Geometry & Topology Monographs 12 (2007) 277–284 **277**

A proof of Waldhausen's uniqueness of splittings of S^3 (after Rubinstein and Scharlemann)

YO'AV RIECK

In [7] J H Rubinstein and M Scharlemann, using Cerf Theory, developed tools for comparing Heegaard splittings of irreducible, non-Haken manifolds. As a corollary of their work they obtained a new proof of Waldhausen's uniqueness of Heegaard splittings of S^3. In this note we use Cerf Theory and develop the tools needed for comparing Heegaard splittings of S^3. This allows us to use Rubinstein and Scharlemann's philosophy and obtain a simpler proof of Waldhausen's Theorem. The combinatorics we use are very similar to the game Hex and requires that Hex has a winner. The paper includes a proof of that fact (Proposition 3.6).

57M99, 57M25

1 Introduction

In [10] F Waldhausen proved uniqueness of non-stabilized Heegaard splittings of S^3.

Theorem 1 (Waldhausen) *Let $\Sigma \subset S^3$ be a Heegaard surface of genus $g > 0$. Then Σ is a stabilization of a Heegaard surface of genus $g-1$.*

In [7] J H Rubinstein and M Scharlemann, using Cerf Theory [2], developed tools for comparing Heegaard splittings of irreducible, non-Haken manifolds. As a corollary of their work they obtained a new proof of Theorem 1. In this note we use Cerf Theory and develop the tools needed for comparing Heegaard splittings of S^3. This allows us to use Rubinstein and Scharlemann's philosophy and obtain a simpler proof of Theorem 1. We assume familiarity with the basic facts and standard terminology of 3–manifold topology and in particular Heegaard splittings; see Scharlemann [8]. For another proof of Waldhausen's Theorem see Johnson [3].

We begin with an outline of the proof. As with many proofs of Theorem 1 we assume the theorem is false and pick Σ to be a minimal genus counterexample; we induct on g, the genus of Σ. A simple application of van Kampen's theorem shows that if $g = 1$ then the meridians of the complementary solid tori intersect minimally once and hence Σ is a stabilization of the genus zero splitting of S^3. The heart of the argument (in the

Published: 3 December 2007 DOI: 10.2140/gtm.2007.12.277

following sections) is to show that if $g > 1$ then Σ weakly reduces. By A Casson and C McA Gordon's seminal work [1] either Σ reduces or S^3 contains an essential surface. As the latter is impossible, Σ must reduce. Cutting S^3 open along the reducing sphere we obtain 2 balls (say B_1 and B_2, resp.) and a once punctured surface in each (say S_1 and S_2, resp.). We attach 3–balls to B_1 and B_2 and cap off S_1 and S_2 with disks. It is easy to see that we obtain two Heegaard splittings of S^3, each of positive genus less than g. By our inductive hypothesis each of them is stabilized. Hence, Σ is stabilized as well.

The remainder of this paper is devoted to showing that if Σ is a Heegaard splitting of S^3 of genus $g > 1$ then Σ weakly reduces.

Acknowledgment

I thank the anonymous referee for helpful suggestions.

2 The Graphic

S^3 is the unit sphere in $\mathbb{R}^4$. As such, it inherits a height function given by the projection onto the x–axis, denoted h_1. S^3 has one maximum at $(0,0,0,1)$, one minimum at $(0,0,0,-1)$ and for any $s \in (-1,1)$ we have that $h_1^{-1}(s)$ is a 2–sphere which we denote S^2_s. This is a special case of a *sweepout*.

Given Σ, we have a sweepout of S^3 corresponding to Σ; this concept was originally introduced in Scharlemann–Rubinstein [7]. Although our description is a little different from that given in [7] it is easy to see that the two are equivalent; for a more detailed treatment similar to this paper, see Rieck [4; 5] and Rieck–Rubinstein [6]. Let $S^3 = U \cup_\Sigma V$ be the Heegaard splitting corresponding to Σ. Let h_2 be a height function on U, $h_2\colon U \to [-1,0]$ so that $\partial U = \Sigma$ is at level 0, a spine of U is at the level -1 and for each $t \in (-1,0]$, $h_2^{-1}(t)$ is a surface parallel to ∂U. Similarly take a height function on V (also denoted h_2) $h_2\colon V \to [0,1]$, so that $\partial V = \Sigma$ is at level 0, a spine of V is at the level 1 and for each $t \in [0,1)$, $h_2^{-1}(t)$ is a surface parallel to ∂V. Pasting the two functions together and obtain a function $h_2\colon S^3 \to [-1,1]$. For $t \in (-1,1)$ we denote $h_2^{-1}(t)$ by Σ_t. By transversality we may assume that the spines of U and V are disjoint from $(0,0,0,1)$ and $(0,0,0,-1)$. For every point $(s,t) \in (-1,1)\times(-1,1)$ we have the two surfaces S^2_s and Σ_t. Cerf Theory says that we can perturb h_1 and h_2 so that the intersection of S^2_s and Σ_t is transverse for almost all $(s,t) \in [-1,1]\times[-1,1]$, and the set for which the intersection is not transverse forms a finite graph (called the *Graphic*) with the following properties.

