Geometry & Topology Monographs 12 (2007) 121–156

Kleinian orbifolds uniformized by $\mathcal{RP}$ groups with an elliptic and a hyperbolic generator

ELENA KLIMENKO
NATALIA KOPTEVA

We consider non-elementary Kleinian groups Γ, without invariant plane, generated by an elliptic and a hyperbolic element, with their axes lying in one plane. We find presentations and a complete list of orbifolds uniformized by such Γ.

30F40; 22E40, 57M12, 57M50

This work is a part of a program to describe all 2–generator Kleinian groups with real parameters. We study $\mathcal{RP}$ groups, that is, marked 2–generator subgroups $\Gamma = \langle f, g\rangle$ of $\mathrm{PSL}(2,\mathbb{C})$ for which the generating pair (f,g) has real parameters $\beta = \beta(f) = \mathrm{tr}^2 f - 4$, $\beta' = \beta(g) = \mathrm{tr}^2 g - 4$ and $\gamma = \gamma(f,g) = \mathrm{tr}[f,g] - 2$ (see Section 1 for exact definitions). Since discreteness questions were answered for elementary groups and for groups with invariant hyperbolic plane (in particular, all Fuchsian groups were described), we concentrate only on the non-elementary $\mathcal{RP}$ groups without invariant plane, which we call *truly spatial* $\mathcal{RP}$ groups.

This paper deals with the most complicated case of $\mathcal{RP}$ groups, the case with one generator elliptic and the other one hyperbolic. It was shown by Klimenko and Kopteva [12] that 'truly spatial' for this class means that the elliptic generator is not a half-turn and the axes of the generators either (1) are disjoint (non-parallel) lines lying in a hyperbolic plane or (2) intersect non-orthogonally at a point of $\mathbb{H}^3$. In terms of parameters, we have here $\beta \in (-4,0)$, $\beta' \in (0,\infty)$ and γ for (1) and (2) belongs to the intervals $(-\infty,0)$ and $(0,-\beta\beta'/4)$, respectively [12, Theorem 1 and Table 1]. In the previous papers by Klimenko [10] and Klimenko and Kopteva [12; 13] necessary and sufficient conditions for discreteness of all such groups were found constructively. Here we use the construction (we reproduce it in Section 2) to determine fundamental polyhedra, presentations and orbifolds for all truly spatial discrete $\mathcal{RP}$ groups with an elliptic and a hyperbolic generators (Section 3).

The other cases of f and g with real traces that generate a truly spatial $\mathcal{RP}$ group and the question when the group is discrete were investigated earlier by Klimenko [9; 11] and Klimenko and Kopteva [14]. The final results including the results of the present paper are collected in Klimenko and Kopteva [15] (mostly without proofs),

Published: 3 December 2007 DOI: 10.2140/gtm.2007.12.121

where parameters, presentations and orbifolds for *all truly spatial* discrete $\mathcal{RP}$ groups *with real traces of the generators* are given.

Acknowledgements

N Kopteva would like to thank Gettysburg College for sincere hospitality during her stay in October 2004 when an essential part of the work was done. Both authors are grateful to Prof F Grunewald and an anonymous referee for various comments on an earlier version of this paper. E Klimenko was partially supported by Gettysburg College and N Kopteva was partially supported by FP6 Marie Curie IIF Fellowship. We also thank the DFG Forschergruppe "Classification of surfaces", Heinrich-Heine-Universität Düsseldorf and the Max-Planck-Institut in Bonn for financial support.

1 Preliminaries

Definitions and notation We identify $\mathrm{PSL}(2,\mathbb{C})$ with the full group of orientation preserving isometries of hyperbolic 3–space $\mathbb{H}^3$.

Let $f,g\in\mathrm{PSL}(2,\mathbb{C})$. The complex numbers $\beta=\beta(f)=\mathrm{tr}^2 f-4$, $\beta'=\beta(g)=\mathrm{tr}^2 g-4$ and $\gamma=\gamma(f,g)=\mathrm{tr}[f,g]-2$, where $[f,g]$ denotes the commutator $fgf^{-1}g^{-1}$, are called the parameters for the pair (f,g) and for the group $\Gamma=\langle f,g\rangle$.

The same 2–generator subgroup Γ of $\mathrm{PSL}(2,\mathbb{C})$ can have different triples of parameters (β,β',γ) depending on the choice of the generating pair (f,g). On the other hand, the triple of parameters (β,β',γ) determines Γ up to conjugacy whenever $\gamma\neq 0$. More precisely, if (f_1,g_1) and (f_2,g_2) both have the same triple of parameters (β,β',γ) with $\gamma\neq 0$, then there is $h\in\mathrm{PSL}(2,\mathbb{C})$ so that $f_2=hf_1h^{-1}$ and either $g_2=hg_1h^{-1}$ or $g_2=hg_1^{-1}h^{-1}$, see Gehring and Martin [5].

Notice that if $\gamma=0$ then Γ is not determined uniquely by the numbers β and β'. There are examples of a discrete group Γ_1 and a non-discrete Γ_2 with $\gamma=0$ and the same pair (β,β'). However, it is known that in this case f and g have a common fixed point in $\partial\mathbb{H}^3$, that is, Γ is elementary. Since we are concerned only with truly spatial groups, we may assume that $\gamma\neq 0$ throughout this paper.

A triple $(\Gamma;f,g)$, where $\Gamma=\langle f,g\rangle$, is called an $\mathcal{RP}$ *group* if the pair (f,g) has real parameters (β,β',γ). Note that the requirement of discreteness is not included in the definition of an $\mathcal{RP}$ group.

We recall that a non-trivial element $f\in\mathrm{PSL}(2,\mathbb{C})$ with real $\beta=\beta(f)$ is *elliptic*, *parabolic*, *hyperbolic* or π*–loxodromic* according to whether $\beta\in[-4,0)$, $\beta=0$,

$\beta \in (0, +\infty)$ or $\beta \in (-\infty, -4)$. If $\beta \notin [-4, \infty)$, ie if $\operatorname{tr} f$ is not real, then f is called *strictly loxodromic*.

An elliptic element f of order n is said to be *primitive* if it is a rotation through $2\pi/n$ (with $\beta = -4\sin^2(\pi/n)$); otherwise, it is called *non-primitive* (and then $\beta = -4\sin^2(\pi q/n)$, where q and n are coprime and $1 < q < n/2$).

A hyperbolic plane divides $\mathbb{H}^3$ into two components; we shall call the closure in $\mathbb{H}^3$ of either of them a *half-space* in $\mathbb{H}^3$. A connected subset P of $\mathbb{H}^3$ with non-empty interior is said to be a *(convex) polyhedron* if it is the intersection of a family $\mathcal{H}$ of half-spaces with the property that each point of P has a neighborhood meeting at most a finite number of boundaries of elements of $\mathcal{H}$. A closed polyhedron with finite number of faces bounded by planes $\alpha_1, \dots, \alpha_k$ is denoted by $\mathcal{P}(\alpha_1, \dots, \alpha_k)$.

We define a *tetrahedron* T to be a polyhedron which in the projective ball model is the intersection of the hyperbolic space $\mathbb{H}^3$ with a Euclidean tetrahedron T_E (possibly with vertices on the sphere $\partial\mathbb{H}^3$ at infinity or beyond it) so that the intersection of each edge of T_E with $\mathbb{H}^3$ is non-empty.

A tetrahedron T (possibly of infinite volume) in $\mathbb{H}^3$ is uniquely determined up to isometry by its dihedral angles. Let T have dihedral angles π/p_1, π/p_2, π/p_3 at the edges of some face and let π/q_1, π/q_2, π/q_3 be dihedral angles of T that are opposite to π/p_1, π/p_2, π/p_3, respectively. Then a standard notation for such a T is $T[p_1, p_2, p_3; q_1, q_2, q_3]$ and a standard notation for the group generated by reflections in the faces of T is G_T.

We denote the reflection in a plane κ by R_κ. The axis of an element $h \in \mathrm{PSL}(2, \mathbb{C})$ with two distinct fixed points in $\partial\mathbb{H}^3$ is denoted by the same h if this does not lead to any confusion.

We use symbols ∞ and $\overline{\infty}$ with the following convention. We assume that $\overline{\infty} > \infty > x$ and $x/\infty = x/\overline{\infty} = 0$ for every real x; $\infty/x = \infty$ and $\overline{\infty}/x = \overline{\infty}$ for every positive real x; in particular, $(\infty, k) = (\overline{\infty}, k) = k$ for every positive integer k. We use $(\cdot, \cdot)$ for $\gcd(\cdot, \cdot)$.

If we denote the dihedral angle between two planes by π/p $(1 < p \le \overline{\infty})$, then the planes intersect when p is finite, they are parallel (that is, their closures in $\overline{\mathbb{H}^3} = \partial\mathbb{H}^3 \cup \mathbb{H}^3$ have just one common point in $\partial\mathbb{H}^3$) when $p = \infty$ and disjoint (that is, the boundaries of the planes do not intersect in $\partial\mathbb{H}^3$) when $p = \overline{\infty}$.

Convention on pictures Since the methods we use here are essentially geometrical, the paper contains many pictures of hyperbolic polyhedra. In those pictures, shaded polygons are not faces of polyhedra, but are drawn to underline the combinatorial structure of the corresponding polyhedron. They are just intersections of the polyhedron with appropriate planes.

If a line on a picture is an edge of a polyhedron, then it is labelled by the dihedral angle at this edge. We often omit labels $\pi/2$. If a line is not an edge of a polyhedron and is labelled by an integer k, then this means that the line is the axis of an elliptic element of order k that belongs to Γ^* (see below). Figure 11 is an exception from this convention. We shall explain labels in Figure 11 in Remark 3.4.

2 Fundamental polyhedra and parameters

From here on f is a primitive elliptic element and g is hyperbolic. The main tool in the study of discreteness of $\Gamma = \langle f, g\rangle$ in Klimenko [10] and Klimenko–Kopteva [12; 13] was a construction of a 'convenient' finite index extension Γ^* of Γ together with a fundamental polyhedron for each discrete Γ^*. In this section, we reproduce the construction of Γ^* and describe the fundamental polyhedra for all discrete Γ^*. This is a preliminary part for Section 3, where we shall work with the groups Γ^* themselves to list the corresponding orbifolds.

2.1 Geometric description of discrete groups for the case of disjoint axes

Theorem 2.1 below gives necessary and sufficient conditions for discreteness of Γ for the case of *disjoint axes* of the generators f and g; a complete proof can be found in Klimenko [10]. We also repeat the geometric construction from [10] and recall fundamental polyhedra for the series of discrete groups Γ^* corresponding to Items (2)(i)–(2)(iii) of Theorem 2.1.

Theorem 2.1 ([10]) *Let $f \in \mathrm{PSL}(2,\mathbb{C})$ be a primitive elliptic element of order $n \geq 3$, $g \in \mathrm{PSL}(2,\mathbb{C})$ be a hyperbolic element and let their axes be disjoint lines lying in a hyperbolic plane.*

(1) *There exists $h \in \mathrm{PSL}(2,\mathbb{C})$ such that $h^2 = fgf^{-1}g^{-1}$ and $(hg)^2 = 1$.*

(2) *$\Gamma = \langle f, g\rangle$ is discrete if and only if one of the following holds:*
   - (i) *h is a hyperbolic, parabolic or primitive elliptic element of order $p \geq 3$;*
   - (ii) *$n \geq 5$ is odd, $h = x^2$, where x is a primitive elliptic element of order n, and $y = hgfx^{-1}f$ is a hyperbolic, parabolic or primitive elliptic element of order $q \geq 4$ or*
   - (iii) *$n = 3$, $h = x^2$, where x is a primitive elliptic element of order 5, and $z = hgf(x^{-1}f)^3$ is a hyperbolic, parabolic or primitive elliptic element of order $r \geq 3$.*

Let f and g be as in Theorem 2.1, and let ω be the plane in which the (disjoint) axes of f and g lie.

Denote by ε the plane that passes through the common perpendicular to the axes of f and g orthogonally to ω. Let α and τ be the planes such that $f = R_\alpha R_\omega$ and $g = R_\tau R_\varepsilon$, and let $\mathcal{P} = \mathcal{P}(\omega, \varepsilon, \alpha, \tau)$. The planes ω and α make a dihedral angle of π/n; the planes ε and τ are disjoint so that the axis of g is their common perpendicular. Moreover, α is orthogonal to ε and τ is orthogonal to ω. The planes α and τ either intersect non-orthogonally or are parallel or disjoint. We denote the dihedral angle of $\mathcal{P}$ between these planes by π/p, $p > 2$, where, by convention, $p = \infty$ if α and τ are parallel and $p = \overline{\infty}$ if they are disjoint.

We consider two finite index extensions of $\Gamma = \langle f, g\rangle$: $\widetilde{\Gamma} = \langle f, g, e\rangle$, where $e = R_\varepsilon R_\omega$, and $\Gamma^* = \langle f, g, e, R_\omega\rangle$. $\widetilde{\Gamma}$ is the orientation preserving subgroup of index 2 in Γ^* and $\widetilde{\Gamma}$ contains Γ as a subgroup of index at most 2. In Section 3, we shall see when $\Gamma = \widetilde{\Gamma}$ and when $\Gamma \neq \widetilde{\Gamma}$.

It was shown in [10] that $h = R_\alpha R_\tau$ is the only element that satisfies both $h^2 = [f, g]$ and $(hg)^2 = 1$. There are three series of discrete groups Γ^* depending on how $\mathcal{P}$ is decomposed into fundamental polyhedra for Γ^*. The series correspond to the conditions (2)(i), (2)(ii) and (2)(iii) of Theorem 2.1.

1. h is a hyperbolic, parabolic or primitive elliptic element of order $p \geq 3$ (that is (2)(i) holds) if and only if the dihedral angle of $\mathcal{P}$ between α and τ is of the form π/p with $p = \overline{\infty}$, $p = \infty$, or $p \in \mathbb{Z}$, $p \geq 3$, respectively. This is the first series of the discrete groups. In this case the polyhedron $\mathcal{P}$ is a fundamental polyhedron for Γ^*. In Figure 1(a) $\mathcal{P}$ is drawn under assumption that $1/n + 1/p > 1/2$.

The other discrete groups appear only if h is the square of a primitive elliptic element $x = R_\kappa R_\tau$, where κ is the bisector of the dihedral angle of $\mathcal{P}$ made by α and τ. Fundamental polyhedra for Γ^* corresponding to these two series are obtained by decomposing $\mathcal{P}$ into smaller polyhedra as follows (see [10] for the proof).

2. Let Γ be determined by the condition (2)(ii). In this case, $n \geq 5$ is odd, the dihedral angle of $\mathcal{P}$ between α and τ is $2\pi/n$, and κ and ω make a dihedral angle of $\pi/3$. Hence, ξ_1, where $\xi_1 = R_\kappa(\omega)$, and ω also make a dihedral angle of $\pi/3$, and ξ_1 and α are orthogonal. The planes ξ_1 and ε either intersect at an angle of π/q, where $q \in \mathbb{Z}$, $q \geq 4$, or are parallel or disjoint ($q = 3$ is not included, because then ε and τ must intersect). One can show that if $y = R_\varepsilon R_{\xi_1}$, then $y = hgfx^{-1}f$. The polyhedron $\mathcal{P}(\omega, \varepsilon, \alpha, \xi_1)$ is a fundamental polyhedron for Γ^*. For $q = 4$ or 5 and $n = 5$, $\mathcal{P}(\omega, \varepsilon, \alpha, \xi_1)$ is a compact tetrahedron. It is denoted by $ABCD$ in Figure 1(b) and shown by bold lines.

