Destabilizing amalgamated Heegaard splittings

JENNIFER SCHULTENS
RICHARD WEIDMANN

We construct a sequence of pairs of 3–manifolds (M_1^n, M_2^n) each with incompressible torus boundary and with the following two properties:

(1) For M^n the result of a carefully chosen glueing of M_1^n and M_2^n along their boundary tori, the genera (g_1^n, g_2^n) of (M_1^n, M_2^n) and the genus g^n of M^n satisfy the inequality

$$\frac{g^n}{g_1^n + g_2^n} < \frac{1}{2}.$$

(2) The result of amalgamating certain unstabilized Heegaard splittings of M_1^n and M_2^n to form a Heegaard splitting of M produces a stabilized Heegaard splitting that can be destabilized successively n times.

57M27

1 Introduction

About 10 years ago, Cameron McA Gordon asked the following question: Can the pairwise connect sum of two 3–manifolds each with an unstabilized Heegaard splitting yield a 3–manifold with a stabilized Heegaard splitting? This question stumped the experts for many years but recently a negative answer to this question has been announced by D Bachman [1] and R Qiu [11].

More generally, one can ask how Heegaard splittings behave under other types of "sums", that is, when the 3–manifolds containing them are glued along positive genus boundary components. How Heegaard genus behaves under these circumstances is one of the many questions investigated by Klaus Johannson in [6] and by the first author in [16]. In both cases, inequalities relating the Heegaard genus of the glued 3–manifold to the Heegaard genera of the original 3–manifolds are obtained. Most strikingly, the inequalities give lower bounds on the Heegaard genus of the glued 3–manifold in terms of the Heegaard genera of the original 3–manifolds. But these lower bounds are fractions of the sum of the genera of the original 3–manifolds. A better bound under more restrictive circumstances has recently been obtained by D Bachman, E Sedgwick and S Schleimer [2].

Published: 3 December 2007 DOI: 10.2140/gtm.2007.12.319

One upshot is that, in general, the phenomenon of "degeneration of Heegaard genus" under glueing of 3–manifolds can't be ruled out. It appears however that under certain, possibly generic circumstances, this phenomenon does not occur. For instance, in [8] Marc Lackenby shows that for a pair of hyperbolic 3–manifolds each with one boundary component and under certain restrictions on the glueing, minimal genus Heegaard splittings of the glued 3–manifold are always obtained from Heegaard splittings of the original 3–manifolds by amalgamation.

It is presently unknown how large "degeneration of Heegaard genus" under glueing can be. Interestingly, the issue of stabilization implicitly arises in the investigation of this phenomenon in [13] and in [16]. The examples given in this note make this issue explicit. In particular, we provide examples that illustrate how "degeneration of Heegaard genus" under glueing corresponds to the existence of stabilizations in the amalgamation of Heegaard splittings of the original 3–manifolds. In doing so, we provide counterexamples to a conjecture of Kobayashi, Qiu, Rieck and Wang [7].

Acknowledgments

The first author is partially supported by the grant DMS-0603736 from the National Science Foundation.

2 Definitions

For standard definitions and results concerning knots, see Burde and Zieschang [3], Lickorish [9] or Rolfsen [12]. For standard definitions and results pertaining to 3–manifolds, see Hempel [4] or Jaco [5].

Definition 1 A *height function* on $\mathbb{S}^3$ is a Morse function with exactly two critical points.

This last assumption guarantees that h induces a foliation of S^3 by spheres, along with one maximum that we denote by ∞ and one minimum that we denote by $-\infty$.

Definition 2 Let K be a knot in S^3. If all minima of $h|_K$ occur below all maxima of $h|_K$, then we say that K is in *bridge position* with respect to h. The *bridge number* of K, $b(K)$, is the minimal number of maxima required for $h|_K$.

Definition 3 If K is in bridge position, then a regular level surface below all maxima and above all minima is called a *bridge surface*.

Definition 4 An *upper disk (lower disk)* is an embedded disk whose interior is disjoint from the knot whose boundary is partitioned into two subarcs, one contained in a bridge surface and one a subarc of the knot that lies above (below) the bridge surface. A *strict upper disk (strict lower disk)* is an upper (lower) disk whose interior lies above (below) the bridge surface.

A *complete set of strict upper (lower) disks* is a set of upper (lower) disks such that each subarc of the knot lying above (below) the bridge surface meets exactly one disk in the set.

Definition 5 A *compression body* is a 3–manifold W obtained from $S \times I$ where S is a closed orientable connected surface by attaching 2–handles to $S \times \{0\} \subset S \times I$ and capping off any resulting 2–sphere boundary components with 3–handles. We denote $S \times \{1\}$ by $\partial_+ W$ and $\partial W - \partial_+ W$ by $\partial_- W$. Dually, a compression body is an orientable 3–manifold obtained from a closed orientable surface $\partial_- W \times I$ or a 3–ball or a union of the two by attaching 1–handles.

