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Incompressible surfaces and $(1,2)$–knots

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We give a description of all $(1,2)$–knots in S^3 which admit a closed meridionally incompressible surface of genus 2 in their complement. That is, we give several constructions of $(1,2)$–knots having a meridionally incompressible surface of genus 2, and then show that any such surface for a $(1,2)$–knot must come from one of the constructions. As an application, we show explicit examples of tunnel number one knots which are not $(1,2)$–knots.

57M25; 57N10

1 Introduction

An important problem in knot theory is that of determining all incompressible surfaces in a given knot complement. The main purpose of this paper is to give a description of all $(1,2)$–knots in S^3 which admit a closed meridionally incompressible surface of genus 2 in their complement. Another purpose is to construct explicit examples of tunnel number one knots that are not $(1,2)$–knots. These are obtained by combining the constructions of the author [4] with the results of this paper.

Let F be a closed surface of genus g standardly embedded in S^3, that is, it bounds a handlebody on each of its sides. Following Doll [2], we say that a knot K has a (g,b)*–presentation* or that it is in a (g,b)*–position*, if K has been isotoped to intersect F transversely in $2b$ points that divide K into $2b$ arcs, so that the b arcs in each side can be isotoped, keeping the endpoints fixed, to disjoint arcs on F. The *genus–g–bridge number* of K, $b_g(K)$, is the smallest integer n for which K has a (g,n)–presentation. The genus–0–bridge number $b_0(K)$ is then the usual bridge number; we say that a knot is an n–bridge knot if $b_0(K) \leq n$. Here we will consider only the case $g=1$. We say that a knot is a $(1,n)$*–knot* if $b_1(K) \leq n$. It is not difficult to see that if K has a (g,b)–presentation, then the tunnel number of K, denoted $\operatorname{tn}(K)$, satisfies $\operatorname{tn}(K) \leq g+b-1$.

Let K be a knot in S^3, and S a closed surface in its complement. We say that S is meridionally compressible if there is an embedded disk D in S^3, with $S \cap D = \partial D$ a nontrivial curve in S, and so that K intersects D at most in one point. Otherwise S is

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called meridionally incompressible. In particular, if S is meridionally incompressible then it is incompressible in $S^3 - K$.

Incompressible surfaces in the complement of knots with a given bridge number have been studied in several cases. Schubert [23] studied incompressible tori in the complement of knots and found a relation between the bridge numbers of a satellite knot K and its companion K'. Hatcher and Thurston [13] proved that there are no closed incompressible surfaces in the complement of 2–bridge knots; this also follows from Gordon and Litherland [11]. On the other hand, Finkelstein and Moriah [10] and Wu [25] proved that the complement of a *generic* n–bridge knot, $n \geq 3$, contains a closed incompressible surface (but in general the surface is meridionally compressible). More recently Ozawa [19] has given a description of all 3–bridge knots whose complement contain a closed incompressible surface of genus 2. It may be difficult to do something similar for 3–bridge knots and surfaces of higher genus. In Eudave-Muñoz and Neumann-Coto [7], some examples are given of 3–bridge knots whose complement contain meridionally incompressible surface of arbitrarily high genus, ie examples of a single 3–bridge knot which contains infinitely many closed meridionally incompressible surfaces in its complement.

Incompressible surfaces in the complement of $(1,1)$–knots have also been studied. $(1,1)$–knots whose complement contain an incompressible torus, ie satellite $(1,1)$–knots, have been classified by Morimoto and Sakuma [17]; see also Eudave-Muñoz [3]. Well, in those papers satellite tunnel number one knots are classified but these turn out to be $(1,1)$–knots, as it is shown in [3]. In [4] a construction is given of $(1,1)$–knots whose complement contain a closed meridionally incompressible surface of genus g, and in [6] it is proved that any $(1,1)$–knot whose complement contains a closed meridionally incompressible surface must come from that construction. It is shown by Saito [22] that the complement of a satellite tunnel number one knot does not contain any meridionally incompressible surface other than the satellite torus; this implies that the knots constructed in [4] are hyperbolic. It follows from work of Gordon and Reid [12] that the complement of a $(1,1)$–knot cannot contain an incompressible planar meridional surface, ie a meridional surface of genus 0 (a meridional surface is a properly embedded surface in a knot exterior whose boundary consists of meridians of the knot). On the other hand, in Eudave-Muñoz and Ramírez-Losada [8] a description is given of all $(1,1)$–knots whose complement contain a meridional and meridionally incompressible surface of genus $g \geq 1$. The complement of any of these knots also contains a closed incompressible surface (but perhaps meridionally compressible) by Culler, Gordon, Luecke and Shalen [1].

Not much is known about incompressible surfaces in the complement of $(1,2)$–knots. In [7], a construction is given of hyperbolic $(1,2)$–knots whose complement contain an acylindrical surface of genus g, $g \geq 2$, ie an incompressible surface which divides the exterior of the knot into manifolds that do not contain essential annuli. The example given in [7] of a 3–bridge knot whose complement contains meridionally incompressible surfaces of arbitrarily high genus can be adapted to produce an example of a $(1,2)$–knot whose complement contains meridionally incompressible surfaces of arbitrarily high genus. To do that just embed the branched surface given in [7, Figure 14] as a surface of type 6, defined in Section 2.6 of this paper. This example shows that it may be difficult to give a description of all $(1,2)$–knots whose complement contain a closed meridionally incompressible surface. However, in this paper we do this for surfaces of genus 2. In Section 2 we give several constructions which produce $(1,2)$–knots whose complement contain a closed meridionally incompressible surface of genus 2. In Section 3 we show that if the complement of a $(1,2)$–knot contains a closed meridionally incompressible surface of genus 2, then the knot and the surface come from one of the given constructions. Contained in that proof is also a description of all $(1,2)$–knots whose complement contain a meridionally incompressible torus; these are some satellites of $(1,1)$–knots. It also follows from the construction that there are $(1,2)$–knots whose complement contains both a closed meridionally incompressible surface of genus 2 and of genus 1.

If K is a $(1,1)$–knot, then it is easy to see that K has tunnel number one. On the other hand, if K has tunnel number one, it seems to be very difficult to determine $b_1(K)$. A priori there should be tunnel number one knots with arbitrarily large b_1, but this has been difficult to prove. Moriah and Rubinstein [16] showed the existence of tunnel number one knots K with $b_1(K) \geq 2$. Morimoto, Sakuma and Yokota also showed this, and gave explicit examples of knots K with tunnel number one and $b_1(K) = 2$ [18]. It was shown in [6] that many of the tunnel number one knots K constructed in [4] are not $(1,1)$–knots. We refine the proof here, and show in Section 4 that some of the knots constructed in [4] are not $(1,2)$–knots. The argument is as follows: the complement of the tunnel number one knots constructed in [4] contains a closed meridionally incompressible surface. We then pick some of them whose complement contain a meridionally incompressible surface of genus 2, and show that the surface comes from none of the constructions of Section 2. This implies that $b_1(K) \geq 3$ for any such knot K. These knots can be explicitly constructed; an example is given in Figure 13. However, our examples seem to satisfy $b_1(K) \geq 4$, and we do not know if one of these knots satisfies $b_1(K) = 3$.

Recently, Johnson and Thompson [14], and independently Minsky, Moriah and Schleimer [15] have shown that for any given n, there exist tunnel number one knots which are not $(1,n)$–knots. In [15] it is in fact shown that for given t and n, there exist tunnel number t knots K so that $b_t(K) > n$. The two papers use similar techniques and prove the existence of such knots, but do not give explicit examples.

Finally, we add that Valdez-Sánchez and Ramírez-Losada [20] have also shown examples of tunnel number one knots K with $b_1(K) = 2$. These knots bound punctured Klein bottles but are not contained in the $(1,1)$–knots bounding Klein bottles determined by the same authors [21]. In Eudave-Muñoz [5] a construction is given of tunnel number one knots which admit a meridional incompressible surface. Using [8], we can show that among these knots there are ones with $b_1(K) = 2$ [9], and expect to prove that there are others with $b_1(k) = 3$.

2 Construction of meridionally incompressible surfaces

Let F be a closed surface of genus g standardly embedded in S^3, and let $I = [0,1]$. Consider a product neighborhood $F \times I$ of F. To say that a knot K has a (g,b)–presentation is equivalent to say that K has been isotoped to lie in $F \times I$, so that $K \cap (F \times \{0\})$ and $K \cap (F \times \{1\})$ consists each of b arcs (or b tangent points), and the rest of the knot consists of $2b$ vertical arcs in $F \times I$, that is, arcs which intersect each leave $F \times \{t\}$, $0 < t < 1$, in the product exactly in one point. Or simply, K is in a (g,b)–position if $K \subset F \times I$, and the projection map $p\colon F \times I \to I$ when restricted to K has exactly b local maxima and b local minima.

Let T be a standard torus in S^3, and let $I = [0,1]$. Consider $T \times I \subset S^3$. $T_0 = T \times \{0\}$ bounds a solid torus R_0, and $T_1 = T \times \{1\}$ bounds a solid torus R_1, such that $S^3 = R_0 \cup (T \times I) \cup R_1$. Think of the solid torus R_1 as containing the point at infinity. By a vertical arc in a product $T \times [a,b]$ we mean an embedded arc which intersects every torus $T \times \{x\}$ in the product in at most one point. By a level simple closed curve we mean a curve which is contained in some level torus $T \times \{x\}$.

In this section we construct knots with a $(1,2)$–presentation whose complement contain a closed meridionally incompressible surface of genus 2. We assume that all knots constructed in this section are contained in $T \times I$.

2.1 Surfaces of type 1

Choose a point e on I, so that $0 < e < 1$. Consider the torus $T_e = T \times \{e\}$. Let γ_0, γ_1 be simple closed curves embedded in the product $T \times (0,e)$ and $T \times (e,1)$ respectively, so that each curve has only one local maximum and one local minimum with respect to the projection to $(0,e)$ or to $(e,1)$. Suppose also that γ_0 (γ_1) is not in a 3–ball contained in $R_0 \cup (T \times [0,e])$ (resp. $(T \times [e,1]) \cup R_1$), that is, it is not a trivial knot in that region.

Let α be a vertical arc in $T\times[0,1]$, joining the maximum point of γ_0 with the minimum of γ_1. Let Γ_1 be the 1–complex consisting of the union of the curves γ_0, γ_1 and the arc α. So Γ_1 is a trivalent graph embedded in S^3.

Let $N(\Gamma_1)$ be a regular neighborhood of Γ_1. This is a genus 2 handlebody. We can assume that $N(\Gamma_1)$ is the union of 2 solid tori, $N(\gamma_0)$ and $N(\gamma_1)$, joined by the 1–handle $N(\alpha)$. Let $S=\partial N(\Gamma_1)$, we say that S is a surface of type 1. In Figure 1 we show in a schematic way a surface of type 1, and also give an explicit example, where both curves γ_i, $i=1,0$, are isotopic to level curves.

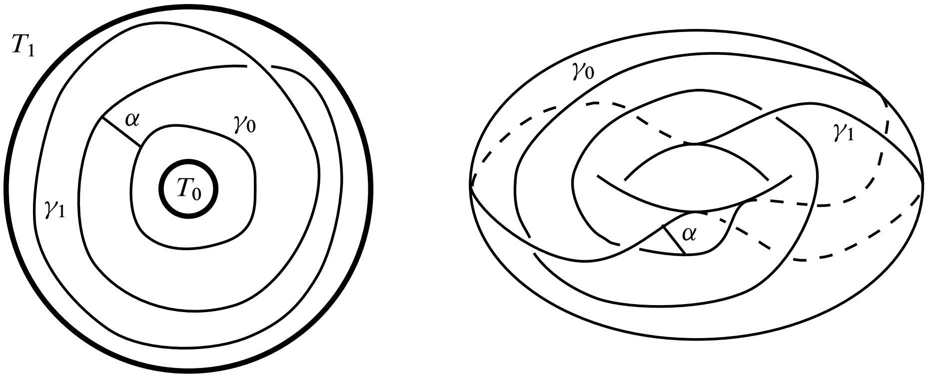

Figure 1

There are three possibilities for the graph Γ_1:

(1) By isotopies of Γ_1, none of the curves γ_0 and γ_1 can be isotoped into a level curve.

(2) By an isotopy of Γ_1, both curves γ_0 and γ_1 can be isotoped into level curves, say γ_i can be isotoped into γ_i', which lies in T_i, $i=0,1$. In this case we assume that γ_i' is not isotopic to the core of R_i, ie γ_i' does not consist of a longitude and several meridians of R_i, and assume also that $\Delta(\gamma_0',\gamma_1')\geq 2$ [4].

(3) By an isotopy of Γ_1, only one of the curves, say γ_1, can be isotoped into a level curve γ_1' contained in T_1. Assume that γ_1' is not isotopic to the core of R_1, ie γ_1' does not consist of a longitude and several meridians of R_1.

Let $E=N(\Gamma_1)\cap T_e$, this is a disk, which is in fact a cocore of the 1–handle $N(\alpha)$. Embed a knot k in $N(\Gamma_1)$ so that it intersects E in four points, $k\cap N(\gamma_1)$ consists of two arcs each having just one local maximum, $k\cap N(\gamma_0)$ consists of two arcs each having just one local minimum, and $k\cap N(\alpha)$ consists of four vertical arcs. Suppose also that $S=\partial N(\Gamma_1)$ is meridionally incompressible in $N(\Gamma_1)-k$. It is not difficult to see that there are plenty of such knots. See Figure 2 for an example.

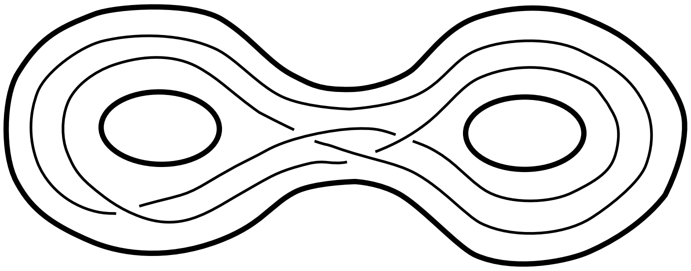

Figure 2

Theorem 2.1 *Let Γ_1 be a graph as above, and k in $N(\Gamma_1)$ as above. Then k is a $(1,2)$–knot, and $S = \partial N(\Gamma_1)$ is meridionally incompressible in $S^3 - k$, except possibly if Γ_1 is as in case (3), ie it can be isotoped so that exactly one of the curves γ_i is a level curve.*

Proof By construction k has a $(1,2)$–presentation, and by hypothesis S is meridionally incompressible in $N(\Gamma_1) - k$. So it remains to prove that S is incompressible in $S^3 - \operatorname{int} N(\Gamma_1)$.

Let $T'_e = T_e - \operatorname{int} N(\Gamma_1)$, this is a once-punctured torus. Suppose D is a compression disk for S, and suppose it intersects transversely the torus T'_e. Let β be a simple closed curve of intersection between D and T'_e, which is innermost in D. So β bounds a disk $D' \subset D$, which is contained, say, in the solid torus $(T \times [e,1]) \cup R_1$. If β is trivial on T'_e, then by cutting D with an innermost disk lying in the disk bounded by β on T'_e, we get a compression disk with fewer intersections with T'_e. If β is essential on T'_e, then it would be parallel to $\partial T'_e$, or it would be a meridian of the solid torus $(T \times [e,1]) \cup R_1$, but in any case the curve γ_1 will be contained in a 3–ball, which is a contradiction.

So suppose D intersects T'_e only in arcs. Let β be such an arc which is outermost on D; it cobounds with an arc $\delta \subset \partial D$ a disk D'. If β is parallel to an arc on $\partial T'_e$, then by cutting D with such an outermost arc lying on T'_e we get another compression disk with fewer intersections with T'_e, so assume that β is an essential arc on T'_e. After isotoping D if necessary, we can assume that the arc δ can be decomposed as $\delta = \delta_1 \cup \delta_2 \cup \delta_3$, where δ_1, δ_3 lie on $\partial N(\alpha)$ and δ_2 lie on $\partial N(\gamma_1)$ (if δ were contained in $\partial N(\alpha)$, then by isotoping D we would get a compression disk whose intersection with T'_e contains a simple closed curve). Let E be a disk contained in $\partial N(\alpha)$ so that $\partial E = \delta_1 \cup \delta_4 \cup \delta_3 \cup \delta_5$, where δ_4 lies on $\partial T'_e$ and δ_5 lies on $\partial N(\gamma_1) \cap \partial N(\alpha)$. So $D' \cup E$ is an annulus, where one boundary component, ie $\beta \cup \delta_4$ lies on T_e, and the other, $\delta_2 \cup \delta_5$, lies on $\partial N(\gamma_1)$. If $\delta_2 \cup \delta_5$ is a meridian of γ_1, then $\beta \cup \delta_4$ is a meridian of the solid torus $(T \times [e,1]) \cup R_1$. Then γ_1 intersects a meridian disk of $(T \times [e,1]) \cup R_1$ in one point, which implies that it is parallel to a knot lying on the torus T_1, and it is isotopic to the core of R_1, which is a contradiction. If $\delta_2 \cup \delta_5$ goes more than once longitudinally on γ_1, then the level curve $\beta \cup \delta_4$ (which is a trivial or a torus knot) would be a cable knot around γ_1. This shows that γ_1 is a core of the solid torus R_1, which is not possible. If $\delta_2 \cup \delta_5$ goes once longitudinally on γ_1, then γ_1 is isotopic to a curve on T_1. So we conclude that either S is incompressible or γ_1 is isotopic to a level curve on T_1.

Suppose now that Γ_1 has been isotoped so that γ_1 is a level curve in T_1 and that γ_0 is a level curve in T_0. The incompressibility of S now follows from [4, Theorem 4.1], because we assume that $\Delta(\gamma_1, \gamma_2) \geq 2$. This completes the proof. □

Note that if one of the curves γ_0, γ_1 does not satisfy the required conditions, then the surface S will be compressible.

It is not difficult to construct examples where γ_1 is a level curve, but the curve γ_0 it is not, and so that S is incompressible in $S^3 - \operatorname{int} N(\Gamma_1)$. But at this writing we do not have a precise description of all such curves. On the other hand, it is also not difficult to construct examples of graphs with such curves so that the surface S is in fact compressible. One such example can be constructed starting with a graph Γ_1 so that $S = \partial N(\Gamma_1)$ is obviously compressible, say a graph where both curves γ_0 and γ_1 are level and $\Delta(\gamma_0, \gamma_1) = 1$. Now slide an endpoint of γ_0 through α and then through γ_1, going around it several times, and then again through α, to get a new curve γ_0' and a new graph Γ_1'. The new curve γ_0' can be chosen so that it has a single local maximum, a local minimum and it is not isotopic to a level curve. However $S' = \partial N(\Gamma_1')$ would be compressible, because the exteriors of Γ_1 and Γ_1' are homeomorphic.