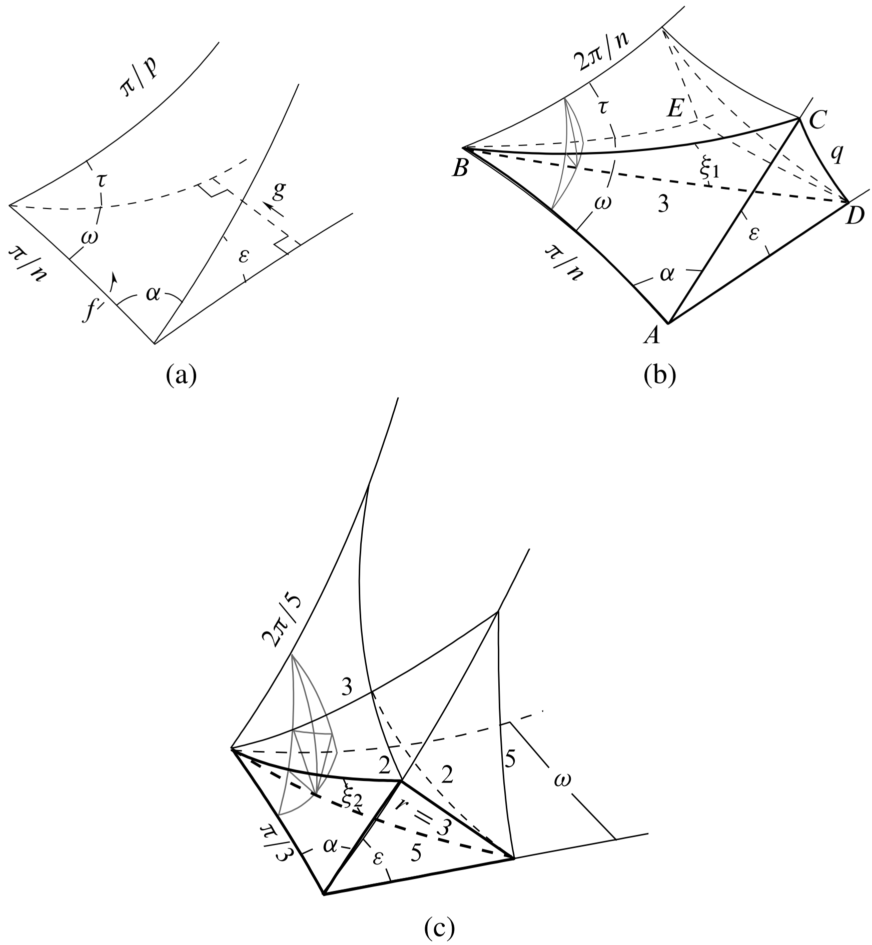


Figure 1: Fundamental polyhedra for Γ^* $(\gamma < 0)$

3. Let Γ be determined by the condition (2)(iii). In this case $n = 3$ and the dihedral angle of $\mathcal{P}$ between α and τ is $2\pi/5$. Denote $\xi_2 = R_\kappa(\omega)$. The planes κ and ω make a dihedral angle of $2\pi/5$ and, hence, ξ_2 and ω make an angle of $\pi/5$. It can be shown that ξ_2 and α are orthogonal. The planes ε and ξ_2 either intersect at an angle of π/r, where $r \in \mathbb{Z}$, $r \geq 3$, or are parallel or disjoint. In this case $z = R_\varepsilon R_{\xi_2} = hgf(x^{-1}f)^3$. The polyhedron $\mathcal{P}(\omega, \varepsilon, \alpha, \xi_2)$ is a fundamental polyhedron for Γ^* (see Figure 1(c), where $\mathcal{P}(\omega, \varepsilon, \alpha, \xi_2)$ is drawn for $r = 3$).

2.2 Parameters for discrete groups in the case of disjoint axes

Let

$$\mathcal{U} = \{u | u = i\pi/p,\ p \in \mathbb{Z},\ p \geq 2\} \cup [0, +\infty).$$

Define a function t: $\mathcal{U} \to \{2, 3, \dots\} \cup \{\infty, \overline{\infty}\}$ as follows:

$$t(u) = \begin{cases} p & \text{if } u = i\pi/p \\ \infty & \text{if } u = 0 \\ \overline{\infty} & \text{if } u > 0. \end{cases}$$

The purpose of introducing the function $t(u)$ is to shorten statements that involve parameters (β, β', γ). We use it in Theorems 2.2, 2.3, 3.1, 3.2 and 3.5.

Now we give a parameter version of Theorem 2.1 with a proof. Theorem 2.2 is new and did not appear before, however, we did use a similar technique in earlier papers.

Theorem 2.2 *Let $(\Gamma; f, g)$ be an $\mathcal{RP}$ group with $\beta = -4\sin^2(\pi/n)$, where $n \geq 3$ is an integer, $\beta' \in (0, +\infty)$ and $\gamma \in (-\infty, 0)$. Then Γ is discrete if and only if one of the following holds:*

(1) $\gamma = -4\cosh^2 u$, *where* $u \in \mathcal{U}$ *and* $t(u) \geq 3$;

(2) $n \geq 5$, $(n, 2) = 1$, $\gamma = -(\beta+2)^2$ *and* $\beta' = 4(\beta+4)\cosh^2 u - 4$, *where* $u \in \mathcal{U}$ *and* $t(u) \geq 4$ *or*

(3) $\beta = -3$, $\gamma = (\sqrt{5}-3)/2$ *and* $\beta' = 2(7+3\sqrt{5})\cosh^2 u - 4$, *where* $u \in \mathcal{U}$ *and* $t(u) \geq 3$.

Proof $\beta = -4\sin^2(\pi/n)$, where $n \in \mathbb{Z}$ and $n \geq 3$, if and only if f is a primitive elliptic element of order $n \geq 3$, and $\beta' \in (0, +\infty)$ if and only if g is hyperbolic. Since $n \geq 3$ and $\gamma \in (-\infty, 0)$, Γ is non-elementary and the axes of f and g are disjoint by Klimenko and Kopteva [12, Theorem 1]. So the hypotheses of Theorem 2.2 and Theorem 2.1 are equivalent.

Let us find explicit values of β' and γ for each of the discrete groups from part (2) of Theorem 2.1. The idea is to use the fundamental polyhedra described in Section 2.1. Since $\gamma = \operatorname{tr}[f, g] - 2$ and h is a square root of $[f, g]$, it is not difficult to get conditions on γ.

We start with (2)(i) in Theorem 2.1. The element $h = R_\alpha R_\tau$ is hyperbolic if and only if the planes α and τ (see Figure 1(a)) are disjoint in $\overline{\mathbb{H}^3}$. Therefore, $\operatorname{tr}[f, g] = \operatorname{tr} h^2 = -2\cosh(2d)$, where d is the hyperbolic distance between α and τ. Here $\operatorname{tr}[f, g]$ must be negative, because γ is negative for all values of β and β' that satisfy the hypotheses of the theorem.

The element h is parabolic if and only if $[f,g]$ is parabolic and if and only if $\operatorname{tr}[f,g] = -2$ which is equivalent to $\gamma = -4$ ($\operatorname{tr}[f,g] = 2$ would give $\gamma = 0$).

Thus, h is hyperbolic or parabolic if and only if

$$\gamma = \operatorname{tr}[f,g] - 2 = -2\cosh(2d) - 2 = -4\cosh^2 d, \quad d \geq 0. \tag{2.1}$$

Now suppose that h is an elliptic element with rotation angle ϕ, where $\phi/2 = \pi/p < \pi/2$ is the dihedral angle of $\mathcal{P}(\omega, \varepsilon, \alpha, \tau)$ made by α and τ. Then $[f,g] = h^2$ is also elliptic with rotation angle 2ϕ. Note that there is another square root $\overline{h}$ of the elliptic commutator $[f,g]$ which has rotation angle $\overline{\phi} = \pi + \phi$. One of the two angles ϕ and $\overline{\phi}$ is a solution to $\operatorname{tr}[f,g] = 2\cos\theta$, the other one to $\operatorname{tr}[f,g] = -2\cos\theta$. Since $\operatorname{tr}[f,g]$ depends on θ continuously and we know that $\operatorname{tr}[f,g]$ must approach -2 as $\phi \to 0$ (geometrically it means that $[f,g]$, and h, approaches a parabolic element as soon as the dihedral angle π/p above approaches 0), we conclude that ϕ is a solution to the second equation, that is, $\operatorname{tr}[f,g] = -2\cos\phi$.

On the other hand, if $\operatorname{tr}[f,g] \in (-2,2)$ is given, we can use the formula $\operatorname{tr}[f,g] = -2\cos\phi$, $0 < \phi < \pi$, to determine the rotation angle ϕ of the element h from Theorem 2.1.

Hence, h is a primitive elliptic element of order p ($p \geq 3$), that is, $\phi = 2\pi/p$, if and only if

$$\gamma = \operatorname{tr}[f,g] - 2 = -2\cos(2\pi/p) - 2 = -4\cos^2(\pi/p), \quad p \in \mathbb{Z}, \quad p \geq 3. \tag{2.2}$$

Now we can combine the formulas (2.1) and (2.2) for γ and write them as

$$\gamma = -4\cosh^2 u, \text{ where } u \in \mathcal{U} \text{ and } t(u) \geq 3.$$

It is clear that for the groups from Item (2)(i) of Theorem 2.1, we have no further restrictions on n and β'. So, (2)(i) of Theorem 2.1 is equivalent to part 1 of Theorem 2.2.

Now consider (2)(ii) of Theorem 2.1. Here $n \geq 5$ is odd and h is the square of a primitive elliptic element of order n (that is, $\phi = 4\pi/n$) if and only if $n \geq 5$, $(n,2) = 1$ and $\gamma = -4\cos^2(2\pi/n) = -(\beta+2)^2$.

So it remains to specify β' for (2)(ii). Now β' depends on the order of the element y defined in Theorem 2.1. Since $g = R_\tau R_\varepsilon$, we have $\beta' = \operatorname{tr}^2 g - 4 = 4\sinh^2 T$, where T is the distance between the planes ε and τ.

Let us show how to calculate T for the case $n=5$ and $4\le q<\infty$. Since the link of the vertex B in Figure 1(b) is a spherical triangle, we get

$$\cos\angle ABE=\frac{\cos(2\pi/n)}{\sin(\pi/n)}\ \text{and}\ \cos\angle ABC=\frac{1}{2\sin(\pi/n)}.$$

Further, since $\triangle ABC$ is a right triangle with $\angle ACB=\pi/q$, we have

$$\cosh AB=\frac{\cos\angle ACB}{\sin\angle ABC}=\frac{2\cos(\pi/q)\sin(\pi/n)}{\sqrt{4\sin^2(\pi/n)-1}}.$$

Since T is the length of the common perpendicular to BE and AD, we can now calculate $\cosh T$ from the plane ω:

$$\cosh T=\sin\angle ABE\cdot\cosh AB.$$

Since

$$\begin{aligned}\sin^2\angle ABE&=\frac{\sin^2(\pi/n)-\cos^2(2\pi/n)}{\sin^2(\pi/n)}=\frac{(\sin^2(\pi/n)-1)(1-4\sin^2(\pi/n))}{\sin^2(\pi/n)}\\&=\frac{\cos^2(\pi/n)(4\sin^2(\pi/n)-1)}{\sin^2(\pi/n)},\end{aligned}$$

we get that

$$\cosh^2 T=4\cos^2(\pi/n)\cos^2(\pi/q)=(\beta+4)\cos^2(\pi/q).$$

Hence, $\beta'=4\sinh^2 T=4(\beta+4)\cos^2(\pi/q)-4$.

Analogous calculations can be done for the other cases (when $n>5$ or $q\ge\infty$). We obtain that for the groups from Item (2)(ii),

$$\beta'=\begin{cases}4(\beta+4)\cos^2(\pi/q)-4 & \text{if } 4\le q<\infty\\ 4(\beta+4)-4 & \text{if } q=\infty\\ 4(\beta+4)\cosh^2 d_1-4 & \text{if } q=\overline{\infty},\end{cases}$$

where d_1 is the distance between ε and ξ_1 if they are disjoint and π/q is the angle between ε and ξ_1 if they intersect. Hence, β' can be written in general form as follows:

$$\beta'=4(\beta+4)\cosh^2 u-4 \text{ where } u\in\mathcal{U} \text{ and } t(u)\ge 4.$$

Finally, for the groups from Item (2)(iii) of Theorem 2.1, we have $n=3$ and $\phi=4\pi/5$ and therefore

$$\beta=-3 \text{ and } \gamma=-4\cos^2(2\pi/5)=(\sqrt{5}-3)/2.$$

Moreover, we can calculate

$$\beta' = \begin{cases} 2(7+3\sqrt{5})\cos^2(\pi/r) - 4 & \text{if } 3 \le r < \infty \\ 2(7+3\sqrt{5}) - 4 & \text{if } r = \infty \\ 2(7+3\sqrt{5})\cosh^2 d_2 - 4 & \text{if } r = \overline{\infty}, \end{cases}$$

where d_2 is the distance between ε and ξ_2 if they are disjoint and π/r is the angle between ε and ξ_2 if they intersect and, hence,

$$\beta' = 2(7+3\sqrt{5})\cosh^2 u - 4, \text{ where } u \in \mathcal{U} \text{ and } t(u) \ge 3.$$

□

2.3 Geometric description of discrete groups for the case of intersecting axes

Now we consider $\Gamma = \langle f, g\rangle$ with f primitive elliptic of order $n > 2$ and g hyperbolic with *non-orthogonally intersecting axes*. In Klimenko–Kopteva [12; 13], criteria for discreteness of such groups were found for n even and odd, respectively. In this section we recall the criteria in terms of parameters and remind the construction of a fundamental polyhedron for each discrete group Γ^*.

Theorem 2.3 ([12] and [13]) *Let $(\Gamma; f, g)$ be an $\mathcal{RP}$ group with $\beta = -4\sin^2(\pi/n)$, where $n \ge 3$ is an integer, $\beta' \in (0,\infty)$ and $\gamma \in (0, -\beta\beta'/4)$. Then Γ is discrete if and only if (β, β', γ) is one of the triples listed in Table 1.*

Remark 2.4 Note that if a formula in Table 1 involves $u \in \mathcal{U}$ such that $(t(u), 2) = 1$, then $t(u)$ is finite and odd, while for $u \in \mathcal{U}$ with $(t(u), 2) = 2$, $t(u)$ can be not only finite (and even), but ∞ or $\overline{\infty}$, which implies that the formula is applicable also to $u \ge 0$. In general, if $(m, k) < k$, then m is finite.

Table 1: All parameters for discrete $\mathcal{RP}$ groups generated by a primitive elliptic element f of order $n \ge 3$ and a hyperbolic element g whose axes intersect non-orthogonally.

