat manifolds and nilmanifolds, have well understood Heegaard splittings as they admit Seifert fibrations. (See for example Moriah and Schultens [19] or Sedgwick [29] for results on Heegaard genus of Seifert fibered spaces.)

Finally, consider a 3-manifold obtained by gluing a solid torus to a manifold X with torus boundary, ie, via Dehn filling. In this case, as in the case of gluing two manifolds along a torus, Heegaard genus can degenerate by an arbitrary amount.

Example 3.3 Let X be the complement of a tunnel number n knot. Perform trivial Dehn filling on ∂X to obtain S^3. As the Heegaard genus of S^3 is 0, it follows that for any n there are manifolds X such that

$$g(X \cup \mathcal{H}(F)) = g(X) - n.$$

Another way Heegaard genus can degenerate under Dehn filling is the following situation.

Example 3.4 Suppose that X has a single torus boundary component T and let $V \cup_H W$ be a Heegaard splitting of X. Assume that T is contained in V, so V is a compression body. Then there exists an essential disk D' in V such that V cut along D' contains a component U homeomorphic to $T \times I$. Suppose there exists an essential disk D in W such that ∂D meets the boundary of U in a single arc δ'. The endpoints of δ' lie on $\partial D'$. Let β' be a loop on ∂U composed of δ' and a properly embedded arc in D'. Then there is a loop β on T and an essential annulus in U whose boundary components are β and β'. Suppose α is a slope on T meeting β in a single point (there is an infinite number of such slopes). Attaching a solid torus $\mathcal{H}(F)$ by gluing a meridian disk to α makes H a stabilized Heegaard surface in the resulting 3-manifold. These slopes correspond to a *destabilization line* in the Dehn filling space of X (the *Dehn filling space of* X is the set of all 3-manifolds obtained by Dehn filling X). Thus in these situations,

$$g(X \cup_\alpha \mathcal{H}(F)) \leq g(X) - 1.$$

3.2 Sufficiently complicated torus gluings

Despite the fact that Heegaard genus can degenerate when gluing along a torus, the following results show that degeneration is in fact a special phenomenon. Recall that X and Y are irreducible and each has a single incompressible torus boundary component (X and Y are called *knot manifolds* in the terminology of Bachman, Schleimer, and Sedgwick [3]).

Theorem 3.5 [3] *Suppose that $\varphi\colon \partial X \to \partial Y$ is a sufficiently complicated homeomorphism. Then the manifold $M(\varphi) = X \cup_\varphi Y$ has no strongly irreducible Heegaard splittings.*

The term *sufficiently complicated* is given in Definition 4.2 in [3] and is a technical statement about the distance φ maps curves on the torus (for example a suitably large power of an Anosov map is sufficiently complicated).

It can be shown using the above theorem that every Heegaard splitting of $M(\varphi)$ is an amalgamation along ∂X, implying that

$$g(M(\varphi)) = g(X) + g(Y) - 1.$$

If $M = T^2 \times_\varphi S^1$ is a solvmanifold, Scharlemann and Cooper's analysis applies here as well.

Theorem 3.6 [6] *If $M = T^2 \times_\varphi S^1$ is a solvmanifold with monodromy φ that cannot be expressed in the form given in Theorem 3.2, then the minimal genus Heegaard splitting of M has genus equal to 3 and is unique up to isotopy.*

Finally, suppose $M = X \cup \mathcal{H}(F)$ is obtained by Dehn filling. Above we gave examples where $g(X)$ degenerates by an arbitrarily large amount upon Dehn filling, and where $g(X)$ can degenerate by at least one for all fillings along slopes corresponding to a destabilization line in the Dehn filling space of X. Following work of Rieck and Rieck–Sedgwick, we see that with mild assumptions on X the above situations are the only possible ways for $g(X)$ to degenerate and are not generic occurrences.

Theorem 3.7 [22; 24] *Let X be an acylindrical manifold with incompressible torus boundary T. Then*

(1) *there are only finitely many slopes on T for which*
$$g(X \cup \mathcal{H}(F)) \leq g(X) - 2,$$

(2) *there are only finitely many destabilization lines in the Dehn filling space of X such that*
$$g(X \cup \mathcal{H}(F)) \leq g(X) - 1.$$

In particular, there are an infinite number of manifolds $X \cup \mathcal{H}(F)$ such that
$$g(X \cup \mathcal{H}(F)) = g(X).$$

Moriah and Rubinstein initially proved a similar theorem for negatively curved manifolds in [18]. Rieck and Sedgwick have proven a more general version of the above theorem for small manifolds in [23], restricting greatly the possibilities for discrepancies between Heegaard splittings of $X \cup \mathcal{H}(F)$ and X of any (not necessarily minimal) genus.