2.2 Surfaces of type 2

Let e be a point on I, so that $0 < e < 1$. Consider the level torus $T_e = T \times \{e\}$. Let γ be a simple closed curve embedded in the level torus T_e. Let α be an arc contained in $T \times I$, with endpoints in γ, so that it has just a local maximum at T_1, and just a local minimum at T_0. Suppose that γ is essential in T_e, or well, it is inessential but bounds a disk in T_e which intersects α in one point. Note that the interior of α intersects T_e in one point, so α is divided into a lower and an upper arc, say α_1 and α_2. Suppose that none of these arcs can be isotoped (in $R_0 \cup (T \times [0,e])$ or $(T \times [e,1]) \cup R_1$), keeping its endpoints fixed, into an arc on T_e with interior disjoint from γ.

Let Γ_2 be the 1–complex consisting of the union of the curve γ and the arc α. So Γ_2 is a trivalent graph embedded in S^3. Let $N(\Gamma_2)$ be a regular neighborhood of Γ_2.

This is a genus 2 handlebody. We can assume that $N(\Gamma_2)$ is the union of the solid tori $N(\gamma)$ and 1–handle $N(\alpha)$, so that these intersect in two disks, E_1 and E_2, where say E_1 is at level $T \times \{e-\epsilon\}$ and E_2 at level $T \times \{e+\epsilon\}$, for some small $\epsilon > 0$. Let $S = \partial N(\Gamma_2)$. We say that S is a surface of type 2. In Figure 3 we show schematically a surface of type 2, and give an explicit example.

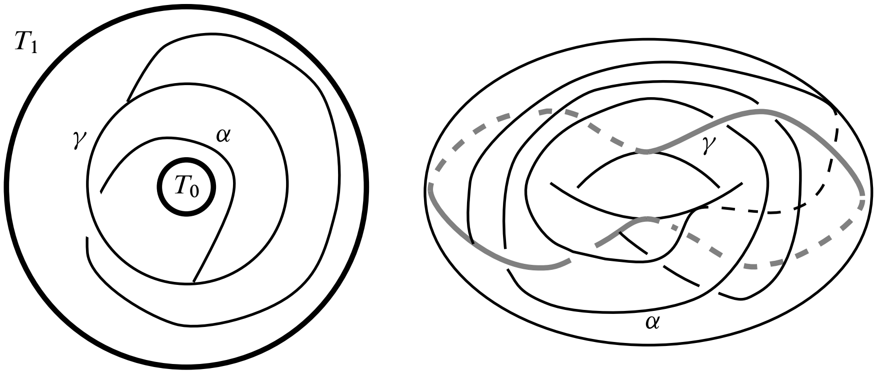

Figure 3

Let k be a knot embedded in $N(\Gamma_2)$ so that k intersects each of E_1 and E_2 in two points, $k \cap N(\alpha)$ consists of two arcs, each with a local maximum and a local minimum in $T \times I$, and $k \cap N(\gamma)$, which is contained in $T \times [e-\epsilon, e+\epsilon]$, consists of two vertical arcs. Suppose also that $S = \partial N(\Gamma_2)$ is meridionally incompressible in $N(\Gamma_2) - k$. To get that it suffices to ask that k is well wrapped in $N(\Gamma_2)$ (ie consider the two arcs of k lying in $N(\gamma)$, get a knot by joining the ends of the arcs lying in E_1 and E_2 with an arc contained in such disks, and then push the knot to the interior of $N(\gamma)$; to be well wrapped just means that the wrapping number of this knot in the solid torus $N(\gamma)$ is ≥ 2 [4]). Note that if such a knot is not well wrapped then S is in fact meridionally compressible.

Theorem 2.2 *Let Γ_2 be a graph as above, and k in $N(\Gamma_2)$ as above. Then $S = \partial N(\Gamma_2)$ is meridionally incompressible in $S^3 - k$, and k is a $(1,2)$–knot.*

Proof By construction k is in a $(1,2)$–position. The surface S is meridionally incompressible in $N(\Gamma_2) - k$ by hypothesis. So it remains to prove that S is incompressible in $S^3 - \operatorname{int} N(\Gamma_2)$.

Let $A = T_e - \operatorname{int} N(\Gamma_2)$. This is a once-punctured annulus if γ is nontrivial in T_e, and it is a once-punctured disk plus a once-punctured torus if γ is trivial in T_e. Suppose D is a compression disk for S. Look at the intersections between D and A. Let β be a simple closed curve of intersection which is innermost in D. If β is nontrivial in T_e, then it is a meridian of the solid torus $(T \times [e, 1]) \cup R_1$, say, which implies that the arc α_2 can be isotoped into T_e disjoint from γ. So suppose that β is trivial in T_e (but perhaps nontrivial in A).

If γ is nontrivial in T_e, then the curve β is trivial in A, and it is easily removed. If γ is trivial in T_e, then β will be trivial in A, except if it is a curve concentric with γ, not contained in the disk bounded by γ. In this case, the disk bounded by β in D union the disk bounded by β in T_e bounds a 3–ball which contains the upper or the lower arc of α, and then as such arc has no local knots (for it has just one maximum or minimum), it can be pushed into T_e, which contradicts our hypothesis. Suppose then that all simple closed curves of intersection between A and D have been removed.

Suppose now that β is an arc of intersection between D and A which is outermost in D. Then β cuts off a disk $D' \subset D$, with $\partial D' = \beta \cup \delta$, where $\delta \subset S$. If β is trivial in A, ie isotopic into a component of ∂A, then by cutting D with the disk in A determined by β (or an innermost one), we get a compression disk with fewer intersections with A. Assume then that β is nontrivial in A. Suppose first that the ends of β lie on $N(\gamma)$; in this case we can assume that δ is disjoint from $\partial N(\alpha)$ (for otherwise the interior of δ would intersect A). If γ is trivial in T_e, then D' is a meridian disk of $(T \times [e, 1]) \cup R_1$, say, which implies that the arc α_2 can be isotoped into T_e. If γ is nontrivial in T_e, then either D' is a meridian disk of $(T \times [e, 1]) \cup R_1$, say, and γ is a curve intersecting a meridian of $(T \times [e, 1]) \cup R_1$ in one point, or $\partial D'$ determines a disk $E \subset T_e$, which contains the point $T_e \cap \alpha$. Again, in both cases this implies that the arc α_2 can be isotoped into T_e. Suppose now that both ends of β lie on $N(\alpha)$. If the arc δ is isotopic into ∂A, then by isotoping D we would get a compression disk whose intersection with T_e contains a simple closed curve. Otherwise, $\delta = \delta_1 \cup \delta_2 \cup \delta_3$, where δ_1 and δ_3 are arcs lying on $\partial N(\alpha)$, and δ_2 is an arc lying on $\partial N(\gamma)$. Note that δ_2 goes around $N(\gamma)$ just once. Let E be a disk contained in $\partial N(\alpha)$, such that $\partial E = \delta_1 \cup \delta_4 \cup \delta_3 \cup \delta_5$, where $\delta_4 \subset \partial N(\alpha) \cap \partial N(\gamma)$, and $\delta_5 \subset \partial A$. Then $D' \cup E$ is an annulus in $(T \times [e, 1]) \cup R_1$, a boundary component of it is a curve parallel to γ, the other component lies on A, and the arc α_2 is an spanning arc of $D' \cup E$. This again shows that α_2 can be isotoped into T_e. Finally, if one endpoint of β lies in $N(\gamma)$ and the other in $N(\alpha)$, then α_2 can be isotoped to lie on T_e. □

Note that if in the 1–complex Γ_2 one of the arcs α_1 or α_2 can be isotoped into an arc on the level torus T_e with interior disjoint from γ, then the surface S will be either compressible, or it can be isotoped to a surface of type 1, so that the knot k remains in a $(1, 2)$–position.

2.3 Surfaces of type 3

Let e be a point on I, so that $0 < e < 1$. Consider the torus $T_e = T \times \{e\}$. Let γ_1 be a simple closed curve embedded in the level torus T_e and let γ_2 be an essential simple closed curve embedded in T_0 which goes around R_0 at least once longitudinally. Let α be an arc contained in $T \times I$, with endpoints in γ_1 and γ_2, so that it has just a local maximum at T_1. Suppose that γ_1 is essential in T_e, or that it is inessential but bounds a disk in T_e which intersects α in one point. Note that the interior of α intersects T_e in one point, so α is divided into a lower and an upper arc, say α_1 and α_2. Suppose that the arc α_2 cannot be isotoped (in $(T \times [e,1]) \cup R_1$), keeping its endpoints fixed, into an arc on T_e with interior disjoint from γ_1.

Let Γ_3 be the 1–complex consisting of the union of the curves γ_1, γ_2 and the arc α. So Γ_3 is a trivalent graph embedded in S^3. Let $N(\Gamma_3)$ be a regular neighborhood of Γ_3. This is a genus 2 handlebody. We can assume that $N(\Gamma_3)$ is the union of the solid tori $N(\gamma_1)$ and $N(\gamma_2)$, joined by the 1–handle $N(\alpha)$, so that these intersect in two disks, E_1 and E_2, where say E_1 is at level $T \times \{e + \epsilon\}$ and E_2 at level $T \times \{\epsilon\}$, for some small $\epsilon > 0$. Let $S = \partial N(\Gamma_3)$, we say that S is a surface of type 3. A surface of type 3 is shown schematically in Figure 4, and an explicit example is also given.

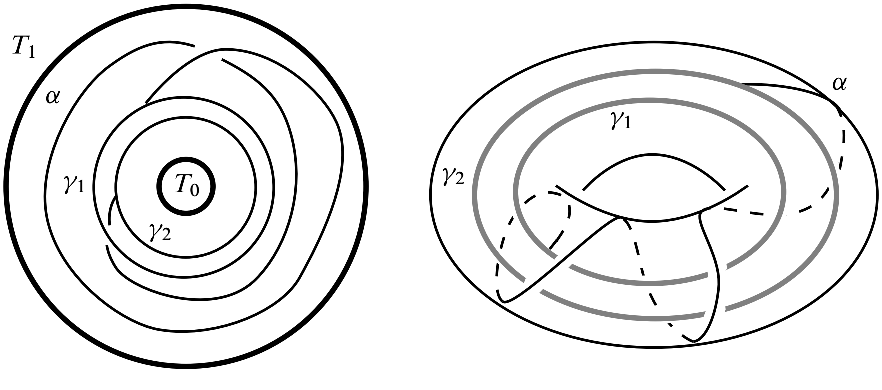

Figure 4

Let k be a knot embedded in $N(\Gamma_3)$ so that k intersects each of E_1 and E_2 in two points, $k \cap N(\alpha)$ consists of two arcs, each with just a local maximum, and $k \cap N(\gamma_i)$, $i = 1, 2$, consists of one arc with just a local minimum. Suppose also that S is meridionally incompressible in $N(\Gamma_3) - k$. To get that it suffices to ask that k is well wrapped in $N(\Gamma_3)$, that is, consider the arc of k_i contained in $N(\gamma_i)$, join its endpoints lying in E_i with an arc in E_i, push the resulting knot into $N(\gamma_i)$, and assume that the wrapping number of such a knot in $N(\gamma_i)$ is ≥ 2.

Theorem 2.3 *Let* Γ_3 *be a graph as above, and* k *in* $N(\Gamma_3)$ *as above. Then* $S = \partial N(\Gamma_3)$ *is meridionally incompressible in* $S^3 - k$*, and* k *is a* $(1,2)$*–knot.*

Proof By construction k is in a $(1,2)$–position. The surface S is meridionally incompressible in $N(\Gamma_3) - k$ by hypothesis. So it remains to prove that S is incompressible in $S^3 - \operatorname{int} N(\Gamma_3)$.

The proof is an innermost disk/outermost arc argument, looking at the intersections of a compression disk D with the surface $T_e - \operatorname{int}(N(\gamma_1) \cup N(\alpha))$. □

Note that if the curves γ_1 and γ_2 have the same slope, then $N(\Gamma_3)$ can be isotoped so that both curves lie on the torus T_0. Note that if the arc α_2 can be isotoped into an arc on the level torus T_e with interior disjoint from γ_1, then the surface S will be either compressible, or it can be isotoped to a surface of type 1, so that the knot k remains in a $(1,2)$–position.

2.4 Surfaces of type 4

Let e, ϵ be points on I so that $0 < \epsilon < e < 1$. Consider the tori $T_e = T \times \{e\}$ and $F = T \times \{\epsilon\}$. Let γ_1 be a simple closed curve embedded in the level torus T_e and let α be an arc contained in $T \times I$, with endpoints in γ_1 and F, so that it has just a local maximum at T_1. Suppose that γ_1 is essential in T_e, or that it is inessential but bounds a disk in T_e which intersects α in one point. Note that the interior of α intersects T_e in one point, so α is divided into a lower and an upper arc, say α_1 and α_2. Suppose that the arc α_2 cannot be isotoped (in $(T \times [e,1]) \cup R_1$), keeping its endpoints fixed, into an arc on T_e with interior disjoint from γ_1.

The torus F bounds a solid torus $F' = R_0 \cup (T \times [0,\epsilon])$. Consider the union $H_4 = N(\gamma_1) \cup N(\alpha) \cup F'$. This is a genus 2 handlebody. This can be seen as the solid tori $N(\gamma_1)$ and F' joined by the 1–handle $N(\alpha)$, so that these intersect in two disks, E_1 and E_2, where say E_1 is at level $T \times \{e + \epsilon\}$, and E_2 is at level $T \times \{\epsilon\}$. Let $S = \partial H_4$; we say that S is a surface of type 4. For an example of a surface of type 4, look at Figure 4, thinking of γ_2 as a fat solid torus engulfing all of R_0.

Let k be a knot embedded in H_4 so that k intersects each of E_1 and E_2 in two points, $k \cap N(\alpha)$ consists of two arcs, each with just a local maximum, $k \cap N(\gamma_1)$ consists of one arc, with just a local minimum, and $k \cap F'$ consists also of one arc, with just a local minimum. Suppose also that $S = \partial H_4$ is meridionally incompressible in H_4. To get that it suffices to ask that k is well wrapped in H_4.

Theorem 2.4 *Let* H_4 *be as above, and* k *in* H_4 *as above. Then* $S = \partial H_4$ *is meridionally incompressible in* $S^3 - k$*, and* k *is a* $(1,2)$*–knot.*

Proof By construction k is in a $(1,2)$–position, and by hypothesis S is meridionally incompressible in $H_4 - k$. So it remains to prove that S is incompressible in the complement $S^3 - \operatorname{int} H_4$. Such a proof is again an innermost disk/outermost arc argument, looking at the intersections between a compression disk D and the surface $T_e - \operatorname{int} N(\gamma_1 \cup \alpha)$. □

Note that a surface of type 4 can be isotoped to look like a surface of type 3, where the curve γ_2 for the new surface of type 3 will be longitudinal. But if this isotopy is done then the knot k constructed for the surface of type 4 may not be in a $(1,2)$–position. But a surface of type 3, where the curve γ_2 is longitudinal, will be in fact a surface of type 4. Note also that if the arc α_2 can be isotoped into an arc on the level torus T_e with interior disjoint from γ_1, then the surface S will be compressible.

2.5 Surfaces of type 5

Consider a sphere Σ that consists of two meridian disks in R_1, say D_1 and D_2, two vertical annuli A_1 and A_2 in $T \times [\epsilon, 1]$, $0 < \epsilon < 1$, and an annulus A_3 in the level torus $T \times \{\epsilon\}$. Let B be a 3–ball bounded by Σ in S^3, say the one which does not contain the point at infinity.

Assume first that the solid torus R_0 is not contained in B. Let γ be a level simple closed curve lying in a level torus $T_e = T \times \{e\}$, $0 < \epsilon < e < 1$, and which lies inside the 3–ball B. Let α_1 be a vertical arc in B with one endpoint in A_1 at a level above T_e, and the other endpoint in γ, and let α_2 be a vertical arc in B with one endpoint in A_2 and the other in A_2 or in A_1. If γ is a trivial curve in the level torus T_e, assume that α_2 intersects in one point the level disk E bounded by γ. Assume that there is no disk D in B with $\partial D = \alpha_2 \cup \delta$, $\delta \subset \Sigma$, and $D \cap (\alpha_1 \cup \gamma) = \varnothing$, that is, the arc α_2 is not isotopic to an arc in Σ. Suppose also that the arc α_1 cannot be isotoped, keeping its endpoints in Σ and γ, so that α_1 lies in the level torus T_e, and the arc α_2 remains being a vertical arc. It is not difficult to construct examples satisfying these conditions.

Let B_1 be the complementary ball in S^3 bounded by Σ. Let $H_5 = B_1 \cup N(\alpha_1) \cup N(\alpha_2) \cup N(\gamma)$. This is a genus 2 handlebody. First, $B_1 \cup N(\alpha_2)$ is a solid torus, formed by the 3–ball B_1 and the 1–handle $N(\alpha_2)$, where $B_1 \cap N(\alpha_2)$ consists of two vertical disks, say E_1 and E_2, where E_1 is at an upper level. So H_5 can be seen as the solid tori $B_1 \cup N(\alpha_2)$ and $N(\gamma)$ joined by the 1–handle $N(\alpha_1)$, where $B_1 \cap N(\alpha_1)$ consists of a vertical disk E_3, and $N(\alpha_1) \cap N(\gamma)$ is a level disk E_4, lying in a level $T \times \{e + \delta\}$, for some small $\delta > 0$. Let $S = \partial H_5$. We say that S is a surface of type 5. Look at Figure 5, left, for an example of a surface of type 5.

Let Σ and B be as above, but suppose now that the solid torus R_0 is contained in the 3–ball B. Let $F = T \times \{\epsilon_1\}$, $0 < \epsilon_1 < \epsilon$, and let $F' = R_0 \cup (T \times [0, \epsilon_1])$. So the solid torus F' is contained in B. Let α_1 be a vertical arc in B with one endpoint in A_1 and the other endpoint in F, and let α_2 be a vertical arc in B with one endpoint in A_2 and the other in A_2 or in A_1. As before, assume that the arc α_2 is not isotopic to an arc in Σ. Again, let B_1 be the complementary ball in S^3 bounded by Σ, and let H_5 be the genus 2 handlebody $H_5 = B_1 \cup N(\alpha_1) \cup N(\alpha_2) \cup F'$. Define disks E_1, E_2, E_3 and E_4 as above. Let $S = \partial H_5$, we also say that S is a surface of type 5.