	$\beta = \beta(f)$	$\gamma = \gamma(f,g)$	$\beta' = \beta(g)$
	$n \ge 4$, $(n,2) = 2$, $u, v \in \mathcal{U}$, $1/n + 1/t(u) < 1/2$		
P_1	$-4\sin^2\frac{\pi}{n}$, $n \ge 4$	$4\cosh^2 u + \beta$, $(t(u),2) = 2$	$\dfrac{4}{\gamma}\cosh^2 v - \dfrac{4\gamma}{\beta}$, $t(v) \ge 3$

Table 1: (continued)

	$\beta=\beta(f)$	$\gamma=\gamma(f,g)$	$\beta'=\beta(g)$
P_2	$-4\sin^2\frac{\pi}{n},\ n\geq 4$	$4\cosh^2 u+\beta$, $(t(u),2)=1$	$\dfrac{4(\gamma-\beta)}{\gamma}\cosh^2 v-\dfrac{4\gamma}{\beta}$, $t(v)\geq 3$
P_3	-2	$2\cos(2\pi/m),\ m\geq 5$, $(m,2)=1$	$\gamma^2+4\gamma$

$n\geq 3,\ (n,2)=1,\ u,v\in\mathcal{U},\ 1/n+1/t(u)<1/2;$

$$S=-2\frac{(\gamma-\beta)^2\cos\frac{\pi}{n}+\gamma(\gamma+\beta)}{\gamma\beta},\ \ T=-2\frac{(\beta+2)^2\cos\frac{\pi}{n}}{\beta+1}-2\frac{(\beta^2+6\beta+4)}{\beta}$$

P_4	$-4\sin^2\frac{\pi}{n},\ n\geq 3$	$4\cosh^2 u+\beta$, $(t(u),2)=2$	$\dfrac{2}{\gamma}(\cosh v-\cos\dfrac{\pi}{n})+S$, $t(v)\geq 2$
P_5	$-4\sin^2\frac{\pi}{n},\ n\geq 3$	$4\cosh^2 u+\beta$, $(t(u),2)=1$	$\dfrac{2(\gamma-\beta)}{\gamma}\cosh v+S$, $t(v)\geq 2$
P_6	$-4\sin^2\frac{\pi}{n},\ n\geq 7$	$(\beta+4)(\beta+1)$	$\dfrac{2(\beta+2)^2}{\beta+1}\cosh v+T,\ t(v)\geq 2$
P_7	$-4\sin^2\dfrac{\pi}{n}$, $n\geq 5,\ (n,3)=1$	$\beta+3$	$\dfrac{2}{\beta}\Big((\beta-3)\cos\dfrac{\pi}{n}-2\beta-3\Big)$
P_8	$-4\sin^2\dfrac{\pi}{n}$, $n\geq 5,\ (n,3)=1$	$2(\beta+3)$	$-\dfrac{6}{\beta}\Big(2\cos\dfrac{\pi}{n}+\beta+2\Big)$
P_9	-3	$2\cos(2\pi/m)-1$, $m\geq 7,\ (m,2)=1$	$\dfrac{2}{\gamma}\Big(\gamma^2+2\gamma+2\Big)$
P_{10}	-3	$2\cos(2\pi/m)-1$, $m\geq 8,\ (m,6)=2$	$\gamma^2+4\gamma$
P_{11}	-3	$2\cos(2\pi/m)$, $m\geq 7,\ (m,4)\leq 2$	2γ
P_{12}	-3	$(\sqrt{5}+1)/2$	$\sqrt{5}$
P_{13}	-3	$(\sqrt{5}-1)/2$	$\sqrt{5}$

Table 1: (continued)

	$\beta=\beta(f)$	$\gamma=\gamma(f,g)$	$\beta'=\beta(g)$
P_{14}	-3	$(\sqrt{5}-1)/2$	$\sqrt{5}-1$
P_{15}	$(\sqrt{5}-5)/2$	$(\sqrt{5}-1)/2$	$\sqrt{5}$
P_{16}	$(\sqrt{5}-5)/2$	$(\sqrt{5}-1)/2$	$(3\sqrt{5}-1)/2$
P_{17}	$(\sqrt{5}-5)/2$	$(\sqrt{5}-1)/2$	$3(\sqrt{5}+1)/2$
P_{18}	$(\sqrt{5}-5)/2$	$(\sqrt{5}+1)/2$	$3(\sqrt{5}+1)/2$
P_{19}	$(\sqrt{5}-5)/2$	$\sqrt{5}+2$	$(5\sqrt{5}+9)/2$

Let f and g be as in Theorem 2.3, that is, let f be a primitive elliptic element of order $n\ge 3$, g be a hyperbolic element and let their axes intersect non-orthogonally. Let ω be the plane containing f and g, and let e be a half-turn whose axis is orthogonal to ω and passes through the point of intersection of f and g.

Again, we define two finite index extensions of $\Gamma=\langle f,g\rangle$ as follows: $\widetilde{\Gamma}=\langle f,g,e\rangle$ and $\Gamma^*=\langle f,g,e,R_\omega\rangle$.

Let e_f and e_g be half-turns such that $f=e_fe$ and $g=e_ge$. The lines e_f and e lie in a plane, denote it by ε, and intersect at an angle of π/n; ε and ω are mutually orthogonal; e_g is orthogonal to ω and intersects g.

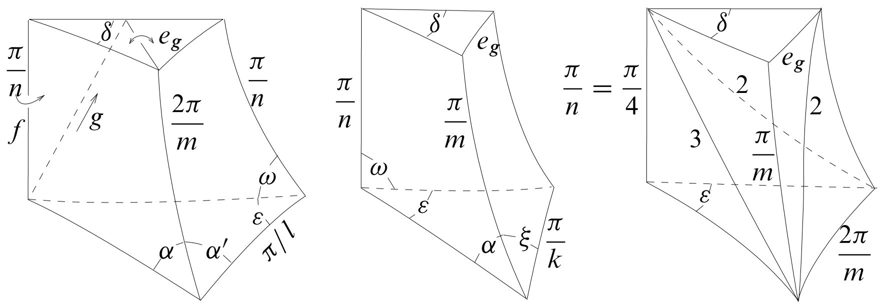
Let α be a hyperbolic plane such that $f=R_\omega R_\alpha$ and let $\alpha'=e_g(\alpha)$. There exists a plane δ which is orthogonal to the planes α, ω and α'. The plane δ passes through the common perpendicular to f and $e_g(f)$ orthogonally to ω. It is clear that $e_g\subset\delta$.

From here on, we describe the cases of even n and odd n separately (n is the order of the elliptic generator f).

$n\ge 4$ is even. Let $\mathcal{P}=\mathcal{P}(\alpha,\omega,\alpha',\delta,\varepsilon)$. The polyhedron $\mathcal{P}$ can be compact or non-compact; in Figure 2(a), $\mathcal{P}$ is drawn as compact.

The polyhedron $\mathcal{P}$ has five right dihedral angles; the dihedral angles formed by ω with α and α' equal π/n. The planes α and α' can either intersect or be parallel or disjoint; the same is true for ε and α'. Denote the angle between ε and α' by π/ℓ, where $\ell\in(2,\infty)\cup\{\infty,\overline{\infty}\}$ and denote the angle between α and α' by $2\pi/m$, where $m\in(2,\infty)\cup\{\infty,\overline{\infty}\}$, $1/n+1/m<1/2$.

For each triple of parameters with n even in Table 1, we know (from the paper [12]) how a fundamental polyhedron for Γ^* looks like, and we describe all such polyhedra below.



(a): P_1 (b): P_2 (c): P_3

Figure 2: Fundamental polyhedra for Γ^* in the case of even n $(0<\gamma<-\beta\beta'/4)$

P$_1$. $\mathcal{P}$ is a fundamental polyhedron for Γ^* if and only if $m \in \mathbb{Z}\cup\{\infty,\overline{\infty}\}$, m is even $(1/m+1/n<1/2)$ and $\ell \in \mathbb{Z}\cup\{\infty,\overline{\infty}\}$ $(\ell \geq 3)$. In terms of the function t, $m=t(u)$ and $\ell=t(v)$ (cf Table 1).

P$_2$. Note that in this case $m=t(u)$ is finite and odd. Let ξ be the bisector of the dihedral angle of $\mathcal{P}$ at the edge $\alpha\cap\alpha'$. It is clear that ξ passes through e_g and is orthogonal to ω. The polyhedron $\mathcal{P}(\alpha,\delta,\xi,\varepsilon,\omega)$ is bounded by reflection planes of Γ^* (see Figure 2(b)) and, therefore, it is a fundamental polyhedron for Γ^* if and only if ξ and ε intersect at an angle of π/k, where $k\geq 3$, or are parallel or disjoint ($k=\infty$ or $k=\overline{\infty}$, respectively). In Table 1, $k=t(v)$ for the parameters P_2.

P$_3$. In this case $n=4$ and the dihedral angle of $\mathcal{P}(\alpha,\delta,\xi,\varepsilon,\omega)$ at the edge $\xi\cap\varepsilon$ is $2\pi/m$, where $m=t(u)$ is odd, $5\leq m<\infty$. The polyhedron $\mathcal{P}(\alpha,\delta,\xi,\varepsilon,\omega)$ is decomposed by reflection planes of Γ^* into three (possibly infinite volume) tetrahedra $T[2,2,4;2,3,m]$, each of which is a fundamental polyhedron for Γ^* (see Figure 2(c)).

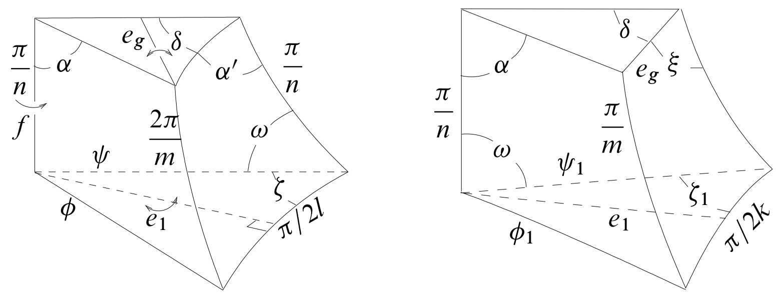
$n\geq 3$ is odd. Denote $e_1=f^{(n-1)/2}e$. Note that e_1 makes angles of $\pi/(2n)$ with α and ω.

We can forget about the plane ε, because now we need another plane, denote it by ζ, for the construction of a fundamental polyhedron for Γ^*. To construct ζ we use an auxiliary plane κ that passes through e_1 orthogonally to α'. The plane ζ then passes through e_1 orthogonally to κ. (Note that ζ is not orthogonal to each of the planes α and ω if $m\neq 2n$.) In fact, the planes ζ and α' can either intersect or be parallel or disjoint. Note that if $\zeta\cap\alpha'\neq\varnothing$ then e_1 is orthogonal to $\zeta\cap\alpha'$. Let $\mathcal{P}=\mathcal{P}(\alpha,\omega,\alpha',\delta,\zeta)$. In Figure 3(a), $\mathcal{P}$ is drawn for the compact case.

Consider the dihedral angles of $\mathcal{P}$. The angles between δ and ω, δ and α, δ and α' are all of $\pi/2$; the angles formed by ω with α and α' equal π/n; since ζ passes through e_1, which is orthogonal to f, the sum of the angles ϕ and ψ formed by ζ

with α and ω, respectively, equals π. The planes α and α' can either intersect or be parallel or disjoint. The same is true for ζ and α'. Denote the angle between α and α' by $2\pi/m$ and the angle between α' and ζ by $\pi/(2\ell)$.

Fundamental polyhedra for groups Γ^* for all triples of parameters with odd n from Table 1 were constructed in [13]. Now we describe them.



(a): P_4 (b): P_5

Figure 3

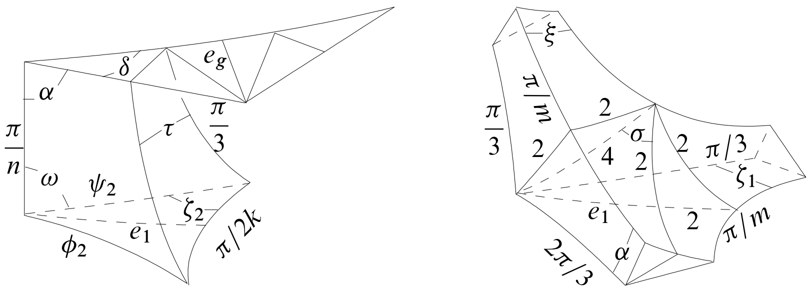
P$_4$. $\mathcal{P}$ itself is a fundamental polyhedron for Γ^* if and only if m is even $(1/m + 1/n < 1/2)$, $m = \infty$ or $m = \overline{\infty}$, and $\ell \in \mathbb{Z} \cup \{\infty, \overline{\infty}\}$, $\ell \geq 2$. In terms of the function t, $m = t(u)$ and $\ell = t(v)$ (cf Table 1).

P$_5$. Let ξ be the bisector of the dihedral angle of $\mathcal{P}$ at the edge $\alpha \cap \alpha'$. Clearly, ξ passes through e_g orthogonally to ω and δ. Construct a plane ζ_1 in a similar way as ζ above (now ξ plays the role of α'). The polyhedron $\mathcal{Q} = \mathcal{P}(\alpha, \delta, \xi, \zeta_1, \omega)$ is a fundamental polyhedron for Γ^* (see Figure 3(b)) if and only if m is odd and ξ and ζ_1 make an angle of $\pi/(2k)$, where $k \geq 2$ is an integer, ∞ or $\overline{\infty}$.

P$_6$. In this case, the dihedral angle of $\mathcal{Q}$ at the edge $\alpha \cap \xi$ equals $2\pi/n$ (ie $m = n/2$), $n \geq 7$ is odd. Let ρ be the bisector of this dihedral angle and let $\tau = R_\rho(\omega)$. The bisector ρ makes an angle of $\pi/3$ with ω and, therefore, so does τ. It is clear that then τ is orthogonal to α (in δ, we have one of Knapp's triangles with one non-primitive and two primitive angles leading to a discrete group [16]). Construct a plane ζ_2 similarly to the planes ζ and ζ_1 above (but using τ). The polyhedron $\mathcal{P}(\alpha, \delta, \omega, \tau, \zeta_2)$ (see Figure 4(a), where we show also a part of the plane δ) is a fundamental polyhedron for Γ^* if and only if ζ_2 and τ intersect at an angle of $\pi/(2k)$, where $k \geq 2$ is an integer, or are parallel or disjoint ($k = \infty$ or $k = \overline{\infty}$, respectively). In Table 1, $t(v)$ corresponds to k.

P$_9$. The dihedral angles of $\mathcal{Q}$ at the edges $\alpha \cap \xi$ and $\zeta_1 \cap \xi$ equal π/m, m is odd. The plane ζ_1 makes dihedral angles of $2\pi/3$ and $\pi/3$ with α and ω, respectively. Let

σ be the bisector of the dihedral angle of $\mathcal{Q}$ at $\alpha \cap \zeta_1$. It is clear that σ is orthogonal to ω. $\mathcal{P}(\alpha, \omega, \xi, \delta, \mu)$, where μ is the plane that passes through $\sigma \cap \omega$ orthogonally to α, is a fundamental polyhedron for Γ^* (see Figure 4(b)). The dihedral angle of $\mathcal{P}(\alpha, \omega, \xi, \delta, \mu)$ at $\mu \cap \omega$ equals $\pi/4$.



(a): P_6 (b): P_9

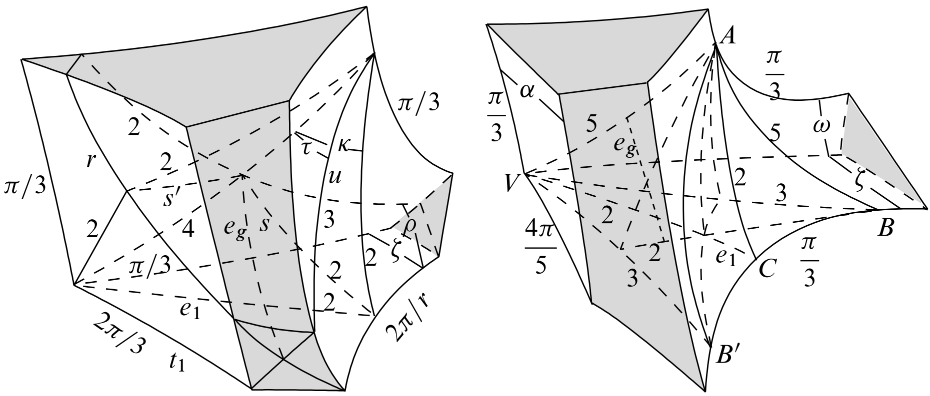
Figure 4

Fundamental polyhedra for remaining discrete groups Γ^* are obtained after decomposition of $\mathcal{P}$ (which is shown in Figure 3(a)) by the planes of reflections from Γ^*, that is, $(m, 2) = 2$, $m \geq 4$ and ℓ is fractional. We first consider the cases where $R_\zeta \in \Gamma^*$.

