Destabilizing amalgamated Heegaard splittings

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We construct a sequence of pairs of 3–manifolds \((M_1^n, M_2^n)\) each with incompressible torus boundary and with the following two properties:

1. For \(M^n\) the result of a carefully chosen glueing of \(M_1^n\) and \(M_2^n\) along their boundary tori, the genera \((g_1^n, g_2^n)\) of \((M_1^n, M_2^n)\) and the genus \(g^n\) of \(M^n\) satisfy the inequality

\[
\frac{g^n}{g_1^n + g_2^n} < \frac{1}{2}.
\]

2. The result of amalgamating certain unstabilized Heegaard splittings of \(M_1^n\) and \(M_2^n\) to form a Heegaard splitting of \(M^n\) produces a stabilized Heegaard splitting that can be destabilized successively \(n\) times.

1 Introduction

About 10 years ago, Cameron McA Gordon asked the following question: Can the pairwise connect sum of two 3–manifolds each with an unstabilized Heegaard splitting yield a 3–manifold with a stabilized Heegaard splitting? This question stumped the experts for many years but recently a negative answer to this question has been announced by D Bachman [1] and R Qiu [11].

More generally, one can ask how Heegaard splittings behave under other types of “sums”, that is, when the 3–manifolds containing them are glued along positive genus boundary components. How Heegaard genus behaves under these circumstances is one of the many questions investigated by Klaus Johannson in [6] and by the first author in [16]. In both cases, inequalities relating the Heegaard genus of the glued 3–manifold to the Heegaard genera of the original 3–manifolds are obtained. Most strikingly, the inequalities give lower bounds on the Heegaard genus of the glued 3–manifold in terms of the Heegaard genera of the original 3–manifolds. But these lower bounds are fractions of the sum of the genera of the original 3–manifolds. A better bound under more restrictive circumstances has recently been obtained by D Bachman, E Sedgwick and S Schleimer [2].
One upshot is that, in general, the phenomenon of “degeneration of Heegaard genus” under glueing of 3–manifolds can’t be ruled out. It appears however that under certain, possibly generic circumstances, this phenomenon does not occur. For instance, in [8] Marc Lackenby shows that for a pair of hyperbolic 3–manifolds each with one boundary component and under certain restrictions on the glueing, minimal genus Heegaard splittings of the glued 3–manifold are always obtained from Heegaard splittings of the original 3–manifolds by amalgamation.

It is presently unknown how large “degeneration of Heegaard genus” under glueing can be. Interestingly, the issue of stabilization implicitly arises in the investigation of this phenomenon in [13] and in [16]. The examples given in this note make this issue explicit. In particular, we provide examples that illustrate how “degeneration of Heegaard genus” under glueing corresponds to the existence of stabilizations in the amalgamation of Heegaard splittings of the original 3–manifolds. In doing so, we provide counterexamples to a conjecture of Kobayashi, Qiu, Rieck and Wang [7].

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2 Definitions

For standard definitions and results concerning knots, see Burde and Zieschang [3], Lickorish [9] or Rolfsen [12]. For standard definitions and results pertaining to 3–manifolds, see Hempel [4] or Jaco [5].

Definition 1 A height function on $S^3$ is a Morse function with exactly two critical points.

This last assumption guarantees that $h$ induces a foliation of $S^3$ by spheres, along with one maximum that we denote by $\infty$ and one minimum that we denote by $-\infty$.

Definition 2 Let $K$ be a knot in $S^3$. If all minima of $h|_K$ occur below all maxima of $h|_K$, then we say that $K$ is in bridge position with respect to $h$. The bridge number of $K$, $b(K)$, is the minimal number of maxima required for $h|_K$.

Definition 3 If $K$ is in bridge position, then a regular level surface below all maxima and above all minima is called a bridge surface.
Definition 4  An upper disk (lower disk) is an embedded disk whose interior is disjoint from the knot whose boundary is partitioned into two subarcs, one contained in a bridge surface and one a subarc of the knot that lies above (below) the bridge surface. A strict upper disk (strict lower disk) is an upper (lower) disk whose interior lies above (below) the bridge surface.

A complete set of strict upper (lower) disks is a set of upper (lower) disks such that each subarc of the knot lying above (below) the bridge surface meets exactly one disk in the set.