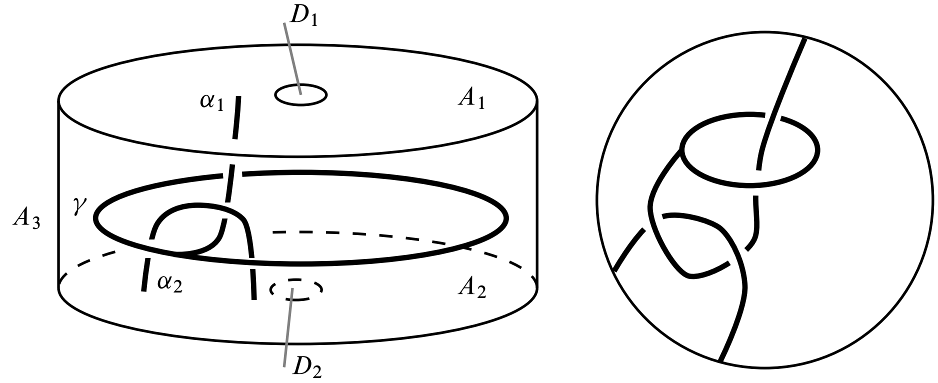

Figure 5

Let k be a knot in H_5, intersecting in two points each of the disks E_i, $i = 1, 2, 3, 4$, and so that $k \cap N(\alpha_2)$ consists of two vertical arcs, $k \cap N(\alpha_1)$ consists of two vertical arcs, $k \cap N(\gamma)$ (or $k \cap F'$) consists of one arc with a single local minimum and which is well wrapped in $N(\gamma)$ (F'), and $k \cap B_1$ consists of three arcs, two of them with a single local maximum and with endpoints in $E_1 \cup E_3$, the other with a single local minimum and with endpoints in E_2. Suppose also that none of these arcs is isotopic in B_1, keeping its endpoints and the other arcs fixed, to an arc lying on some E_i, $i = 1, 2, 3$.

Note that the two constructions of surfaces of type 5 produce surfaces which are isotopic in S^3, but if such an isotopy is performed transforming one surface then the corresponding knot k may not longer be in a $(1,2)$–position.

Theorem 2.5 *Let* S *and* k *be as above.* S *is meridionally incompressible in* $S^3 - k$*, and* k *is a* $(1,2)$*–knot.*

Proof By construction k is in a $(1,2)$–position. We have to show that S is incompressible in $B - \operatorname{int} H_5$ and that S is meridionally incompressible in $H_5 - k$. It is not difficult to prove that S is meridionally incompressible in $H_5 - k$. To do that consider the disks E_1, E_2, E_3 and E_4; these are disks which intersects k in two points. Look at the intersections between a compression disk D and $E_1 \cup E_2 \cup E_3 \cup E_4$. Using the hypothesis on k, we conclude that D and $E_1 \cup E_2 \cup E_3 \cup E_4$ can be made disjoint, which then implies that D cannot exist.

Suppose that the solid torus R_0 is not contained in the 3–ball B, the proof for the remaining case is similar. Note that if the lower endpoint of the arc α_2 lies in the annulus A_1, and it is at a level below T_e, then it can be isotoped, going through the annulus A_3, so that both of its endpoints lie in A_2. This isotopy can be performed, moving H_5 and k, but so that k remains in a $(1,2)$–position. If γ is a nontrivial curve in the level torus T_e, and both endpoints of α_2 lie in A_2, then γ can be isotoped so that it lies at a level below the lower endpoint of α_2. Assume these isotopies have been performed, if possible.

Let $E = T_e \cap (B - \operatorname{int} H_5)$. If γ is a nontrivial curve in the level torus T_e, then E consist of two annuli. And if γ is a trivial curve in the level torus then E consists of a punctured annulus and a punctured disk. The proof is now an innermost disk/outermost arc argument, looking at the intersections between E and a compression disk. □

Note that if the arcs α_1 and α_2 do not satisfy the required hypothesis, then the surface S will be compressible.

2.6 Surfaces of type 6

Let γ be a knot in $T \times I$ in a $(1,1)$–position, so that it has a local maximum at a level just below T_1, and a local minimum at a level just above T_0. Assume that γ is not isotopic in $T \times I$ to a meridian or a longitude of a level torus. Let $N(\gamma)$ be a neighborhood of γ. Let α_1 be a trivial curve, in a level torus $T \times \{e\}$, $0 < e < 1$, which bounds a level disk E such that $E \subset \operatorname{int} N(\gamma)$. Let α_2 be an arc contained in $N(\gamma)$ with an endpoint in $\partial N(\gamma)$, lying at a level f, with $0 < f < e$, and the other point in α_1. Suppose also that α_2 has a single local maximum in $T \times I$, that α_2 intersects in one point the disk E bounded by α_1, and that $\alpha_1 \cup \alpha_2$ intersects each meridian of $N(\gamma)$, that is, we have something like in Figure 6. Assume that $N(\alpha_1) \cap N(\alpha_2)$ consists of a level disk E_1 lying in a level torus $T \times \{e + \epsilon\}$, for some $\epsilon > 0$, and that $N(\alpha_2) \cap \partial N(\gamma)$ is a disk E_2 lying in the level torus $T \times \{f\}$. Let $M_6 = N(\gamma) - \operatorname{int} N(\alpha_1) \cup N(\alpha_2)$. Note that $S = \partial M_6$ is a genus 2 surface. We say that S is a surface of type 6.

Let k be a knot contained in $S^3 - M_6$, which intersects each of E_1, E_2 in two points, and so that $k \cap N(\alpha_2)$ consists of two arcs each having a single local maximum, $k \cap N(\alpha_1)$ consists of one arc, well wrapped in $N(\alpha_1)$, and having just one local minimum, and $k \cap (S^3 - \operatorname{int} N(\gamma))$ is one arc, with a single local minimum and which goes around a longitude of R_0 at least once. If γ is a trivial knot assume further that k is well wrapped in $S^3 - \operatorname{int} N(\gamma)$. See Figure 6.

Theorem 2.6 *Let S and k be as above. S is meridionally incompressible in $S^3 - k$, and k is a $(1,2)$–knot. Furthermore, if γ is a nontrivial knot, then S does not bound a handlebody in S^3.*

Proof We have to show that S is incompressible in M_6 and that S is meridionally incompressible in $(S^3 - M_6) - k$. It is not difficult to prove that S is meridionally incompressible in $(S^3 - M_6) - k$. To do that look at the intersections between a compression disk D and $E_1 \cup E_2$. Using the facts that k is well wrapped in $N(\alpha_1 \cup \alpha_2)$ and that k goes at least once longitudinally around R_0, or that k is well wrapped in $S^3 - \operatorname{int} N(\gamma)$, we conclude that D and $E_1 \cup E_2$ can be made disjoint, which then implies that D cannot exist.

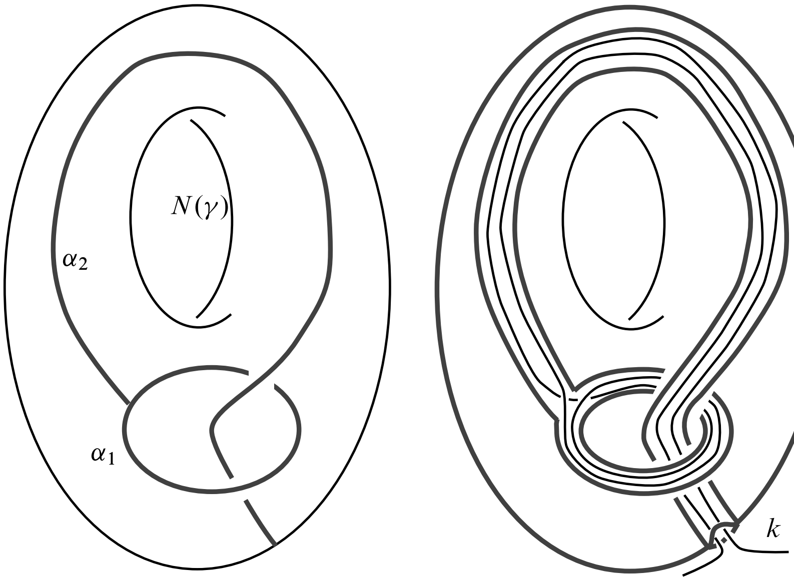

Figure 6

It remains to prove that S is incompressible in M_6. Note that there are two nonseparating annuli properly embedded in M_6, say A_1 and A_2, so that the boundary of A_1 consists of a longitude of α_1 and the boundary of a cocore of the 1–handle $N(\alpha_2)$, and the boundary of A_2 consists of a meridian of γ and the boundary of a cocore of the 1–handle $N(\alpha_2)$. Note that anyone of the boundary components of A_1 and A_2 is a nontrivial curve in M_6, because $\alpha_1 \cup \alpha_2$ is not contained in a 3–ball inside $N(\gamma)$. In particular this shows that the annuli A_1 and A_2 are incompressible. Note also that $A_1 \cup A_2$ does not separate M_6. Suppose that D is a compression disk for S and look at the intersections between D and $A_1 \cup A_2$. Simple closed curves of intersection are easily removed, for these have to be trivial in the annuli. So the intersection consists only of arcs. Let β be an outermost arc of intersection in D, so it cuts off a disk $D' \subset D$, with $\partial D' = \beta \cup \delta$. If the endpoints of β lie on different boundary components of A_1 or A_2, then by inspection we see that there cannot be an arc δ in S, with interior disjoint from the annuli, joining these two points. So if such β exists, it must have endpoints in the same boundary component of one of the annuli, so it bounds a disk D'' in the annulus, and by cutting D with D'' (or with one outermost disk contained in D''), we get a disk with fewer intersections with the annuli. Note also that D cannot be disjoint from the annuli, for otherwise $\alpha_1 \cup \alpha_2$ will be contained in a 3–ball inside $N(\gamma)$.

Finally note that if γ is a nontrivial knot, then S does not bound a handlebody in S^3, for in one it side bounds the disk sum of $S^3 - \operatorname{int} N(\gamma)$ with $N(\alpha_1) \cup N(\alpha_2)$, and the other side it bounds the manifold M_6, which is not a handlebody for it has incompressible boundary. □

Note that a knot k whose complement has a surface of type 6 will in general also have a surface of type 4. To see that, consider the union of the curve α_1, the arc α_2 and R_0 (where the arc α_2 has been prolonged to touch R_0). So, let $H_4 = N(\alpha_1 \cup \alpha_2 \cup R_0)$. Note that $S' = \partial H_4$ is a surface of type 4, and if H_4 is thin then $S \cap S' = \varnothing$. The surface S' will be in fact meridionally incompressible in $S^3 - k$, except if k goes around R_0 exactly once longitudinally.

2.7 Surfaces of type 7

Let γ be a simple closed curve in $T \times \{1/2\}$ of slope p/q, $|p| \geq 1$ (in the usual coordinates for the solid torus $R_0 \cup (T \times \{1/2\})$. Let $N_1 = N_1(\gamma)$ and $N_2 = N_2(\gamma)$ be two regular neighborhoods of γ, with $N_1 \subset N_2$, and $N_1 \subset T \times [1/4, 3/4]$, $N_2 \subset T \times [1/8, 15/16]$. Let α be an arc contained in $N_2 - \operatorname{int} N_1$, connecting ∂N_1 with ∂N_2, and so that α has just one local maximum. Suppose also that α cannot be isotoped, keeping its endpoints in ∂N_1 and ∂N_2, to an arc lying on a level torus $T \times \{y\}$. The arc α can be isotoped so that it looks like the union of the arcs $\alpha_1 \cup \alpha_2$, where α_1 is an arc of the form $\{x\} \times [1/2, 7/8]$, going from ∂N_1 to the local maximum, and α_2 is a descending arc, which wraps around N_2 and around α_1, until it finishes at a point in ∂N_2. See Figure 7 for an example. By isotoping and sliding the arc α, and maintaining it with a single maximum, we can assume that its endpoints lie in the same level, say at level $1/2$. We can connect its endpoints with an arc β contained in an annulus A (which is one of the components of $(N_2 - \operatorname{int} N_1) \cap (T \times \{1/2\})$), and in fact there are many of such arcs. Assuming that the arc α cannot be isotoped to be level, is equivalent to assuming that the knot $\ell = \alpha \cup \beta$ is never trivial in $N_2 - \operatorname{int} N_1$, for any of the choices of β. We can think of the knot ℓ as lying in $A \times I$, so that ℓ has just a maximum in $A \times I$. By embedding $A \times I$ as a standard solid torus in S^3, with $\partial A \times \{0\}$ being a preferred longitude of such a solid torus, we get a 2–bridge link, formed by ℓ and a meridian of the solid torus $A \times I$. The assumption on the arcs is equivalent to asking that the corresponding 2–bridge link is never a split link.

We can assume that a neighborhood of α, $N(\alpha)$, is contained in $N_2 - \operatorname{int} N_1$, and that $N_1 \cap N(\alpha)$ consists of a disk E_1 and that $N_2 \cap N(\alpha)$ is a disk E_2. Let $M_7 = N_2 - \operatorname{int}(N_1 \cup N(\alpha))$, and let $S = \partial M_7$. We say that S is a surface of type 7. Let k be a knot contained in $S^3 - M_7$, which intersects each of E_1, E_2 in two points, and so that $k \cap N(\alpha)$ consists of two arcs, each with a single local maximum, and each with an endpoint in E_1 and the other in E_2. $k \cap N_1$ is an arc with endpoints in E_1, with a single local minimum, and which is well wrapped in N_1, and $k \cap (S^3 - N_2)$ is an arc with endpoints in E_2, with a single local minimum, and which goes around R_0 at least once longitudinally. If γ is a trivial knot assume further that k is well wrapped in $S^3 - \operatorname{int} N_2$.

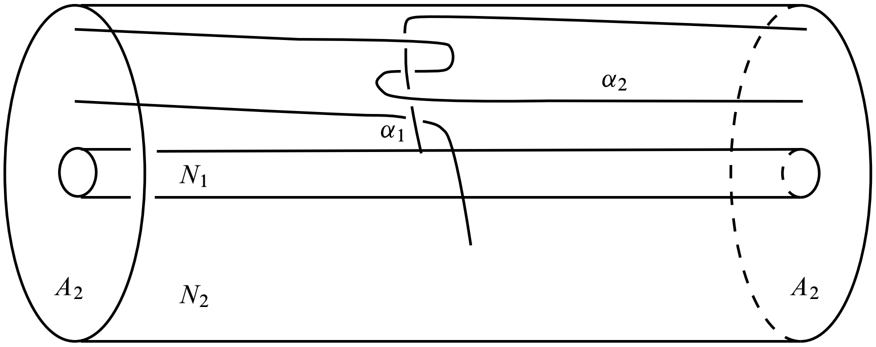

Figure 7

Theorem 2.7 *Let S and k be as above. S is meridionally incompressible in $S^3 - k$, and k is a $(1,2)$–knot. Furthermore, if γ is a nontrivial knot, then S does not bound a handlebody in S^3.*

Proof It is not difficult to prove that S is meridionally incompressible in $(S^3 - M_7) - k$. Look at the intersections between a compression disk D and the disks E_1 and E_2.

Using the hypothesis on k, we conclude that D and E_1, E_2 can be made disjoint, which then implies that D cannot exist.

It remains to prove that S is incompressible in M_7. $M_7 \cap (T \times \{1/2\})$ consists of two annuli, and α must be disjoint from one of these annuli, for otherwise it will contain more than one local maxima. So let A_1 be one of such annuli, and suppose that A_1 is disjoint from α (the other annulus was denoted before by A). A_1 is a nonseparating annulus in M_7. Let A_2 be an annulus in $N_2 - \operatorname{int} N_1$ consisting of a meridian disk of N_2 minus a meridian disk of N_1. A_1 and A_2 intersect in a single arc which is essential in both annuli. The arc α must intersect A_2, for otherwise it will be contained in a 3–ball and it would be isotopic to a level arc. So suppose that α and A_2 intersect transversely and that this intersection is minimal. $A_2 \cap M_7$ is then a punctured annulus, which we call also A_2.

Suppose that S is compressible in M_7, and let D be a compression disk. Look at the intersections between D and $A_1 \cup A_2$. Simple closed curves of intersection between D and A_1 are easily removed, for no such curve can be essential in A_1. For the same reason, if there is a simple closed curve of intersection between D and A_2, it must bound a disk in A_2; if the disk intersects α, then an isotopy reduces the number of points of intersection between A_2 and α, otherwise such an isotopy reduces the number of intersection curves between D and A_2. So the intersection consists only of arcs. By isotoping D, we can assume that it is disjoint from the arc of intersection between A_1 and A_2. Let β be an outermost arc of intersection between D and $A_1 \cup A_2$, and suppose that β lies on A_1. If β is inessential in A_1 then it is easily removed. So suppose β has endpoints on different components of A_1. The arc β cuts off a disk $D' \subset D$, with $\partial D = \beta \cup \delta$. But the arc δ must pass through $N(\alpha)$, for otherwise cannot connect points on different components of ∂A_1. Then δ must intersect A_2, which contradicts the fact that β is outermost. So any outermost arc of intersection β must lie on A_2. In any of the possible cases, the disk D' determined by β can be used to isotope α, reducing the number of points of intersection of α with A_2. So the disk D must be disjoint from A_1 and A_2. Then it is not difficult to see that this is not possible.

Finally, if γ is a nontrivial knot, then S does not bound a handlebody in S^3, for in one side it bounds the disk sum of $S^3 - \operatorname{int} N_2$ with $N(\alpha) \cup N_1$, and in the other side it bounds M_7, which is not a handlebody for it has incompressible boundary. □

Note that if the curve γ is a curve of slope $1/q$ on $T \times \{1/2\}$, then S will be isotopic to a surface of type 4. Note that a knot k whose complement has a surface of type 7 will in general also have a surface of type 4. To see that, consider the union of the solid torus N_1, the arc α and R_0 (where the arc α has been prolonged to touch R_0).

So, let $H_4 = N_1 \cup N(\alpha) \cup R_0$. Note that $S' = \partial H_4$ is a surface of type 4, and that if H_4 is thin enough then $S \cap S' = \varnothing$. The surface S' will be in fact meridionally incompressible in $S^3 - k$, except if k goes around R_0 exactly once longitudinally.