A compact convex polyhedron in $\mathbb{H}^3$ whose skeleton is a trivalent graph is uniquely determined by its dihedral angles up to isometry of $\mathbb{H}^3$, see Hodgson and Rivin [7]. Given n, m and ℓ, all the dihedral angles of the polyhedron $\mathcal{P} \cup e_1(\mathcal{P})$ are defined. Therefore, the dihedral angle ϕ of $\mathcal{P}$ at $\alpha \cap \zeta$ can be obtained. So to determine a compact $\mathcal{P}$ it is sufficient to indicate only n, m and ℓ, but we shall also give the value of ϕ for convenience. If $\mathcal{P}$ has infinite volume, but $\ell < \infty$ and $m < \infty$ (then $2/m + 1/n + 1/\ell < 1$), $\mathcal{P}$ is also determined by the values of n, m and ℓ, since we can cut off a compact polyhedron from $\mathcal{P} \cup e_1(\mathcal{P})$ by a plane orthogonal to ζ, α, α' and a plane orthogonal to ζ, ω, α'.

There are no discrete groups for which $m = \infty$ or $\ell \geq \infty$ except for those with parameters of type P_4. When $m = \overline{\infty}$ we also indicate the distance d between α and α' to determine $\mathcal{P}$. In fact, given d, one can find ϕ, but we shall give ϕ explicitly for convenience.

In all of these cases $R_\delta \notin \Gamma^*$, so we do not show δ (but indicate e_g) in figures in order to simplify the picture. By the same reason we draw only those parts of the decomposition (including ω) that are important for the reconstruction of the action of Γ^* and help to determine positions of e_1 and e_g.



(a): P_{11} (b): P_{12}

Figure 5

$\mathbf{P}_{11}$. $n=3$, $m=\overline{\infty}$, $\ell=r/4$, $(r,4)\le 2$, $r\ge 7$, $\phi=2\pi/3$ and $\cosh d = 2\cos^2(\pi/r)-1/2$. $\mathcal{P}(\alpha,\alpha',\omega,\zeta)$ is decomposed into tetrahedra $T=T[2,3,r;2,2,4]$ each of which is a fundamental polyhedron for Γ^* (Figure 5(a)). $\Gamma^*=G_T$.

$\mathbf{P}_{12}$. $n=3$, $m=\overline{\infty}$, $\ell=3/2$, $\phi=4\pi/5$ and $\cosh d=(3+\sqrt{5})/4$. $\mathcal{P}(\alpha,\alpha',\omega,\zeta)$ is decomposed into tetrahedra $T=T[2,2,3;2,5,3]$. A half of T is a fundamental polyhedron for Γ^* (Figure 5(b)). $\Gamma^*=\langle G_T, e_g\rangle$.

$\mathbf{P}_{13}$. $n=3$, $m=10$, $\ell=3/2$, $\phi=3\pi/5$. $\mathcal{P}(\alpha,\alpha',\omega,\zeta)$ is decomposed into tetrahedra $T=T[2,3,5;2,3,2]$. A half of T is a fundamental polyhedron for Γ^* (Figure 6(a)). $\Gamma^*=\langle G_T, e_g\rangle$.

$\mathbf{P}_{14}$. $n=3$, $m=10$, $\ell=5/4$, $\phi=2\pi/3$. $\mathcal{P}(\alpha,\alpha',\omega,\zeta)$ is decomposed into tetrahedra $T=T[2,3,5;2,2,4]$ each of which is a fundamental polyhedron for Γ^* (Figure 6(b)). $\Gamma^*=G_T$.

$\mathbf{P}_{15}$. $n=5$, $m=4$, $\ell=3/2$, $\phi=\pi/5$. $\mathcal{P}(\alpha,\alpha',\omega,\zeta)$ is decomposed into tetrahedra $T=T[2,3,5;2,3,2]$. A half of T is a fundamental polyhedron for Γ^* (Figure 6(c)). $\Gamma^*=\langle G_T, e_g\rangle$.

$\mathbf{P}_{17}$. $n=5$, $m=4$, $\ell=5/2$, $\phi=\pi/3$. $\mathcal{P}(\alpha,\alpha',\omega,\zeta)$ is decomposed into tetrahedra $T=T[2,3,5;2,2,5]$. A half of T is a fundamental polyhedron for Γ^* (Figure 6(d)). $\Gamma^*=\langle G_T, e_g\rangle$.

$\mathbf{P}_{18}$. $n=5$, $m=6$, $\ell=5/4$, $\phi=\pi/3$. $\mathcal{P}(\alpha,\alpha',\omega,\zeta)$ is decomposed into tetrahedra $T=T[2,3,5;2,2,5]$. A half of T is a fundamental polyhedron for Γ^* (Figure 6(e)). $\Gamma^*=\langle G_T, e_g\rangle$.

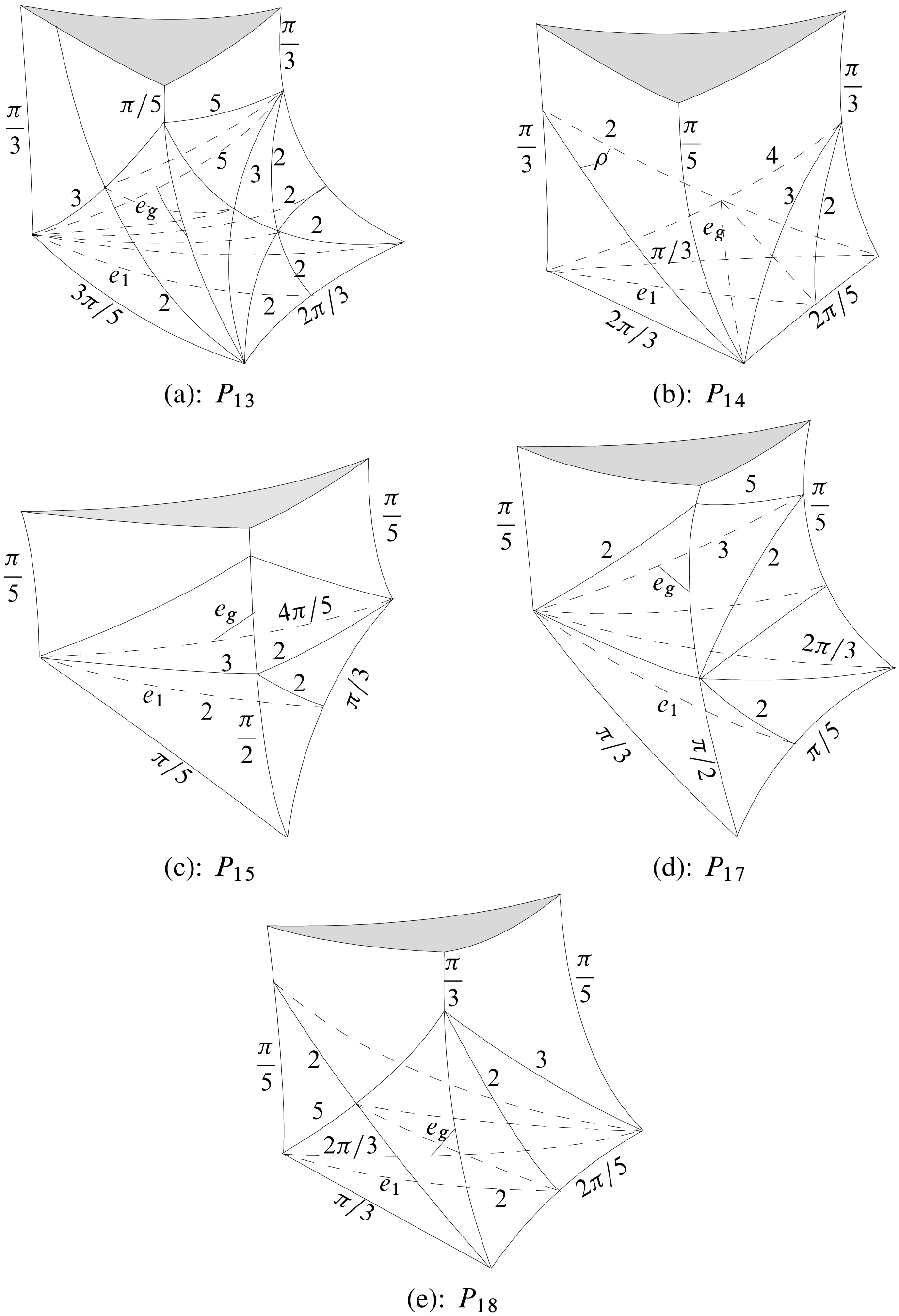
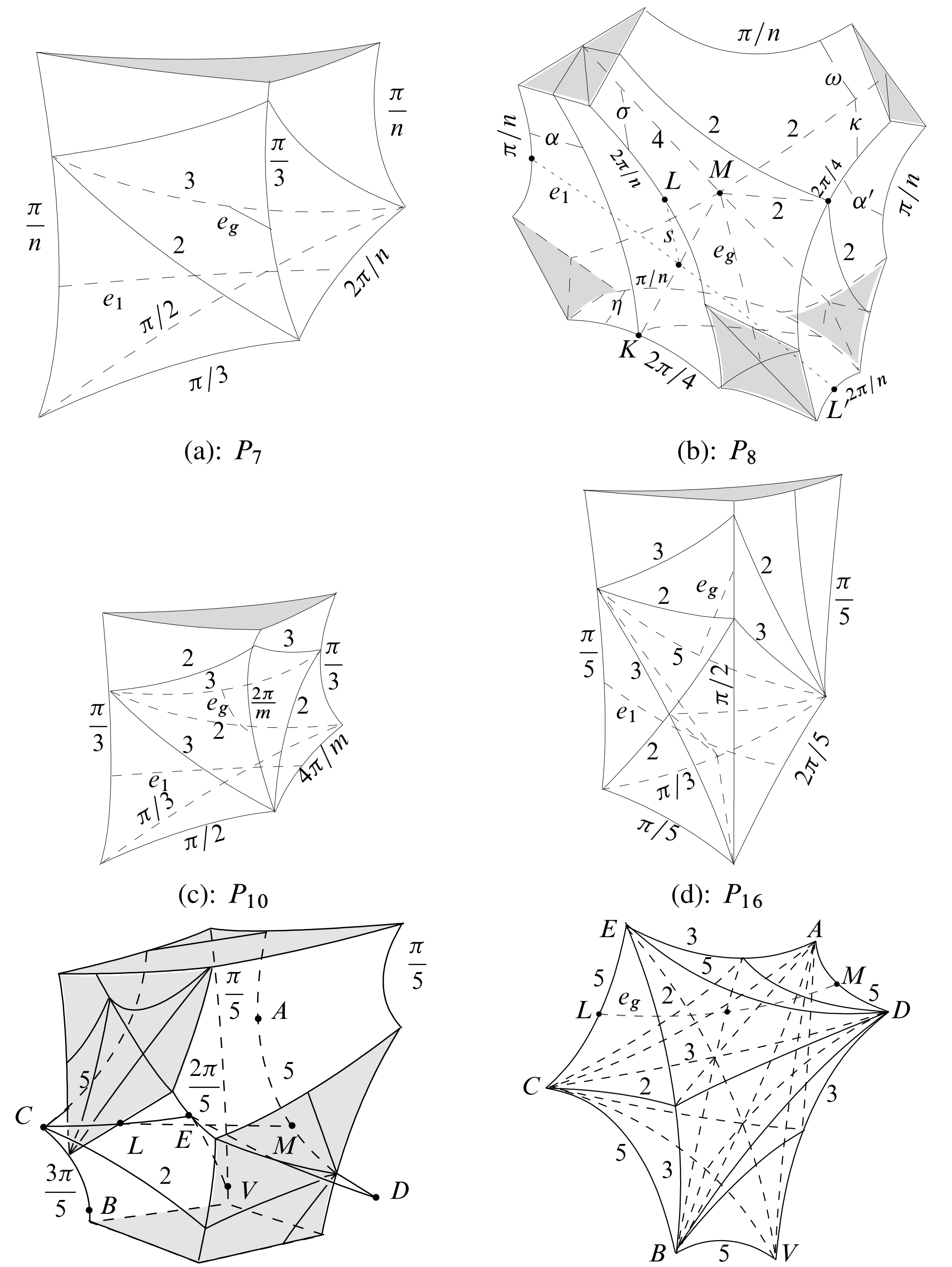

π/3
π/3
π/5
5
5
3
2
3
2
2
e_g
e_1
3π/5
2
2
2
2π/3
(a): P_13
π/3
2
ρ
π/5
4
π/3
3
2
e_g
π/3
e_1
2π/3
2π/5
(b): P_14
π/5
π/5
e_g
4π/5
3
2
2
e_1
2
π/2
π/3
π/5
(c): P_15
5
π/5
π/5
2
3
2
e_g
2π/3
e_1
2
π/3
π/2
π/5
(d): P_17
π/3
π/5
π/5
2
3
2
5
2π/3
e_g
e_1
2
2π/5
π/3
(e): P_18

Figure 6



(a): P_7

(b): P_8

(c): P_{10}

(d): P_{16}

(e): P_{19}

Figure 7

Now consider discrete groups for which $R_\zeta \notin \Gamma^*$. In all these cases $\ell = p/3$, where $(p,3)=1$. Let η be the plane through $\alpha' \cap \zeta$ that makes a dihedral angle of $2\pi/p$ with α' and let $\overline{\mathcal{P}} = \mathcal{P}(\alpha, \alpha', \omega, \delta, \eta)$. Denote by θ_1 and θ_2 dihedral angles of $\overline{\mathcal{P}}$ at $\eta \cap \alpha$ and $\eta \cap \omega$, respectively.

If $\mathcal{P}$ is compact or non-compact with $m < \infty$, $\overline{\mathcal{P}}$ is determined by values n, m, ℓ, θ_1 and θ_2. For $m = \overline{\infty}$, we give the distance d between α and α'.

$\mathbf{P}_7$. $n \geq 5$, $(n,3)=1$, $m=6$, $\ell = n/3$, $\theta_1 = \pi/3$, $\theta_2 = \pi/2$. $\mathcal{P}(\alpha, \alpha', \omega, \eta)$ is decomposed into tetrahedra $T = T[2,3,n;2,3,n]$. A quarter of T is a fundamental polyhedron for Γ^* (Figure 7(a)). $\Gamma^* = \langle G_T, e_1, e_g \rangle$.

$\mathbf{P}_8$. $n \geq 5$, $(n,3)=1$, $m = \overline{\infty}$, $\ell = n/3$, $\theta_1 = \pi/2$, $\theta_2 = \pi/n$ and $\cosh d = 2\cos^2(\pi/n)$. $\mathcal{P}(\alpha, \alpha', \omega, \eta)$ is decomposed into tetrahedra $T = T[2,2,4;2,n,4]$. A half of T is a fundamental polyhedron for Γ^* (Figure 7(b)). $\Gamma^* = \langle G_T, e_1 \rangle$.

$\mathbf{P}_{10}$. $n=3$, $m \geq 8$ is even, $(m,3)=1$, $\ell = m/6$, $\theta_1 = \pi/2$, $\theta_2 = \pi/3$. $\mathcal{P}(\alpha, \alpha', \omega, \eta)$ is decomposed into tetrahedra $T = T[2,3,m/2;2,3,3]$. A half of T is a fundamental polyhedron for Γ^* (Figure 7(c)). $\Gamma^* = \langle G_T, e_g \rangle = \langle G_T, e_1 \rangle$.

$\mathbf{P}_{16}$. $n=5$, $m=4$, $\ell = 5/3$, $\theta_1 = \pi/5$, $\theta_2 = 2\pi/3$. $\mathcal{P}(\alpha, \alpha', \omega, \eta)$ is decomposed into tetrahedra $T = T[2,3,5;2,3,2]$. A half of T is a fundamental polyhedron for Γ^* (Figure 7(d)). $\Gamma^* = \langle G_T, e_g \rangle = \langle G_T, e_1 \rangle$.

$\mathbf{P}_{19}$. $n=5$, $m = \overline{\infty}$, $\ell = 5/3$, $\theta_1 = 3\pi/5$ and $\cosh d = (5+\sqrt{5})/4$. The planes η and ω are disjoint. $\mathcal{P}(\alpha, \alpha', \omega, \eta)$ is decomposed into tetrahedra $T = T[2,2,3;2,5,3]$. A half of T is a fundamental polyhedron for Γ^*, see Figure 7(e), where $LM = e_g$ and $VE = e_1$. $\Gamma^* = \langle G_T, e_g \rangle$.

