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On the degeneration ratio of tunnel numbers and free tangle decompositions of knots

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In this paper, we introduce a notion called n/k–free tangle and study the degeneration ratio of tunnel numbers of knots.

57M25, 57N10

1 Introduction

Let K be a knot in the 3–sphere S^3, $t(K)$ the tunnel number of K and $K_1 \# K_2$ the connected sum of two knots K_1 and K_2, where $t(K)$ is the minimal genus -1 among all Heegaard splittings which contain K as a core of a handle. Concerning the relationship between $t(K_1)+t(K_2)$ and $t(K_1 \# K_2)$, we showed in Morimoto [2] that there are infinitely many tunnel number two knots K such that $t(K \# K')$ is two again for any 2–bridge knots K'. These are the first examples whose tunnel numbers go down under connected sum, ie, "2+1 = 2". Subsequently, Kobayashi showed in Kobayashi [1], by taking connected sum of those knots, that there are infinitely many pairs of knots (K_1, K_2) such that $t(K_1 \# K_2) < t(K_1) + t(K_2) - n$ for any integer $n > 0$. This shows that tunnel numbers of knots have arbitrarily high degeneration.

Contrary to these phenomena, Scharlemann and Schultens introduced in [5] a notion called *degeneration ratio* which is a ratio of $t(K_1 \# K_2)$ and $t(K_1)+t(K_2)$, and showed in [5] that $\dfrac{t(K_1 \# K_2)}{t(K_1)+t(K_2)} \geq \dfrac{2}{5}$ for any prime knots K_1 and K_2. We note that Scharlemann and Schultens's original degeneration ratio is $1 - \dfrac{t(K_1 \# K_2)}{t(K_1)+t(K_2)}$, but we use the above one for convenience.

The degeneration ratio of our first example in Morimoto [2] is $\dfrac{2}{3}$ because $t(K_1) = 2$, $t(K_2) = 1$ and $t(K_1 \# K_2) = 2$. In fact, this is the smallest example among all we know so far. In this article, we introduce a notion called n/k–free tangle and study the existence of a pair (K_1, K_2) such that $\dfrac{t(K_1 \# K_2)}{t(K_1)+t(K_2)} < \dfrac{2}{3}$.

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Throughout the present paper, we will work in the piecewise linear category. For a manifold X and a subcomplex Y in X, we denote a regular neighborhood of Y in X by $N(Y, X)$ or simply $N(Y)$.

2 Free tangles

Let M be a compact 3–manifold with boundary, and $T = t_1 \cup t_2 \cup \cdots \cup t_n$ the mutually disjoint arcs properly embedded in M. Then we say that T is a *trivial arc system* if there are mutually disjoint disks $\Delta_1, \Delta_2, \ldots, \Delta_n$ in M such that $\partial \Delta_i = t_i \cup t_i'$ $(i = 1, 2, \ldots, n)$, where t_i' is an arc in ∂M.

Let $M = B$ be a 3–ball, then the pair (B, T) is called an *n–string tangle*. We say that (B, T) is *trivial* if T is a trivial arc system in B. We say that (B, T) is *essential* if $\mathrm{cl}(\partial B - N(T))$ is incompressible in $\mathrm{cl}(B - N(T))$ in the case when $n > 1$ or (B, T) is not trivial in the case when $n = 1$, where $N(T)$ is a regular neighborhood of T in B. We also say that (B, T) is *free* if $\mathrm{cl}(B - N(T))$ is a handlebody.

Definition 2.1 (C–trivialization arc system) Let (B, T) be an n–string tangle, and let T' be a subfamily of T. Then we say that T' is a *C–trivialization arc system* if $T - T'$ is a trivial arc system in the 3–manifold $\mathrm{cl}(B - N(T'))$.

Definition 2.2 (n/k–free tangle) Suppose (B, T) is an n–string free tangle, and let k be an integer with $0 \le k \le n$. Then we say that (B, T) is a *n/k–free tangle* if the following conditions hold:

(1) there is a subfamily $T' \subset T$ with $\#(T') = k$ such that T' is a C–trivialization arc system,

(2) T'' is not a C–trivialization arc system for any subfamily $T'' \subset T$ with $\#(T'') < k$.

Remark 1 (1) $n/0$–free tangle is a trivial tangle. (2) We say that n/n–free tangle is a *full* free tangle. Examples of a $2/0$–free tangle, a $2/1$–free tangle and a $2/2$–free tangle are illustrated in Figure 1. (3) If T' is a C–trivialization arc system in an n–string free tangle (B, T), then $\mathrm{cl}(B - N(T'))$ is a handlebody. Because $T - T'$ is a trivial arc system in $\mathrm{cl}(B - N(T'))$ and $\mathrm{cl}(B - N(T') - N(T - T')) = \mathrm{cl}(B - N(T))$ is a handlebody.