2.8 Surfaces of type 8

Let R be a torus in S^3 constructed as follows. Let A_1, A_2, $\dots$, A_n be n annuli properly embedded in R_1, all with slope p/q, $|p| \geq 2$, $|q| \geq 2$ (in the usual coordinates of the solid torus $R_0 \cup (T \times [0, 1])$). Suppose the annuli are nested, and say, A_1 is the innermost one. Let B_1^1, B_1^2, $\dots$, B_n^1, B_n^2 be $2n$ vertical annuli in $T \times I$, so that $\partial A_i \subset \partial(B_i^1 \cup B_i^2)$, $1 \leq i \leq n$. Let C_1, $\dots$, C_n be n annuli properly embedded in R_0, which are nested, whose boundaries coincide with the boundaries of the B_i^j's, so that C_1 is the innermost annulus and $\partial C_1 \subset \partial(B_1^2 \cup B_2^2)$. Note that the union of the A_i's, the B_i^j's and the C_i's is an embedded torus, denoted by R.

In the special case when $n = 1$, assume that the annulus C_1 is chosen so that the torus R does not bound a solid torus contained in $T \times I$. In the special case when $n = 2$, it is enough to assume that $|p| \geq 2$, $|q| \geq 1$. In this case we can take the annuli C_1 and C_2 to be nested or non-nested.

Note that R is a standard torus in S^3, except in the case when $n = 2$, $|q| \geq 2$, and the annuli C_1, C_2 are non-nested. In that case R is isotopic to the boundary of a regular neighborhood of the (p, q)–torus knot.

Let α be an arc in $T \times I$, with one endpoint in B_1^1, the other in B_1^2, with interior disjoint from the B_i^j's, and so that it has a single local maximum in $T \times I$. Let E be the annulus in T_0, with $\partial E \subset \partial(B_1^1 \cup B_1^2)$, and whose interior is disjoint from the B_i^j's. So $A_1 \cup B_1^1 \cup B_1^2 \cup E$ bounds a solid torus P which contains the arc α. By sliding the arc α, we can assume that its endpoints lie in the same level, say at level e, $0 < e < 1$. We assume that the arc α cannot be isotoped, keeping its endpoints fixed, to a level arc lying in $T \times \{e\}$.

Let R' be the solid torus bounded by R which does not contain the arc α. Let $H_8 = R' \cup N(\alpha)$. This is a genus 2 handlebody; it can be seen as the solid torus R' union the 1–handle $N(\alpha)$, where $R' \cap N(\alpha) = (B_1^1 \cup B_1^2) \cap N(\alpha)$ consists of two disks, say E_1 and E_2. Let $S = \partial H_8$. We say that S is a surface of type 8. See Figure 8, left.

Let k be a knot in H_8, intersecting in two points each of E_1, E_2, and so that $k \cap N(\alpha)$ consists of two arcs, each with a single local maximum, and $k \cap R'$ consists of two arcs, each with a single local minimum, and each going at least once longitudinally around R', ie none of the arcs can be isotoped into an arc lying in E_1 or E_2.

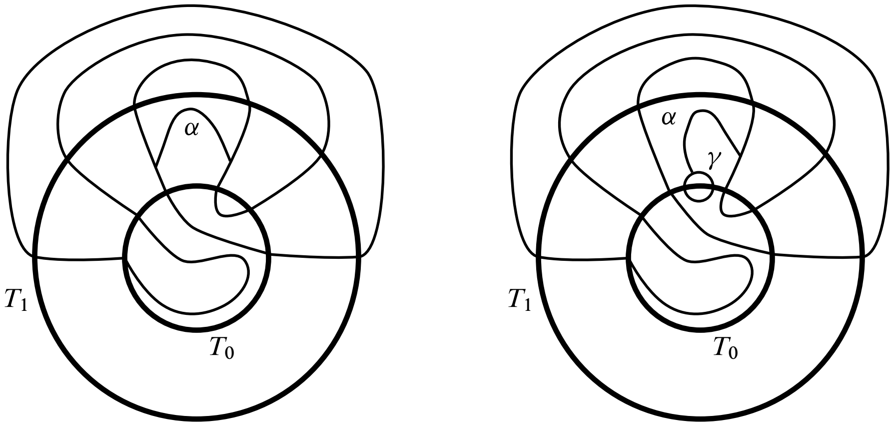

Figure 8

Theorem 2.8 *Let S and k be as above. S is meridionally incompressible in S^3-k, and k is a $(1,2)$–knot.*

Proof By construction k is in a $(1,2)$–position. We have to show that S is incompressible in $S^3-\operatorname{int} H_8$ and that S is meridionally incompressible in H_8-k. It is not difficult to prove that S is meridionally incompressible in H_8-k. To do that consider the disks E_1, E_2, and look at the intersections between a compression disk D and $E_1 \cup E_2$. Using the hypothesis on k, we conclude that D and $E_1 \cup E_2$ can be made disjoint, which then implies that D cannot exist.

Suppose now that S is compressible in $S^3-\operatorname{int} H_8$, and let D be a compression disk. Look at the intersections between D and the annulus E defined above. Simple closed curves of intersection are easily removed. Let β be an outermost arc of intersection in D, which cuts a disk $D' \subset D$, and say $\partial D' = \beta \cup \delta$. If δ is trivial in E, we cut D with the disk in E determined by δ (or with an innermost one), getting a compression disk with fewer intersections with E. If δ is essential in E, there are two cases. If D' is contained in the solid torus P, this will imply that the arc α can be isotoped to be level. If D' is not contained in P, this would imply that the slope p/q of the annulus E would satisfy $|p|=1$ if n is even, or that $|q|=1$ if n is odd, contrary to the hypothesis. So assume that D and E do not intersect. Let F be a meridian disk of the solid torus P, this can intersect the arc α in many points, but suppose α has been isotoped so that its intersection with F is minimal. The intersection of F with $S^3-\operatorname{int} H_8$ is a punctured disk, which we call F again. Now look at the intersections between F and D. An innermost disk or outermost arc of intersection can be used to reduce the number of points of intersection between F and α, which is not possible. So D must be disjoint from F, but this is not possible. $\square$

Note that a knot k which has a surface of type 8 may also have a surface of type 3. Prolong the arc α on both ends until it touches R_0 in two points, but so that intersects the B_i^j's only in two points, one lying in E_1, the other in E_2. Take curves γ_1 and γ_2 in R_0 of slope p/q, disjoint from the C_i's, so that one endpoint of α lies in γ_1 and the other in γ_2. $\Gamma_3 = \alpha \cup \gamma_1 \cup \gamma_2$ is a 1–complex as defined in Section 2.3, and $S' = \partial N(\Gamma_3)$ is disjoint from S, just take $N(\gamma_3)$ thin enough. Note that if $n \geq 3$, the knot k can be isotoped so that k is contained in $N(\Gamma_3)$; so if k is well wrapped in $N(\Gamma_3)$, S' will also be meridionally incompressible in $S^3 - k$. If $n = 1, 2$ then this construction may also work, depending if the knot k can or cannot be isotoped into $N(\Gamma_3)$. However, if $n = 2$ and the annuli C_1 and C_2 are nested, a similar construction yields a surface of type 4, and if $n = 1$, then we get a surface of type 1. In both cases the new surface may be meridionally incompressible.

Note that if n is even, and the slope of the annulus A_1 is $1/q$, then with a little work it can be shown that the surface S is in fact compressible. If n is even, and the slope of A_1 is $p/1$, then the surface S is isotopic to a surface with $n = 2$. If n is odd, and the slope of the annulus A_1 is $1/q$, then the surface S is isotopic to a surface with $n = 2$. If n is odd, and the slope of A_1 is $p/1$, then the surface S is compressible.

Also note that if $n = 2$, and the slope of the annulus A_1 is $p/1$, then both versions of a surface of type 8 are identical, ie the annuli C_1 and C_2 can be isotoped to be nested or non-nested.

2.9 Surfaces of type 9

Let R be a torus in S^3 constructed as follows. Let $A_1, A_2, \ldots, A_n$ be n annuli properly embedded in R_1, all with slope p/q, $|p| \geq 2$, $|q| \geq 2$ (in the usual coordinates of the solid torus $R_0 \cup (T \times [0,1])$). Suppose the annuli are nested, and say, A_1 is the innermost one. Let $B_1^1, B_1^2, \ldots, B_n^1, B_n^2$ be $2n$ vertical annuli in $T \times I$, so that $\partial A_i \subset \partial(B_i^1 \cup B_i^2)$, $1 \leq i \leq n$. Let $C_1, \ldots, C_n$ be n annuli properly embedded in R_0, which are nested, whose boundaries coincide with the boundaries of the B_i^j's, so that C_1 is the innermost annulus and $\partial C_1 \subset \partial(B_1^2 \cup B_2^2)$. Note that the union of the A_i's, the B_i^j's and the C_i's is an embedded torus, denoted by R. In the special case when $n = 1$, assume that the annulus C_1 was chosen so that the torus R does not bound a solid torus contained in $T \times I$. Note that in any case, R is a standard torus in S^3.

Let E be the annulus in T_0, with $\partial E \subset \partial(B_1^1 \cup B_1^2)$, and whose interior is disjoint from the B_i^j's. Let γ be a simple closed curve which is a core of the annulus E, and let $N(\gamma)$ a small neighborhood of γ, disjoint from the B_i^j's. Let α be an arc in $T \times I$, with one endpoint in B_1^1, the other in $N(\gamma)$, with interior disjoint from the B_i^j's, and so that it has a single local maximum in $T \times I$. By sliding the arc α, we can assume that its endpoints lie in the same level, say at level ϵ, $0 < \epsilon < 1$. We assume that the arc α cannot be isotoped, keeping its endpoints fixed, to a level arc lying in $T \times \{\epsilon\}$.

Let R' be the solid torus bounded by R which does not contain the arc α. Let $H_9 = R' \cup N(\alpha) \cup N(\gamma)$. This is a genus 2 handlebody; it can be seen as the solid tori R' and $N(\gamma)$ joined by the 1–handle $N(\alpha)$, where $R' \cap N(\alpha) = B_1^1 \cap N(\alpha)$ consists of a disk, say E_1, and $N(\alpha) \cap N(\gamma)$ consists of a disk E_2, which lies in level torus $T \times \{\epsilon\}$. Let $S = \partial H_9$. We say that S is a surface of type 9. See Figure 8, right.

Let k be a knot in H_9, intersecting in two points each of E_1, E_2, and so that $k \cap N(\alpha)$ consists of two arcs, each with a single local maximum, and $k \cap R'$ consists of one arc with a single local minimum, which is well wrapped in the solid torus R'. Furthermore, $k \cap N(\gamma)$ consists of one arc with a single local minimum and which is well wrapped in $N(\gamma)$.

Theorem 2.9 *Let S and k be as above. S is meridionally incompressible in $S^3 - k$, and k is a $(1,2)$–knot.*

Proof By construction k is in a $(1,2)$–position. We have to show that S is incompressible in $S^3 - \operatorname{int} H_9$ and that S is meridionally incompressible in $H_9 - k$. It is not difficult to prove that S is meridionally incompressible in $H_9 - k$. To do that consider the disks E_1, E_2, and look at the intersections between a compression disk D and $E_1 \cup E_2$. Using the hypothesis on k, we conclude that D and $E_1 \cup E_2$ can be made disjoint, which then implies that D cannot exist.

Suppose now that S is compressible in $S^3 - \operatorname{int} H_9$, and let D be a compression disk. Let $E' = E - \operatorname{int} N(\gamma)$, these are two annuli. Look at the intersections between D and the annuli E'. An argument as in the proof of Theorem 2.7 or Theorem 2.8 yields a contradiction. □

Note that a knot k whose complement has a surface of type 9 may also have a surface of type 3. The construction is identical to the one done in the previous section, just after the proof of Theorem 2.8.

Note that if n is even, and the slope of the annulus A_1 is $1/q$, then the surface S is isotopic to a surface of type 7. If n is even, and the slope of A_1 is $p/1$, then the surface S is isotopic to a surface of type 3. If n is odd, and the slope of the annulus A_1 is $1/q$, then the surface S is isotopic to a surface of type 3. If n is odd, and the slope of A_1 is $p/1$, then the surface S is isotopic to a surface of type 7.

2.10 Incompressible tori and $(1,2)$–knots

It is not difficult to construct $(1,2)$–knots whose complement contains a closed meridionally incompressible torus. There are three cases.

Let K be a nontrivial knot in a $(1,1)$–position. Embed a knot k in $N(K)$ so that the wrapping number of k in $N(K)$ is 2, and that k is in a $(1,2)$–position. Clearly $S = \partial N(K)$ is meridionally incompressible and k is a $(1,2)$–knot. These knots inside $N(K)$ look like in Figure 9, left.

Let K be a nontrivial knot contained in the standard torus T in S^3. Assume that $N(K)$ is of the form $N(K) = A \times I \subset T \times I$, where A is an annulus in T. Embed a knot k in $N(K)$ so that the wrapping number of k in $N(K)$ is ≥ 2, and that k is in a $(1,2)$–position. Clearly $S = \partial N(K)$ is meridionally incompressible and k is a $(1,2)$–knot. These knots inside $N(K)$ look like in Figure 9, right.

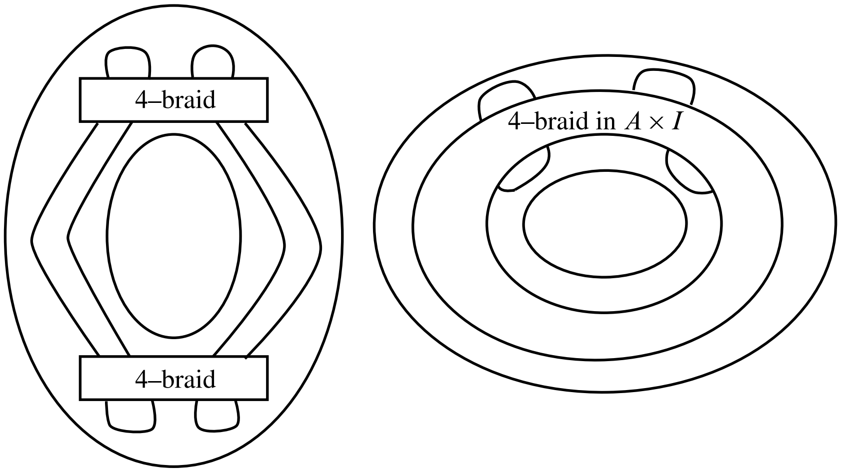

Figure 9

Note that some of the knots constructed above could contain both, a meridionally incompressible surface of genus 2, and one of genus 1. Namely, let S be a surface of type 1, and let H be the handlebody bounded by S. Then we can find a knot $K \subset H$, so that K is in a $(1,1)$–position, and S is incompressible in $H - K$ (but it is meridionally compressible). Now let k be any 2–cable of K contained in $N(K)$. Then both surfaces $\partial N(K)$ and S are meridionally incompressible in the complement of k. This construction may not work for the remaining types of surfaces, for it seems that there is no $(1,1)$–knot embedded in the regions bounded by that surfaces, so that the surfaces are incompressible in the complement of that knot.

Let S be a surface of type 7, and let N be a big solid torus containing S, that is, a big neighborhood of the torus knot used in the construction of the surface of type 7. If the knot k constructed in Section 2.7 is chosen so that k lies in N, then both surfaces S and ∂N will be meridionally incompressible.

This construction, that is, of a big torus containing the surface, can also be done for surfaces of type 1, 2, 3, 4, and 6, if the surfaces and the knots are chosen adequately. But it may fail in the remaining cases. This is because it seems that a surface of type 5, 8 or 9 cannot be confined inside a solid torus N, so that ∂N remains incompressible in $S^3 - k$; except in the special case in surfaces of type 8, where $n = 2$, and the lower annuli are non-nested.

We showed before that knots with a surface of type 6 or 7 usually contain a surface of type 4, and knots with a surface of type 8 usually contain a surface of type 3. It follows that there are $(1,2)$–knots which contain 3 meridionally incompressible surfaces, one of genus 1, and two of genus 2 (one of type 4 and one of type 6 or 7, or one of type 3 and one of type 8).

2.11 Further remarks

All the knots k constructed in this section have a $(1,2)$–presentation, that is, $b_1(k) \leq 2$. So we could ask if they really have $b_1(k) = 2$. In [6], all $(1,1)$–knots containing a closed meridionally incompressible surface are described, and it is shown that the surfaces are the boundary of a regular neighborhood of what is called a toroidal graph. So to show that the present knots are not $(1,1)$–knots it suffices to show that the surfaces constructed here do not satisfy the conditions given in [6]. This is clear for some of the surfaces of type 6 and 7, the ones that do not bound a handlebody. It is intuitively obvious for the remaining cases, but a little more work is required to show that.

3 Characterization of meridionally incompressible surfaces

In this section we prove the following theorem.

Theorem 3.1 *Let K be a $(1,2)$–knot and let S be a genus 2 meridionally incompressible surface in the complement of K. Then K and S come from the construction of Section 2, that is, K and S can be isotoped so that S looks as one of the surfaces of types 1, 2, 3, 4, 5, 6, 7, 8 and 9 constructed in Section 2.*

Let T be a standard torus in S^3, and let $I=[0,1]$. Consider $T\times I\subset S^3$. $T_0=T\times\{0\}$ bounds a solid torus R_0, and $T_1 = T\times\{1\}$ bounds a solid torus R_1, such that $S^3 = R_0\cup(T\times I)\cup R_1$. Let k be a $(1,2)$–knot, and assume that k lies in $T\times I$, such that $k\cap T_0 = k_0$ consists of two arcs, $k\cap T_1 = k_1$ consists of two arcs, and $k\cap(T\times(0,1))$ consists of four vertical arcs.

Suppose there is a closed surface S in $S^3-\operatorname{int} N(k)$, which is incompressible and meridionally incompressible. Assume that S intersects T_0 and T_1 transversely. Let $S_0 = S\cap R_0$, $S_1 = S\cap R_1$, and $\widetilde{S} = S\cap(T\times I)$. Let $\pi\colon T\times I\to I$ be the height function, where we choose 0 to be the lowest point, and 1 the highest. We may assume that the height function on $\widetilde{S}$ is a Morse function. So there is a finite set of different points $X=\{x_1,x_2,\dots,x_m\}$ in I, so that $\widetilde{S}$ is tangent to $T\times\{x_i\}$ at exactly one point, and this singularity can be a local maximum, a local minimum, or a simple saddle. Suppose that $1>x_1>x_2>\dots>x_m>0$, that is, we numerate the singular points starting from the upper level. For any $y\notin X$, $T\times\{y\}$ intersects $\widetilde{S}$ transversely, so for any such y, $\widetilde{S}\cap(T\times\{y\})$ consists of a finite collection of simple closed curves called level curves, and at a saddle point x_i, either one level curve of $\widetilde{S}$ splits into two level curves, or two level curves are fused into one curve.