3 Kleinian orbifolds and their fundamental groups

Let Γ be a non-elementary Kleinian group, and let $\Omega(\Gamma)$ be the discontinuity set of Γ. Following Boileau and Porti [2], we say that the *Kleinian orbifold* $Q(\Gamma) = (\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$ is an orientable 3–orbifold with a complete hyperbolic structure on its interior $\mathbb{H}^3/\Gamma$ and a conformal structure on its boundary $\Omega(\Gamma)/\Gamma$.

In this section we shall describe the Kleinian orbifold $Q(\Gamma)$ and a presentation for each truly spatial discrete $\mathcal{RP}$ group $(\Gamma; f, g)$ with f elliptic and g hyperbolic. Since a fundamental polyhedron for Γ^* (a finite index extension of Γ) was shown, it remains to construct a fundamental polyhedron for Γ itself and identify the equivalent points on the boundary of the new polyhedron to get the corresponding orbifold.

In figures, we schematically draw singular sets and boundary components of the orbifolds using fat vertices and fat edges. In fact, each picture gives rise to an infinite series of orbifolds which might be compact or non-compact of finite or infinite volume.

We say that a finite 3–regular graph $\Sigma(Q)$ with fat vertices and fat edges embedded in a topological space X *represents the singular set and/or boundary components of* $Q = Q(\Gamma)$ if:

(1) non-fat edges of $\Sigma(Q)$ are labelled by positive integers greater than 1,

(2) fat edges of $\Sigma(Q)$ are labelled by positive integers greater than 1 or symbols ∞ and $\overline{\infty}$,

(3) the endpoints of a fat edge are fat vertices,

(4) if p, q and r are labels of the edges incident to a non-fat vertex, then $1/p + 1/q + 1/r > 1$.

If an edge has no label then the label is meant to be 2. To reproduce the orbifold Q from a graph $\Sigma(Q)$ we first work out all fat vertices and then all fat edges according to labels assigned as follows.

Let $v \in \Sigma(Q)$ be a fat vertex and p, q and r be the labels of the edges incident to v.

Suppose that all $p, q, r < \infty$. If $1/p + 1/q + 1/r > 1$ then the vertex v is a singular point of Q and the local group of v is one of the finite groups D_{2n}, S_4, A_4, A_5. If $1/p + 1/q + 1/r = 1$ then v represents a puncture. A cusp neighborhood of v is a quotient of a horoball in $\mathbb{H}^3$ by a Euclidean triangle group $(2,3,6)$, $(2,4,4)$ or $(3,3,3)$. In case $1/p + 1/q + 1/r < 1$ the vertex v must be removed together with its open neighborhood, which means that Q has a boundary component.

If one of the indices, say p, equals ∞ and $1/p + 1/q + 1/r = 1$, then $q = r = 2$ and v is a puncture.

For all the other p, q, r, the vertex v is removed together with its open neighborhood.

Now we proceed with the edges. If an edge e (fat or non-fat) is labelled by an integer $p < \infty$, then e is a part of the singular set of the orbifold Q and consists of cone points of index p.

Fat edges labelled by ∞ represent cusps of Q. A cusp neighborhood is the quotient of a horoball by an elementary parabolic group. Topologically it is $F \times [0,\infty)$, where F is a Euclidean orbifold called the cross-section of the cusp (see eg Boileau–Maillot–Porti [1] for geometric structures on orbifolds).

If e is labelled by $\overline{\infty}$, then it must be deleted together with its open regular neighborhood. More details on how to 'decode' an orbifold with fat edges and vertices are given in Klimenko–Kopteva [15]. We do not discuss them here since fundamental polyhedra for all Γ will be found, so it is not difficult to reconstruct the orbifolds.

Denote:

(1) $GT[n,m;q] = \langle f, g \mid f^n, g^m, [f,g]^q \rangle$,

(2) $PH[n,m,q] = \langle x, y, z \mid x^n, y^2, z^2, (xz)^2, [x,y]^m, (yxyz)^q \rangle$,

(3) $H[p;n,m;q] = \langle x, y, s \mid s^2, x^n, y^m, (xy^{-1})^p, (sxsy^{-1})^q, (sx^{-1}y)^2 \rangle$,

(4) $P[n,m,q] = \langle w, x, y, z \mid w^n, x^2, y^2, z^2, (wx)^2, (wy)^2, (yz)^2, (zx)^q, (zw)^m \rangle$,

(5) $\mathrm{Tet}[p_1, p_2, p_3; q_1, q_2, q_3] =$
$\langle x, y, z \mid x^{p_1}, y^{p_2}, z^{p_3}, (yz^{-1})^{q_1}, (zx^{-1})^{q_2}, (xy^{-1})^{q_3} \rangle$, where, for simplicity, the group $\mathrm{Tet}[2,2,n;2,q,m]$ is denoted by $\mathrm{Tet}[n,m;q]$,

(6) $GTet_1[n,m,q] = \langle x, y, z \mid x^n, y^2, (xy)^m, [y,z]^q, [x,z] \rangle$,

(7) $GTet_2[n,m,q] = \langle x, y, z \mid x^n, y^2, (xy)^m, (xz^{-1}y^{-1}zy)^q, [x,z] \rangle$,

(8) $\mathcal{S}_2[n,m,q] = \langle x, L \mid x^n, (xLxL^{-1})^m, (xL^2x^{-1}L^{-2})^q \rangle$,

(9) $\mathcal{S}_3[n,m,q] = \langle x, L \mid x^n, (xLxL^{-1})^m, (xLxLxL^{-2})^q \rangle$,

(10) $R[n,m;q] = \langle u, v \mid (uv)^n, (uv^{-1})^m, [u,v]^q \rangle$.

In the presentations (1)–(10), the exponents $n, m, q, \dots$ may be integers (greater than 1), ∞ or $\overline{\infty}$. We employ the symbols ∞ and $\overline{\infty}$ in the following way. If we have relations of the form $w^n = 1$, where $n = \overline{\infty}$, we remove them from the presentation (in fact, this means that the element w is hyperbolic in the Kleinian group). Further, if we keep the relations $w^\infty = 1$, we get a Kleinian group presentation where parabolics are indicated. To get an abstract group presentation, we need to remove such relations as well.

The reader can find the orbifolds that correspond to the above presentations in Figures 8, 10 and 12.

We start with description of presentations and orbifolds for all truly spatial discrete groups generated by a primitive elliptic and a hyperbolic elements with *disjoint axes* (Theorem 3.1). All such orbifolds are embedded in $\mathbb{S}^3$.

As usual, we can also apply the theorem when the elliptic generator is non-primitive, using recalculation formulas for parameters as follows. Suppose that f is a non-primitive elliptic of finite order n, that is, $\beta(f) = -4\sin^2(q\pi/n)$, where $(q,n) = 1$ and $1 < q < n/2$. Then there exists an integer r so that f^r is primitive of the same order. Obviously, $\langle f, g \rangle = \langle f^r, g \rangle$ and $\beta(f^r) = -4\sin^2(\pi/n)$. By Gehring and Martin [6], $\gamma(f,g) = \gamma(f^r, g) \cdot \beta(f)/\beta(f^r)$.

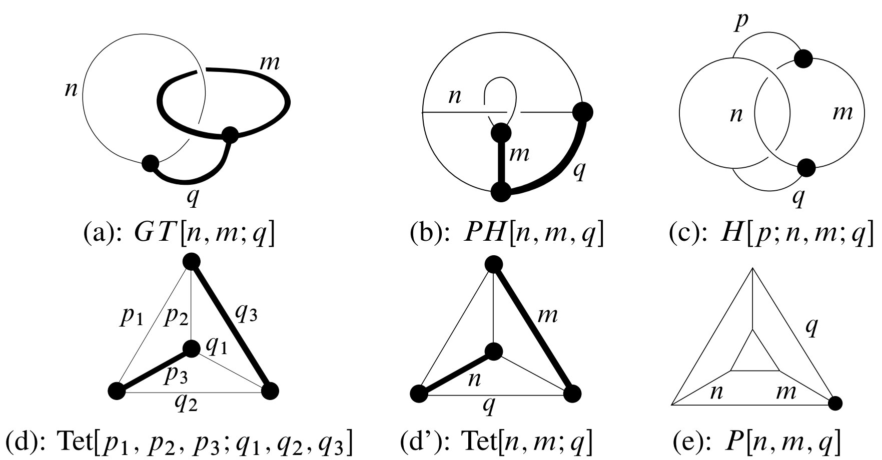


Figure 8: Orbifolds embedded in $\mathbb{S}^3$

Theorem 3.1 *Let* $(\Gamma; f, g)$ *be a discrete* $\mathcal{RP}$ *group with* $\beta = -4\sin^2(\pi/n)$, $n \geq 3$, $\beta' \in (0, +\infty)$ *and* $\gamma \in (-\infty, 0)$. *Then one of the following occurs.*

(1) $\gamma = -4\cosh^2 u$, *where* $u \in \mathcal{U}$, $(t(u), 2) = 2$ *and* $t(u) \geq 4$; Γ *is isomorphic to* $GT[n, \overline{\infty}; t(u)/2]$.

(2) $\gamma = -4\cosh^2 u$, *where* $u \in \mathcal{U}$, $(t(u), 2) = 1$ *and* $t(u) \geq 3$; Γ *is isomorphic to* $\mathrm{Tet}[n, \overline{\infty}; t(u)]$.

(3) $n \geq 5$, $(n, 2) = 1$, $\gamma = -(\beta+2)^2$ *and* $\beta' = 4(\beta+4)\cosh^2 u - 4$, *where* $u \in \mathcal{U}$ *and* $t(u) \geq 4$; Γ *is isomorphic to* $\mathrm{Tet}[n, t(u); 3]$.

(4) $\beta = -3$, $\gamma = (\sqrt{5}-3)/2$ *and* $\beta' = 2(7+3\sqrt{5})\cosh^2 u - 4$, *where* $u \in \mathcal{U}$ *and* $t(u) \geq 3$; Γ *is isomorphic to* $\mathrm{Tet}[3, t(u); 5]$.

Proof All parameters for discrete groups in the statement of Theorem 3.1 are described in Theorem 2.2. We shall obtain a presentation for each discrete group by using the Poincaré polyhedron theorem, see eg Epstein and Petronio [4].

Let Γ have parameters as in part (1) of Theorem 2.2. In Section 2.1, a fundamental polyhedron for the group Γ^* was described. Since $\widetilde{\Gamma}$ is the orientation preserving index 2 subgroup of Γ^*, we can take $\widetilde{\mathcal{P}} = \mathcal{P}(\varepsilon, \alpha, \tau, R_\omega(\alpha))$ as a fundamental polyhedron for $\widetilde{\Gamma}$ (see Figure 9(a)). In our notation $p = t(u)$.

Let $e_g = R_\tau R_\omega$. It is clear that $e_g = ge$. By applying the Poincaré polyhedron theorem to $\widetilde{\mathcal{P}}$ and face pairing transformations e, e_g and f, we get

$$\widetilde{\Gamma} = \langle e, e_g, f \,|\, e^2, e_g^2, f^n, (fe)^2, (fe_g)^p \rangle,$$

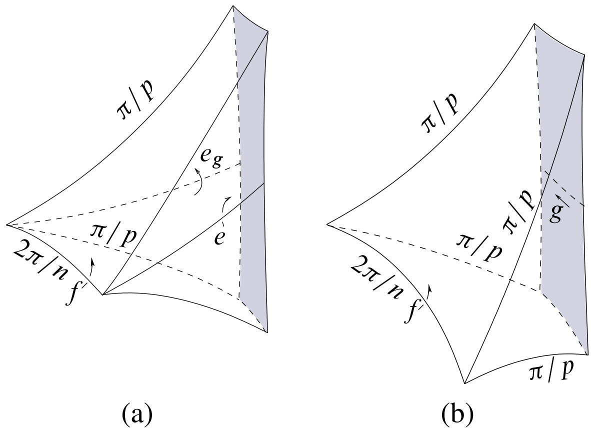


Figure 9: Fundamental polyhedra for $\widetilde{\Gamma}$ and Γ in case of disjoint axes

where p is an integer, ∞ or $\overline{\infty}$. Since $g = e_g e$,

$$\widetilde{\Gamma} = \langle f, g, e \,|\, f^n, e^2, (fe)^2, (ge)^2, (fge)^p \rangle.$$

If p is odd, then from the relations for $\widetilde{\Gamma}$ it follows that $e = (fgf^{-1}g^{-1})^{(p-1)/2} fg$. Hence, in this case $\widetilde{\Gamma} = \Gamma$ and $\Gamma \cong \mathrm{Tet}[n, \overline{\infty}; p]$. The isomorphism is given by $f \mapsto z$, $g \mapsto xy^{-1}$ and $e \mapsto y$. Identifying faces of $\widetilde{\mathcal{P}}$, we get the orbifold $Q(\Gamma)$ shown in Figure 8(d').

If p is even, ∞ or $\overline{\infty}$, then Γ is a subgroup of index 2 in $\widetilde{\Gamma}$. To see this, we apply the Poincaré theorem to the polyhedron $\mathcal{P}(\alpha, \tau, R_\omega(\alpha), R_\varepsilon(\tau))$ (see Figure 9(b)). Then

$$\Gamma = \langle f, g \,|\, f^n, (fgf^{-1}g^{-1})^{p/2} \rangle \cong GT[n, \overline{\infty}; p/2].$$

The orbifold $Q(\Gamma)$ is shown in Figure 8(a).

Now consider the groups with parameters from part (2) of Theorem 2.2. In this case $t(u) = q$ from Theorem 2.1. By applying the Poincaré theorem to the polyhedron $\mathcal{P}(\varepsilon, \alpha, \xi_1, R_\omega(\alpha), R_\omega(\xi_1))$ and the group generated by f, e and s, where $s = R_\omega R_{\xi_1}$, we get the following presentation for the group $\langle f, e, s \rangle$:

$$\langle f, e, s \,|\, f^n, e^2, s^3, (fe)^2, (fs)^2, (se)^q \rangle.$$

Since $x = R_\kappa R_\tau$, we have $x^2 = h$ and $x = fs^{-1}$. Therefore, $g = e_g e = f^{-1}he = f^{-1}x^2 e = f^{-1}(fs^{-1})^2 e = s^{-1}fs^{-1}e$ and hence $\Gamma \subseteq \langle f, e, s \rangle$.

Since $h^n = 1$, n is odd and $h^2 = [f, g]$, we have that $h = [f, g]^{-(n-1)/2} \in \Gamma$. Further, $e_g = f^{-1}h$ and so $e = e_g g = f^{-1}hg \in \Gamma$. From $x^n = 1$ we have that $x = h^{-(n-1)/2} \in$

Γ and, therefore, $s = x^{-1} f \in \Gamma$. Then $\langle f, e, s\rangle \subseteq \Gamma$ and so we have shown that $\Gamma = \langle f, e, s\rangle$.

Mapping $x \mapsto s^{-1} f^{-1}$, $y \mapsto fe$, $z \mapsto f$, we see that $\langle f, e, s\rangle = \Gamma$ is isomorphic to the group $\mathrm{Tet}[n, q; 3]$. Therefore, $\Gamma \cong \mathrm{Tet}[n, q; 3]$, where $q \geq 4$ is an integer, ∞ or $\overline{\infty}$.

The orbifold $Q(\Gamma)$ is shown in Figure 8(d').

Similarly, one can show that the groups with parameters from part (3) of Theorem 2.2 are isomorphic to $\mathrm{Tet}[3, t(u); 5]$, where $t(u) \geq 3$ is an integer, ∞ or $\overline{\infty}$. □

Let $T(p)$, $p \in \mathbb{Z}$, be a Seifert fibered solid torus obtained from a trivially fibered solid torus $D^2 \times \mathbb{S}^1$ by cutting it along $D^2 \times \{x\}$ for some $x \in \mathbb{S}^1$, rotating one of the discs through $2\pi/p$ and gluing back together.