Definition 5  A compression body is a 3–manifold $W$ obtained from $S \times I$ where $S$ is a closed orientable connected surface by attaching 2–handles to $S \times \{0\} \subset S \times I$ and capping off any resulting 2–sphere boundary components with 3–handles. We denote $S \times \{1\}$ by $\partial_+ W$ and $\partial W - \partial_+ W$ by $\partial_- W$. Dually, a compression body is an orientable 3–manifold obtained from a closed orientable surface $\partial_- W \times I$ or a 3–ball or a union of the two by attaching 1–handles.

In the case where $\partial_- W = \emptyset$, we also call $W$ a handlebody.

Definition 6  Let $\mathcal{A} = \{a_1, \ldots, a_k\}$ be a collection of annuli in a compression body $W$. Then $\mathcal{A}$ is a primitive collection if there is a collection $\mathcal{D} = \{D_1, \ldots, D_k\}$ of pairwise disjoint properly embedded disks in $W$ such that $a_i$ meets $D_i$ in a single spanning arc and $a_i \cap D_j = \emptyset$ for $j \neq i$.

Definition 7  A set of defining disks for a compression body $W$ is a set of disks $\{D_1, \ldots, D_n\}$ properly embedded in $W$ with $\partial D_i \subset \partial_+ W$ for $i = 1, \ldots, n$ such that the result of cutting $W$ along $D_1 \cup \cdots \cup D_n$ is homeomorphic to $\partial_- W \times I$ or a 3–ball in the case that $W$ is a handlebody.

Definition 8  A Heegaard splitting of a 3–manifold $M$ is a pair $(V, W)$ in which $V$, $W$ are compression bodies and such that $M = V \cup W$ and $V \cap W = \partial_+ V = \partial_+ W = S$. We call $S$ the splitting surface or Heegaard surface. Two Heegaard splittings are considered equivalent if their splitting surfaces are isotopic.

The genus of $M$, denoted by $g(M)$, is the smallest possible genus of the splitting surface of a Heegaard splitting for $M$.

Definition 9  Let $(V, W)$ be a Heegaard splitting. A Heegaard splitting is called stabilized if there is a pair of properly embedded disks $(D, E)$ with $D \subset V$ and $E \subset W$ such that $|\partial D \cap \partial E| = 1$. We call the pair of disks $(D, E)$ a stabilizing pair of disks. A Heegaard splitting is unstabilized if it is not stabilized.
**Definition 10** Destabilizing a Heegaard splitting \((V, W)\) is the act of creating a Heegaard splitting from \((V, W)\) by performing ambient 2–surgery on \(S\) along the cocore of a 1–handle in either \(V\) or \(W\).

Note that the result of performing ambient 2–surgery on \(S\) along the cocore of a 1–handle in either \(V\) or \(W\) is not necessarily a Heegaard splitting. In order for this operation to be a destabilization, the result is required to be a Heegaard splitting. This definition is equivalent to presupposing a stabilizing pair of disks \(D; E\) and cutting along \(D\). (Here \(D\) is the cocore of a 1–handle of \(V\) and the existence of \(E\) guarantees that the result of cutting along \(D\) is a Heegaard splitting.)

**Definition 11** Let \(M\) be a compact orientable Seifert fibered space with quotient space an orientable orbifold \(Q\). Denote the genus of the surface underlying \(Q\) by \(g\) and the number of cone points by \(n\). Assume further that \(M\) (and hence \(Q\)) has exactly one boundary component. (This simplifying assumption is met in all examples considered here.)

Let \(a_1, \ldots, a_{2g}, b_1, \ldots, b_{n-1}\) be a disjoint collection of arcs in \(Q\) that cut \(Q\) into disks each containing at most one cone point. In the case of the once punctured torus, such a collection of arcs is shown in Figure 1. In the case of an orbifold with underlying surface a disk and with four cone points, such a collection of arcs is shown in Figure 2. If the underlying surface of \(Q\) is a disk, we further assume that each arc \(b_i\) cuts off a subdisk from \(Q\) containing exactly one cone point.