We say that a knot K has an *n–string free tangle decomposition* if (S^3, K) is decomposed into two n–string free tangles $(B_1, T_1) \cup (B_2, T_2)$.

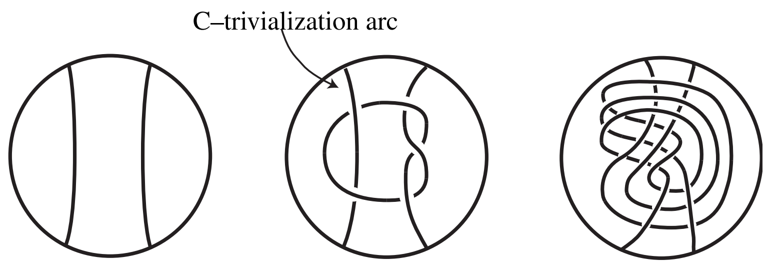

Figure 1

Proposition 2.3 Let K be a knot in S^3 which has an n–string free tangle decomposition $(S^3, K) = (B_1, T_1) \cup (B_2, T_2)$. Suppose at least one of (B_1, T_1) and (B_2, T_2) is an n/k–free tangle for some k with $0 \leq k \leq n$, then $t(K) \leq n + k - 1$.

Proof We may assume that (B_1, T_1) is an n/k–free tangle, and put $T_1 = t_1^1 \cup t_2^1 \cup \cdots \cup t_n^1$. Then we can put $T_1' = t_1^1 \cup \cdots \cup t_k^1$ to be a C–trivialization arc system, and $T_1' = \varnothing$ if $k = 0$. Let $\alpha_1, \ldots, \alpha_{n-1}, \beta_1, \ldots, \beta_k$ be the arcs in ∂B_1 as in Figure 2 so that α_i connects a point of ∂t_i^1 and a point of ∂t_{i+1}^1 $(i = 1, 2, \ldots, n-1)$, β_1 connects the two points of ∂t_1^1 and β_i connects a point of ∂t_{i-1}^1 and a point of ∂t_i^1 $(i = 2, \ldots, k)$.

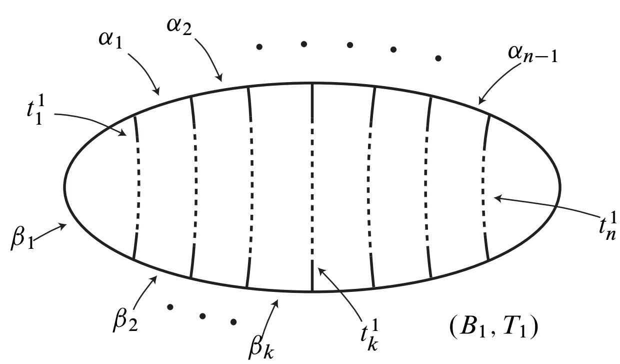

Figure 2

Put $D = \mathrm{cl}(\partial B_1 - N(\alpha_1 \cup \cdots \cup \alpha_{n-1} \cup \beta_1 \cup \cdots \cup \beta_k))$, where $N(\alpha_1 \cup \cdots \cup \alpha_{n-1} \cup \beta_1 \cup \cdots \cup \beta_k)$ is a regular neighborhood of $\alpha_1 \cup \cdots \cup \alpha_{n-1} \cup \beta_1 \cup \cdots \cup \beta_k$ in S^3. Then D is a disk in ∂B_1 and D is a disk in ∂B_2 too. We note that $T_1 - T_1' = t_{k+1}^1 \cup \cdots \cup t_n^1$ is a trivial arc system in the genus k handlebody $\mathrm{cl}(B_1 - N(T_1'))$, and one end point of ∂t_i^1 $(i = k+1, \ldots, n)$ is in D.

Put $W_1 = N(K) \cup N(\alpha_1 \cup \cdots \cup \alpha_{n-1} \cup \beta_1 \cup \cdots \cup \beta_k)$, then W_1 is a genus $n+k$ handlebody. Put $W_2 = \mathrm{cl}(S^3 - W_1)$ and put $D' = \mathrm{cl}(D - N(t_{k+1}^1 \cup \cdots \cup t_n^1))$. Then, by the above note, $W_2 = \mathrm{cl}(B_1 - N(T_1) - N(\alpha_1 \cup \cdots \cup \alpha_{n-1} \cup \beta_1 \cup \cdots \cup \beta_k)) \cup_{D'} \mathrm{cl}(B_2 - N(T_2) - N(\alpha_1 \cup \cdots \cup \alpha_{n-1} \cup \beta_1 \cup \cdots \cup \beta_k))$ is a genus $n+n-(n-k) = n+k$ handlebody. Hence (W_1, W_2) is a genus $n+k$ Heegaard splitting of S^3 such that W_1 contains K as a core of a handle. This shows that $t(K) \leq n+k-1$. □

Corollary 2.4 (Morimoto [4]) If K has an n–string free tangle decomposition, then $t(K) \leq 2n-1$.