Denote by $\mathcal{S}(p)$ a space obtained by gluing $T(p)$ with its mirror symmetric copy along their boundaries fiber to fiber. Clearly, $\mathcal{S}(p)$ is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$ and is p-fold covered by trivially fibered $\mathbb{S}^2 \times \mathbb{S}^1$. There are two critical fibers[1] in $\mathcal{S}(p)$ whose 'length' is p times shorter than the 'length' of a regular fiber.

Next two theorems describe presentations and orbifolds for all truly spatial discrete groups $\Gamma = \langle f, g\rangle$ whose generators have *intersecting axes*, g is hyperbolic and f is primitive elliptic of even order (Theorem 3.2) or odd order (Theorem 3.5). In both theorems there are series of orbifolds embedded into $\mathbb{S}^3$ and $\mathbb{S}^2 \times \mathbb{S}^1$; in case when f has odd order some orbifolds are embedded into $\mathbb{R}P^3$.

Again for non-primitive elliptics we can use recalculation formulas for parameters to apply Theorem 3.2 or Theorem 3.5 (see the paragraph before Theorem 3.1).

Theorem 3.2 *Let $(\Gamma; f, g)$ be a discrete $\mathcal{RP}$ group so that $\beta = -4\sin^2(\pi/n)$, $n \geq 4$, $(n, 2) = 2$, $\beta' \in (0, +\infty)$ and $\gamma \in (0, -\beta\beta'/4)$. Then $\gamma = 4\cosh^2 u + \beta$, where $u \in \mathcal{U}$ and $1/n + 1/t(u) < 1/2$, and one of the following occurs.*

(1) *$(t(u), 2) = 2$ and $\beta' = 4(\cosh^2 v)/\gamma - 4\gamma/\beta$, where $v \in \mathcal{U}$, $t(v) \geq 3$ and $(t(v), 2) = 1$; Γ is isomorphic to $PH[n, t(u)/2, t(v)]$.*

(2) *$(t(u), 2) = 2$ and $\beta' = 4(\cosh^2 v)/\gamma - 4\gamma/\beta$, where $v \in \mathcal{U}$, $t(v) \geq 4$ and $(t(v), 2) = 2$; Γ is isomorphic to $\mathcal{S}_2[n, t(u)/2, t(v)/2]$.*

(3) *$(t(u), 2) = 1$ and $\beta' = 4(\gamma - \beta)(\cosh^2 v)/\gamma - 4\gamma/\beta$, where $v \in \mathcal{U}$, $t(v) \geq 3$ and $(t(v), 2) = 1$; Γ is isomorphic to $P[n, t(u), t(v)]$.*

[1] A critical fiber is also called a *singular* fiber. We use the word 'critical' in order not to confuse it with components of the singular sets of orbifolds.

(4) $(t(u),2)=1$ *and* $\beta'=4(\gamma-\beta)(\cosh^2 v)/\gamma-4\gamma/\beta$, *where* $v\in\mathcal{U}$, $t(v)\geq 4$ *and* $(t(v),2)=2$; Γ *is isomorphic to* $GTet_1[n,t(u),t(v)/2]$.

(5) $\beta=-2$, $(t(u),2)=1$, $t(u)\geq 5$ *and* $\beta'=\gamma^2+4\gamma$; Γ *is isomorphic to* $\mathrm{Tet}[4,t(u);3]$.

Proof The idea of the proof is the same as for Theorem 3.1. We refer now to the part of Section 2.3 where n is even.

1. Let Γ have parameters as in row P_1 of Table 1. A fundamental polyhedron $\mathcal{P}(\alpha,\alpha',\delta,\varepsilon,\omega)$ for Γ^* is shown in Figure 2(a). A fundamental polyhedron for $\widetilde{\Gamma}$ is $\mathcal{P}(\alpha,\alpha',R_\omega(\alpha),R_\omega(\alpha'),\delta,\varepsilon)$, whose faces are identified by face pairing transformations f, $f'=R_\omega R_{\alpha'}$, $e_2=f^{n/2}e$ and e_g. (We doubled the fundamental polyhedron for Γ^* shown in Figure 2(a).) By the Poincaré polyhedron theorem, we get that

$$\widetilde{\Gamma}=\langle f,f',e_g,e_2\,|\,f^n,(f')^n,e_g^2,e_2^2,(fe_2)^2,e_gf^{-1}e_gf',(f^{-1}f')^{m/2},(e_2f')^\ell\rangle.$$

Since $e_g=ge$ and $e_2=f^{n/2}e$, we have that

$$\widetilde{\Gamma}=\langle f,g,e\,|\,f^n,e^2,(fe)^2,(ge)^2,(gfg^{-1}f)^{m/2},(f^{n/2}g^{-1}fge)^\ell\rangle.$$

If ℓ is odd, $e\in\langle f,g\rangle$. Therefore, in this case $\Gamma=\widetilde{\Gamma}\cong PH[n,m/2,\ell]$, where $m/2$ is an integer, ∞ or $\overline{\infty}$ and ℓ is odd; the orbifold $Q(\Gamma)$ is shown in Figure 8(b).

Suppose now that ℓ is even. Consider the polyhedron $\mathcal{P}'$ bounded by α, α', ε, $R_\omega(\alpha)$, $R_\omega(\alpha')$ and $e_g(\varepsilon)$. The π–loxodromic element $L=e_ge_2$ identifies the faces of $\mathcal{P}'$ lying in ε and $e_g(\varepsilon)$. Applying the Poincaré polyhedron theorem to $\mathcal{P}'$ and the transformations f, f' and L, we get that $\langle f,f',L\rangle$ is discrete and $\mathcal{P}'$ is a fundamental polyhedron for it. It follows, in particular, that $|\widetilde{\Gamma}:\Gamma|=2$ for even ℓ. Moreover, $\langle f,f',L\rangle$ has the following presentation:

$$\langle f,f',L\,|\,f^n,(f')^n,(f^{-1}f')^{m/2},L^{-1}f'Lf,(L^{-1}fLf')^{\ell/2}\rangle.$$

Since $f'=Lf^{-1}L^{-1}$, the group $\langle f,f',L\rangle$ is generated by f and L and is isomorphic to $\mathcal{S}_2[n,m/2,\ell/2]$. Further, since $L=e_ge_2=gf^{n/2}$, the group $\langle f,L\rangle$ coincides with Γ. Therefore, $\Gamma\cong\mathcal{S}_2[n,m/2,\ell/2]$, where $m/2$ and $\ell/2$ are integers, ∞ or $\overline{\infty}$; the orbifold $Q(\Gamma)$ is shown in Figure 10(a), see also Remark 3.3 and Remark 3.4 after the proof.

2. Now let Γ have parameters as in row P_2 of Table 1. A fundamental polyhedron for $\widetilde{\Gamma}$ is $\mathcal{P}(\alpha,\delta,\varepsilon,\xi,R_\omega(\alpha))$, whose faces are identified by f, e_2, $y=R_\delta R_\omega$ and $z=R_\omega R_\xi$. Then

$$\widetilde{\Gamma}=\langle f,e_2,y,z\,|\,f^n,e_2^2,y^2,z^2,(yz)^2,(yf)^2,(fe_2)^2,(ze_2)^k,(fz)^m\rangle.$$

Using the facts that $e_g = yz = ge$ and $yfy = f^{-1}$, we get $zfz = (zy)(yfy)(yz) = gef^{-1}ge = gfg^{-1}$. Therefore, since m is odd, $z = z(f,g)$. Furthermore, since $e_2 = f^{n/2}e$, $\langle f, e_2, y, z\rangle = \widetilde{\Gamma}$. Similarly to part 1 above, if k is odd then $\widetilde{\Gamma} = \Gamma$ since in this case $e = e(f,g) \in \Gamma$. Further, the group $\langle f, e_2, y, z\rangle$ is obviously isomorphic to $P[n,m,k]$, where $m < \infty$ is also odd. The orbifold $Q(\Gamma)$ is shown in Figure 8(e).

If k is even, Γ is an index 2 subgroup in $\widetilde{\Gamma}$. The polyhedron $\mathcal{P}(\alpha, \varepsilon, \xi, R_\omega(\alpha), R_\delta(\varepsilon))$, whose faces are identified by f, z and $u = ye_2 = zgf^{n/2} \in \Gamma$, satisfies the hypotheses of the Poincaré polyhedron theorem. Then $\langle f, z, u\rangle$ is discrete and has presentation

$$\langle f, z, u \,|\, f^n, z^2, (zf)^m, [z,u]^{k/2}, [f,u]\rangle$$

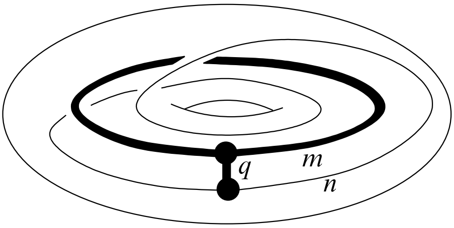
and $\mathcal{P}(\alpha, \varepsilon, \xi, R_\omega(\alpha), R_\delta(\varepsilon))$ is a fundamental polyhedron for this group.

Obviously, $\langle f, z, u\rangle$ is isomorphic to $GTet_1[n,m,k/2]$. On the other hand, since $u = zgf^{n/2}$, we have $\langle f, z, u\rangle = \langle f, g, z\rangle$. Moreover, $z = z(f,g)$ because m is odd. Hence, $\langle f, g, z\rangle = \Gamma$ and, therefore, Γ is isomorphic to $GTet_1[n,m,k/2]$, where $m < \infty$ is odd and $k/2$ is an integer, ∞ or $\overline{\infty}$. The orbifold $Q(\Gamma)$ is shown in Figure 10(d).

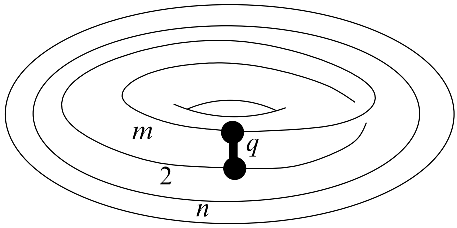
3. If Γ has parameters as in row P_3 of Table 1, it is easy to show that $\Gamma = \widetilde{\Gamma}$ and Γ is isomorphic to a tetrahedron group Tet$[4, m; 3]$, where $5 \le m < \infty$ is odd. □

Remark 3.3 Note that when $Q = Q(\mathcal{S}_2[n, m/2, \ell/2])$, due to the action of the face pairing transformation of the fundamental polyhedron, Q is embedded in a Seifert fiber space $\mathcal{S}(2)$ and the singular set is placed in $\mathcal{S}(2)$ in such a way that the axis of order m (if $m < \infty$) lies on a critical fiber of $\mathcal{S}(2)$ and the axis of order n lies on a regular one. In Figure 10(a) we draw only the solid torus that contains singular points (or boundary components). The other fibered torus is meant to be attached and is not shown.

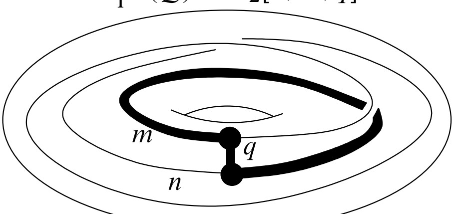
Remark 3.4 As an illustration of the orbifold covering $Q(\Gamma) \to Q(\widetilde{\Gamma})$, consider the case when parameters of Γ are as in row P_1 of Table 1 and $t(u) = \ell$ is even. Denote $Q = Q(\Gamma)$ and $\widetilde{Q} = Q(\widetilde{\Gamma})$, where $\Gamma \cong \mathcal{S}_2[n, m/2, \ell/2]$ and $\widetilde{\Gamma} \cong PH[n, m/2, \ell]$. Let us show the structure of the orbifold covering $\pi\colon Q \to \widetilde{Q}$. Assume for simplicity that $m, \ell < \infty$. Draw the orbifold Q (same as in Figure 10(a), but with the change of indices $q \mapsto \ell/2$, $m \mapsto m/2$) in the spherical shell $\mathbb{S}^2 \times I$ as shown in Figure 11; keep in mind that the inner and outer spheres are identified. In Figure 11, the labels on the upper left and the lower right pictures are integers and denote the cone singularities. The labels on the central pictures (which show the structure of the covering) are of the form $2\pi/k$; they indicate cone/dihedral angles.



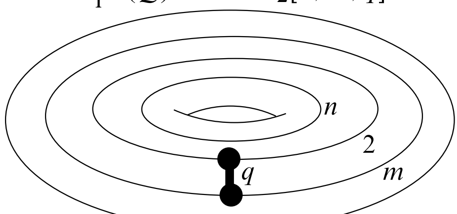
(a) Orbifolds embedded in $\mathcal{S}(2)$; $\pi_1^{\text{orb}}(Q) \cong \mathcal{S}_2[n, m, q]$



(b) Orbifolds embedded in $\mathcal{S}(2)$; $\pi_1^{\text{orb}}(Q) \cong GTet_2[n, m, q]$



(c) Orbifolds embedded in $\mathcal{S}(3)$; $\pi_1^{\text{orb}}(Q) \cong \mathcal{S}_3[n, m, q]$



(d) Orbifolds embedded in $\mathbb{S}^2 \times \mathbb{S}^1$; $\pi_1^{\text{orb}}(Q) \cong GTet_1[n, m, q]$

Figure 10: Orbifolds embedded in Seifert fiber spaces; only the torus that contains cone points or boundary components is shown.

Let σ be a circle in the xy-plane such that the inversion in the sphere for which σ is a big circle identifies the inner and the outer spheres. Let s be the orientation preserving automorphism of Q induced by the composition of this inversion and the reflection in the xy–plane. Thus, s is of order 2 with the axis σ. Then s determines $\pi\colon Q \to \widetilde{Q}$ and $\langle \pi_1^{\text{orb}}(Q), s\rangle = \pi_1^{\text{orb}}(\widetilde{Q})$. The underlying space of $\widetilde{Q}$ is $\mathbb{S}^3$.

Theorem 3.5 *Let $(\Gamma; f, g)$ be a discrete $\mathcal{RP}$ group so that $\beta = -4\sin^2(\pi/n)$, $n \geq 3$, $(n, 2) = 1$, $\beta' \in (0, +\infty)$ and $\gamma \in (0, -\beta\beta'/4)$. Then one of the following occurs.*

(1) $\gamma = 4\cosh^2 u + \beta$, *where* $u \in \mathcal{U}$, $(t(u), 2) = 2$, $1/n + 1/t(u) < 1/2$, *and* $\beta' = \frac{2}{\gamma}(\cosh v - \cos(\pi/n)) - \frac{2}{\gamma\beta}\big((\gamma-\beta)^2\cos(\pi/n) + \gamma(\gamma+\beta)\big)$, *where* $v \in \mathcal{U}$; Γ *is isomorphic to* $\mathcal{S}_3[n, t(u)/2, t(v)]$.

(2) $\gamma = 4\cosh^2 u + \beta$, *where* $u \in \mathcal{U}$, $(t(u), 2) = 1$, $1/n + 1/t(u) < 1/2$, *and* $\beta' = \frac{2(\gamma-\beta)}{\gamma}\cosh v - \frac{2}{\gamma\beta}\big((\gamma-\beta)^2\cos(\pi/n) + \gamma(\gamma+\beta)\big)$, *where* $v \in \mathcal{U}$; Γ *is isomorphic to* $GTet_2[n, t(u), t(v)]$.

(3) $n \geq 7$, $\gamma = (\beta+4)(\beta+1)$ *and* $\beta' = 2(\beta+2)^2(\cosh v - \cos(\pi/n))/(\beta+1) - 2\left(\beta^2 + 6\beta + 4\right)/\beta$, *where* $v \in \mathcal{U}$; Γ *is isomorphic to* $GTet_2[n, 3, t(v)]$.