Abusing notation slightly, denote a collection of arcs in \(M\) that projects to \(a_1, \ldots, a_{2g}, b_1, \ldots, b_{n-1}\) also by \(a_1, \ldots, a_{2g}, b_1, \ldots, b_{n-1}\). Now take \(V\) to be a regular neighborhood of \(a_1, \ldots, a_{2g}, b_1, \ldots, b_{n-1}\) together with a regular neighborhood of \(\partial Q \times S^1\). Take \(W\) to be the closure of the complement of \(V\) in \(M\). It is an easy exercise to show that \((V, W)\) is a Heegaard splitting of \(M\). Such a Heegaard splitting is called a vertical Heegaard splitting of \(M\). If \(Q\) has no cone points, that is, if \(M = Q \times S^1\), then this splitting is also called the standard Heegaard splitting of \(M\).

**Definition 12** A tunnel system for a knot \(K\) in \(S^3\) is a collection of arcs \(t_1, \ldots, t_n\) such that the complement of \(K \cup t_1 \cup \cdots \cup t_n\) is a handlebody. The tunnel number of a knot \(K\) is the least number of arcs required for a tunnel system of \(K\).

**Definition 13** Suppose \(K\) is in bridge position and that there are \(n\) maxima. We may assume temporarily that all maxima occur in the same level surface \(L\). The maxima may be connected by a system of \(n-1\) disjoint arcs in \(L\). It is an easy exercise to show that this set of arcs is a tunnel system. It is called an upper tunnel system.
The same exercise shows that there is a set of defining disks $D$ for the complement of $K \cup t_1 \cup \cdots \cup t_n$ of the following type: Each component of $D$ has interior below $L$, furthermore, below $L$, its boundary runs once along exactly one component of $K - K \cap L$. This set of disks is called a complete set of lower disks for the upper tunnel system.

**Definition 14** Suppose $t_1, \ldots, t_n$ is a tunnel system for a knot $K$ in $S^3$. Denote the complement of $K$ by $M$. Take $V$ to be a regular neighborhood of $\partial M \cup t_1 \cup \cdots \cup t_n$ and take $W$ to the closure of the complement of $V$. Then $(V, W)$ is a Heegaard splitting called the Heegaard splitting corresponding to the tunnel system $t_1, \ldots, t_n$. 

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The definition of amalgamation is a lengthy one. In the last 15 years, this term has been used in the following context: A pair of 3–manifolds \( M_1, M_2 \) each with a Heegaard splitting are identified along components of their boundary. This results in a 3–manifold \( M \). The Heegaard splittings of \( M_1, M_2 \) can be used to construct a canonical Heegaard splitting of \( M \) called the amalgamation of the two Heegaard splittings. One assumes that in each of \( M_1, M_2 \) the boundary components along which the glueing occurs are contained in a single compression body. Roughly speaking, then, the collars of the boundary components lying in this compression body are discarded and the remnants of the two compression bodies in \( M_1 – \) (collars) identified to the remnants of the two compression bodies in \( M_2 – \) (collars). This is done in such a way that the 1–handles that are attached to the collar on such a boundary component in \( M_1 \) become attached to the compression body in \( M_2 \) that does not meet any of the boundary components along which the glueing takes place and vice versa. For a formal definition see below.

**Definition 15** Let \( M_1, M_2 \) be 3–manifolds with \( R \) a closed subsurface of \( \partial M_1 \), and \( S \) a closed subsurface of \( \partial M_2 \). Suppose that \( R \) is homeomorphic to \( S \) via a homeomorphism \( h \). Further, let \( (X_1, Y_1), (X_2, Y_2) \) be Heegaard splittings of \( M_1, M_2 \). Suppose further that \( N(R) \subset X_1, N(S) \subset X_2 \). Then, for some \( R' \subset \partial M_1 \setminus R \) and \( S' \subset \partial M_2 \setminus S \), \( X_1 = N(R \cup R') \cup (1\text{–handles}) \) and \( X_2 = N(S \cup S') \cup (1\text{–handles}) \). Here \( N(R) \) is homeomorphic to \( R \times I \) via a homeomorphism \( f \) and \( N(S) \) is homeomorphic to \( S \times I \) via a homeomorphism \( g \). Let \( \sim \) be the equivalence relation on \( M_1 \cup M_2 \) generated by