3.3 Stabilization and amalgamation along a torus

In considering Question 1.2, the result of Schultens and Weidmann given in Theorem 3.1 shows that for any n there exist examples of unstabilized Heegaard splittings that can be amalgamated to give a splitting that destabilizes n times. It seems, however, that this situation is special.

Conjecture 3.8 *Let $M = X \cup_{T^2} Y$ where X and Y each have a single incompressible torus boundary component. There is a complexity on maps $\varphi\colon \partial X \to \partial Y$ and an integer $n(X,Y)$ such that if the complexity of φ is greater than n then the amalgamation of any unstabilized splittings of X and Y is unstabilized.*

4 Higher genus gluings

Although the results are similar we consider the case when $g(F) \geq 2$ separately because the techniques and the implications of the theorems are different. For example, in the previous section the conclusion of Theorem 3.5 is that when $g(F) = 1$ and the gluing map is "sufficiently complicated", then M contains *no* strongly irreducible Heegaard splittings. The results presented in Section 4.2 show that when $g(F) \geq 2$ and the gluing map is "sufficiently complicated", then M contains *no minimal genus* strongly irreducible Heegaard splittings.

4.1 Degeneration is possible

As with the case when F is a torus, it is also possible for Heegaard genus to degenerate when F is a surface of genus at least 2 as shown by the following result of Kobayashi, Qiu, Rieck and Wang.

Theorem 4.1 [12] *There exists a 3-manifold M containing connected, separating incompressible surfaces F_n of arbitrarily large genus such that amalgamating two minimal genus Heegaard splittings of X_n and Y_n along F_n yields a $g(F_n) - 3$ times stabilized Heegaard splitting of M.*

As a consequence it follows that

$$g(M) \leq g(X_n) + g(Y_n) - 2g(F_n) + 3.$$

An interesting aspect of these examples is that this degeneration occurs in the same 3-manifold M, ie, M does not depend on n.

For manifolds of the form $F \times_\varphi S^1$, degeneration is also possible.

Example 4.2 Let M be a Seifert fibered space with base B a sphere and containing three exceptional fibers of multiplicities n, $2n$, $2n$, where n is an integer greater than 2. Assume that the Euler number of M is 0, so that there is some horizontal surface F in M (see for example Hatcher [8, Proposition 2.2]). By Jaco [9, Theorem VI.34], $M = F \times_\varphi S^1$. Moreover, the surface F branch covers B (a sphere) and by an Euler characteristic argument yields the following equation (see [8]):

$$\begin{aligned}\chi(F) &= m\chi(B) - m\left(\frac{2n-1}{2n} + \frac{2n-1}{2n} + \frac{n-1}{n}\right) \\ &= 2m - m\frac{6n-4}{2n}\end{aligned}$$

where m is the degree of the cover. As the least common multiple of the multiplicities of the fibers divides m, it follows that $2n$ divides m. Moreover, the assumption that $n \geq 3$ implies that $2-(6n-4)/2n$ is negative. Thus

$$\begin{aligned}\chi(F) &= 2m - m\frac{6n-4}{2n} \\ &\leq 4n-(6n-4) \\ &= 4-2n.\end{aligned}$$

Taking $\chi(F) = 2-2g(F)$ and solving, we obtain

$$g(F) \geq n-1.$$

Thus, a Heegaard splitting of M which is an amalgamation along F has genus at least $2n-1$. It is well known, however, that $g(M) = 2$ (see for example [4]). Therefore, given an integer $n \geq 3$ there is a 3-manifold M such that $g(M) \leq 2g(F)+1-(2n-3)$, implying Heegaard genus can degenerate by an arbitrary amount.

Finally, for manifolds of the form $X \cup_F \mathcal{H}(F)$, as before degeneration can occur. This can be seen by modifying Examples 3.3 and 3.4.

Example 4.3 Let $\mathcal{H}(F)$ be a knotted handlebody in S^3 whose complement X has incompressible boundary. Then there is a way of gluing $\mathcal{H}(F)$ to X such that the resulting manifold is S^3. Thus if $g(F) = n$,

$$g(X \cup \mathcal{H}(F)) \leq g(X) - n.$$

Example 4.4 Suppose that X has a single boundary component F, where $g(F) \geq 2$. Let $V \cup_H W$ be a minimal genus Heegaard splitting of X. Assume that F is contained in V, so V is a compression body. For each loop α on F one can find an essential annulus in V which meets F in α and meets H in a loop α_H. Now let D be a compressing disk for H in W and suppose there is a loop α on F such that α_H meets ∂D in a point. Attaching a handlebody $\mathcal{H}(F)$ in such a way so that α now bounds a disk makes H a stabilized Heegaard surface in the resulting 3-manifold.