Figure 1: Obtaining the board

(1) For (s,t) on an edge of the graphic, $S^2_s \cap \Sigma_t$ contains exactly one non-degenerate critical point (either center or a saddle).

(2) At a valence 4 vertex the corresponding surfaces have exactly two non-degenerate critical points. A valence 4 vertex can be seen as a point where two arcs of the graphic cross each other, each corresponding to a single non-degenerate critical point.

(3) There is one other type of vertex (called a *Birth-Death* vertex) that has valence 2. Birth-death vertices do not play a role in our study and we will not describe them here.

The closure of a component of $[-1,1] \times [-1,1]$ cut open along the Graphic is called a *region*. Given a region, the intersection of the surfaces that correspond to a point in the region does not depend in the choice of point in any essential way.

3 The labels I, E and a friendly game of Hex

We label the regions. A region is labeled E (standing for "essential") whenever the intersection of surfaces corresponding to a point in the region contains a curve that is essential in Σ_t; otherwise, the label I (standing for "inessential") is used. By definition each region has exactly one label.

In order to enjoy a game of Hex we modify the Graphic as follow: if a valence 4 vertex is adjacent to two E–regions and two I–regions and the labels alternate when going cyclically around it, we split the Graphic and introduce a short edge separating the I regions; see Figure 1 where the northern and southern regions are I–regions and the western and eastern regions are E–regions. The graph obtained is called the *Board*. Note that there is a natural correspondence between regions of the Graphic and those of the Board; using this correspondence the regions of the Board inherit labels from the Graphic.

The reason for creating the Board is the following.

Proposition/Definition 3.1 *By the* border *we mean the union of the edges of the Board that separate E–regions from I–regions. In $(-1,1)\times(-1,1)$ the border forms an embedded 1–manifold.*

Proof Away from the vertices the proposition clearly holds. Let v be a valence 4 vertex. If all regions around v have the same label v isn't on the border. If one region has one label and three have the other label, the border is locally an interval (with a corner). If two regions adjacent to v are labeled E and two are labeled I, then by construction of the Board the labels do not alternate. Hence the Border cuts across such a vertex smoothly. At a valence 3 vertex v either all three regions have the same label (and v is not on the Border) or two regions with one label meet a third region with the other label (and the border is locally an interval). □

Remark 3.2 In the original game of Hex every vertex has valence 3. Hence the border there forms a 1–manifold as well.

We now pick two volunteers to play Hex. The goal of the first volunteer, Ivan, is finding a chain of regions (say $R_1,\dots,R_n$, $n\geq 1$) labeled I that connects the left edge of the Board (points with $s=-1$) with its right ($s=1$). Similarly, the goal of the second volunteer, Esmeralda, is finding a chain of regions labeled E that connects the bottom edge of the board ($t=-1$) with its top ($t=1$). In both cases, the region R_{i-1} shares an edge with R_i ($i=2,\dots,n$).

The next proposition is quite special to S^3.

Proposition 3.3 *Ivan can't win.*

Proof Suppose Ivan wins and let $R_1,\dots,R_n$ (for some n) be a chain of regions, starting at the left ($s=-1$) and ending at the right ($s=1$) (note that it is possible that R_1 meets the left edge in a single point only, and similarly for R_n and the right edge). Consider the corresponding regions in the Graphic (still denoted R_i). The cost: since some edges of the Board are crushed, it is now possible that R_{i-1} shares only a valence 4 vertex with R_i. Given $s\in[-1,1]$ we color $h_1^{-1}([-1,s])$ yellow and $h_1^{-1}([s,1])$ green.

The proof of the following lemma is an easy innermost disk argument and is left to the reader.

Lemma/Definition 3.4 (regarding I–regions) *If (s,t) is in an I–region then the entire surface Σ_t (except perhaps for parts contained in a disk) is either yellow or green (resp.); we say that Σ_t is* essentially yellow *(green resp.).*