1. \( x \sim y \) if \( x, y \in \eta(R) \) and \( p_1 \cdot f(x) = p_1 \cdot f(y) \),
2. \( x \sim y \) if \( x, y \in \eta(S) \) and \( p_1 \cdot g(x) = p_1 \cdot g(y) \),
3. \( x \sim y \) if \( x \in R, y \in S \) and \( h(x) = y \),

where \( p_1 \) is projection onto the first coordinate. Perform isotopies so that for \( D \) an attaching disk for a 1–handle in \( X_1, D' \) an attaching disk for a 1–handle in \( X_2, [D] \cap [D'] = \emptyset \). Set \( M = (M_1 \cup M_2)/ \sim, X = (X_1 \cup Y_2)/ \sim, \) and \( Y = (Y_1 \cup X_2)/ \sim \). In particular, \( (N(R) \cup N(S))/ \sim \cong R, S \). Then \( X = Y_2 \cup N(R') \cup (1\text{–handles}) \), where the 1–handles are attached to \( \partial_+ Y_2 \) and connect \( \partial N(R') \) to \( \partial_+ Y_2 \). Hence \( X \) is a compression body. Analogously, \( Y \) is a compression body. So \( (X, Y) \) is a Heegaard splitting of \( M \). The splitting \( (X, Y) \) is called the *amalgamation* of \( (X_1, Y_1) \) and \( (X_2, Y_2) \) along \( R, S \) via \( h \).

### 3 A single destabilization

We first consider a concrete example that illustrates the issues under discussion. Let \( T_i \) be a punctured torus for \( i = 1, 2 \). As 3–manifolds \( M_1, M_2 \) we take \( T_i \times S^1 \) for
\[ i = 1, 2. \] Note that \( \partial M_i \) is a torus, for \( i = 1, 2. \) We take \( M \) to be the result of glueing \( M_1 \) to \( M_2 \) in such a way that \((\partial T_1) \times \{1\} \) and \((\partial T_2) \times \{p\}\) have intersection number one on the resulting torus.

We describe two distinct Heegaard splittings for \( M \):

**Example 16** Let \( S^1 = I_1 \cup I_2 \) be a decomposition of \( S^1 \) into two intervals that meet at their endpoints. Let \( V_i = T_i \times I_1 \) and \( W_i = T_i \times I_2 \), for \( i = 1, 2. \) Then \( V_i \) and \( W_i \) are genus 2 handlebodies. Denote the annulus in which \( V_i \) meets \( \partial M_i \) by \( A_i \) and that in which \( W_i \) meets \( \partial M_i \) by \( B_i \). Due to the choice of glueing of \( \partial M_1 \) and \( \partial M_2 \) that results in \( M \), \( A_1 \) meets \( A_2 \) in a (square) disk. As do \( B_1 \) and \( B_2 \). In other words, \( V = V_1 \cup V_2 \) is homeomorphic to the result of taking the disjoint union of \( V_1 \) and \( V_2 \) and joining the two components by a 1-handle. In particular, it is a genus 4 handlebody. The same is true for \( W = W_1 \cup W_2 \). Thus \((V, W)\) is a genus 4 Heegaard splitting of \( M \).

**Example 17** Let \((X_i, Y_i)\) be the standard Heegaard splitting of \( M_i \) (see Definition 11), for \( i = 1, 2. \) And let \((X, Y)\) be the amalgamation of \((X_1, Y_1)\) and \((X_2, Y_2)\)

**Theorem 18** The genus of \( M_i \) is three for \( i = 1, 2 \) and the genus of \( M \) is four.

**Proof** Recall that the rank, that is, the smallest number of generators, of the fundamental group of a 3–manifold provides a lower bound for the genus of a Heegaard splitting of that 3–manifold. Here

\[ \pi_1(M_i) = F_2 \oplus \mathbb{Z} \]

Abelianizing yields a free abelian group of rank 3. Thus rank \( \pi_1(M_i) = 3 \) and hence the Heegaard splitting constructed in Example 17 has minimal genus.