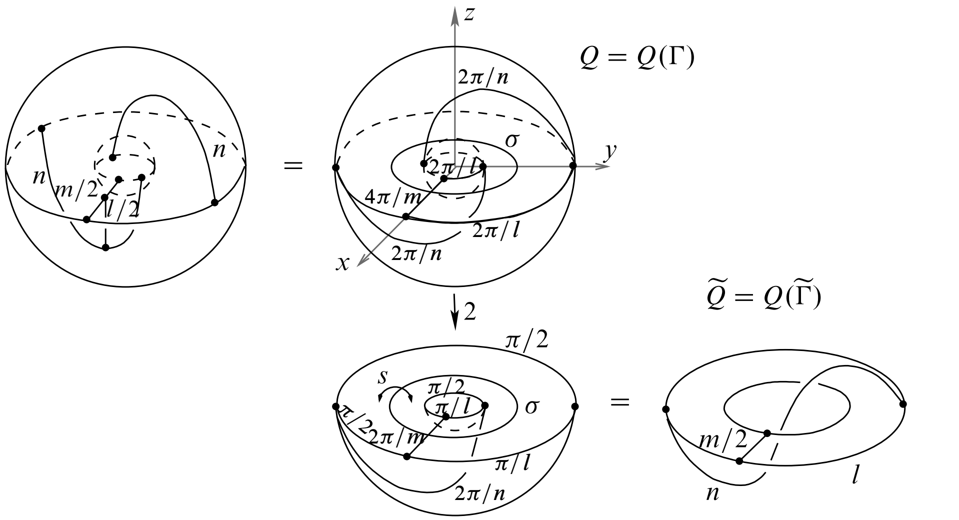


Figure 11: Branched covering $\pi\colon Q \to \widetilde{Q}$

(4) $\beta = -3$, $\gamma = 2\cos(2\pi/m) - 1$, *where* $m \geq 7$, $(m, 2) = 1$, *and* $\beta' = 2(\gamma^2 + 2\gamma + 2)/\gamma$; Γ *is isomorphic to* $GTet_1[m, 3, 2]$.

(5) $n \geq 5$, $(n, 3) = 1$, $\gamma = \beta + 3$ *and* $\beta' = 2\left((\beta - 3)\cos(\pi/n) - 2\beta - 3\right)/\beta$; Γ *is isomorphic to* $H[2; 3, n; 2]$.

(6) $(\beta, \gamma, \beta') = ((\sqrt{5} - 5)/2, (\sqrt{5} \pm 1)/2, 3(\sqrt{5} + 1)/2)$; Γ *is isomorphic to* $H[2; 2, 5; 3]$.

(7) $(\beta, \gamma, \beta') = (-3, (\sqrt{5} \pm 1)/2, \sqrt{5})$ *or* $(\beta, \gamma, \beta') = ((\sqrt{5} - 5)/2, (\sqrt{5} - 1)/2, \sqrt{5})$, *or* $(\beta, \gamma, \beta') = ((\sqrt{5} - 5)/2, \sqrt{5} + 2, (5\sqrt{5} + 9)/2)$; *in all cases* Γ *is isomorphic to* $H[2; 2, 3; 5]$.

(8) $(\beta, \gamma, \beta') = ((\sqrt{5} - 5)/2, (\sqrt{5} - 1)/2, (3\sqrt{5} - 1)/2)$; Γ *is isomorphic to* Tet$[3, 3; 5]$.

(9) $\beta = -3$, $\gamma = 2\cos(2\pi/m)$, *where* $m \geq 5$, $(m, 4) = 1$, *and* $\beta' = 2\gamma$; Γ *is isomorphic to* Tet$[4, m; 3]$.

(10) $n \geq 5$, $(n, 3) = 1$, $\gamma = 2(\beta + 3)$ *and* $\beta' = -6(2\cos(\pi/n) + \beta + 2)/\beta$; Γ *is isomorphic to* $R[n, 2; 2]$.

(11) $\beta = -3$, $\gamma = 2\cos(2\pi/m)$, *where* $m \geq 8$, $(m, 4) = 2$, *and* $\beta' = 2\gamma$; Γ *is isomorphic to* $H[m; 3, 3; 2]$.

(12) $\beta = -3$, $\gamma = 2\cos(2\pi/m) - 1$, *where* $m \geq 4$, $(m, 3) = 1$, *and* $\beta' = \gamma^2 + 4\gamma$; Γ *is isomorphic to* Tet$[2, 3, 3; 2, 3, m]$.

Proof Now we shall use fundamental polyhedra for Γ^* described in Section 2.3 for n odd and the Poincaré theorem to find a presentation for Γ.

1. Let Γ have parameters as in row P_4 of Table 1. Consider the polyhedron bounded by α, α', $R_\omega(\alpha)$, $R_\omega(\alpha')$, ζ, $R_\omega(\zeta)$, $e_g(\zeta)$ and $R_\omega(e_g(\zeta))$, which is the union of four copies of $\mathcal{P}$ shown in Figure 3(a). Its faces are identified by f, $f' = e_g f^{-1} e_g$ and two loxodromic elements $L = e_g e_1$ and $L' = e_g e_1 f^{-1} = L f^{-1}$. Using the Poincaré polyhedron theorem, one can show that

$$\langle f, f', L, L'\rangle = \langle f, L \,|\, f^n, (LfL^{-1}f)^{m/2}, (fL^{-1}fL^{-1}fL^2)^{\ell}\rangle.$$

Obviously, $\langle f, L\rangle \cong \mathcal{S}_3[n, m/2, \ell]$. Further, since $L = e_g e_1 = gef^{(n-1)/2}e = gf^{-(n-1)/2}$, the group $\langle f, L\rangle$ coincides with Γ. Hence, Γ is isomorphic to $\mathcal{S}_3[n,m/2,\ell]$, where m is even $(1/n + 1/m \le 1/2)$, $m = \infty$ or $m = \overline{\infty}$ and $\ell \ge 2$ is an integer. The orbifold $Q(\Gamma)$ is shown in Figure 10(c).

2. Let Γ have parameters as in row P_5. Denote $y = R_\delta R_\omega$, $\zeta_1' = R_\omega(\zeta_1)$ and consider the polyhedron $\mathcal{P}(\alpha, R_\omega(\alpha), \xi, \zeta_1, \zeta_1', y(\zeta_1), y(\zeta_1'))$, which is the union of four copies of the polyhedron $\mathcal{Q}$ shown in Figure 3(b). Its faces are identified by the transformations f, $v = ye_g$, $u = ye_1$ and $u' = yfe_1$. As usual, we apply the Poincaré polyhedron theorem to get

$$\langle f, v, u, u'\rangle = \langle f, v, u \,|\, f^n, v^2, (fv)^m, (fvuvu^{-1})^k, [f, u]\rangle.$$

We see that $\langle f, v, u\rangle \cong GTet_2[n, m, k]$, where the isomorphism is given by $f \mapsto x^{-1}$, $v \mapsto y$, $u \mapsto z^{-1}$. So it remains to show that $\langle f, v, u\rangle$ is actually generated by f and g.

First, note that since the axis of y is orthogonal to the axis of f, $yfy = f^{-1}$. Now since m is odd and $v^2 = 1$, we can write

$$\begin{aligned} 1 = (fv)^m &= (fvfv)^{(m-1)/2} fv = (fe_g yfye_g)^{(m-1)/2} fv \\ &= (fe_g f^{-1} e_g)^{(m-1)/2} fv = (fgfg^{-1})^{(m-1)/2} fv. \end{aligned}$$

Therefore, $v = (fgfg^{-1})^{(m-1)/2} f \in \Gamma$. Further, $u = ye_1 = vgef^{(n-1)/2}e = vgf^{-(n-1)/2}$ and hence $u \in \Gamma$. So we have shown that $\langle f, v, u\rangle \subseteq \Gamma$. On the other hand, $g = vuf^{(n-1)/2}$ and hence $\Gamma \subseteq \langle f, v, u\rangle$. Then $\Gamma = \langle f, v, u\rangle \cong GTet_2[n, m, k]$, where m is odd $(1/n + 1/m < 1/2)$ and $k \ge 2$ is an integer, ∞ or $\overline{\infty}$.

Now suppose that Γ has parameters as in row P_6. This case is similar to the case of parameters P_5, but technically it is more complicated.

Denote $t = R_\delta R_\tau$, $y = R_\alpha R_\delta$ and $v = yt$. Consider the polyhedron bounded by ω, $R_\alpha(\omega)$, τ, ζ_2, $R_\alpha(\zeta_2)$, $y(\zeta_2)$ and $y(R_\alpha(\zeta_2))$, which is the union of four copies of the polyhedron shown in Figure 4(a). Its faces are identified by f, v, $u = ye_1$ and $u'' = ye_1 f$. Again, by the Poincaré polyhedron theorem, we get

$$\langle f, v, u, u''\rangle = \langle f, v, u | f^n, v^2, (fv)^3, (fvuvu^{-1})^k, [f, u]\rangle \cong GTet_2[n, 3, k].$$

Further,

$$u = ye_1 = (ye_g)(e_g e_1) = (R_\alpha R_\delta)(R_\delta R_\xi)(gef^{(n-1)/2}e) = h_1^{-1} g f^{-(n-1)/2},$$

where $h_1 = R_\xi R_\alpha$. Note that $h_1 = vf^{-2}v$. Then

$$g = h_1 u f^{(n-1)/2} = vf^{-2}vuf^{(n-1)/2}.$$

Hence, Γ is a subgroup of $\langle f, v, u\rangle$. Now one can apply the Todd–Coxeter algorithm, see eg Johnson [8] to $\langle f, v, u\rangle$ and its subgroup Γ generated by f and $g = vf^{-2}vuf^{(n-1)/2}$ to show that $|\langle f, v, u\rangle : \Gamma| = 1$, ie, $\langle f, v, u\rangle = \Gamma$.

The orbifold with the fundamental group $GTet_2[n, m, k]$ is shown in Figure 10(b).

3. If Γ has parameters as in row P_9, we consider the polyhedron bounded by ω, ξ, $R_\alpha(\omega)$, $R_\alpha(\xi)$, μ and $R_\delta(\mu)$ (compare with Figure 4(b)) whose faces are identified by f, $h_1 = R_\xi R_\alpha$ and $z = R_\delta R_\mu$. Then $\langle f, h_1, z\rangle$ has the presentation

$$\langle f, h_1, z | f^3, h_1^m, (f^{-1}h_1)^2, [f, z]^2, [h_1, z]\rangle \tag{3.3}$$

Hence, $\langle f, h_1, z\rangle \cong GTet_1[m, 3, 2]$, where the isomorphism is given by $f \mapsto xy$, $h_1 \mapsto x$, $z \mapsto z$. Let us show that $\langle f, h_1, z\rangle = \Gamma$.

Denote $a = R_\mu R_\alpha$, $b = R_\delta R_\alpha$ and $s = R_\alpha R_{\zeta_1}$. Then $z = ba$. Since the axis of b is orthogonal to f, we have $bfb = f^{-1}$ and, since μ is orthogonal to α, we have that $a^2 = 1$. From the decomposition of the link made by α, ω and ζ_1 by the reflection planes, we obtain

$$e_1 = sas^{-1} \text{ and } s = afa.$$

Therefore, $e = f^{-1}e_1 = f^{-1}sas^{-1} = f^{-1}afaf^{-1}a$ and

$$g = e_g e = h_1 b f^{-1}afaf^{-1}a = h_1 fzfz^{-1}fz.$$

So we have shown that Γ is a subgroup of $\langle f, h_1, z\rangle$. Now it is sufficient to apply the Todd–Coxeter algorithm to the group $\langle f, h_1, z\rangle$ given by presentation (3.3) and its subgroup $\langle f, g\rangle$ to see that $\langle f, h_1, z\rangle = \Gamma$.

Thus $\Gamma \cong GTet_1[m, 3, 2]$ and the orbifold $Q(\Gamma)$ is shown in Figure 10(d).

4. Consider the groups with parameters as in rows P_{11}–P_{15}, P_{17}, P_{18}. In all of these cases $R_\zeta \in \Gamma^*$. We know a fundamental polyhedron for Γ^* and the structure of Γ^* in each case. Since all these polyhedra are obtained as decompositions of $\mathcal{P}$ into smaller polyhedra, they have common properties. Namely, (P1) the elements $f' = R_{\alpha'} R_\omega$ and $h = R_{\alpha'} R_\alpha$ belong to Γ. Indeed,

$$f' = R_{\alpha'} R_\omega = e_g R_\alpha e_g R_\omega = (e_g R_\alpha R_\omega)(R_\omega e_g R_\omega) \\ = e_g f^{-1} e_g = g f g^{-1}$$

and $h = f' f = g f g^{-1} f$.

(P2) the elements $h_2 = R_{\alpha'} R_\zeta$, $t_1 = R_\alpha R_\zeta$ and $t_2 = R_\omega R_\zeta$ belong to Γ. Denote $\alpha'' = e_1(\alpha')$. Note that $R_{\alpha''} = e_1 R_{\alpha'} e_1$ and $R_\omega = e_1 R_\alpha e_1$. Then

$$h_2^2 = R_{\alpha'} R_{\alpha''} = (R_{\alpha'} R_\alpha)(R_\alpha R_{\alpha''}) = h R_\alpha e_1 R_{\alpha'} e_1 = h e_1 R_\omega R_{\alpha'} e_1 \\ = h e_1 g f^{-1} g^{-1} e_1 = g f g^{-1} f^{-(n-1)/2} g^{-1} f g f^{-(n-1)/2}.$$

Since h_2 always has odd order for the groups with parameters P_{11}–P_{15}, P_{17}, P_{18}, the fact that $h_2^2 \in \Gamma$ implies $h_2 \in \Gamma$. Further, since $t_1 = (R_\alpha R_{\alpha'})(R_{\alpha'} R_\zeta) = h^{-1} h_2$ and $t_2 = R_\omega R_\zeta = e_1 R_\alpha e_1 R_\zeta = e_1 R_\alpha R_\zeta e_1 = e_1 t_1 e_1$, both t_1 and t_2 belong to Γ.

For the groups with parameters P_{12}, P_{13}, P_{15}, P_{17} or P_{18}, $\Gamma^* = \langle G_T, e_g \rangle$, where e_g coincides with the axis of a $\mathbb{Z}_2$–symmetry of T (see Figures 5(b), 6(a), 6(c)–(e)). Then $\widetilde{\Gamma} = \langle \Delta_T, e_g \rangle$, where Δ_T is the orientation preserving subgroup of G_T. Proceeding as in the proof of the property (P2), one can show that the rotations from Δ_T belong to Γ. In particular, since e_1 passes through an edge of T, $e_1 \in \Gamma$ and, therefore, $e \in \Gamma$. Thus, $\Gamma = \widetilde{\Gamma}$. If T is a compact tetrahedron, it was shown by Derevnin and Mednykh [3] that each $\langle \Delta_T, e_g \rangle$ is isomorphic to some $H[p; n, m; q]$. It is easy to see that the same is true for non-compact T. It remains to find p, n, m and q, which is not difficult to do since the position of e_g is known in each case. For example, if Γ has parameters P_{12}, $\Gamma \cong H[2; 2, 3; 5]$.

Now consider the groups with parameters as in row P_{14} (see Figure 6(b)). In this case $T = T[2, 2, 4; 2, 3, 5]$. Denote by ρ the reflection plane through e_g and $\alpha' \cap \zeta$ and let $\overline{h}_2 = R_\rho R_\zeta$. Then $\overline{h}_2^2 = h_2$.