By the above proposition, we can ask if the estimate in the proposition is best possible.

Problem 2.5 For any $n > 1$ and k with $0 \leq k \leq n$, are there knots K satisfying the following conditions:

(1) K has an n–string free tangle decomposition with at least one n/k–free tangle,

(2) $t(K) = n+k-1$?

In particular, we want to ask the following.

Problem 2.6 For any $n > 1$, are there knots K satisfying the following conditions:

(1) K has an n–string free tangle decomposition,

(2) $t(K) = 2n-1$?

3 Degeneration ratio

Proposition 3.1 Let K_1 be a knot which has an n–string free tangle decomposition for $n > 1$, and K_2 a knot which has an $(n+1)/0$–free tangle decomposition (ie $n+1$–bridge decomposition). Then $t(K_1 \# K_2) \leq 2n-1$.

Proof Suppose $(S_1^3, K_1) = (B_1, T_1) \cup (B_2, T_2)$ is an n–string free tangle decomposition and $(S_2^3, K_2) = (C_1, S_1) \cup (C_2, S_2)$ is an $(n+1)/0$–free tangle decomposition, where $T_1 = t_1^1 \cup t_2^1 \cup \cdots \cup t_n^1$, $T_2 = t_1^2 \cup t_2^2 \cup \cdots \cup t_n^2$, $S_1 = s_1^1 \cup s_2^1 \cup \cdots \cup s_{n+1}^1$ and $S_2 = s_1^1 \cup s_2^2 \cup \cdots \cup s_{n+1}^2$. Let $N_i^j = N(t_i^j)$ be a regular neighborhood of t_i^j in B_i such that $N(K_1) = N_1^1 \cup N_2^1 \cup \cdots \cup N_n^1 \cup N_1^2 \cup N_2^2 \cup \cdots \cup N_n^2$ is a regular neighborhood of K_1 in S_1^3, and let $M_i^j = N(s_i^j)$ be a regular neighborhood of s_i^j in C_i such that $N(K_2) = M_1^1 \cup M_2^1 \cup \cdots \cup M_{n+1}^1 \cup M_1^2 \cup M_2^2 \cup \cdots \cup M_{n+1}^2$ is a regular neighborhood of K_2 in S_2^3.

Divide t_n^2 into three arcs $t_{n0}^2 \cup t_{n1}^2 \cup t_{n2}^2$ such that $t_{n0}^2 \cap t_{n2}^2 = \varnothing$, and divide N_n^2 into three pieces $N_{n0}^2 \cup N_{n1}^2 \cup N_{n2}^2$ according as $t_{n0}^2 \cup t_{n1}^2 \cup t_{n2}^2$. Put $N = N_1^1 \cup N_2^1 \cup \cdots \cup N_n^1 \cup N_1^2 \cup N_2^2 \cup \cdots \cup N_{n-1}^2 \cup N_{n0}^2 \cup N_{n2}^2$, and put $M = M_1^1 \cup M_2^1 \cup \cdots \cup M_n^1 \cup M_1^2 \cup M_2^2 \cup \cdots \cup M_{n+1}^2$, ie $N = \mathrm{cl}(N(K_1) - N_{n1}^2)$ and $M = \mathrm{cl}(N(K_2) - M_{n+1}^1)$. Note that $N \cap N_{n1}^2$ consists of two 2–disks and $M \cap M_{n+1}^1$ consists of two 2–disks. Then N is a 3–ball in S_1^3 and $(N, N \cap K_1)$ is a 1–string trivial tangle, and M is a 3–ball in S_2^3 and $(M, M \cap K_2)$ is a 1–string trivial tangle. We make a connected sum of (S_1^3, K_1) and (S_2^3, K_2) as follows. First, by changing the letters if necessary, we may assume that t_i^1 connects t_i^2 and t_{i+1}^2 $(i = 1, 2, \ldots, n-1)$ and t_n^1 connects t_n^2 and t_1^2, and that s_i^1 connects s_i^2 and s_{i+1}^2 $(i = 1, 2, \ldots, n-1)$, s_n^1 connects s_{n+1}^2 and s_1^2 and s_{n+1}^1 connects s_n^2 and s_{n+1}^2. Hence we can identify N and M by the following map $f\colon N \to M$.

$$\begin{aligned}
f(N_i^1) &= M_i^1 \quad (i = 1, 2, \ldots, n)\\
f(N_i^2) &= M_i^2 \quad (i = 1, 2, \ldots, n-1)\\
f(N_{n0}^2) &= M_n^2\\
f(N_{n2}^2) &= M_{n+1}^2.
\end{aligned}$$

Put $g = f|_{\partial N}\colon \partial N \to \partial M$, then by this glueing map, we get the connected sum $(S^3, K_1 \# K_2) = \mathrm{cl}(S_1^3 - N) \cup_g \mathrm{cl}(S_2^3 - M)$, where $K_1 \# K_2 = (N_{n1}^2 \cap K_1) \cup (M_{n+1}^1 \cap K_2)$ as in Figure 3 $(n = 4)$.