The Seifert–Van Kampen Theorem yields a presentation of \( \pi_1(M) \) as

\[ \pi_1(M_1) *_{\mathbb{Z}^2} \pi_1(M_2). \]

Quotienting out the normal closure of the amalgamated subgroup yields \( \mathbb{Z}^2 * \mathbb{Z}^2 \) as this kills the fibre and the commutator of the generating pair of the free base group on both sides. It follows that

\[ \text{rank } \pi_1(M) \geq \text{rank } \mathbb{Z}^2 * \mathbb{Z}^2 = 4. \]

Hence the Heegaard splitting in Example 16 has minimal genus. \( \square \)

The fact that the minimal genus Heegaard splitting is less than the genus of a minimal genus amalgamation in these examples illustrates a phenomenon known as “degeneration of Heegaard genus” under glueing.
Theorem 19  The Heegaard splitting $(X, Y)$ of $M$ is stabilized.

Proof  For $i = 1, 2$, choose arcs $a_1^i, a_2^i$ in $T_i \subset M_i$ as in Definition 11. Then $T_i - (N(a_1^i) \cup N(a_2^i))$ is a disk $D_i$. It’s boundary meets $\partial M_i$ as in Figure 3. After the amalgamation, a copy of $D_i$ survives in $M_i \subset M$, for $i = 1, 2$. How $\partial D_1$ and $\partial D_2$ intersect is pictured in Figure 4. Thus $(D_1, D_2)$ are a stabilizing pair of disks.  

Corollary 20  The Heegaard splitting $(X, Y)$ of $M$ can be destabilized exactly once.

Exercise 21  Show that destabilizing the Heegaard splitting in Example 17 yields the Heegaard splitting in Example 16.

4 Arbitrarily many destabilizations

We now construct a sequence of pairs of 3–manifolds that exhibit a more general phenomenon. More specifically, for each \( n \), we construct a pair \( (M^n_1, M^n_2) \) of 3–manifolds as follows: Given \( n \), take \( M^n_1 \) to be a Seifert fibered space with base orbifold a disk with \( n + 1 \) cone points. We denote the natural quotient map from \( M^n_1 \) to the base orbifold by \( p_n \). Take \( K^n \) to be a knot that has bridge number \( n \) and tunnel number \( n - 1 \). The existence of such knots is guaranteed by [10, Theorem 0.1]. Indeed, in [10], M Lustig and Y Moriah define the class of generalized Montesinos knots. The referenced theorem provides very technical but nevertheless achievable sufficient conditions under which such a knot has bridge number \( n \) and tunnel number \( n - 1 \). Take \( M^n_2 \) to be the complement of \( K^n \) in \( S^3 \).

Glue \( M^n_1 \) to \( M^n_2 \) in such a way that a fiber of \( M^n_1 \) is identified with a meridian of \( M^n_2 \). Denote the 3–manifold obtained in this way by \( M^n \). Consider the following Heegaard splittings of \( M^n \):

Example 22  Let \( b_1, \ldots, b_n \) be a collection of arcs that cut the base orbifold of \( M^n_1 \) into disks each with exactly one cone point. Bicolor these disks red and blue, that is, color these disks in such a way that disks abutting along an arc are given distinct colors. The preimage of these arcs in \( M^n_1 \) is a collection of annuli that cut \( M^n_1 \) into solid tori. These tori inherit colors from the bicoloring of the disks to which they project. Take \( V^n_1 \) to be the union of the red tori and \( W^n_1 \) to be the union of the blue tori.

Let \( P \) be a bridge sphere for \( K^n \). Then \( P \) divides \( M^n_2 \) into two components that we label \( V^n_2 \) and \( W^n_2 \). We can clearly assume that the \( 2n \) meridional boundary curves of \( P \cap M^n_2 \) match up with the boundary curves of the annuli \( b_1, \ldots, b_n \). Now set \( V^n = V^n_1 \cup V^n_2 \) and \( W^n = W^n_1 \cup W^n_2 \).

Lemma 23  The decomposition \( (V^n, W^n) \) is a Heegaard splitting of \( M^n \).

We first prove an auxilliary lemma. It is well known, but we include it here for completeness.