In the case where $\partial_- W = \varnothing$, we also call W a handlebody.

Definition 6 Let $\mathcal{A} = \{a_1, \dots, a_k\}$ be a collection of annuli in a compression body W. Then $\mathcal{A}$ is a *primitive collection* if there is a collection $\mathcal{D} = \{D_1, \dots, D_k\}$ of pairwise disjoint properly embedded disks in W such that a_i meets D_i in a single spanning arc and $a_i \cap D_j = \varnothing$ for $j \neq i$.

Definition 7 A set of *defining disks* for a compression body W is a set of disks $\{D_1, \dots, D_n\}$ properly embedded in W with $\partial D_i \subset \partial_+ W$ for $i = 1, \dots, n$ such that the result of cutting W along $D_1 \cup \cdots \cup D_n$ is homeomorphic to $\partial_- W \times I$ or a 3–ball in the case that W is a handlebody.

Definition 8 A *Heegaard splitting* of a 3–manifold M is a pair (V, W) in which V, W are compression bodies and such that $M = V \cup W$ and $V \cap W = \partial_+ V = \partial_+ W = S$. We call S the *splitting surface* or *Heegaard surface*. Two Heegaard splittings are considered *equivalent* if their splitting surfaces are isotopic.

The *genus* of M, denoted by $g(M)$, is the smallest possible genus of the splitting surface of a Heegaard splitting for M.

Definition 9 Let (V, W) be a Heegaard splitting. A Heegaard splitting is called *stabilized* if there is a pair of properly embedded disks (D, E) with $D \subset V$ and $E \subset W$ such that $\#|\partial D \cap \partial E| = 1$. We call the pair of disks (D, E) a *stabilizing pair of disks*. A Heegaard splitting is *unstabilized* if it is not stabilized.

Definition 10 *Destabilizing* a Heegaard splitting (V, W) is the act of creating a Heegaard splitting from (V, W) by performing ambient 2–surgery on S along the cocore of a 1–handle in either V or W.

Note that the result of performing ambient 2–surgery on S along the cocore of a 1–handle in either V or W is not necessarily a Heegaard splitting. In order for this operation to be a destabilization, the result is required to be a Heegaard splitting. This definition is equivalent to presupposing a stabilizing pair of disks (D, E) and cutting along D. (Here D is the cocore of a 1–handle of V and the existence of E guarantees that the result of cutting along D is a Heegaard splitting.)

Definition 11 Let M be a compact orientable Seifert fibered space with quotient space an orientable orbifold Q. Denote the genus of the surface underlying Q by g and the number of cone points by n. Assume further that M (and hence Q) has exactly one boundary component. (This simplifying assumption is met in all examples considered here.)

Let $a_1, \ldots, a_{2g}, b_1, \ldots, b_{n-1}$ be a disjoint collection of arcs in Q that cut Q into disks each containing at most one cone point. In the case of the once punctured torus, such a collection of arcs is shown in Figure 1. In the case of an orbifold with underlying surface a disk and with four cone points, such a collection of arcs is shown in Figure 2. If the underlying surface of Q is a disk, we further assume that each arc b_i cuts off a subdisk from Q containing exactly one cone point.

Abusing notation slightly, denote a collection of arcs in M that projects to $a_1, \ldots, a_{2g}, b_1, \ldots, b_{n-1}$ also by $a_1, \ldots, a_{2g}, b_1, \ldots, b_{n-1}$. Now take V to be a regular neighborhood of $a_1, \ldots, a_{2g}, b_1, \ldots, b_{n-1}$ together with a regular neighborhood of $\partial Q \times \mathbb{S}^1$. Take W to be the closure of the complement of V in M. It is an easy exercise to show that (V, W) is a Heegaard splitting of M. Such a Heegaard splitting is called a *vertical* Heegaard splitting of M. If Q has no cone points, that is, if $M = Q \times S^1$, then this splitting is also called the *standard* Heegaard splitting of M.

Definition 12 A *tunnel system* for a knot K in $\mathbb{S}^3$ is a collection of arcs $t_1, \ldots, t_n$ such that the complement of $K \cup t_1 \cup \cdots \cup t_n$ is a handlebody. The *tunnel number* of a knot K is the least number of arcs required for a tunnel system of K.

Definition 13 Suppose K is in bridge position and that there are n maxima. We may assume temporarily that all maxima occur in the same level surface L. The maxima may be connected by a system of $n-1$ disjoint arcs in L. It is an easy exercise to show that this set of arcs is a tunnel system. It is called an *upper tunnel system*.