Put $B_1' = \mathrm{cl}(B_1 - N)$, $C_1' = \mathrm{cl}(C_1 - M) \cup N_{n1}^2$. Glue $\partial B_1' \cap \partial N$ and $\partial C_1' \cap \partial M$ with g, and put $W_1 = B_1' \cup_g C_1'$. Then, since B_1' is a genus n handlebody, and since $\{s_1^1, s_2^1, \ldots, s_n^1\}$ is a trivial arc system in C_1 and N_{n1}^2 is a 1–handle for C_1, we see that W_1 is a genus $n + (n-1) + 1 = 2n$ handlebody. On the other hand, put $B_2' = \mathrm{cl}(B_2 - (N \cup N_{n1}^2))$, $C_2' = \mathrm{cl}(C_2 - M)$. Glue $\partial B_2' \cap \partial N$ and $\partial C_2' \cap \partial M$ with g, and put $W_2 = B_2' \cup_g C_2'$. Then, since B_2' is a genus n handlebody, and since $\{s_1^2, s_2^2, \ldots, s_{n+1}^2\}$ is a trivial arc system in C_2, we see that W_2 is a genus $n + n = 2n$

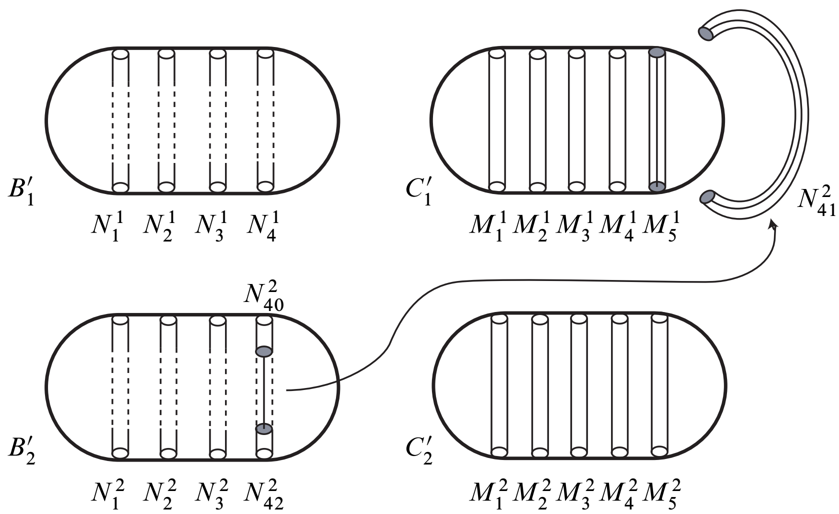

Figure 3

handlebody. Hence (W_1, W_2) is a genus $2n$ Heegaard splitting of S^3, and $K_1 \# K_2$ is a central loop of a handle of W_1. This shows that $t(K_1 \# K_2) \leq 2n-1$, and completes the proof of Proposition 3.1. □

Suppose there is a knot K_1 which has an n-string free tangle decomposition with $t(K_1) = 2n-1$ (cf Problem 2.6). Let K_2 be a knot which has an $(n+1)/0$–free tangle decomposition with $t(K_2) = n$ (such a knot indeed exists). Then $t(K_1) + t(K_2) = 2n-1+n = 3n-1$, and by Proposition 3.1, $t(K_1 \# K_2) \leq 2n-1$. Hence $\dfrac{t(K_1 \# K_2)}{t(K_1)+t(K_2)} \leq \dfrac{2n-1}{3n-1}$.

In particular, suppose there is a knot K_1 which has a 2–string free tangle decomposition with $t(K_1) = 3$. Then, since there is a knot K_2 which has a 3/0–free tangle (3–bridge) decomposition with $t(K_1) = 2$, we have $t(K_1) = 3$, $t(K_2) = 2$ and $t(K_1 \# K_2) \leq 3$ by Proposition 3.1. Moreover, if $t(K_1 \# K_2) = 2$ then $t(K_1) = 1$ or $t(K_2) = 1$ by [M1, Theorem], a contradiction. Hence $t(K_1 \# K_2) = 3$. This shows that $\dfrac{t(K_1 \# K_2)}{t(K_1)+t(K_2)} = \dfrac{3}{3+2} = \dfrac{3}{5} < \dfrac{2}{3}$. Hence, we need to solve the following problem (a special case of Problem 2.6).