It is clear that Δ_T is generated by e_1, t_1 and $\overline{h}_2$ and has the presentation

$$\langle e_1, t_1, \overline{h}_2 | e_1^2, t_1^3, \overline{h}_2^5, (e_1 t_1)^4, (e_1 \overline{h}_2)^2, (t_1^{-1} \overline{h}_2)^2 \rangle \cong \mathrm{Tet}[4, 5; 3].$$

Let us show that $\Delta_T = \Gamma$. From the link of the vertex made by α, ω and ζ, we see that $f = t_2 t_1^{-1} = e_1 t_1 e_1 t_1^{-1}$. Since $e_g = \overline{h}_2 t_1$, $g = e_g e = \overline{h}_2 t_1 f^{-1} e_1 = \overline{h}_2 t_1^2 e_1 t_1^{-1}$. Therefore, $\langle f, g \rangle$ is a subgroup of Δ_T. Furthermore, since $\overline{h}_2$ is of odd order, the Todd–Coxeter method gives us that $\langle f, g \rangle$ coincides with Δ_T. Thus $\Gamma \cong \mathrm{Tet}[4, 5; 3]$.

We remark that the case of the groups with parameters as in row P_{11} with $(r,4)=1$ is analogous to the case of the groups with parameters as in row P_{14} with the difference that h is hyperbolic and $T=T[2,2,4;2,3,r]$ is an infinite volume tetrahedron. The group Γ is then isomorphic to Tet$[4,r;3]$, where $r\geq 7$ is odd.

Consider the groups with parameters P_{11} with $(r,4)=2$. The consideration is quite delicate so we shall do it in detail.

Let κ be the reflection plane passing through e_1 orthogonally to ζ (see Figure 5(a)), let τ be the plane through e_g and t_1, and let ρ again be the plane through e_g and $\alpha'\cup\zeta$. Denote $s=R_\rho R_\kappa$, $s'=R_\tau s R_\tau$, $u=R_\tau R_{\alpha'}$ and consider $\mathcal{P}(\alpha,\zeta,\kappa,\alpha', R_\tau(\alpha'), R_\tau(\kappa))$. Its faces are identified by s, s', t_1 and u. Then by the Poincaré polyhedron theorem we get the presentation

$$G=\langle s,s',t_1,u \mid s^2,(s')^2,t_1^3,u^3,(t_1u)^{r/2},ust_1s',(ss')^2\rangle.$$

Since $s'=ust_1$,

$$G=\langle s,t_1,u \mid s^2,t_1^3,u^3,(t_1u)^{r/2},(sust_1)^2,(ust_1)^2\rangle\cong H[r/2;3,3;2],$$

where $r/2\geq 5$ and $r/2$ is odd. We claim that $G=\Gamma$.

Note that $R_\omega=R_\kappa R_\tau R_\kappa$ and $R_\alpha=R_\tau R_\zeta R_\tau$. Therefore,

$$\begin{aligned} f=R_\omega R_\alpha &= R_\kappa R_\tau R_\kappa R_\tau R_\zeta R_\tau = ((R_\kappa R_\rho)(R_\rho R_\tau))^2(R_\zeta R_\tau)\\ &=(se_g)^2t_1=ss't_1, \end{aligned}$$

because $e_g s e_g = s'$. Hence, $f=sust_1^2$. Denote as before $\overline{h}_2=R_\rho R_\zeta$ and $h_2=\overline{h}_2^{\,2}$. Since $e_1=\overline{h}_2^{\,-1}s$, we get $e=e_1f=\overline{h}_2^{\,-1}ust_1^2$. Therefore, since $\overline{h}_2^{\,-2}=h_2=u^{-1}t_1^{-1}$ and $e_g=t_1^{-1}\overline{h}_2^{\,-1}$, we obtain that

$$g=e_ge=t_1^{-1}\overline{h}_2^{\,-2}ust_1^2=u^2st_1^2.$$

So we have proved that Γ is a subgroup of G.

On the other hand, since $h_2\in\Gamma$ and $t_1\in\Gamma$ (see the property (P2)), we get that $u=t_1^{-1}h_2^{-1}\in\Gamma$ and, therefore, $s=u^{-2}gt_1^{-2}=ugt_1\in\Gamma$. Thus, $\Gamma=G$.

5. Consider the remaining cases of groups, with parameters as in rows P_7, P_8, P_{10}, P_{16} and P_{19}. In all of these cases $R_\zeta\notin\Gamma^*$. As in part 4 of the proof, the elements $h=R_{\alpha'}R_\alpha$ and $h_3=R_{\alpha'}R_{\alpha''}$ belong to Γ. Denote $\sigma=e_1(\eta)$, $a=R_\eta R_\alpha$ and $b=R_\alpha R_\sigma$.

Suppose Γ has parameters as in row P_7. Consider the tetrahedron $T=T[2,3,n;2,3,n]$ bounded by α, ω, η and σ. Denote $s=R_\sigma e_g R_\sigma$ and $t=e_1 s$. Then t passes through the "midpoints" of the edges with dihedral angles of $\pi/2$ and all e_1, s and t are the axes of $\mathbb{Z}_2$–symmetries of T.

It is clear that the group Δ_T, which is the orientation preserving subgroup of G_T, is generated by f, a and b. Let $H = \langle \Delta_T, t\rangle$. We leave a proof of the fact that $H \cong H[2;3,n;2]$ as an exercise for the reader, but we prove that $H = \Gamma$. Let $\sigma' = e_g(\sigma)$. Then $R_{\sigma'} = e_g R_\sigma e_g$ and $R_\sigma R_{\sigma'} = R_\omega R_\sigma$. Therefore,

$$e_1 e_g = (e_1 s)(s e_g) = ts R_\sigma R_\sigma e_g = t R_\sigma e_g R_\sigma e_g = t R_\sigma R_{\sigma'} = t R_\omega R_\sigma = tfb.$$

Hence, $g = e_g e = e_g e_1 f^{(n-1)/2} = (tfb)^{-1} f^{(n-1)/2} \in H$. So $\Gamma \subseteq H$.

On the other hand, $e_1 e_g = f^{(n-1)/2} g^{-1}$. Denote $\overline{h}_3 = R_\sigma R_\eta = R_{\alpha'} R_\sigma$. We know that $\overline{h}_3^3 = h_3 \in \Gamma$. Since $(n,3) = 1$, $\overline{h}_3 \in \Gamma$. Then $b = (R_\alpha R_{\alpha'})(R_{\alpha'} R_\sigma) = h^{-1}\overline{h}_3 \in \Gamma$, $a = \overline{h}_3^{-1} b^{-1} \in \Gamma$ and $t = f^{(n-1)/2} g^{-1} b^{-1} f^{-1} \in \Gamma$. Thus, $H = \langle f, a, b, t\rangle$ is a subgroup of Γ. So $\Gamma = H \cong H[2;3,n;2]$.

Suppose Γ has parameters as in row P_8. Let now κ be the reflection plane such that $e_g = R_\kappa R_\sigma$ and let $s = R_\kappa e_1 R_\kappa$. Let ρ be the plane through M, K and L, and let $\tau = R_\kappa(\rho)$ (see Figure 7(b)). Then τ passes through M, K and L', where $L' = R_\kappa(L)$, and s lies in ρ. Moreover, the sum of the angles that ρ makes with α and σ equals π and ρ intersects τ orthogonally. Consider the polyhedron $\mathcal{P} = \mathcal{P}(\alpha, \eta, \alpha', R_\sigma(\alpha), \rho, \tau, R_\sigma(\rho), R_\sigma(\tau))$. Its faces are identified by $\overline{h}_3$, $z = e_g \overline{h}_3 e_g$, $u = s e_g$ and $v = e_1 e_g$. Using the Poincaré polyhedron theorem, we get the presentation

$$H = \langle z, \overline{h}_3, u, v \,|\, z^n, \overline{h}_3^n, (\overline{h}_3 z)^2, v u \overline{h}_3, u v z^{-1}, (u v^{-1})^2\rangle.$$

Since $z = uv$ and $\overline{h}_3 = u^{-1} v^{-1}$,

$$H = \langle u, v \,|\, (uv)^n, (uv^{-1})^2, [u,v]^2\rangle \cong R[n,2;2].$$

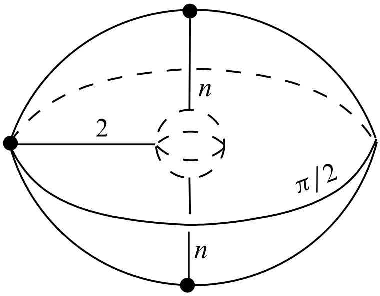
Let us show that $H = \Gamma$. First, note that since $(n,3) = 1$, $\overline{h}_3 \in \Gamma$. Therefore, since $v = e_1 e_g = f^{(n-1)/2} g^{-1} \in \Gamma$ and $u = v^{-1}\overline{h}_3 \in \Gamma$, $H \subseteq \Gamma$.

In order to express f and g in terms of u and v, we recall that $R_\sigma = e_1 R_\eta e_1$ and $R_\omega = e_1 R_\alpha e_1$ and note that $z = R_\kappa \overline{h}_3^{-1} R_\kappa = R_\alpha R_\sigma$. Then

$$s\overline{h}_3 s = R_\kappa e_1 R_\kappa \overline{h}_3 R_\kappa e_1 R_\kappa = R_\kappa e_1 z^{-1} e_1 R_\kappa = R_\kappa e_1 R_\sigma R_\alpha e_1 R_\kappa = R_\kappa R_\eta R_\omega R_\kappa.$$

Furthermore, $R_\alpha = R_\kappa R_\eta R_\kappa$ and, since κ is orthogonal to ω, $R_\omega R_\kappa = R_\kappa R_\omega$. Hence, $s\overline{h}_3 s = R_\alpha R_\kappa R_\omega R_\kappa = R_\alpha R_\omega = f^{-1}$. On the other hand, $s\overline{h}_3 s = s e_g z e_g s = u^2 v u^{-1}$. Thus, $f = u v^{-1} u^{-2}$ and $g = v^{-1} f^{(n-1)/2} = v^{-1}(u v^{-1} u^{-2})^{(n-1)/2}$. So we have shown that $\Gamma \subseteq H$ and, hence, $\Gamma = H$.

Gluing the faces of $\mathcal{P}$ by $\overline{h}_3$, z, u and v, we obtain the orbifold embedded in $\mathbb{R}P^3$ (see Figure 12).



$\pi_1^{\mathrm{orb}}(Q) \cong R[n,2;2]$

Figure 12: Underlying space is $\mathbb{R}P^3 \backslash \mathbb{B}^3$

Suppose that Γ has parameters as in row P_{10}. Consider the tetrahedron $T = T[2,3,3;2,3,m/2]$ bounded by α, ω, η and σ. The group Δ_T has the presentation

$$\Delta_T = \langle f,a,b \,|\, f^3, a^2, b^3, (af)^3, (bf)^2, (ab)^{m/2} \rangle \cong \mathrm{Tet}[2,3,3;2,3,m/2].$$

We shall show that $\Delta_T = \Gamma$. Note that $e_g = b^{-1}e_1 b$. Then

$$e_1 b e_1 = e_1 R_\alpha R_\sigma e_1 = e_1 R_\alpha e_1 R_\eta = R_\omega R_\eta = fa.$$

Therefore,

$$g = e_g e_1 f = b^{-1} e_1 b e_1 f = b^{-1} faf.$$

Hence, $\Gamma = \langle f,g \rangle$ is a subgroup of Δ_T. Applying the Todd–Coxeter algorithm, we see that, since $(m/2,3) = 1$, $\langle f,g \rangle = \Delta_T$.

Similarly, one can show that in the case of the parameters of type P_{16}, $\Gamma = \Delta_T$, where $T = T[2,2,3;2,5,3]$ and $\Delta_T \cong \mathrm{Tet}[3,3;5]$.

Suppose that Γ has parameters as in row P_{19}. In this case $\Gamma^* = \langle G_T, e_g \rangle$, where $T = T[2,2,3;2,5,3]$. Then $\widetilde{\Gamma} = \langle \Delta_T, e_g \rangle$. Notice that all rotations from Δ_T belong to Γ, in particular, $e_1 \in \Gamma$. Hence, $e \in \Gamma$ and, therefore, $\Gamma = \widetilde{\Gamma}$. It was shown by Derevnin and Mednykh [3] that $\langle \Delta_T, e_g \rangle \cong H[2;2,3;5]$. Thus, $\Gamma \cong H[2;2,3;5]$. □

Remark 3.6 When $Q = Q(\mathcal{S}_3[n,m,q])$, the singular set of Q is placed into $\mathcal{S}(3)$ in such a way that the curve consisting of cone points of indices n and m lies on a regular fiber. When $Q = Q(GTet_2[n,m,q])$, the curve consisting of cone points of indices m and 2 lies on a regular fiber and the singular component of index n lies on a critical fiber.

In Figure 12, $\mathbb{R}P^3$ is shown as a lens with antipodal points on the boundary identified. The angle at the edge of the lens is $\pi/2$ and, therefore, the edge is mapped onto a singular loop with index 2.

References

[1] **M Boileau**, **S Maillot**, **J Porti**, *Three-dimensional orbifolds and their geometric structures*, Panoramas et Synthèses 15, Société Mathématique de France, Paris (2003) MR2060653

[2] **M Boileau**, **J Porti**, *Geometrization of 3–orbifolds of cyclic type*, Astérisque (2001) 208 MR1844891 Appendix A: Limit of hyperbolicity for spherical 3–orbifolds by M Heusener and J Porti

[3] **D A Derevnin**, **A D Mednykh**, *Discrete extensions of the Lanner groups*, Dokl. Akad. Nauk 361 (1998) 439–442 MR1693083 Translation in: Dokl. Math. 58 (1998), no. 1, 78–80

[4] **D A Epstein**, **C Petronio**, *An exposition of Poincaré's polyhedron theorem*, Enseign. Math. (2) 40 (1994) 113–170 MR1279064

[5] **F W Gehring**, **G J Martin**, *Stability and extremality in Jørgensen's inequality*, Complex Variables Theory Appl. 12 (1989) 277–282 MR1040927

[6] **F W Gehring**, **G J Martin**, *Chebyshev polynomials and discrete groups*, from: "Proceedings of the Conference on Complex Analysis (Tianjin, 1992)", Conf. Proc. Lecture Notes Anal., I, Int. Press, Cambridge, MA (1994) 114–125 MR1343502

[7] **C D Hodgson**, **I Rivin**, *A characterization of compact convex polyhedra in hyperbolic 3-space*, Invent. Math. 111 (1993) 77–111 MR1193599

[8] **D L Johnson**, *Presentations of groups*, London Mathematical Society Student Texts 15, Cambridge University Press, Cambridge (1990) MR1056695

[9] **E Klimenko**, *Discrete groups in the three-dimensional Lobachevsky space that are generated by two rotations*, Sibirsk. Mat. Zh. 30 (1989) 95–100 MR995026

[10] **E Klimenko**, *Some remarks on subgroups of* $\mathrm{PSL}(2,\mathbf{C})$, Questions Answers Gen. Topology 8 (1990) 371–381 MR1065285

[11] **E Klimenko**, *Some examples of discrete groups and hyperbolic orbifolds of infinite volume*, J. Lie Theory 11 (2001) 491–503 MR1851804

[12] **E Klimenko**, **N Kopteva**, *Discreteness criteria for $\mathcal{RP}$ groups*, Israel J. Math. 128 (2002) 247–265 MR1910384

[13] **E Klimenko**, **N Kopteva**, *All discrete $\mathcal{RP}$ groups whose generators have real traces*, Internat. J. Algebra Comput. 15 (2005) 577–618 MR2151429

[14] **E Y Klimenko**, **N V Kopteva**, *Discrete $\mathcal{RP}$–groups with a parabolic generator*, Sibirsk. Mat. Zh. 46 (2005) 1069–1076 MR2195032

[15] **E Klimenko**, **N Kopteva**, *Two-generator Kleinian orbifolds* (2006) arXiv:math.GT/0606066

[16] **A W Knapp**, *Doubly generated Fuchsian groups*, Michigan Math. J. 15 (1969) 289–304 MR0248231

Max Planck Institute for Mathematics
Vivatsgasse 7, 53111 Bonn, Germany

Sobolev Institute of Mathematics
Acad. Koptyug ave., 4, Novosibirsk 630090, Russia

klimenko@mpim-bonn.mpg.de, natasha@math.nsc.ru

Received: 6 October 2006 Revised: 16 March 2007