4.2 Sufficiently complicated higher genus gluings

A *simple* 3-manifold is a 3-manifold that is compact, orientable, irreducible, atoroidal, acylindrical and has incompressible boundary. The following result of Lackenby shows that when two simple 3-manifolds are glued along a surface F with $g(F) \geq 2$ via a "sufficiently complicated" map, then as in the torus case there is no degeneration of Heegaard genus.

Theorem 4.5 [14] *Let X and Y be simple 3-manifolds, and let $h: \partial X \to F$ and $h': F \to \partial Y$ be homeomorphisms with some connected surface F of genus at least two. Let $\psi: F \to F$ be a pseudo-Anosov homeomorphism. Then, provided $|n|$ is sufficiently large,*

$$g(X \cup_{h'\psi^n h} Y) = g(X) + g(Y) - g(F).$$

Furthermore, any minimal genus Heegaard splitting for $X \cup_{h'\psi^n h} Y$ is obtained from splittings of X and Y by amalgamation, and hence is weakly reducible.

The intuition behind the proof of Lackenby's theorem is as follows. When the map φ is sufficiently complicated then geometrically M has a "long neck" region homeomorphic to $F \times (0,1)$. A result of Pitts and Rubinstein [20] implies that a strongly irreducible Heegaard splitting surface H is isotopic to a minimal surface or to two copies of a double cover of a non-orientable minimal surface attached by a tube. In either case, if such a surface passes through the long neck region then it must have large area. By the Gauss-Bonnet theorem this implies H has large genus. The conclusion is that if the map φ is complicated enough then any strongly irreducible Heegaard splitting has genus higher than the genus of an amalgamated splitting. From here it is not difficult to show that any splitting (strongly irreducible or not) which is not an amalgamation of splittings of X and Y is not minimal genus.

Souto has generalized this technique using the notion of distance in the curve complex.

Theorem 4.6 [30] *Let X and Y be simple 3-manifolds and suppose ∂X and ∂Y are connected and homeomorphic with genus at least two. Fix an essential simple closed curve $\alpha \subset \partial X$ and $\alpha' \subset \partial Y$. Then there is a constant n_0 such that every minimal genus Heegaard splitting of $X \cup_\psi Y$ is constructed by amalgamating splittings of X and Y and hence*

$$g(X \cup_\psi Y) = g(X) + g(Y) - g(F)$$

for every diffeomorphism $\psi : \partial X \to \partial Y$ with $d_{(\partial Y)}(\psi(\alpha), \alpha') \geq n_0$, where $d_{(\partial Y)}(\beta, \gamma)$ denotes the distance of essential simple closed curves β and γ in the curve complex of ∂Y.

Like Lackenby, Souto uses geometry to establish the above result. T Li has obtained a combinatorial proof of a similar theorem [16].

Next suppose $M = F \times_\varphi S^1$. The following theorem of Lackenby indicates that, generically, the minimal genus Heegaard splittings of manifolds of the form $F \times_\varphi S^1$ are formed by amalgamating splittings of $F \times I$.

Theorem 4.7 [15] *Let M be a closed, orientable 3-manifold that fibers over the circle with pseudo-Anosov monodromy. Let $\{M_i \to M\}$ be the cyclic covers dual to the fiber. Then, for all but finitely many i, M_i has an irreducible, weakly reducible, minimal genus Heegaard splitting.*

This implies that for all but finitely many i,

$$g(M_i) = 2g(F) + 1.$$

Note that a stronger version of the above theorem has been proved by Rubinstein [25]. Also see a generalization by Brittenham and Rieck [5].

Bachman and Schleimer have improved this result using the notion of distance in the curve complex. Suppose that $M = F \times_\varphi S^1$ is formed using monodromy $\varphi\colon F \to F$. Define $d(\varphi)$ to be the minimum distance that φ moves a vertex in the curve complex of F.

Theorem 4.8 [2] *Any Heegaard surface H in $F \times_\varphi S^1$ with $-\chi(H) < d(\varphi)$ is an amalgamation of splittings of $F \times I$.*

Finally, consider the case that M is of the form $X \cup_F \mathcal{H}(F)$. As a generalization of Thurston's hyperbolic Dehn surgery theorem, Lackenby has shown in [13] that if X is simple and $\varphi\colon \partial X \to \partial\mathcal{H}(F)$ is "sufficiently complicated" then $X \cup_\varphi \mathcal{H}(F)$ is irreducible, atoroidal, word hyperbolic and not Seifert fibered. Lackenby then asks if the structure of the Heegaard splittings of these manifolds can also be understood.