We replace the labels I by labels $I(G)$ and $I(Y)$ as follows: I–regions with essentially green surfaces are labeled $I(G)$ and I–regions with essentially yellow surfaces are labeled $I(Y)$. Of course no surface is essentially green and essentially yellow simultaneously; this, together with Lemma/Definition 3.4, establishes that every I–region gets exactly one label. In addition, it is easy to see that I–regions with s very close to -1 are labeled $I(G)$ and I–regions with s very close to 1 are labeled $I(Y)$. Considering the chain of I-regions $R_1, \dots, R_n$, we see that R_1 is labeled $I(G)$ and R_n is labeled $I(Y)$. Let i be the first index with R_i labeled $I(Y)$. Thus R_{i-1} is labeled $I(G)$ and R_i is labeled $I(Y)$. If R_{i-1} and R_i share an edge then passing from one to the other we cross a single critical point, either a center or a saddle. In either case, no essential curve is introduced or removed (recall we are crossing from one I–region to another) and therefore labels cannot change. Thus we may assume that we cross a valence 4 vertex (say v), corresponding to 2 singular points (say s_1 and s_2). By construction of the Board v was obtained from pinching an edge of the Board and the remaining two regions adjacent to v are E–regions (recall Figure 1); since crossing a center doesn't change an I–region to an E–region we see that both s_1 and s_2 are saddles.

Moving out of R_{i-1} by crossing s_1, we arrive at a region labeled E; thus crossing the saddle has the effect of changing a single inessential curve into two parallel essential curves bounding an annulus. Since the surface was essentially green prior to crossing s_1, the annulus between the parallel curves is essentially yellow (ie the annulus is yellow except perhaps for regions contained in a disk). Crossing s_2 into R_i the label becomes I; hence the 2 parallel curves are pinched together to become a single inessential curve. If the pinching is done inside the essentially yellow annulus (thus turning it into a disk) the surface becomes essentially green; hence the boundary of the annulus is pinched outside the essentially yellow annulus. We obtain an essentially yellow once-punctured torus T or pair of pants P. Since R_i is an I-region the boundary of T (resp. P) is inessential; hence $g = 1$ (resp. $g = 0$), contradicting our assumptions. This establishes Proposition 3.3. □

Remark 3.5 (About the game Hex) In the following proposition we prove that Hex has a winner. The only properties of the game we are using are: (1) the Border is an embedded 1–manifold and (2) the four corners of the Board are adjacent to exactly one region each. It is easy to see that these conditions hold for the traditional game Hex (recall Remark 3.2), hence the proof of Proposition 3.6 shows the well-known fact that in that game too there is a winner.

Proposition 3.6 *Hex has a winner.*

Proof (We work on the Board.) First observe that there is exactly one region adjacent to each corner of $[-1,1]\times[-1,1]$ (these regions correspond to disjoint surfaces S^2_s and Σ_t).

By Proposition/Definition 3.1 the Border forms an embedded 1–manifold in $(-1,1)\times(-1,1)$ and therefore has four types of components (we note that distinct components of the Border may share a point on the boundary):

(1) simple closed curves,

(2) arcs connecting an edge to itself,

(3) arcs connecting an edge to an adjacent edge and

(4) arcs connecting an edge to an opposite edge.

We note that curves of type (1) do not play a role in the proof. Suppose there is a arc of type (4), say connecting the top edge to the bottom edge. Then on one side of that arc the regions are all labeled E and therefore Esmeralda wins. Similarly, if there is an arc connecting left edge to the right edge Ivan wins. We may therefore assume there are no arcs of type (4). In that case, by induction on the arcs of type (2) and (3), we can easily prove that some region R is adjacent to all four edges. If R is labeled I then Ivan wins and if it is labeled E then Esmeralda wins. $\square$

By Proposition 3.3 and Proposition 3.6 Esmeralda wins. Esmeralda's victory is given by a path of regions in the Board, say $R_1,\dots,R_n$, $n\geq 1$, all labeled E and connecting the bottom of the board to its top. We consider the corresponding regions in the Graphic, still denoted $R_1,\dots,R_n$. Observe that by construction of the Board, R_{i-1} still shares an edge with R_i ($i=2,\dots,n$). (As in Proposition 3.3 R_1 (R_n resp.) may have only one point on the bottom edge (top edge resp.).)

4 The weak reduction

We complete the proof by finding a weak reduction; this is a standard argument in Cerf Theory, originally due to Rubinstein and Scharlemann [7]. Denote the handlebodies obtained by cutting M along Σ_t by U_t and V_t. First we show the following Lemma.

Lemma 4.1 (regarding E–regions) *Let S^2_s and Σ_t be surfaces corresponding to a region labeled E. Then some curve of $S^2_s\cap\Sigma_t$ bounds a meridian disk in U_t or V_t.*

Proof Let $\mathcal{C} \subset S^2_s \cap \Sigma_t$ denote the collection of curves of $S^2_s \cap \Sigma_t$ that are essential in Σ_t. Since the label is E, $\mathcal{C} \neq \varnothing$. Consider $\mathcal{C}$ as an embedded 1-manifold in S^2_s and let $c \subset \mathcal{C}$ be an innermost curve and $D \subset S^2_s$ the innermost disk it bounds. Then in its interior D intersects Σ_t in a (possibly empty) collection of curves that are inessential in Σ_t; a standard disk swap argument gets a disk disjoint from Σ_t with boundary c. $\square$

Since R_1 contains points with t arbitrarily close to -1 (where Σ_t collapses to a spine of U_t) it is easy to see that curves of $S^2_s \cap \Sigma_t$ bound meridians of U_t; likewise curves of $S^2_s \cap \Sigma_t$ in R_n bound meridians of V_t. By Lemma 4.1 every region R_i has curves of $S^2_s \cap \Sigma_t$ that bound meridians of U_t or V_t. Let i be the lowest index so that R_i has a curve of $S^2_s \cap \Sigma_t$ bounds a meridian of V_t. We arrive at the following dichotomy.