Lemma 24  Suppose \( X \) and \( Y \) are handlebodies. Let \( A \) be a collection of \( k \) essential annuli in \( \partial X \) and let \( B \) be a primitive collection of \( k \) annuli in \( \partial Y \). Glue \( X \) to \( Y \) by identifying \( A \) and \( B \). Denote the result by \( E \). Then \( E \) is a handlebody.
Proof Since $B$ is a primitive collection of $k$ annuli in $\partial Y$, there is a collection $\mathcal{Y}$ of $k$ disjoint essential disks such that each annulus meets one of the disks in exactly one arc and is disjoint from the other disks. Cutting $Y'$ along $\mathcal{Y}$ yields a handlebody $Y'$ and cuts each component of $B$ into a disk. The remnants of $\mathcal{Y} \cup B$ on $\partial Y'$ are disks. Thus a set of defining disks for $Y'$ can be isotoped to be disjoint from the remnants of $\mathcal{Y} \cup B$ on $\partial Y'$. Hence they can be used to augment $\mathcal{Y}$ to a set of defining disks $\mathcal{Y}'$ of $Y$.

Choose a set of defining disks $X$ for $X$. We may assume that each component of $X$ meets each component of $A$ in spanning arcs. (Note that each component of $A$ is met by a non zero number of such arcs, because it is essential.) In $E$ we can place a copy of the appropriate element of $\mathcal{Y}$ along each such spanning arc. Thus in $E$, the components of $X$ can be extended into $Y \subset E$ by parallel copies of elements of $\mathcal{Y}$ to an embedded disk. Denote the set of disks resulting from $X$ via these extensions along with a set of defining disks for $Y'$ by $E$.

Now the result of cutting $E$ along $E$ is a 3–ball since it can also be obtained by glueing 3–balls (the result of cutting $Y$ along $E \cap Y$) to a 3–ball (the result of cutting $X$ along $X'$) along disks. It follows that $E$ is a handlebody.

We now prove Lemma 23. Fortunately, the hard work has already been accomplished.

Proof of Lemma 23 To see that $(V^n, W^n)$ is a Heegaard splitting, consider the following: Each component of $V_1^n$ and $W_1^n$ is a solid torus. In particular, it is a handlebody. Furthermore, both $V_2^n$ and $W_2^n$ are genus $n$ handlebodies each meeting $\partial M_2^n$ in a primitive collection of $n$ annuli. More specifically, we can take a complete set of strict upper disks or a complete set of strict lower disks, respectively, to be the required collection of disks. See Figure 5.

It thus follows from Lemma 24 that $V^n$ and $W^n$ are handlebodies. Thus $(V^n, W^n)$ is a Heegaard splitting.

Example 25 Take $(X_1^n, Y_1^n)$ to be a vertical Heegaard splitting of $M_1^n$. Take $t_1, \ldots, t_{n-1}$ to be an upper tunnel system of $M_2^n$ and take $(X_2^n, Y_2^n)$ to be the Heegaard splitting corresponding to $t_1, \ldots, t_{n-1}$. Now take $(X^n, Y^n)$ to be the Heegaard splitting of $M^n$ resulting from the amalgamation of $(X_1^n, Y_1^n)$ and $(Y_1^n, Y_2^n)$.

Theorem 26 For $M_1^n, M_2^n, M^n$ as above,

$$\text{genus}(M_1^n) + \text{genus}(M_2^n) - \text{genus}(M^n) > n$$
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Figure 5: The submanifold $V^1_2$, or $W^1_2$ of $M^1_2$ with a collection of disks meeting primitive annuli as required.

and

$$\frac{\text{genus}(M^n)}{\text{genus}(M^n_1) + \text{genus}(M^n_2)} < \frac{1}{2}$$

Proof Note first that $\pi_1(M^n_1)$ maps onto the fundamental group of the base orbifold which is a free product of $n + 1$ cyclic groups. Thus $\pi_1(M^n_1)$ is of rank $n + 1$ by Grushko’s theorem. It follows that the genus of $M^n_1$ is $n + 1$. Furthermore, since the tunnel number of $K^n$ is $n - 1$, the genus of $M^n_2$ is $n$. The Heegaard splitting constructed in Example 22 bears witness to the fact that the Heegaard genus of $M^n$ is at most $n$.

Again, the manifold pairs $M^n_1, M^n_2$ exhibit the phenomenon of “degeneration of Heegaard genus” under gluing.