Question 4.9 [13] How does Heegaard genus degenerate under handlebody gluing?

We pose the following conjecture as an answer to Lackenby's question. This conjecture is similar in nature to Theorem 3.7.

Conjecture 4.10 *Suppose that X has a single boundary component F, where $g(F) \geq 2$. Let $V \cup_H W$ be a minimal genus Heegaard splitting of X. Assume that F is contained in V, so V is a compression body. Let $\mathcal{W}$ denote the set of vertices of the curve complex of H that correspond to the boundaries of disks in W. For each loop α on F one can find an essential annulus in V which meets F in α and meets H in a loop α_H. Now glue a handlebody $\mathcal{H}(F)$ to ∂X. Let $\mathcal{V}_F$ denote the vertices of the curve complex of H defined as follows: if α bounds a disk in $\mathcal{H}(F)$ then $\alpha_H \in \mathcal{V}_F$. If the distance between $\mathcal{W}$ and $\mathcal{V}_F$ is large enough then H is a minimal genus Heegaard splitting of $X \cup_F \mathcal{H}(F)$.*

4.3 Stabilization and amalgamation along a higher genus surface

Again we consider Question 1.2. As with the torus case, we have only a conjecture.

Conjecture 4.11 *Let $M = X \cup_\varphi Y$ where X and Y each have a single incompressible boundary component of genus at least two. Then there is a complexity on maps $\varphi\colon \partial X \to \partial Y$ and an integer $n(X, Y, g)$ such that if the complexity of φ is greater than n the amalgamation of any unstabilized splittings of X and Y whose genera are less than g is unstabilized.*

Note the subtle difference between this conjecture and Conjecture 3.8. In Conjecture 3.8 we posit that if the gluing map is "sufficiently complicated" then the amalgamation of *any* two unstabilized splittings is unstabilized. Here we conjecture that the same is true only if the splittings have low genus compared with the complexity of the gluing map.

4.4 Table of degeneration of Heegaard genus

In the following table we summarize the results of Sections 2 – 4. The columns correspond to the genus of F and the rows to the type of gluing used to construct M. We take "D" to mean "degeneration is possible", "ND" to mean "no degeneration", and "NDSC" to mean "no degeneration if the gluing map φ is sufficiently complicated" in the appropriate contexts. In parentheses we provide the number of the theorem, corollary or example associated to the result.

	$g(F) = 0$	$g(F) = 1$	$g(F) \geq 2$
$X \cup_F Y$	ND 2.2	D 3.1 NDSC 3.5	D 4.1 NDSC (4.5, 4.6)
$F \times_\varphi S^1$	ND	D 3.2 NDSC 3.6	D 4.2 NDSC (4.7, 4.8)
$X \cup_F \mathcal{H}(F)$	ND	D (3.3, 3.4) NDSC 3.7	D 4.3 NDSC ??

5 Lower bounds on the degeneration

In the previous sections we discussed several situations in which Heegaard genus can degenerate by an arbitrary amount. In this section we state results that bound the amount by which Heegaard genus can degenerate in terms of the genus of the gluing surface and the Heegaard genera of the pieces. We will focus on the case that M is obtained by gluing X and Y together along a connected, orientable surface F, ie, $M = X \cup_F Y$. Whereas some of the results on Heegaard genus degeneration in the previous sections are obtained by amalgamating unstabilized Heegaard splittings and getting stabilized splittings, the results in this section are obtained by finding lower bounds on the genus of the possible Heegaard splittings one can construct in a given manifold.

As more restrictions are placed on the component manifolds X and Y, there are better known bounds. The least restrictive class of manifolds was studied by Schultens [27]. Suppose X and Y are irreducible 3-manifolds, and let n_X and n_Y denote the number of non-parallel essential annuli that can be simultaneously embedded in X and Y, respectively. Then Schultens obtains the bound

$$g(X \cup_F Y) \geq \frac{1}{5}(g(X) + g(Y) - 8g(F) + 11 - 4(n_X + n_Y)).$$

If, in addition, the manifolds X and Y are assumed the be atoroidal and acylindrical, then previously Johannson [10] had obtained the bound

$$g(X \cup_F Y) \geq \frac{1}{5}(g(X) + g(Y) - 2g(F)).$$

Most recently the first author, in conjunction with Schleimer and Sedgwick [3], added the restriction that the component manifolds X and Y are *small* (ie, irreducible and every incompressible surface is parallel to a boundary component). This allowed them to obtain the bound

$$g(X \cup_F Y) \geq \frac{1}{2}(g(X) + g(Y) - 2g(F)).$$

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