For example, any of the knots and surfaces constructed in Section 2 can be put in this position, after doing an appropriate isotopy.

Define the complexity of S by the pair $c(S) = (|S_0|+|S_1|+|\widetilde{S}|, |X|)$ (where $|Y|$ denotes the number of points if Y is a finite set, or the number of connected components if it is a surface, and give to such pairs the lexicographical order). Assume that S has been isotoped so that $c(S)$ is minimal.

Claim 3.2 *The surfaces* S_0*,* S_1 *and* $\widetilde{S}$ *are incompressible and meridionally incompressible in* R_0*,* R_1*, and* $T\times I-\operatorname{int} N(k)$ *respectively.*

Proof If there is a meridian compression disk for one of the surfaces, then it will be also a meridian compression disk for S. Suppose then one of the surfaces is compressible, say $\widetilde{S}$, and let D be a compression disk, which is disjoint from k. Then ∂D is essential in $\widetilde{S}$ but inessential in S. By cutting S along D we get a surface S' and a sphere E. Note that S and S' are isotopic in $M-k$. For S' we can similarly define the surfaces S_0', S_1' and $\widetilde{S}'$. Note that $|S_i| = |S_i'|+|E\cap R_i|$, $i=1,\ 0$, then either $|S_0'|<|S_0|$ or $|S_1'|<|S_1|$, for E intersects at least one of R_0, R_1. Also $|\widetilde{S}'|\le|\widetilde{S}|$, so $c(S')<c(S)$, but this contradicts the minimality of $c(S)$. □

This implies that S_0 is a collection of trivial disks, meridian disks and incompressible annuli in R_0. If a component of S_0 is a trivial disk E, then ∂E bounds a disk on T_0

which contains at least a component of k_0, for otherwise $|S_0|$ could be reduced. If a component of S_0 is an incompressible annulus A, then A is parallel to an annulus $A' \subset T_0$, and A' must contain a component of k_0, for otherwise $|S_0|$ could be reduced. Note that the slope of ∂A can consist of one longitude and several meridians of R_0; in this case A would also be parallel to $T_0 - A'$, and then the other component of k_0 would be in $T_0 - A'$. Note also that S_0 cannot contain both incompressible annuli and meridian disks. A similar thing can be said for S_1.

Claim 3.3 *$\widetilde{S}$ does not have any local maximum or minimum.*

Proof The proof is similar to that of [6, Claim 2]. It consists in taking the maximum at the lowest level and then in pushing it down, getting that either the surface is compressible or that $\widetilde{S}$ has a component which is parallel to a subsurface in T_0. By pushing it into R_0 the complexity of S is reduced. □

The proof of the claim also implies that if S is in a position where $\widetilde{S}$ has a maximum or a minimum, then S can be isotoped to a position of lower complexity.

Note that if at a certain nonsingular level $\{y\}$, there is a curve of intersection γ which is trivial in the level torus $T \times \{y\}$, then γ bounds a disk in the level torus which intersects k in two or more points, for otherwise $\widetilde{S}$ will be compressible, meridionally compressible, or it would have a local maximum or minimum.

Claim 3.4 *Only the following types of saddle points are possible.*

1. *A saddle changing a trivial simple closed curve into two non-nested trivial simple closed curves.*
2. *A saddle changing a trivial simple closed curve into two nested trivial simple closed curves.*
3. *A saddle changing two non-nested trivial simple closed curves into a trivial simple closed curve.*
4. *A saddle changing two nested trivial simple closed curves into a trivial simple closed curve.*
5. *A saddle changing a trivial simple closed curve into two essential simple closed curves.*
6. *A saddle changing two parallel essential curves into a trivial curve.*
7. *A saddle changing an essential curve γ into a curve with the same slope as γ, and a trivial curve.*

(8) *A saddle changing an essential curve* γ *and a trivial curve into an essential curve with the same slope as* γ*.*

Proof See Figure 10 and Figure 11. At a saddle, either one level curve of $\widetilde{S}$ splits into two level curves, or two level curves are joined into one level curve. If a level curve is trivial in the corresponding level torus and it bounds a disk intersecting k in four points and at a saddle the curve joins with itself, then the result must be either two non-nested trivial curves each bounding a disk intersecting k in two points, or two essential simple closed curves, for otherwise $\widetilde{S}$ would be compressible or meridionally compressible; in this case the singularity is of type 1 or 5.

If a level curve is trivial in the corresponding level torus and it bounds a disk intersecting k in less than four points and at a saddle the curve joins with itself, then the result must be either two essential simple closed curves, or two nested trivial curves; in the latter case, the original curve must bound a disk intersecting k in two points, and one of the new curves bound a disk intersecting k in four points, for otherwise $\widetilde{S}$ would be compressible or meridionally compressible. So we have a singularity of type 2 or 5.

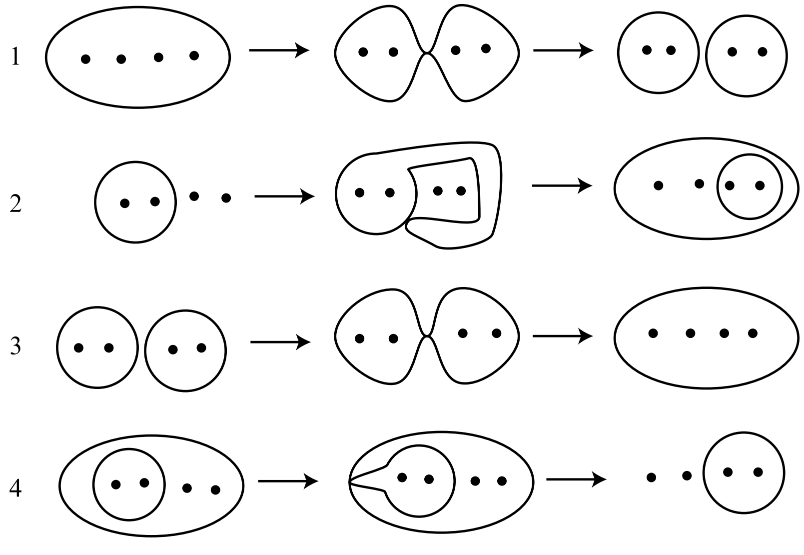

Figure 10

If two trivial level curves are joined into one and are non-nested, then each bounds a disk intersecting k twice, and the new curve bounds a disk intersecting k in four points. This is a singularity of type 3. If two trivial level curves are joined into one and are nested, then the innermost one bounds a disk intersecting k twice, the outermost one bounds a disk intersecting k in four points, and the new one bounds a disk intersecting k twice. This gives a singularity of type 4.

If in a level there are nontrivial curves of intersection, then there is an even number of them, for S is separating. So if a curve is nontrivial and at the saddle joins with itself, then the result is a curve with the same slope as the original and a trivial curve, for the saddle must join points on the same side of the curve. This is a singularity of type 7.

Finally note that there may be singularities of types 6 and 8. □

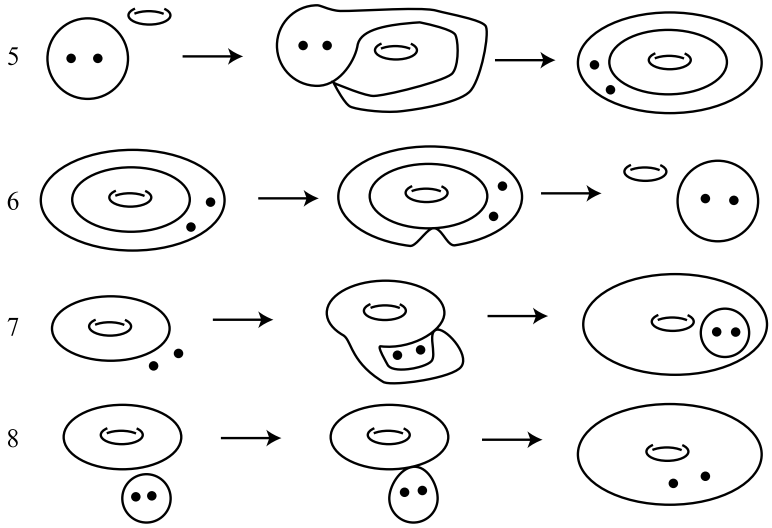

Figure 11

In Figure 12 we show locally how the surface S looks in the neighborhood of a singularity.

Claim 3.5 *Suppose that at a certain non-singular level there is a curve γ of intersection which is trivial in the level torus and bounds a disk which intersects k in two points. If γ is a trivial curve in S, then γ bounds a disk $D \subset S$, which consists of an annulus contained in $\widetilde{S}$ with no singular points and a disk component of S_i which is trivial in R_i $(i = 1, 0)$.*

Proof The curve γ bounds a disk E in the level torus $T \times \{y\}$, and also bounds a disk $D \subset S$. Suppose that D is not as required. A collar neighborhood of γ in D lies below $T \times \{y\}$, say. Suppose first that $E \cap D = \partial E = \partial D$, but note that E may contain more curves of intersection with S. Now, $D \cup E$ bounds a 3–ball B, which intersects

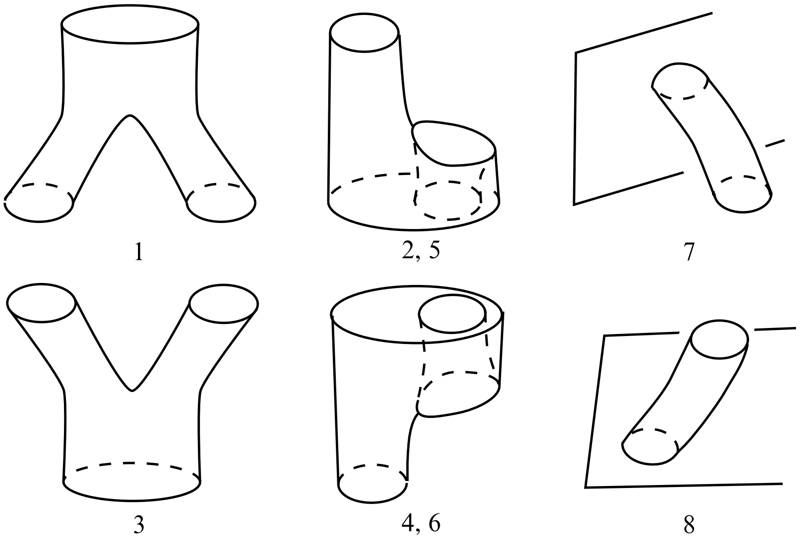

Figure 12

k in a spanning arc k'. The 3–ball B can be isotoped to lie in a product neighborhood of E, so that k' is isotoped to an arc in this product neighborhood, and it preserves its singular points. Note that D may contain disks of S_0 or S_1 which are meridian disks of R_0 or R_1, and then an arc of k may wrap around a meridional annulus in T_0 or T_1; during the isotopy, such arc must be arranged so that it now lies on a trivial annulus contained in some level torus. Now pull B down, eliminating singular points of D, until D consists of an annulus in $\widetilde{S}$ without singular points and it intersects R_0 in a trivial disk. If $int\ B \cap S$ is non-empty, any component of $B \cap S$ must be a disk, for otherwise a component of $\widetilde{S}$, S_0 or S_1 would be compressible. So any component of $B \cap S$ can be arranged to be a disk consisting of an annulus in $\widetilde{S}$ without singular points and a trivial disk in R_0. At the end of this procedure we have reduced $c(S)$. Now the arc k' can be rearranged so that the knot k is in a $(1,2)$–position.

Suppose now that E and D intersect in some simple closed curves. Look at the whole collection of curves $E \cap S$; these curves are concentric in E, for each bounds a disk intersecting k twice, and each curve is trivial in S. Look at the intersection curves in S, and among the innermost ones, take the one which is outermost in E. Let α be this curve, which bounds a disk $D' \subset S$. As $D' \cap E = \partial D'$, we can assume, by an argument as in the previous paragraph, that D' does not contain any singular points of $\widetilde{S}$ and that contain a single component of S_0 or S_1. The curve α in E, with another curve of intersection, say β (perhaps $\beta = \gamma$), cobound an annulus E' in E, with interior disjoint from S. Consider a copy of D' and E' to form a disk D_1 with interior disjoint from S, and with $\partial D_1 = \beta$. Cut S with D_1, getting a new surface S' isotopic to S. If the curves α and β are concentric in S, the region bounded by them in S contains at least a component of S_0 or S_1, which is eliminated in S', so this reduces $c(S)$. If the curves α and β are non-concentric in S, the disk D'' bounded by β in S must intersect the disk E in some simple closed curves and then must contain at least two components of S_0 or S_1. By replacing D'' with D_1 we reduce the complexity of S, for D_1 contains only a component of S_0 or S_1. We may have introduced a new local maximum or minimum in S' at level $\{y\}$. This new singularity can be eliminated as in Claim 3.3, getting a surface with lower complexity. □

Claim 3.6 *Suppose that at a certain nonsingular level there is a curve γ of intersection which is trivial in the level torus and bounds a disk which intersects k in four points, and its interior is disjoint from S. If γ is a trivial curve in S, then γ bounds a disk $D \subset S$, which consists of an annulus contained in $\widetilde{S}$ with no singular points and a disk component of S_i trivial in R_i $(i = 1, 0)$.*

Proof It is similar to the previous claim. □

Claim 3.7 *Suppose that at a certain level torus $T \times \{y\}$, there is a singularity x_i of type 1, 2 or 7. So there is a curve γ of intersection, which contains the singular point, it is trivial in the level torus, bounds a disk E, so that the other singular curve is not contained in E. Assume that E intersects k in two points. At a level just below y, there is a curve γ' of intersection, which is parallel to γ in S, and which bound a disk E' in that level, also intersecting k in two points. Then γ' is a nontrivial curve in S.*

Proof Suppose that γ' is trivial in S. Then by Claim 3.5, γ', and in fact γ, bounds a disk $D \subset S$, which consists of an annulus contained in $\widetilde{S}$ with no singular points and a disk component of S_0 trivial in R_0. Now, E and D bound a 3–ball, which intersects k in a spanning arc k'. So D can be isotoped into E, and then the singularity x_i is eliminated. The arc k' can be isotoped to lie in a product neighborhood of E, preserving its singular points, and now by finding a vertical path from $T \times \{y\}$ to T_0 disjoint from S, k' can be rearranged to be in a $(1,2)$–position. This contradicts the minimality of $c(S)$. □

Claim 3.8 *Suppose that at a certain nonsingular level there is a curve γ of intersection which is essential in the level torus. If γ is a trivial curve in S, then γ is a meridian or a longitude of the level torus, and it bounds a disk $D \subset S$, which contains a meridian disk of S_0 or S_1.*

Proof Let $T \times \{y\}$ be the nonsingular level at which γ lies. Let D be the disk in S bounded by γ. The disk D intersects $T \times \{y\}$ in ∂D and possibly in a collection of simple closed curves, so the slope of ∂D on γ, seen it as a knot in S^3, is the same as the slope of $(T \times \{y\}) - \operatorname{int} N(\gamma)$ in γ, so this has to be 0, and then γ must be a meridian or a longitude of $T \times \{y\}$. Now, D must contain a meridian disk of S_0 or S_1, for otherwise it will be contained in S^3 minus the cores of R_0 and R_1, but γ is not trivial in that product region. □

Claim 3.9 *Suppose that at a certain nonsingular level, there are two concentric trivial curves of intersection,* γ_1 *and* γ_2*, which bound an annulus* A *in such level, which is disjoint from* k*. Suppose the curves are nontrivial in the surface* S*. Then the curves cannot be parallel in* S*.*

Proof Suppose the curves γ_1 and γ_2 are parallel in S, then they bound an annulus $A' \subset S$. Now A and A' bound a solid torus disjoint from k, and as γ_i is a trivial curve, it must be that A and A' are in fact isotopic. If A has more intersections with S, then we just have more annuli contained in that solid torus, and could take an innermost one. The annulus A' may have some singular points of $\widetilde{S}$, if not then it contains an annulus of S_0 or S_1, which is parallel to an annulus in T_0 or T_1 disjoint from k, which is not possible. So isotope A' to A, reducing the complexity of S, but putting it in a position, in which it has a maximum or minimum. This can be eliminated as in Claim 3.3, reducing then the complexity of S. □

Claim 3.10 *Suppose that at a certain nonsingular level, there are two curves of intersection,* γ_1 *and* γ_2*, where* γ_1 *is trivial and* γ_2 *is essential in the level torus. If the curves are parallel in* S*, then* γ_2 *is a meridian or longitude of the level torus, and* $\gamma_1 \cup \gamma_2$ *bounds an annulus containing a meridian disk of* R_0 *or* R_1*.*

Proof If γ_1 and γ_2 are parallel in S, then there is an annulus $A \subset S$, $\partial A = \gamma_1 \cup \gamma_2$. As γ_1 is trivial in the level torus, by an argument as in Claim 3.8 we see that this is possible only if γ_2 is a meridian or a longitude of the level torus, but then A would intersect the core of R_0 or R_1, implying that one of S_0 or S_1 must contain a meridian disk of R_0 or R_1. □

Claim 3.11 *Suppose that at a certain nonsingular level torus* $T \times \{y\}$*, there are two curves of intersection,* γ_1 *and* γ_2*, which are essential in the level torus, and bound an annulus intersecting* k *twice. Suppose that the curves are parallel in* S*, and cobound an annulus* $A \subset S$ *which is contained in* $R_0 \cup (T \times [0, y])$*. Then either* A *has no singular points of* S *and contains just one annulus component of* S_0*, or it has just a type 6 singularity and contains just one disk component of* S_0*.*

Proof Suppose A has singular points of S, and look at the first one. If it is of type 7, then it will contradict Claim 3.7. So it must be of type 6, changing the two curves into a trivial curve γ_3. Then by Claim 3.5 there cannot be more singularities in A, and γ_3 bounds a disk in S which contains a disk of S_0. □

Claim 3.12 *Suppose that at a certain nonsingular level torus $T \times \{y\}$, there are two curves of intersection, γ_1 and γ_2, which are trivial in the level torus, are non-nested and each bounds a disk intersecting k twice. Suppose that the curves are parallel in S, and cobound an annulus $A \subset S$ which is contained in $R_0 \cup (T \times [0, y])$. Then A has just a type 3 singularity and contains just one disk component of S_0.*

Proof Note that A must have some singularities. The first one can be of type 2 or 3. If it is of type 2, it will contradict Claim 3.7, so assume it is of type 3. So, after this singularity we get a trivial curve γ_3 in a level torus, which bounds a disk intersecting k in four points, and γ_3 is trivial in S. By Claim 3.6, γ_3 bounds a disk in S which has no singular points and contain one disk component of S_0, so the annulus is as desired. □

Note that up to this point the arguments apply for any meridionally incompressible surface in the complement of a $(1, 2)$–knot. The next claims will make use of the hypothesis that $\text{genus}(S) = 2$.