(1) ($i = 1$) Surfaces corresponding to R_i contain a curve of $S^2_s \cap \Sigma_t$ that bounds a meridian in U_t and a curve that bounds a meridian in V_t.

(2) ($i > 1$) Surfaces corresponding to R_{i-1} contain a curve of $S^2_s \cap \Sigma_t$ that bounds a meridian in U_t and surfaces corresponding to R_i contain a curve of $S^2_s \cap \Sigma_t$ that bounds a meridian in V_t.

In Case (1) we directly see a weak reduction or reduction (if both meridian disks bound the same curve).

In Case (2), we note that crossing from R_{i-1} to R_i corresponds to crossing one critical point, either a saddle or a center. In either case the set of essential curves in $S^2_s \cap \Sigma_t$ corresponding to R_{i-1} can be isotoped to be disjoint from those corresponding to R_i; hence Σ reduces or weakly reduces.

Since reduction implies a weak reduction, we find a weak reduction in every case above. This completes the proof of Theorem 1.

Remark 4.2 If $g = 1$ it is clearly impossible to find a weak reduction. Reading through the proof, we find exactly one place where the assumption $g > 1$ was used: in the proof that Ivan can't win (Proposition 3.3). We conclude that if we run the Cerf-theoretic argument in that case, it is actually Ivan who wins and Esmeralda who loses.

As a concluding remark we mention that it is quite possible that Waldhausen never intended to study Heegaard splittings of S^3, but rather prove the Poincaré Conjecture. If we replace S^3 with a homotopy 3–sphere the argument above fails miserably, since the "weak reduction" we will obtain consists of immersed disks (small problem, in light of Papakyriakopoulos's work) that might intersect each other (and hence will not give a weak reduction at all, even if each disk is embedded). Even if these problems are miraculously overcome, the best we can hope for is a "reduction" of the Heegaard surface via an immersed sphere that intersects the Heegaard surface in a single (probably not simple) closed curve. This is equivalent to the following condition: *the intersection of the kernels of the two maps induced on the fundamental group of* Σ *by its inclusion into* U_t *and* V_t *is non-trivial*. This is apparently not the right way to go: it was proven by J R Stallings in his paper "How not to prove the Poincaré conjecture" [9].

References

[1] **A J Casson**, **C M Gordon**, *Reducing Heegaard splittings*, Topology Appl. 27 (1987) 275–283 MR918537

[2] **J Cerf**, *Sur les difféomorphismes de la sphère de dimension trois* $(\Gamma_4 = 0)$, Lecture Notes in Mathematics, No. 53, Springer, Berlin (1968) MR0229250

[3] **J Johnson**, *Locally unknotted spines of Heegaard splittings*, Algebr. Geom. Topol. 5 (2005) 1573–1584 MR2186110

[4] **Y Rieck**, *Heegaard structures of manifolds in the Dehn filling space*, Topology 39 (2000) 619–641 MR1746912

[5] **Y Rieck**, *An-annular complexes in 3–manifolds*, Kyungpook Math. J. 45 (2005) 549–559 MR2205956

[6] **Y Rieck**, **J H Rubinstein**, *Invariant Heegaard surfaces in manifolds with involutions and the Heegaard genus of double covers* (2006) `arXiv:math.GT/0607145`

[7] **H Rubinstein**, **M Scharlemann**, *Comparing Heegaard splittings of non-Haken 3–manifolds*, Topology 35 (1996) 1005–1026 MR1404921

[8] **M Scharlemann**, *Heegaard splittings of compact 3–manifolds*, from: "Handbook of geometric topology", North-Holland, Amsterdam (2002) 921–953 MR1886684 `arXiv:math.GT/0007144`

[9] **J R Stallings**, *How not to prove the Poincaré conjecture*, Ann. of Math. Stud. 60 (1966) 83–88 available at `http://math.berkeley.edu/~stall`

[10] **F Waldhausen**, *Heegaard–Zerlegungen der 3–Sphäre*, Topology 7 (1968) 195–203 MR0227992

Department of mathematical Sciences, University of Arkansas
Fayetteville, AR 72701, USA

`yoav@uark.edu`

Received: 19 October 2005 Revised: 16 July 2006