Problem 3.2 Are there (or find) knots K satisfying the following conditions

(1) K has a 2–string free tangle decomposition,

(2) $t(K) = 3$?

Remark 2 If there is a knot K satisfying the conditions in the above problem, then by Proposition 2.3, both tangles in the free tangle decomposition are full free tangles. However, the converse is not true, because there is a knot K which has a 2–string full free tangle decomposition but $t(K) = 2$ as follows.

Let (B_1, T_1) be a 2/2–free tangle illustrated in Figure 1(iii). Then (B_1, T_1) is a 2–string full free tangle. Let (B_2, T_2) be a copy of (B_1, T_1) and put $(S^3, K) = (B_1, T_1) \cup (B_2, T_2)$ with a half twist. Then, by taking a half twist, K is a knot (not a link) in S^3 which has a 2–string full free tangle decomposition. However, by a little observation, we see that $t(K) = 2$. This shows that the converse is not true.

Proposition 3.3 Let K_1 be a knot which has an n–string free tangle decomposition with at least one $n/(n-1)$–free tangle for $n > 1$, and K_2 a knot which has an $n/0$–free tangle decomposition (ie n–bridge decomposition). Then $t(K_1 \# K_2) \le 2n - 2$.

Proof Suppose $(S_1^3, K_1) = (B_1, T_1) \cup (B_2, T_2)$ is an n–string free tangle decomposition with an $n/(n-1)$–free tangle, say (B_1, T_1), and $(S_2^3, K_2) = (C_1, S_1) \cup (C_2, S_2)$ is an $n/0$–free tangle decomposition, where $T_1 = t_1^1 \cup t_2^1 \cup \cdots \cup t_n^1$, $T_2 = t_1^2 \cup t_2^2 \cup \cdots \cup t_n^2$, $S_1 = s_1^1 \cup s_2^1 \cup \cdots \cup s_n^1$ and $S_2 = s_1^1 \cup s_2^2 \cup \cdots \cup s_n^2$. Let $N_i^j = N(t_i^j)$ be a regular neighborhood of t_i^j in B_i such that $N(K_1) = N_1^1 \cup N_2^1 \cup \cdots \cup N_n^1 \cup N_1^2 \cup N_2^2 \cup \cdots \cup N_n^2$ is a regular neighborhood of K_1 in S_1^3, and let $M_i^j = N(s_i^j)$ be a regular neighborhood of s_i^j in C_i such that $N(K_2) = M_1^1 \cup M_2^1 \cup \cdots \cup M_n^1 \cup M_1^2 \cup M_2^2 \cup \cdots \cup M_n^2$ is a regular neighborhood of K_2 in S_2^3.

By changing the letters if necessary, we may assume that t_i^1 connects t_i^2 and t_{i+1}^2 $(i = 1, 2, \ldots, n-1)$ and t_n^1 connects t_n^2 and t_1^2, and that s_i^1 connects s_i^2 and s_{i+1}^2 $(i = 1, 2, \ldots, n-1)$ and s_n^1 connects s_n^2 and s_1^2. Moreover, since (B_1, T_1) is a $n/(n-1)$–free tangle, we may assume that $t_1^1 \cup t_2^1 \cup \cdots \cup t_{n-1}^1$ is a C–trivialization arc system in B_1, ie $\mathrm{cl}(B_1 - N(t_1^1 \cup t_2^1 \cup \cdots \cup t_{n-1}^1))$ is a handlebody and t_n^1 is a trivial arc in the handlebody.

Put $N = N_1^1 \cup N_2^1 \cup \cdots \cup N_{n-1}^1 \cup N_1^2 \cup N_2^2 \cup \cdots \cup N_n^2$, and put $M = M_1^1 \cup M_2^1 \cup \cdots \cup M_{n-1}^1 \cup M_1^2 \cup M_2^2 \cup \cdots \cup M_n^2$, ie $N = \mathrm{cl}(N(K_1) - N_n^1)$ and $M = \mathrm{cl}(N(K_2) - M_n^1)$. Then N is a 3–ball in S_1^3 and $(N, N \cap K_1)$ is a 1–string trivial tangle, and M is a 3–ball in S_2^3 and $(M, M \cap K_2)$ is a 1–string trivial tangle. Hence we can identify N and M by the following map $f\colon N \to M$.

$$f(N_i^1) = M_i^1 \quad (i = 1, 2, \ldots, n-1)$$
$$f(N_i^2) = M_i^2 \quad (i = 1, 2, \ldots, n).$$

Put $g = f|_{\partial N}\colon \partial N \to \partial M$, then by this glueing map, we get the connected sum $(S^3, K_1\#K_2) = \mathrm{cl}(S^3_1 - N) \cup_g \mathrm{cl}(S^3_2 - M)$, where $K_1\#K_2 = (N^1_n \cap K_1) \cup (M^1_n \cap K_2)$ as in Figure 4 $(n = 4)$.