Claim 3.13 *Suppose that a certain nonsingular level there is a collection of concentric trivial curves, where the innermost and the outermost one bound an annulus A in such level, which is disjoint from k. Suppose the curves are nontrivial in the surface S. Then the collection consists of at most 3 curves.*

Proof If there are more than 3 curves, then as they are nontrivial in S, and S has genus 2, necessarily two of them will be parallel, contradicting Claim 3.9. □

These claims imply that most of the curves we see in a nonsingular level are essential curves in S.

Claim 3.14 *S_1 (S_0) does not contain both meridian disks and trivial disks of R_1 (R_0).*

Proof Suppose S_1 contains meridian disks and trivial disks; it must consist of an even number of meridian disks, for it is separating. If $\widetilde{S}$ has no singular points, then it would be a collection of annuli, and S would be a sphere; suppose then that there are some singularities on $\widetilde{S}$.

If the first singularity is of type 2, then there is a trivial disk whose boundary joins with itself. Note that this singularity can be pushed to lie on R_1, changing the trivial disk into an annulus bounded by trivial curves. One of these curves bounds a disk E intersecting k in two points, corresponding to one of the arcs of k lying in R_1. As there are meridian disks, the arc can be rearranged to lie in E, but then S will be compressible, as ∂E must be essential in S by Claim 3.7.

If the first singularity is of type 3, then it can be pushed to lie on R_1, changing two trivial disks into a trivial disk, then reducing the complexity of S.

If the first singularity is of type 4, again it can be pushed to lie on R_1, changing two trivial disks into a trivial disk, then reducing the complexity of S.

If the first singularity is of type 5, then it can be pushed to lie on R_1, changing a trivial disk into an annulus A, whose boundary consists of curves parallel to the boundaries of the meridian disks. At least one of the arcs of k_1 must lie in the region between A and T_1. So there is a disk D in R_1, with $D \cap A = \partial D$, which is an essential curve on A. The remaining arc of k_1, say k' can be arranged so that intersect D in at most one point. If k' intersects D in one point then S is meridionally compressible, and if it is disjoint from D then S is compressible, unless ∂A is an inessential curve in S, but in this case by cutting S with D we get a surface S' isotopic to S, but with $c(S') < c(S)$.

If the first singularity is of type 6, then again it can be pushed to lie on R_1, changing two meridian disks into a trivial disk. The arcs of k_1 can be rearranged to be in the required position. So we have reduced the complexity of S.

If the first singularity is of type 7, then there is a meridian disk whose boundary joins with itself. Push this singularity to R_1, getting an annulus whose boundary consists of a meridian of R_1 and a trivial curve α_1 which bounds a disk E in T_1 intersecting k_1. Note that because there are trivial disks in S_1, the curve α_1 bounds a disk D in R_1 disjoint from k_1, and then S will be compressible, unless α_1 is an inessential curve in S, but this is not possible by Claim 3.7.

If the first singularity is of type 8, then it can be pushed to lie on R_1, changing a meridian disk and a trivial disk into a meridian disk. Again, the arcs of k_1 can be rearranged to be in the required position. So we have reduced the complexity of S.

Therefore assume that the first singularity is of type 1. Then we have a collection of nested trivial disks, each bounding a disk intersecting k in four points. Possibly we have a sequence of type 1 singularities, starting with the innermost curve. In each singularity the disk changes into an annulus. If there is a type 2, 3, 4, 5, 6, 7 or 8 singularity before all trivial disks are transformed, then with the same arguments as above we see that there will be a compression disk for S, or there is an isotopy which reduces the complexity of S. So we must have a sequence of type 1 singularities which transforms the disks into a collection of nested annuli. Note that the new level curves obtained are nontrivial in S, by Claim 3.7. Then by Claim 3.13 there are at most 3 of these singularities, so there are at most 3 nested annuli.

Up to this point, the annuli can be seen as lying around an arc with one maximum in a region $A \times [y, 1]$ (where $A \subset T_1$ is an annulus), together with a ball bounded by two meridians disks of R_1, ie this region is just a 3–ball B'. In the level torus $T \times \{y\}$ there are 3 sets of curves of intersection with S, consisting of curves α_i's, which are trivial and concentric in the level torus, essential and nonparallel in S; curves β_i's, which are trivial and concentric in the level torus, essential and nonparallel in S, and such that α_j is parallel to β_j in S, for each j; curves γ_i's, which are essential in the level torus, are trivial in S, and bound a meridian disk in the solid torus $(T \times [y, 1]) \cup R_1$. Note also that the curves α_i's and β_i's lie in the same component of $(T \times \{y\}) - \cup\{\gamma_i\}$, say in an annulus between γ_1 and γ_2.

The next singularity could be of types 2, 3, 5, 6, 7 or 8. It is not difficult to see that if the next singularity is of type 2, 3 or 5, then there will be a compression disk for S. If it is of type 6, then two meridian disks are changed into a disk E. The path followed by the singularity can be complicated, but because the nested annuli lie around an arc with just one maximum, these can be isotoped (in the 3–ball B' determined by $A \times [y, 1]$ and the 3–ball bounded by two meridians disks of R_1), so that the path and the annuli look simple, and then it is not difficult to see that the disk E can be isotoped so that $E \cap R_1$ is a new trivial disk in S_1, reducing the complexity of S. If the next singularity is of type 8, then it can be pushed into R_1, changing a meridian disk and a trivial disk into a meridian disk, then reducing the complexity of S.

So the only possibility left is that the next singularity is of type 7, where the path of the saddle encircles the nested annuli. So a curve, say γ_1, is split into a nontrivial curve γ_1' and a trivial curve β_j, which is concentric with the curves β_i's, say. So if there are 3 nested annuli, we will have four parallel curves, which contradicts Claim 3.13. Suppose then that there are two nested annuli, and then curves α_1, α_2, and β_1, β_2. After the singularity of type 7 we have one more curve, denoted β_3, and the curves β_i's lie between γ_1' and γ_n, say.

So far, we have 3 singularities, x_1, x_2 of type 1 and x_3 of type 7. Look at the next singularity x_4, it may be of types 5, 6, 7 or 8. If it is of type 5, then the surface S will be compressible. If it is of type 6 or 8, then such singularity can be pushed to a level above the singularity x_3. So the singularity x_4 is again of type 7. After this singularity we will have one more trivial curve of intersection, which cannot be concentric with the β_i 's by Claim 3.13, so it must be concentric with the α_i 's. There are two cases, depending on which of γ_1' or γ_2 is split by the singularity. Suppose first that γ_1' is split into γ_1'' and α_3. Note that γ_1'' is the boundary of a disk D (formed by a meridian disk bounded by γ_2 and an annulus between γ_1'' and γ_2), so S is compressible, except if γ_1'' is trivial in S, but in that case, by cutting S with D, we get a new surface S' isotopic to S, its embedding has a local minimum at $T \times [0,1]$, so that by isotoping it we get a surface with lower complexity than S.

Suppose then that the singularity x_4 splits γ_2 into curves γ_2' and α_3. If there are more than 2 curves γ_i, by the same arguments the next singularity would be of type 7, which again yields a contradiction with Claim 3.13. So, up to this level, say y_1, we have only the curves of intersection γ_1', γ_2', α_1, α_2, α_3, β_1, β_2, β_3. Note that the curves γ_1' and β_3 are parallel in S, and γ_2' and α_3 are also parallel in S. Note that because S is a surface of genus 2, these four curves must be parallel in S (for α_3 (β_3) is nonparallel to α_1 or α_2 (β_1 or β_2)). So in the solid torus $R_0 \cup (T \times [0,y_1])$, there is an annulus A bounded by two of these curves. In fact, $S \cap (R_0 \cup (T \times [0,y_1]))$ consists of the annulus A plus two pair of pants. Note that ∂A cannot consist of γ_i' and one of α_3 or β_3, for γ_i' is a longitude of the solid torus $R_0 \cup (T \times [0,y_1])$. If $\partial A = \alpha_3 \cup \beta_3$, then as A is a separating surface, it leaves in one side the curves α_1, α_2, β_1, β_2, and in the other side the curves γ_1', γ_2', so there cannot be two pairs of pants with these six curves as their boundary. So we have $\partial A = \gamma_1' \cup \gamma_2'$, but this annulus is isotopic to an annulus in $T \times \{y\}$, by an isotopy that leaves fixed the knot and the rest of the surface. Then $c(S)$ was not minimal.

Suppose then that S_0 contains just one trivial disk. The first singularity is of type 1, and after that level we have curves α_1, β_1 and the γ_i 's. By the same arguments as above, the next two singularities must be of type 7. If there are 6 or more curves γ_i, there will be 4 or more curves β_i, which is not possible. So suppose we have just curves γ_1, γ_2, γ_3, and γ_4. By arguments as above, there are 4 singularities of type 7, after which at a level y_1 we get curves $\{\alpha_1,\alpha_2,\alpha_3\}$, $\{\beta_1,\beta_2,\beta_3\}$, and $\{\gamma_1',\gamma_2',\gamma_3',\gamma_4'\}$, where the following pair of curves are parallel in S, $\{\alpha_1,\beta_1\}$, $\{\beta_2,\gamma_1'\}$, $\{\beta_3,\gamma_4'\}$, $\{\alpha_2,\gamma_2'\}$, $\{\alpha_3,\gamma_3'\}$. The curves α_i 's and β_i 's lie in an annulus in the level torus between the curves γ_2' and γ_3'. Now in S we have two sets of 4 parallel curves, and $S \cap (R_0 \times [0,y_1])$ consists of two annuli and two pair of pants. An argument as above yields a contradiction.

Suppose now we have just two curves γ_1 and γ_2. After the singularity of type 1 we have curves α_1, β_1, γ_1 and γ_2. Arguments as above show that the next two singularities are of type 7. There are two possibilities. In one case, in a level y_1 just after these singularities we get curves α_1, β_1, β_2, β_3, γ_1' and γ_2', where the pairs $\{\beta_2,\gamma_1'\}$ and $\{\beta_3,\gamma_2'\}$ are parallel curves in S. So in S we have 3 pairs of parallel curves, which implies that $S \cap (R_0 \cup (T \times [0,y_1]))$ consists of two pairs of pants, say

P_1 and P_2. One of the pair of pants must have the curves γ_1' and γ_2'. By the position of the curves in the level torus $T \times \{y_1\}$, it is not possible that $\partial P_1 = \gamma_1' \cup \gamma_2' \cup \beta_1$ and $\partial P_2 = \alpha_1 \cup \beta_2 \cup \beta_3$, so we must have $\partial P_1 = \gamma_1' \cup \gamma_2' \cup \alpha_1$ and $\partial P_2 = \beta_1 \cup \beta_2 \cup \beta_3$. By the position of the curves, we see that the curve β_3 must bound a disk disjoint from S and disjoint from the knot, so S will be compressible.

The final case is that after the two type 7 singularities, at a level y_1, we have curves α_1, α_2, β_1, β_2, γ_1' and γ_2', where the α_i's and the β_i's lie in the same level annulus between γ_1' and γ_2', and the pairs of curves $\{\alpha_2, \gamma_2'\}$, $\{\beta_2, \gamma_1'\}$ are parallel in S. If γ_1' and γ_2' are parallel in S, we will have a set of 4 parallel edges and an argument as above yields a contradiction. If they are nonparallel in S, then $S \cap (R_0 \times [0, y_1])$ consists of two pairs of pants, say P_1 and P_2. One of the pair of pants must have the curves γ_1' and γ_2'. So the only possibility (up to interchanging α_i's and β_i's) is that $\partial P_1 = \gamma_1' \cup \gamma_2' \cup \beta_1$ and $\partial P_2 = \alpha_1 \cup \alpha_2 \cup \beta_2$, but by the position of the curves this is not possible. □

Claim 3.15 *Suppose S_1 contains meridian disks of R_1, then S_1 contains exactly two meridian disks of R_1, and S is a surface of type 5.*

Proof Suppose that ∂S_1 consists of curves $\gamma_1, \gamma_2, \ldots, \gamma_n$, with n an even number. Look at the first singularity, it may be of type 6 or 7, for there are no trivial disks in S_1. If it is of type 6, then two curves, say γ_1 and γ_2 are fused into a curve α_1, which is trivial in a level torus and bounds a disk in such level torus intersecting k in 2 or 4 points. But the curve γ_1 is trivial in S, so by Claim 3.5 or Claim 3.6, S can be isotoped to a position with lower complexity.

So assume the first singularity is of type 7. So a curve, say γ_1, is split into a nontrivial curve γ_1' and a trivial curve β_1, which bounds a disk in a level torus $T \times \{y_1\}$ intersecting k in two points (for if it intersects k in 3 or 4 points, there is a meridional compression or compression disk for S, which is formed by the union of an annulus between γ_1' and γ_2 and a meridian disk bounded by γ_2). If the arcs of k_1 lie in different components of $T_1 - S_1$, then again after passing the first singularity, we see that there is a compression disk for S. So both arcs of k_1 lie in the annulus determined by γ_1 and γ_2, say. After the first singularity, the curve β_1 lies in an annulus determined by the curves γ_1' and γ_n. The second singularity may be of types 5, 6, 7 or 8. If it is of type 5, then the surface S will be compressible. If it is of type 6 or 8, then the surface will be compressible or such singularity can be pushed to a level above the first singularity. So the second singularity is again of type 7.

Suppose first that the second singularity splits the curve γ_1' into a nontrivial curve γ_1'' and a trivial curve β_2. There are two cases, depending if the curves β_1 and β_2 are concentric or not. Suppose first that the curves β_1 and β_2 are concentric. They lie in a level annulus determined by γ_1'' and γ_2. Note that γ_1'' is the boundary of a disk D (formed by a meridian disk bounded by γ_n and an annulus between γ_1'' and γ_n), so S is compressible, except if γ_1'' is trivial in S, but in that case, by cutting S with D, we get a new surface S' isotopic to S, its embedding has a local minimum at $T \times [0,1]$, so that by isotoping it we get a surface with lower complexity than S.

Suppose now that the curves β_1 and β_2 are nonconcentric. They lie in a level annulus determined by γ_1'' and γ_n. Note that γ_1'' is the boundary of a disk D (formed by a meridian disk bounded by γ_2 and an annulus between γ_1'' and γ_2). So as before, S is compressible or it can be isotoped to a surface with lower complexity.

Therefore the second singularity splits a curve other than γ_1' into two curves. By these arguments, and as in the proof of the previous claim, we have a sequence of type 7 singularities which create two sets of concentric trivial curves, α_i's and β_i's. If $n \geq 8$, there will in a certain level 4 of such concentric curves, contradicting Claim 3.13. If $n = 6$, there will be in a certain level torus $T \times \{y_1\}$, two sets of concentric trivial curves $\{\alpha_1, \alpha_2, \alpha_3\}$, $\{\beta_1, \beta_2, \beta_3\}$, plus 6 essential curves. Then in the surface S we have two sets of 4 parallel edges. As in the proof of Claim 3.14, this yields a contradiction. If $n = 4$ there are two possibilities, one is that in a level torus $T \times \{y_1\}$ there are two sets of concentric trivial curves $\{\alpha_1, \alpha_2\}$, $\{\beta_1, \beta_2\}$, plus 4 essential curves, or well two sets of concentric trivial curves $\{\alpha_1\}$, $\{\beta_1, \beta_2, \beta_3\}$, plus 4 essential curves. An argument as in the proof of Claim 3.14 yields a contradiction.

So we must have $n = 2$, and the first two singularities are of type 7. There are two possibilities for the curves obtained after these singularities, either we get two trivial concentric curves, or two trivial curves which are nonconcentric, and in any case we get essential curves γ_1' and γ_2'. Suppose first that after the singularities we get trivial concentric curves β_1 and β_2, plus the essential curves γ_1' and γ_2'. Note that β_1 and β_2 are essential and nonparallel in S, by Claim 3.7 and Claim 3.9. The third singularity can be of types 2, 5, 6, 7 or 8. Let $T \times \{y_1\}$ be a level torus just below this singularity, and let $W = R_0 \cup (T \times [0, y_1])$, this is a solid torus. If the third singularity is of type 2, the curve β_2 is split into curves α_1 and α_2, where α_1 bounds a disk intersecting k in two points and α_2 encircles β_1 and α_1; the curve α_1 is essential in S by Claim 3.7, and the curve α_2 is also essential, for otherwise by Claim 3.5, α_2 bounds a disk D with no singularities, which then implies that γ_1' and γ_2' cobound an annulus which can be isotoped into $T \times I$, contradicting the minimality of $c(S)$. The surface $S \cap W$ consists of an annulus and a pair of pants. Because of the position of the curves, the annulus bounds the curves α_1 and β_1, and the pair of pants has as boundary the curves γ_1', γ_2' and α_2. So, by Claim 3.12, in the annulus there must be a singularity of type

3, but this singularity can be interchanged with the type 2 singularity, to become a singularity between β_1 and β_2, which implies that S is compressible.

If the third singularity is of type 5, the curve β_2 is split into curves α_1 and α_2, which are parallel in the level torus $T\times\{y_1\}$ to γ'_1, and which are essential in S, for otherwise α_1 or α_2 will bound a disk contained in W, which is not possible. The surface $S\cap W$ consists of an annulus and a pair of pants. Because of the position of the curves, the annulus bounds, say, the curves α_1 and γ'_1, and the pair of pants has as boundary the curves γ'_2, α_2 and β_1. So in the pair of pants there must be a singularity of type 8, joining β_1 and α_2, but this singularity can be interchanged with the type 5 singularity, to become a singularity between β_1 and β_2, which implies that S is compressible.

If the third singularity is of type 6, the curves γ'_1 and γ'_2 are fused into a curve α_1, which is essential in S, and in the level torus $T\times\{y_1\}$ cobound an annulus with β_2 which intersects k in two points. Now the curves α_1, β_1 and β_2 must bound a pair of pants in the solid torus W. Then the curve α_1 bounds a disk disjoint from k, so S is compressible.