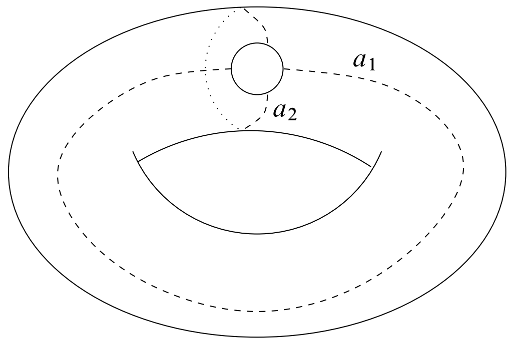

Figure 1: Arcs a_1, a_2 for a punctured torus

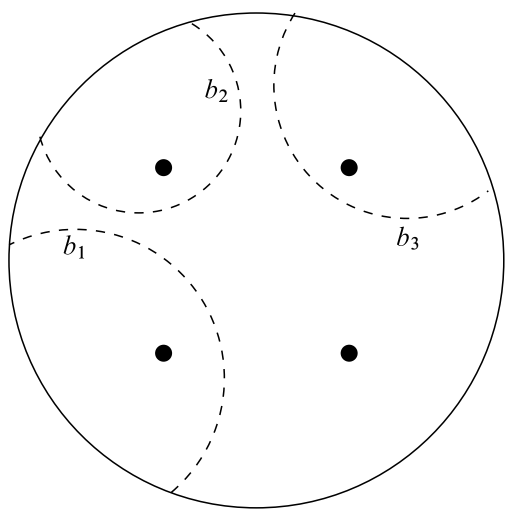

Figure 2: Arcs b_1, b_2, b_3 for an orbifold with four cone points

The same exercise shows that there is a set of defining disks $\mathcal{D}$ for the complement of $K \cup t_1 \cup \cdots \cup t_n$ of the following type: Each component of $\mathcal{D}$ has interior below L, furthermore, below L, its boundary runs once along exactly one component of $K - K \cap L$. This set of disks is called a *complete set of lower disks* for the upper tunnel system.

Definition 14 Suppose $t_1, \ldots, t_n$ is a tunnel system for a knot K in $\mathbb{S}^3$. Denote the complement of K by M. Take V to be a regular neighborhood of $\partial M \cup t_1 \cup \cdots \cup t_n$ and take W to the closure of the complement of V. Then (V, W) is a Heegaard splitting called the *Heegaard splitting corresponding to the tunnel system* $t_1, \ldots, t_n$.

The definition of amalgamation is a lengthy one. In the last 15 years, this term has been used in the following context: A pair of 3–manifolds M_1, M_2 each with a Heegaard splitting are identified along components of their boundary. This results in a 3–manifold M. The Heegaard splittings of M_1, M_2 can be used to construct a canonical Heegaard splitting of M called the amalgamation of the two Heegaard splittings. One assumes that in each of M_1, M_2 the boundary components along which the glueing occurs are contained in a single compression body. Roughly speaking, then, the collars of the boundary components lying in this compression body are discarded and the remnants of the two compression bodies in $M_1 - (\text{collars})$ identified to the remnants of the two compression bodies in $M_2 - (\text{collars})$. This is done in such a way that the 1–handles that are attached to the collar on such a boundary component in M_1 become attached to the compression body in M_2 that does not meet any of the boundary components along which the glueing takes place and vice versa. For a formal definition see below.

Definition 15 Let M_1, M_2 be 3–manifolds with R a closed subsurface of ∂M_1, and S a closed subsurface of ∂M_2. Suppose that R is homeomorphic to S via a homeomorphism h. Further, let $(X_1, Y_1), (X_2, Y_2)$ be Heegaard splittings of M_1, M_2. Suppose further that $N(R) \subset X_1, N(S) \subset X_2$. Then, for some $R' \subset \partial M_1 \backslash R$ and $S' \subset \partial M_2 \backslash S$, $X_1 = N(R \cup R') \cup (\text{1–handles})$ and $X_2 = N(S \cup S') \cup (\text{1–handles})$. Here $N(R)$ is homeomorphic to $R \times I$ via a homeomorphism f and $N(S)$ is homeomorphic to $S \times I$ via a homeomorphism g. Let $\sim$ be the equivalence relation on $M_1 \cup M_2$ generated by