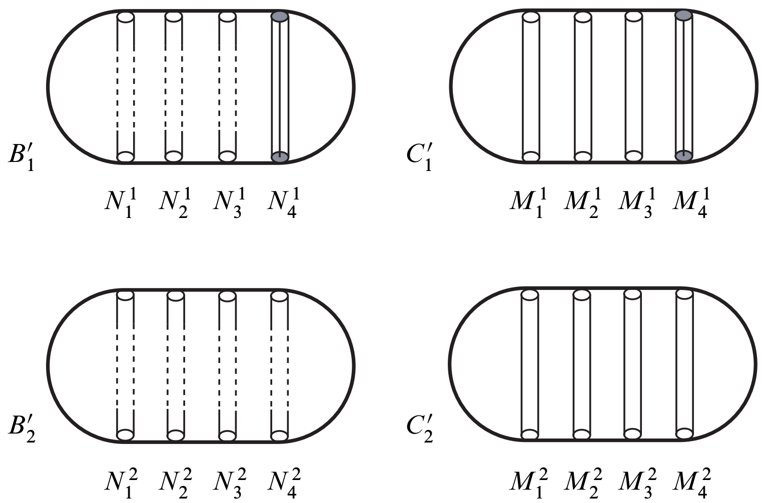

Figure 4

Put $B_1' = \mathrm{cl}(B_1 - N)$, $C_1' = \mathrm{cl}(C_1 - M)$. Glue $\partial B_1' \cap \partial N$ and $\partial C_1' \cap \partial M$ with g, and put $W_1 = B_1' \cup_g C_1'$. Then, since B_1' is a genus $n-1$ handlebody and t^1_n is a trivial arc in the handlebody, and since $\{s^1_1, s^1_2, \dots, s^1_n\}$ is a trivial arc system in C_1 and $N^1_n \cap M^1_n$ consists of two 2–disks, we see that W_1 is a genus $(n-1) + (n-1) + 1 = 2n-1$ handlebody. On the other hand, put $B_2' = \mathrm{cl}(B_2 - N)$, $C_2' = \mathrm{cl}(C_2 - M)$. Glue $\partial B_2' \cap \partial N$ and $\partial C_2' \cap \partial M$ with g, and put $W_2 = B_2' \cup_g C_2'$. Then, since B_2' is a genus n handlebody, and since $\{s^2_1, s^2_2, \dots, s^2_n\}$ is a trivial arc system in C_2, we see that W_2 is a genus $n + (n-1) = 2n-1$ handlebody. Hence (W_1, W_2) is a genus $2n-1$ Heegaard splitting of S^3, and $K_1\#K_2$ is a central loop of a handle of W_1. This shows that $t(K_1\#K_2) \le 2n-2$, and completes the proof of Proposition 3.3. □

Suppose there is a knot K_1 which has an n–string free tangle decomposition with at least one $n/(n-1)$–free tangle and $t(K_1) = 2n-2$ (cf Problem 2.5), and let K_2 be a knot which has an $n/0$–free tangle decomposition with $t(K_2) = n-1$ (such a knot indeed exists). Then $t(K_1) + t(K_2) = (2n-2) + (n-1) = 3n-3$, and by Proposition 3.3, $t(K_1\#K_2) \le 2n-2$. Hence $\dfrac{t(K_1\#K_2)}{t(K_1)+t(K_2)} \le \dfrac{2n-2}{3n-3} = \dfrac{2(n-1)}{3(n-1)} = \dfrac{2}{3}$.

In particular, in the case when $n=2$, there indeed exists a knot K_1 which has a 2–string free tangle decomposition with at least one 2/1–free tangle and $t(K)=2$ (cf Figure 1(ii)), and let K_2 be a 2–bridge knot. Then $t(K_1)=2$, $t(K_2)=1$ and $t(K_1\#K_2)=2$ by Proposition 3.3. Hence $\dfrac{t(K_1\#K_2)}{t(K_1)+t(K_2)}=\dfrac{2}{2+1}=\dfrac{2}{3}$. This is the first example whose tunnel numbers go down under connected sum introduced in Morimoto [2; 3].

In general, for any $n>1$ and k with $0\le k\le n$, we have the following Theorem.