If the third singularity is of type 7, the curve γ'_1 (or γ'_2) is split into a curve γ''_1 and a trivial curve α_1. Again in $S\cap W$, there is a pair of pants and an annulus bounded by the curves γ''_1, γ'_2, α_1, β_1 and β_2. The only possibility is that the pair of pants bounds γ''_1, γ'_2 and β_2, and the annulus bounds α_1 and β_1, so α_1 is nonconcentric with β_2 in the level torus, by Claim 3.9. Note that the pair of pants can be isotoped into the level torus $T\times\{y_1\}$, so that $c(S)$ can be reduced.

If the third singularity is of type 8, the curve β_2 and γ'_1 (or γ'_2) are fused into a curve γ''_1. The surface $S\cap W$ consists of a pair of pants with boundary curves γ''_1, γ'_2 and β_1. Then there must be one more singularity of type 8. If it is between β_1 and γ''_1, it can be interchanged with the previous singularity, getting then a singularity between β_1 and β_2, which shows that S is compressible. If the singularity is between β_1 and γ'_2, then after isotoping S in R_0, the singularities can be interchanged, showing that S is compressible.

So assume that after the first two type 7 singularities we have two nonconcentric trivial curves α_1 and β_1, and two essential curves γ'_1 and γ'_2. Note that the pairs of curves $\{\alpha_1,\gamma'_2\}$ and $\{\beta_1,\gamma'_1\}$ are parallel in S, and that curves from different pairs are nonparallel in S; this is because if they are parallel, then there is an annulus cobounded by α_1 and β_1, and a twice punctured torus bounded by γ'_1 and γ'_2, but in this case S would be compressible. The next singularity can be of types 2, 3, 5, 6, 7 or 8. Let $T\times\{y_1\}$ be a level torus just below this singularity, and let $W=R_0\cup(T\times[0,y_1])$.

If the third singularity is of type 3, the curves α_1 and β_1 are fused into a curve α_2 which bounds a disk in the level torus $T\times\{y_1\}$ intersecting k in four points. One possibility is that the surface $S \cap W$ is a pair of pants bounded by γ_1', γ_2' and α_2. Note that such a pair of pants can be isotoped into $T \times \{y_1\}$, then $c(S)$ is not minimal. The other possibility is that $S \cap W$ consists of a once-punctured torus bounded by α_2 and an annulus bounded by γ_1' and γ_2'. But note that because the curves γ_1' and γ_2' are longitudinal in W, such an annulus bounded by γ_1' and γ_2' can be isotoped into $T \times \{y_1\}$, reducing $c(S)$.

If the third singularity is of type 7, the curve γ_1' is split into γ_1'' and β_2, where now β_2 is concentric with β_1. After that level $S \cap W$ consists of an annulus and a pair of pants, and by the configuration of the curves this is not possible.

If the third singularity is of type 6, the curves γ_1' and γ_2' are fused into a curve α_2 which is trivial in the level torus and encircles α_1 and β_1. Below that level, there are two possibilities for the surface $S \cap W$. If it is a pair of pants bounded by α_1, β_1 and α_2, then it can be isotoped to the level torus, and so $c(S)$ is not minimal. The other case is that $S \cap W$ is an annulus bounded by α_2 and α_1 and a once-punctured torus bounded by β_1. In that case, the fourth singularity is of type 4, splitting β_1 into β_2 and β_3. Note that the third and fourth singularities can be interchanged, and the type 6 singularity can be pushed to R_0, reducing $c(S)$.

If the third singularity is of type 8, the curves γ_1' and β_1, say, are fused into a curve γ_1''. After that level the surface $S \cap W$ is a pair of pants or an annulus and a once-punctured torus. If it is a pair of pants, then it can be isotoped to the level torus, contradicting the minimality of $c(S)$. So, it consists of an annulus and a once-punctured torus. Then the once-punctured torus is bounded by α_1 and the annulus by γ_1'' and γ_2'. So there must be a singularity of type 5, and the part of S below a certain level torus $T \times \{y_2\}$ consists of two annuli. By Claim 3.11, S_0 consists of two annuli (which are isotopic to nested annuli, and anyone of them could be the innermost one), or there are two more singularities of type 6 and S_0 consists of two nested trivial disks (again, anyone of them could be the innermost one). If there is just one more singularity of type 6, then it can be pushed into R_0, reducing $c(S)$. Note that the singularities of type 8 and 5 can be interchanged, and that the singularity of type 8 and one of the singularities of type 6 can be interchanged, giving one of type 6 and one of type 4. In any case we get a surface of type 5.

If the third singularity is of type 2, then similar arguments show that there is one more singularity of type 4, followed by one of type 8, concluding with a trivial disk and an annulus in R_0, or well, there is one more singularity of type 6, and S_0 consists of two nested trivial disks. Again, the singularities of type 8 and 6 can be interchanged, giving one of type 6 and one of type 4. In any case this is a surface of type 5.

If the third singularity is of type 5, then similar arguments show that there is one more singularity of type 8, concluding with two annuli or two disks in R_0. This is a surface of type 5.

Summarizing, the following sequences of singularity types are possible, all producing a surface of type 5.

(a) 7, 7, 2, 4, 8, and S_0 consists of a trivial disk and an annulus, which are nested.

(b) 7, 7, 2, 4, 8, 6, or 7, 7, 2, 4, 6, 4, and S_0 consists of two nested trivial disks.

(c) 7, 7, 8, 5, or 7, 7, 5, 8, and S_0 consists of two nested annuli.

(d) 7, 7, 8, 5, 6, 6, or 7, 7, 5, 8, 6, 6, or 7, 7, 5, 6, 8, 6, or 7, 7, 5, 6, 6, 4, and S_0 consists of two nested trivial disks. □

Note that the proof of this claim also shows that if S_1 contains meridian disks, then S_0 has two components, two disks, two annuli or a disk and an annulus.

Claim 3.16 *Suppose that S_0 and S_1 consist only of trivial disks and annuli. Suppose that at a certain nonsingular level there is a curve γ of intersection which is essential in the level torus. Then γ is an essential curve in S.*

Proof This follows from Claim 3.8. □

Claim 3.17 *Suppose S_1 consists of just one trivial disk. Then S is a surface of type 1, 2, 3 or 4.*

Proof As S_1 contains a single disk, S_0 cannot contain meridian disks by the remark following Claim 3.15. So by Claim 3.16, any level curve of intersection which is essential in a level torus, it must be essential in S.

Let D be the disk component of S_1, then ∂D bounds a disk in T_1 which contains both arcs of k_1, for otherwise S will be nonseparating. If the first singularity is of type 5, then it can be pushed to lie in R_1, changing D into an annulus, but this reduces the complexity of S, which is not possible. So the first singularity must be of type 1, which splits the trivial curve into two trivial curves γ_1 and γ_2, each bounding a disk intersecting k in two points.

The next singularity has to be of type 3 or of type 2 or 5. Suppose first that it is of type 3. Then the curves γ_1 and γ_2 are fused into a trivial curve γ_3, which bounds a disk in a level torus intersecting k in four points. Up to this level the surface obtained is a once-punctured torus. The next singularity can be of type 1 or 5. Suppose it is of type 5;

so γ_3 is split into two curves γ_4 and γ_5, which are essential in the corresponding level torus. These curves are essential in S, so they must be parallel in S, for otherwise the genus of S is greater than 2. Then by Claim 3.11, we can assume that γ_4 and γ_5 lie on T_0 and bound an annulus component of S_0. Then S is as surface of type 1, as defined in Section 2.1.

Suppose now the third singularity is of type 1, so γ_3 is split into two trivial curves γ_4 and γ_5, each bounding a disk in a level torus intersecting k in two points. Note that γ_4 and γ_5 are essential in S by Claim 3.7, and that γ_4 and γ_5 must be parallel in S, because $\text{genus}(S) = 2$. Then by Claim 3.12, the next singular point must be of type 3, fusing γ_4 and γ_5 into a trivial curve γ_6. So, γ_6 must lie at T_0, and it bounds a disk in R_0. So, S is a surface of type 1.

Suppose now that the second singularity is of type 5. So, the curve γ_1 splits into two curves γ_3 and γ_4, which are essential in the corresponding level torus. The third singularity can be of type 5, 6, 7 or 8.

If the third singularity is of type 8, say fusing γ_2 and γ_3, then the second and third singularities can be interchanged, so that the new second singularity is of type 3 and we are in the previous case.

If the third singularity is of type 5, then γ_2 is split into two curves γ_5 and γ_6, which are essential in a level torus, say $T \times \{y_1\}$. Then as γ_3, γ_4, γ_5 and γ_6 are essential in S, among them there must be two pairs of parallel curves of S, which bound annuli A_1 and A_2 in S. So A_1 and A_2 are contained in $R_0 \cup (T \times [0, y_1])$. If A_1 bounds γ_3 and γ_6 and A_2 bounds γ_4 and γ_5, then one of the annuli, say A_1, can be isotoped to lie in $T \times I$, reducing $c(S)$. So assume that A_1 bounds γ_3 and γ_4 and that A_2 bounds γ_5 and γ_6. By Claim 3.11, we can assume that either there are no more singularities in S and that S_0 consists of two annuli, or that there two more singularities of type 6 and S_0 consists of two disks (which have to be nested, for otherwise $c(S)$ could be reduced). Note also that if there is just one more singularity of type 6, then it can be pushed into R_0, reducing $c(S)$. If the annuli A_1 and A_2 are non-nested in $R_0 \cup (T \times [0, y_1])$ we have a surface of type 3, and if the annuli are nested then we have a surface of type 4.

If the third singularity is of type 6. Then the two curves γ_3 and γ_4 are fused into a trivial curve γ_5. The surface up to that level is a twice punctured torus. The curve γ_2 is essential in S by Claim 3.7. If γ_5 is essential in S, then γ_2 and γ_5 must be parallel in S. By Claim 3.12, the next singularity must be of type 3, fusing γ_2 and γ_5 into a trivial curve γ_6 which must bound a disk in R_0. So we have a surface of type 2. If γ_5 is trivial in S, then it bounds a disk in S with no more singularities by Claim 3.5. It follows that the next singularity must be of type 5, changing γ_2 into two curves γ_6 and γ_7. These curves must be parallel in S, so that we may assume that in a level torus $T \times \{y_1\}$ there are 3 curves, γ_6, γ_7 and γ_5. The part of S contained in $R_0 \cup (T \times [0, y_1])$ consists of a disk and an annulus. If the disk and the annulus are non-nested in $R_0 \cup (T \times [0, y_1])$, then we can assume that S_0 consists of a trivial disk and an annulus and we have a surface of type 3. If the disk and annulus are nested then we have a surface of type 4, but in this case there may be one more singularity of type 6, so S_0 consists of a disk and one annulus, or of two disks.

Finally suppose that the third singularity is of type 7. Then the curve γ_3, say, is split into two curves γ_5 and γ_6, where γ_5 is concentric with γ_2 and γ_6 is parallel to γ_4. The curve γ_5 is nontrivial in S, by Claim 3.7. Also, the curves γ_2 and γ_5 are nonparallel in S by Claim 3.9. The curves γ_4 and γ_6 are also nontrivial in S by Claim 3.16. The curve γ_4 (or γ_6) cannot be parallel in S to γ_3 or γ_5, by Claim 3.10. Then γ_4 and γ_6 must be parallel in S, for otherwise $\text{genus}(S) > 2$. Note that the surface up to a level just below the third singularity, union with the annulus bounded by γ_4 and γ_6 is a twice punctured torus, so that for S to be a genus 2 surface, the curves γ_2 and γ_5 must be parallel in S, a contradiction.

Suppose now that the second singularity is of type 2. So, the curve γ_1 splits into two curves γ_3 and γ_4, which are trivial and concentric with γ_2 in the corresponding level torus. The third singularity can be of type 4 or 5. If it is of type 4, then the curves γ_3 and γ_4 are fused into a curve γ_5 which is trivial in the corresponding level torus. The surface up to that level is a twice punctured torus. The curve γ_2 is essential in S. If γ_5 is essential in S, then γ_2 and γ_5 must be parallel in S. So the next singularity must be of type 3, fusing γ_2 and γ_5 into a trivial curve γ_6 which must bound a disk in R_0. So we have a surface of type 2. If γ_5 is trivial in S, then the next singularity must be of type 5, changing γ_2 into two curves γ_6 and γ_7. These curves must be parallel in S. So in a level torus $T \times \{y_1\}$ there are 3 curves, γ_6, γ_7 and γ_5, and the part of S below that level consists of a trivial disk and an annulus. If the annulus and the disk are non-nested we have a surface of type 3, and if they are nested, we have a surface of type 4, but in this last case we may have one more singularity of type 6. Finally, suppose the third singularity is of type 5. In this case the curve γ_4 transforms into two curves γ_5 and γ_6, which are essential in the corresponding level torus. This situation is identical to the situation in the preceding paragraph, so it is not possible.

Summarizing, the following sequences of singularity types are possible:

(a) 1, 3, 5, or 1, 5, 8, S_0 consists of an annulus, and S is a surface of type 1.

(b) 1, 3, 1, 3, S_0 consists of a trivial disk, and S is a surface of type 1.

(c) 1, 5, 5, S_0 consists of two non-nested annuli, and S is a surface of type 3.

(d) 1, 5, 5, or 1, 5, 5, 6, 6, S_0 consists of two nested annuli, or of two nested disks, and S is a surface of type 4.

(e) 1, 5, 6, 3, S_0 consists of a trivial disk, and S is a surface of type 2.

(f) 1, 5, 6, 5, S_0 consists of a trivial disk and an annulus, which are non-nested, and S is a surface of type 3.

(g) 1, 5, 6, 5, or 1, 5, 6, 5, 6, S_0 consists of a trivial disk and an annulus, which are nested, or of two nested trivial disks, and S is a surface of type 4.

(h) 1, 2, 4, 3, S_0 consists of a trivial disk, and S is a surface of type 2.

(i) 1, 2, 4, 5, S_0 consists of a trivial disk and an annulus, which are non-nested and S is a surface of type 3.

(j) 1, 2, 4, 5, or 1, 2, 4, 5, 6, S_0 consists of a trivial disk and an annulus, which are nested, or of two nested trivial disks, and S is a surface of type 4. □

Claim 3.18 *Suppose* S_1 *consists of two trivial disks. Then* S *is a surface of type 4, 5, 6 or 7.*

Proof Suppose that the disk components of S_1 are non-nested, so each curve bounds a disk in T_1 containing one arc of k_1. The first singularity must be of type 2, 3 or 5. Note that in any case such singularity can be pushed into R_1, showing that S is compressible, or well, that a disk is changed by an annulus, or that the two disks are changed by one disk. In any case the complexity of S is reduced. So suppose the disks are nested.

Suppose first that each of these disks bounds a disk in T_1 containing the two arcs of k_1. So just below T_1, we have two concentric curves bounding a disk which intersects k in 4 points. The first two singularities must be of type 1, so just after these singularities we have two pairs of parallel curves, say $\{\gamma_1,\gamma_2\}$ and $\{\gamma_3,\gamma_4\}$, each bounding a disk intersecting k in two points. So up to this level the surface is just two nested annuli. The third singularity must be of type 2, 3 or 5. Suppose first it is of type 5. So, say, the curve γ_1 is split into two curves γ_5 and γ_6, which are essential in the corresponding level torus. Note that the curves γ_2, γ_5 and γ_6 are nontrivial in S (by Claim 3.7 and Claim 3.16). By Claim 3.9 the curves γ_3 and γ_4 are nonparallel in S, and note also that the curves γ_5 and γ_6 cannot be parallel in S to one of the curves γ_3 or γ_4, by Claim 3.10. So γ_5 and γ_6 must be parallel in S for otherwise the genus of S is > 2. So γ_5 and γ_6 cobound an annulus in S. Then there should be a singularity of type 5 joining the curve γ_2 with itself. This shows that the genus of S is > 2.

Suppose now that the third singularity is of type 2, so, say, the curve γ_1 is split into two curves γ_5 and γ_6, which are trivial in the corresponding level torus. The next singularity must be of type 5, changing the curve γ_6, say, into curves γ_7 and γ_8, which are essential in a level torus. Note that γ_3, γ_4 and γ_5 are nonparallel in S, by Claim 3.9, and none of them is parallel in S to γ_7, by Claim 3.10. This implies that the genus of S is > 2.

So the third singularity must be of type 3. In this singularity the curves γ_1 and γ_3 are fused into a curve γ_5; this is a trivial curve bounding in its interior two nonconcentric curves. Note that γ_5 must be essential in S, for otherwise S will be disconnected. If the next singularity is of type 3 again, then γ_2 and γ_4 are fused into a curve γ_6, but then γ_5 and γ_6 must be parallel in S, contradicting Claim 3.9.

If the fourth singularity is of type 4, then it can be interchanged with the third singularity, showing then that the surface is compressible. If the fourth singularity is of type 5, the curve γ_5 is split into two curves which will be essential in S, but then there will be more than 3 essential nonparallel curves in S, which is not possible.

So the fourth singularity must be of type 2, splitting γ_2, say, into γ_6 and γ_7. The next singularity must be of type 4, fusing γ_6 and γ_7 into a curve γ_8, for otherwise S will be compressible. Note that the curves γ_3, γ_4, γ_5, γ_6 and γ_7 are essential in S, and that γ_3 and γ_4 are nonparallel in S. If γ_8 is nontrivial in S, then S is disconnected or have genus > 2. Then γ_8 is trivial and γ_6, γ_7 are parallel in S. We must also have that γ_4 and γ_5 are parallel in S. So we must have one more singularity of type 4, fusing γ_4 and γ_5 into a trivial curve γ_9. Then we can assume that γ_8 and γ_9 lie on T_0, and bound nested disks of S_0 enclosing just one of the arcs of k_0. This shows that S is a surface of type 6.

Summarizing, we have the following case:

(a) The sequence of singularity types is 1, 1, 3, 2, 4, 4, S_0 consists of nested two trivial disks, and S is a surface of type 6.

Suppose now that S_1 consists of two nested disks which enclose just one of the two arcs of k_1. A similar argument shows that these are the possible cases for the sequences of types of singularities:

(b) 2, 2, 4, 1, 3, 3, S_0 consists of two nested disks, and S is a surface of type 6, but in an inverted position, ie changing the roles of R_0 and R_1.

(c) 5, 5, 6, 7, 3, or 5, 5, 7, 6, 3, S_0 consists of a disk and an annulus, which are nested, and S is a surface of type 7, but in an inverted position.

(d) 5, 6, 5, 6, 3, or 5, 5, 6, 6, 3, or 5, 6, 2, 4, 3, S_0 consists of a trivial disk and S is a surface of type 4, which looks inverted.