Note that the genus of a Heegaard splitting of $M^n$ resulting from an amalgamation of minimal genus Heegaard splittings is $2n$. In particular, the genus of $(X^n, Y^n)$ is $2n$.

Theorem 27 There are $n$ disjoint pairs of stabilizing disks for $(X^n, Y^n)$. In other words, the Heegaard splitting $(X^n, Y^n)$ of $M^n$ can be destabilized successively at least $n$ times. Specifically, the Heegaard splitting obtained from $(X^n, Y^n)$ is the result of stabilizing $(V^n, W^n)$ $n$ times.

Proof Recall that $M^n_2$ is the complement of $K^n$ and that $Y^n_2$ is the complement of $K^n$ together with an upper tunnel system. See Figure 6. Recall also that after amalgamation, a collar of $Y^n_2$ is a subset of $X^n$.

Denote the torus resulting from the identification of $\partial M_1^n$ and $\partial M_2^n$ by $T$. Recall that after the amalgamation, the torus $T$ minus the attaching disks for the 1–handles with cores $b_1, \ldots, b_n$ to one side and the upper tunnel system to the other side lies in the splitting surface $F^n$ of $(X^n, Y^n)$. We isotope $n$ essential subannuli of $T$ into $M_1^n$ and denote the resulting annuli by $U_1, \ldots, U_n$. We isotope the other $n$ subannuli of $T$ into $M_2^n$ and denote the result by $A_1, \ldots, A_n$. We subdivide $T$ into these subannuli in such a way that $U_1, \ldots, U_n$ are vertical in $M_1^n$ and $A_1, \ldots, A_n$ are meridional in $M_2^n$. Furthermore, we subdivide $T$ into these subannuli in such a way that $U_i$ meets the endpoints of exactly two distinct components of $b_1, \ldots, b_n$. See Figures 7 and 8.

Consider the portion of $F^n$ lying in $M_2^n$. See Figure 8. It is a punctured sphere. Moreover, it is isotopic to a punctured sphere that consists of a level disk with $2n$ punctures and an upper hemisphere. See Figure 9. Now note that the portion of $S^3$ above a bridge sphere that coincides with this level punctured disk and above the upper hemisphere is a 3–ball. (Replacing the upper hemisphere of this sphere with a level disk is equivalent to isotoping the upper hemisphere of this sphere through infinity. For details, see [15, Lemma 1].) Thus the portion of $F^n$ lying in $M_2^n$ is isotopic to a bridge sphere. It is hence as required in $M_2^n$.

It now suffices to verify that the portion of $F^n$ lying in $M_2^n$ admits the required pairs of disks. After a small isotopy, $b_1, \ldots, b_n$ lie in the interior of $M_1^n$. We then see that the portion of $F^n$ lying in $M_1^n$ may be reconstructed from $n$ vertical annuli and one torus by ambient 1–surgery along arcs dual to $b_1, \ldots, b_n$. See Figure 10. (Compare to Figure 7.)
Comparing the decomposition here with \((V^n, W^n)\), we see that the splitting surface \(F^n\) is entirely contained in a collar of one of the handlebodies \(V^n, W^n\), say \(V^n\). Furthermore, it induces a Heegaard splitting \((X_v^n, Y_v^n)\) of \(V^n\) as follows: Take \(X_v^n\) to be \(X^n \cap V^n = X^n\) and take \(Y_v^n\) to be the collar of \(\partial V^n\) together with \(Y^n \cap V^n\).
Then $Y^n_v = (\text{collar of } V^n) \cup (\text{solid torus}) \cup (1\text{-handles})$ and hence $X^n_v$ and $Y^n_v$ are both handlebodies.

However, the genus of $F^n$ is $2n$ and the genus of $\partial V^n$ is $n$. It thus follows from Scharlemann and Thompson [14, Lemma 2.7] that $(X^n_v, Y^n_v)$ and thus $(X^n, Y^n)$ is
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stabilized. By applying [14, Lemma 2.7] to locate a stabilizing pair of disks and using one of the disks to destabilize $n$ times in succession, we locate the $n$ pairs of stabilizing disks required.

References

[1] D Bachman, Connected sums of unstabilized Heegaard splittings are unstabilized, preprint


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