Theorem 3.4 *Let K_1 be a knot which has an n–string free tangle decomposition with at least one n/k–free tangle, and K_2 a knot which has a $(k+1)/0$–free tangle decomposition (ie, $(k+1)$–bridge decomposition). Then $t(K_1\#K_2)\le n+k-1$.*

Proof If $k=n$ or $n-1$, then this is the same as Proposition 3.1 or Proposition 3.3 respectively. Hence we assume $k<n-1$.

Suppose $(S^3_1,K_1)=(B_1,T_1)\cup(B_2,T_2)$ is an n–string free tangle decomposition with an n/k–free tangle, say (B_1,T_1), and $(S^3_2,K_2)=(C_1,S_1)\cup(C_2,S_2)$ is an $(k+1)/0$–free tangle decomposition, where $T_1=t^1_1\cup t^1_2\cup\cdots\cup t^1_n$, $T_2=t^2_1\cup t^2_2\cup\cdots\cup t^2_n$, $S_1=s^1_1\cup s^1_2\cup\cdots\cup s^1_{k+1}$ and $S_2=s^1_1\cup s^2_2\cup\cdots\cup s^2_{k+1}$. Let $N^j_i=N(t^j_i)$ be a regular neighborhood of t^j_i in B_i such that $N(K_1)=N^1_1\cup N^1_2\cup\cdots\cup N^1_n\cup N^2_1\cup N^2_2\cup\cdots\cup N^2_n$ is a regular neighborhood of K_1 in S^3_1, and let $M^j_i=N(s^j_i)$ be a regular neighborhood of s^j_i in C_i such that $N(K_2)=M^1_1\cup M^1_2\cup\cdots\cup M^1_{k+1}\cup M^2_1\cup M^2_2\cup\cdots\cup M^2_{k+1}$ is a regular neighborhood of K_2 in S^3_2.

By changing the letters if necessary, we may assume that t^1_i connects t^2_i and t^2_{i+1} $(i=1,2,\dots,n-1)$ and t^1_n connects t^2_n and t^2_1, and that s^1_i connects s^2_i and s^2_{i+1} $(i=1,2,\dots,k)$ and s^1_{k+1} connects s^2_{k+1} and s^2_1. Moreover, since (B_1,T_1) is a n/k–free tangle, we may assume that $t^1_1\cup t^1_2\cup\cdots\cup t^1_k$ is a C–trivialization arc system in B_1, ie, $\mathrm{cl}(B_1-N(t^1_1\cup t^1_2\cup\cdots\cup t^1_k))$ is a handlebody and $t^1_{k+1}\cup\cdots\cup t^1_n$ is a trivial arc system in the handlebody.

Put $N=N^1_1\cup N^1_2\cup\cdots\cup N^1_k\cup N^2_1\cup N^2_2\cup\cdots\cup N^2_{k+1}$, and put $M=M^1_1\cup M^1_2\cup\cdots\cup M^1_k\cup M^2_1\cup M^2_2\cup\cdots\cup M^2_{k+1}$, ie $N=\mathrm{cl}(N(K_1)-(N^1_{k+1}\cup\cdots\cup N^1_n\cup N^2_{k+2}\cup\cdots\cup N^2_n))$ and $M=\mathrm{cl}(N(K_2)-M^1_{k+1})$. Then N is a 3–ball in S^3_1 and $(N,N\cap K_1)$ is a 1–string trivial tangle, and M is a 3-ball in S^3_2 and $(M,M\cap K_2)$ is a 1–string trivial tangle. Hence we can identify N and M by the following map $f\colon N\to M$.

$$f(N^1_i)=M^1_i\quad(i=1,2,\dots,k)$$
$$f(N^2_i)=M^2_i\quad(i=1,2,\dots,k+1).$$

Put $g = f|_{\partial N}\colon \partial N \to \partial M$, then by this glueing map, we get the connected sum $(S^3, K_1\#K_2) = \mathrm{cl}(S^3_1 - N) \cup_g \mathrm{cl}(S^3_2 - M)$, where $K_1\#K_2 = (((N^1_{k+1} \cup \cdots \cup N^1_n) \cup (N_{k+2} \cup \cdots \cup N^2_n)) \cap K_1) \cup (M^1_{k+1} \cap K_2)$ as in Figure 5 $(n = 6, k = 3)$.