(e) 5, 5, 6, 7, 8, 8, or 5, 5, 7, 6, 8, 8, or, 5, 7, 5, 6, 8, 8, or 2, 5, 5, 6, 8, 8, or 5, 7, 2, 4, 8, 8, or 2, 5, 2, 4, 8, 8, and S_0 consists of two meridian disks and S is a surface of type 5, which looks inverted. □

Claim 3.19 *Suppose* S_1 *consists of just one annulus. Then* S *is a surface of type 1 or 8.*

Proof The annulus in S_1 then determines an annulus in T_1 which contains both arcs of k_1. An argument as above shows that there are several possibilities for the sequences of types of singularities, these are:

(a) 6, 1, 3, or 7, 6, 3, S_0 consists of a trivial disk, and S is a surface of type 1.

(b) 6, 5, S_0 consists of an annulus, and S is a surface of type 1.

(c) 7, 7, 3, S_0 consists of a trivial disk and an annulus, which are nested, and S is a surface of type 8, which looks inverted. □

Claim 3.20 *Suppose* S_1 *consists of two non-nested annuli (and nonisotopic to nested annuli). Then* S *is a surface of type 3, 8 or 9.*

Proof Each of the annuli on S_1 determines an annulus in T_1 which contains an arc of k_1. The possible sequences of types of singularities are:

(a) 6, 6, 3, S_0 consists of a trivial disk, and S is a surface of type 3.

(b) 7, 7, 3, S_0 consists of a trivial disk and two annuli, which are nested, and S is a surface of type 8, which looks inverted.

(c) 6, 7, 3, or 7, 6, 3, S_0 consists of a disk and an annulus, which are nested, and S is a surface of type 9, which looks inverted. □

Claim 3.21 *Suppose* S_1 *consists of two nested annuli. Then* S *is a surface of type 4, 5, 7 or 8.*

Proof One possibility is that there is an annulus in T_1 containing both arcs of k_1, but by tracking the singularities as above, we can see that this case is not possible. So the arcs of k_1 are on different components of $T_1 - S_1$. The possible cases for the sequence of singularity types are:

(a) 6, 6, 3, S_0 consists of a trivial disk, and S is a surface of type 4.

(b) 6, 7, 8, 8, or 7, 6, 8, 8, S_0 consists of two meridian disks, and S is a surface of type 5, but it looks inverted.

(c) 7, 6, 3, or 6, 7, 3, S_0 consists of a trivial disk and an annulus, which are nested, and S is a surface of type 7, but it looks inverted.

(d) 7, 7, 3, S_0 consists of a trivial disk and two annuli, which are nested, and S is a surface of type 8, which looks inverted. □

Claim 3.22 *Suppose S_1 consists of an annulus and a disk, which are non-nested (and cannot be isotoped to be nested). Then S is a surface of type 3.*

Proof Here the annulus in S_1 determines an annulus in T_1 which contains an arc of k_1, and the trivial disk determines a disk in T_1 which contains the other arc of k_1. The possible sequences of types of singularities are:

(a) 6, 5, 6, 3, and S_0 consists of a trivial disk.

(b) 6, 2, 4, 3, and S_0 consists of a trivial disk. □

Claim 3.23 *Suppose S_1 consists of one annulus and one disk, which are nested. Then S is a surface of type 4, 5, 7, 8 or 9.*

Proof If the disk in S_1 determines a disk in T_1 containing just one of the arcs of k_1, then the possible sequences of types of singularities are:

(a) 6, 2, 4, 3, or 6, 5, 6, 3, S_0 consists of a trivial disk and S is a surface of type 4, but it looks inverted.

(b) 7, 2, 4, 8, 8, S_0 consists of two meridian disks, and S is a surface of type 5, which looks inverted.

If the disk in S_1 determines a disk in T_1 containing both arcs of k_1 then the possible sequences of types of singularities are:

(c) 1, 8, 5, or 1, 5, 8, S_0 consists of two nested annuli, and S is a surface of type 7.

(d) 1, 8, 5, 6, 6, or 1, 5, 8, 6, 6, S_0 consists of two nested annuli, and S is a surface of type 7.

(e) 1, 8, 8, S_0 consists of an annulus, and S is a surface of type 8.

(f) 1, 8, 5 , or 1, 5, 8, S_0 consists of two non-nested annuli, and S is a surface of type 9. □

Claim 3.24 *If $|S_1| \geq 3$, then S is a surface of type 8 or 9.*

Proof An argument as in Claim 3.18, shows that it is not possible that S_1 contain 3 or more trivial disks, or two disks and some annuli. If S_0 contains a trivial disk and two or more annuli, an argument as in previous claims shows that the only possibility is that the trivial disks bounds a disk in T_1 containing both arcs of k_1, and that the disk and the annuli are nested. There are two possibilities:

(a) S_1 consists of a disk and n annuli, which are nested, the sequence of singularity types is 1, 8, 8, S_0 consists of n nested annuli, and S is a surface of type 8.

(b) S_1 consists of a disk and n annuli, which are nested, the sequence of singularity types is 1, 8, 5 (or 1, 5, 8), S_0 consists of $n+1$ annuli, two of them are innermost, the others are nested around the innermost ones, and S is a surface of type 9.

Suppose now that S_1 consists only of annuli. By similar arguments as in previous claims, it can be shown that there are two possibilities:

(c) S_1 consists of n annuli which are nested. The innermost one determines an annulus in T_1 which contains one arc of k_1. The other arc of k_1 is between the second and third annulus. The sequence of singularity types is 7, 7, 3, S_0 consists of a trivial disk and n nested annuli, and S is a surface of type 8, which looks inverted.

(d) S_1 consists of n annuli, two of them are innermost, each determining an annulus in T_0 containing an arc of k_1, the other annuli are nested around both of the innermost annuli. The sequence of singularity types is 6, 7, 3 (or 7, 6, 3), S_0 consists of a trivial disk and $n-1$ annuli, which are nested, and S is a surface of type 9, which looks inverted. □

This completes the proof of Theorem 3.1.

The same arguments as in the previous claims can be applied when S is a surface of genus 1, and in fact the arguments are simpler. Again, the surface S can be divided into pieces S_1, S_0, and $\widetilde{S}$, and it can be shown that S_1 consists of just one disk or one annulus. So we have the following result.

Theorem 3.25 *Let K be a $(1,2)$–knot and S a genus 1 meridionally incompressible surface in the complement of K. Then K and S can be isotoped so that S look as one of the surfaces constructed in Section 2.10.*

4 Knots which are not $(1,2)$–knots

We recall the construction of [4, Section 6], which produces tunnel number one knots whose complement contain a genus 2 closed meridionally incompressible surface which does not bound a handlebody in S^3.

Let K be a satellite tunnel number one knot in S^3, and let S be the closed incompressible surface of genus 1 contained in the complement of K; then S divides S^3 into two parts, denoted by M_1 and M_2, where, say, K lies in M_2. In fact, it follows from [17] (or [3]) that M_1 is the exterior of a torus knot, $M_2 - \operatorname{int} N(K)$ is homeomorphic to the exterior of a 2–bridge link and that a fiber of M_1 is glued to a meridian of $M_2 - \operatorname{int} N(K)$. Let $\beta = \beta_1 \cup \beta_2$ be an unknotting tunnel for K, where β_1 is a simple closed curve, and β_2 is an arc joining K and β_1. The tunnel β can be chosen so that β_1 is disjoint from S, and that β_2 intersects S transversely in one point, so β_1 lies in M_1. The surface S then divides β_2 in two arcs, β_2' and β_2'', where β_2' joins K and S, and β_2'' joins S and β_1.

Let γ be a simple closed curve contained in $\partial N(K)$. Assume that the arc β_2' connects S with a point in $\partial N(K)$, so that such a point lies on γ. Consider the manifold $M = M_2 - \operatorname{int} N(K)$. This is a compact, irreducible 3–manifold, whose boundary consists of two incompressible surfaces, S and $\partial N(K)$. The curve γ lies on $\partial N(K)$. We assume that γ goes at least twice longitudinally around $N(K)$, and that γ is a nonreducing curve for M, ie $M(\gamma)$, the manifold obtained by doing Dehn filling on M with slope γ, has incompressible boundary.

Note that if K is not a cable knot, then any such curve γ is a nonreducing curve, for it is at distance ≥ 2 from a meridian of K [24]. If K is a cable knot, then there is a properly embedded annulus in M, with one of its boundary components lying in S and the other one lying in $\partial N(K)$, which we denote by γ_1. It follows from [1, Theorem 2.4.3] that $M(\gamma)$ is ∂–reducible if and only if $\Delta(\gamma, \gamma_1) \leq 1$.

Let $H = M_1 \cup N(\gamma) \cup N(\beta_2')$. We can assume that H is made of the union of the solid torus $N(\gamma)$ and the manifold M_1, which are joined by the 1–handle $N(\beta_2')$. Let $W = S^3 - \operatorname{int} H$. It follows from [4, Theorem 6.3] that $\Sigma = \partial W = \partial H$ is incompressible in W.

Let K^* be a knot such that $K^* \subset H$, $K^* = K_1 \cup K_2$, where K_1 is an arc contained in $\partial N(K) \cap N(\gamma)$, and K_2 is an arc contained in $N(\beta_2') \cup M_1$, which is obtained by sliding β_1 over β_2, ie K_2 is an unknotting tunnel for K. In other words, K^* is an iterate of K and β, as defined in [4, Section 6], and in particular K^* is a tunnel number one knot. Assume further that the wrapping number of K_1 in $N(K)$ is ≥ 2, that is, if we connect the endpoint of K_1 with an arc lying in $N(\beta_2') \cap N(\gamma)$, we should get a knot whose wrapping number in $N(K)$ is ≥ 2. It follows from [4, Theorem 6.4] that Σ is meridionally incompressible in $S^3 - K^*$.

Note that the wrapping number of γ in $N(K)$ is ≥ 2, and that the wrapping number of K_1 in $N(\gamma)$ is also ≥ 2, so the wrapping number of K_1 in $N(K)$ is ≥ 4. As K is a $(1,1)$–knot, then it seems that K^* is a $(1,n)$–knot with $n \geq 4$.

Here, we show the following.

Theorem 4.1 *A knot K^* constructed as above is not a $(1,2)$–knot, ie $b_1(K^*) \geq 3$.*

Proof Note that the surface Σ does not bound a handlebody in S^3. In fact, to one side it bounds the manifold W, which has incompressible boundary, and to the other side it bounds the manifold H, which is the disk sum of a solid torus (ie $N(\gamma)$) and the exterior of a torus knot (ie M_1). This shows immediately that K is not a $(1,1)$–knot, for any meridionally incompressible surface in the complement of a $(1,1)$–knot bounds a handlebody in S^3 [6]. It follows also that Σ cannot be a surface of type 1, 2, 3, 4, 5, 8 or 9 for any such surface bounds a handlebody in S^3. So, if we show that Σ cannot be a surface of type 6 or 7, then we will show that K^* cannot be a $(1,2)$–knot.

By construction $K^* \subset H$, and there is a disk D_W properly embedded in H, which intersects K^* in two points, separates H and ∂D_W is essential in ∂H. The disk D_W is just the cocore of $N(\beta'_2)$, and we can assume that D_W divides K^* into the arcs K_1 and K_2 defined above. We claim that if D is another disk properly embedded in H, intersecting K^* in two points, separating H, and with ∂D essential in ∂H, then D must be isotopic to D_W. To see that, look at the intersections between D_W and D, and by doing an innermost disk/outermost arc argument, remove all curves and arcs of intersection. To do that we use the following facts: (a) Σ is meridionally incompressible in $H - K^*$; (b) the arc obtained by sliding β_1 over β''_2 is not isotopic into the surface S, for it is an unknotting tunnel for M_1; (c) the knot in $N(\gamma)$ (M_1) obtained from the arc K_1 (K_2), by joining the endpoints of K_1 (K_2) lying in D_W and then pushing it into $N(\gamma)$ (M_1), does not have local knots in $N(\gamma)$ (M_1). If D and D_W are disjoint, then it is not difficult to see that they are isotopic in H.

Let S_6 be a surface of type 6, and let K_6 be a $(1,2)$–knot in the complement of S_6, so that the surface S_6 is meridionally incompressible in $S^3 - K_6$. It follows from Section 2.6 that S_6 bounds a manifold M_6 which has incompressible boundary (this is a single manifold). Let $H_6 = S^3 - \operatorname{int} M_6$. It follows from Section 2.6 that H_6 is the disk sum of the exterior, $S^3 - \operatorname{int} N(\gamma_6)$, of a certain knot γ_6, and the solid torus $N(\alpha_1)$. We assume that γ_6 is a nontrivial knot, for otherwise H_6 is a handlebody. Here there is also a disk D_6, separating H_6, with ∂D_6 essential in ∂H_6, and which intersects K_6 in two points. This is just a cocore of $N(\alpha_2)$. Note however that the disk D_6 may not be unique, it will depend on the way the corresponding arc of K_6 is embedded in $S^3 - \operatorname{int} N(\gamma_6)$.

Let S_7 be a surface of type 7, and let K_7 be a $(1,2)$–knot in the complement of S_7, so that the surface S_7 is meridionally incompressible in $S^3 - K_7$. It follows from Section 2.7 that S_7 bounds a manifold M_7 which has incompressible boundary (where this is a family of manifolds constructed in a similar manner). Let $H_7 = S^3 - \operatorname{int} M_7$. It follows from Section 2.7 that H_7 is the disk sum of the solid torus N_1 and the exterior of a torus knot ($S^3 - \operatorname{int} N_2$), which we are assuming is nontrivial. Here there is also a disk D_7, separating H_7, with ∂D_7 essential in ∂H_7, and which intersects

K_7 in two points. This is just a cocore of $N(\alpha)$. Note however that the disk D_7 may not be unique, it depends on the way the corresponding arc of K_7 is embedded in $S^3 - \operatorname{int} N_2$.

If K^* is a $(1,2)$–knot, there must be a homeomorphism $h : (S^3, K^*) \to (S^3, K_i)$, $i = 6$ or 7, where K_i is a $(1,2)$–knot having a surface of type 6 or 7. The image of the surface Σ must be a surface of type 6 or 7, and the image of W must be a manifold M_6 or M_7. But in the complement of W there is a unique disk D_W with certain properties, so in M_6 or M_7 there must be also such a disk, ie the disk D_6 or D_7 must be the unique disk with that properties. Then the homeomorphism must take the disk D_W onto a disk parallel to D_6 or D_7. This implies that $W \cup N(D_W)$ must be homeomorphic to $M_7 \cup N(D_6)$ or to $M_7 \cup N(D_7)$.

Note that $W \cup N(D_W)$ is just the manifold $M_2 - \operatorname{int} N(\gamma)$, which is an irreducible manifold with incompressible boundary, and not homeomorphic to $T^2 \times I$, T^2 a torus. But $M_6 \cup N(D_6)$ is homeomorphic to $N(\gamma_6) - \operatorname{int} N(\alpha_1)$, which is a reducible manifold with compressible boundary. And note that $M_7 \cup N(D_7)$ is the manifold $(N_2 - \operatorname{int}(N_1 \cup N(\alpha)) \cup N(\alpha)$, which is just $N_2 - \operatorname{int} N_1$, and it is homeomorphic to $T^2 \times I$. So we got different manifolds in each case.

We conclude that Σ cannot be a surface of type 6 or 7. □

We give now an explicit example of a knot K^*. Suppose that K is the $(-13,2)$–cable of the left hand trefoil knot. This is a tunnel number one knot. The cabling annulus of K has slope $(-26,1)$ on K. So let γ be a curve on $\partial N(K)$ of slope $(-55,2)$. Now take an arc K_1 on $\partial N(K)$ which goes around $N(\gamma)$ twice, and connect it with an unknotting arc K_2 for K. Such a $K^* = K_1 \cup K_2$ is shown in Figure 13. As said above, there is a surface Σ in the complement of K^* which is meridionally incompressible; this surface is implicit in Figure 13. The surface Σ bounds a manifold W, in this example the manifold W is homeomorphic to the manifold shown in Figure 14, ie the exterior of $\gamma \cup \beta_2'$ in the solid torus. To see a real picture, just embed W appropriately in the neighborhood of a trefoil knot, or of any torus knot.

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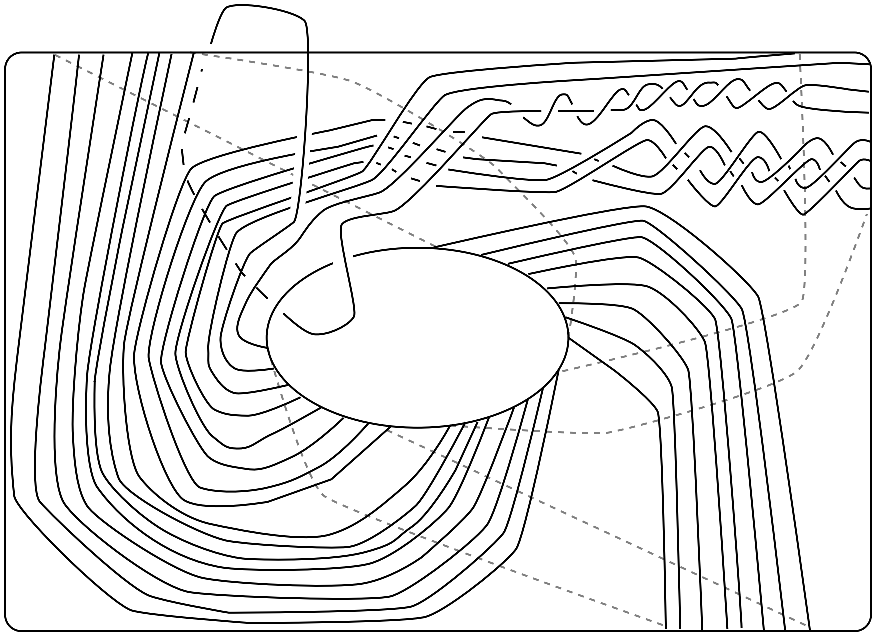

Figure 13

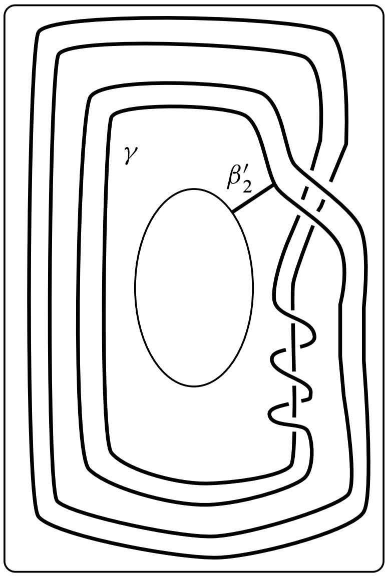

Figure 14

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