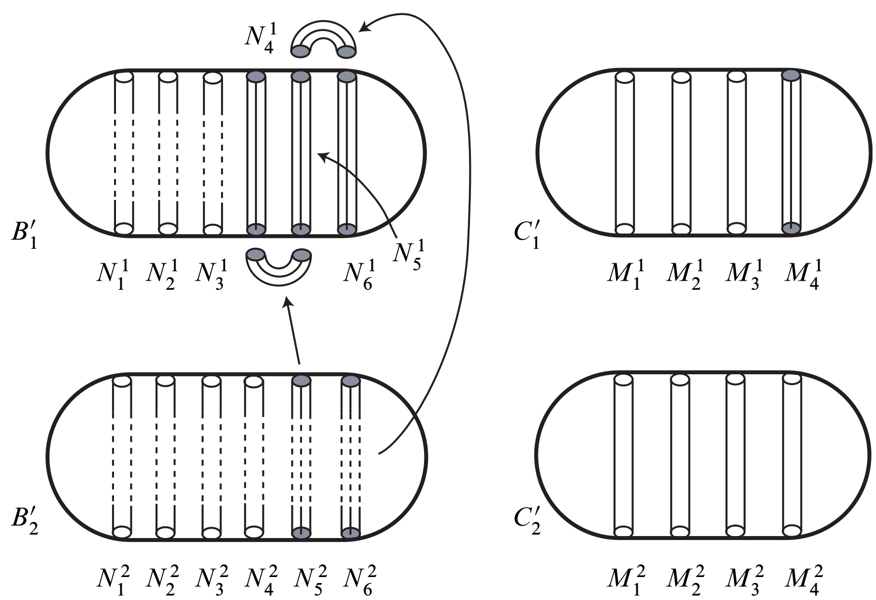

Figure 5

Put $B'_1 = \mathrm{cl}(B_1 - N) \cup (N^2_{k+2} \cup \cdots \cup N^2_n)$, $C'_1 = \mathrm{cl}(C_1 - M)$. Glue $\partial B'_1 \cap \partial N$ and $\partial C'_1 \cap \partial M$ with g, and put $W_1 = B'_1 \cup_g C'_1$. Then, since B'_1 is a genus $n-1$ handlebody and $t^1_{k+1} \cup \cdots \cup t^1_n \cup t^2_{k+2} \cup \cdots \cup t^2_n$ is a trivial arc in the handlebody, and since $\{s^1_1, s^1_2, \ldots, s^1_{k+1}\}$ is a trivial arc system in C_1 and $((N^1_{k+1} \cup \cdots \cup N^1_n) \cup (N_{k+2} \cup \cdots \cup N^2_n)) \cap M^1_{k+1}$ consists of two 2–disks, we see that W_1 is a genus $(n-1)+(k-1)+2 = n+k$ handlebody. On the other hand, put $B'_2 = \mathrm{cl}(B_2 - N - N^2_n)$, $C'_2 = \mathrm{cl}(C_2 - M)$. Glue $\partial B'_2 \cap \partial N$ and $\partial C'_2 \cap \partial M$ with g, and put $W_2 = B'_2 \cup_g C'_2$. Then, since B'_2 is a genus n handlebody, and since $\{s^2_1, s^2_2, \ldots, s^2_{k+1}\}$ is a trivial arc system in C_2, we see that W_2 is a genus $n+k$ handlebody. Hence (W_1, W_2) is a genus $n+k$ Heegaard splitting of S^3, and $K_1\#K_2$ is a central loop of a handle of W_1. This shows that $t(K_1\#K_2) \le n+k-1$, and completes the proof of Theorem 3.4. □

Suppose there is a knot K_1 which has an n–string free tangle decomposition with at least one n/k–free tangle and $t(K_1) = n+k-1$ (cf Problem 2.5), and let K_2 be a knot which has a $(k+1)/0$–free tangle decomposition with $t(K_2) = k$ (such a knot indeed exists). Then $t(K_1)+t(K_2)=n+2k-1$, and by Theorem 3.4, $t(K_1\#K_2)\le n+k-1$. Hence $\dfrac{t(K_1\#K_2)}{t(K_1)+t(K_2)}\le\dfrac{n+k-1}{n+2k-1}$.

Put $\ell=n-k$, then $0\le\ell\le n$, $k=n-\ell$, $n+k-1=2n-\ell-1$ and $n+2k-1=3n-2\ell-1$. Hence $\dfrac{t(K_1\#K_2)}{t(K_1)+t(K_2)}\le\dfrac{2n-\ell-1}{3n-2\ell-1}$.

If $\ell=0$ $(k=n)$, then $\dfrac{2n-\ell-1}{3n-2\ell-1}=\dfrac{2n-1}{3n-1}\to\dfrac{2}{3}(-0)$ as $(n\to\infty)$.

If $\ell=1$ $(k=n-1)$, then $\dfrac{2n-\ell-1}{3n-2\ell-1}=\dfrac{2(n-1)}{3(n-1)}=\dfrac{2}{3}$.

If $\ell>1$ $(k<n-1)$, then $\dfrac{2n-\ell-1}{3n-2\ell-1}\to\dfrac{2}{3}(+0)$ as $(n\to\infty)$.

Therefore, we see that the least degeneration ratio can be gotten by the method in this paper is $\dfrac{3}{5}$ in the case when $n=2$ and $\ell=0$ $(k=2)$.

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