(1) $x \sim y$ if $x, y \in \eta(R)$ and $p_1 \cdot f(x) = p_1 \cdot f(y)$,

(2) $x \sim y$ if $x, y \in \eta(S)$ and $p_1 \cdot g(x) = p_1 \cdot g(y)$,

(3) $x \sim y$ if $x \in R$, $y \in S$ and $h(x) = y$,

where p_1 is projection onto the first coordinate. Perform isotopies so that for D an attaching disk for a 1–handle in X_1, D' an attaching disk for a 1–handle in X_2, $[D] \cap [D'] = \varnothing$. Set $M = (M_1 \cup M_2)/\sim$, $X = (X_1 \cup Y_2)/\sim$, and $Y = (Y_1 \cup X_2)/\sim$. In particular, $(N(R) \cup N(S)/\sim) \cong R, S$. Then $X = Y_2 \cup N(R') \cup (\text{1–handles})$, where the 1–handles are attached to $\partial_+ Y_2$ and connect $\partial N(R')$ to $\partial_+ Y_2$. Hence X is a compression body. Analogously, Y is a compression body. So (X, Y) is a Heegaard splitting of M. The splitting (X, Y) is called the *amalgamation* of (X_1, Y_1) and (X_2, Y_2) along R, S via h.

3 A single destabilization

We first consider a concrete example that illustrates the issues under discussion. Let T_i be a punctured torus for $i = 1, 2$. As 3–manifolds M_1, M_2 we take $T_i \times \mathbb{S}^1$ for

$i = 1, 2$. Note that ∂M_i is a torus, for $i = 1, 2$. We take M to be the result of glueing M_1 to M_2 in such a way that $(\partial T_1) \times \{1\}$ and $(\partial T_2) \times \{p\}$ have intersection number one on the resulting torus.

We describe two distinct Hegaard splittings for M:

Example 16 Let $\mathbb{S}^1 = I_1 \cup I_2$ be a decomposition of $\mathbb{S}^1$ into two intervals that meet at their endpoints. Let $V_i = T_i \times I_1$ and $W_i = T_i \times I_2$, for $i = 1, 2$. Then V_i and W_i are genus 2 handlebodies. Denote the annulus in which V_i meets ∂M_i by A_i and that in which W_i meets ∂M_i by B_i. Due to the choice of glueing of ∂M_1 and ∂M_2 that results in M, A_1 meets A_2 in a (square) disk. As do B_1 and B_2. In other words, $V = V_1 \cup V_2$ is homeomorphic to the result of taking the disjoint union of V_1 and V_2 and joining the two components by a 1–handle. In particular, it is a genus 4 handlebody. The same is true for $W = W_1 \cup W_2$. Thus (V, W) is a genus 4 Heegaard splitting of M.

Example 17 Let (X_i, Y_i) be the standard Heegaard splitting of M_i (see Definition 11), for $i = 1, 2$. And let (X, Y) be the amalgamation of (X_1, Y_1) and (X_2, Y_2)

Theorem 18 *The genus of M_i is three for $i = 1, 2$ and the genus of M is four.*

Proof Recall that the rank, that is, the smallest number of generators, of the fundamental group of a 3–manifold provides a lower bound for the genus of a Heegaard splitting of that 3–manifold. Here

$$\pi_1(M_i) = F_2 \oplus \mathbb{Z}$$

Abelianizing yields a free abelian group of rank 3. Thus rank $\pi_1(M_i) = 3$ and hence the Heegaard splitting constructed in Example 17 has minimal genus.

The Seifert–Van Kampen Theorem yields a presentation of $\pi_1(M)$ as

$$\pi_1(M_1) *_{\mathbb{Z}^2} \pi_1(M_2).$$

Quotienting out the normal closure of the amalgamated subgroup yields $\mathbb{Z}^2 * \mathbb{Z}^2$ as this kills the fibre and the commutator of the generating pair of the free base group on both sides. It follows that

$$\text{rank } \pi_1(M) \geq \text{rank } \mathbb{Z}^2 * \mathbb{Z}^2 = 4.$$

Hence the Heegaard splitting in Example 16 has minimal genus. □

The fact that the minimal genus Heegaard splitting is less than the genus of a minimal genus amalgamation in these examples illustrates a phenomenon known as "degeneration of Heegaard genus" under glueing.

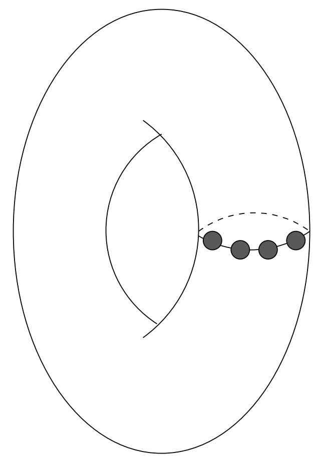

Figure 3: The boundary of D_i as it appears on ∂M_i

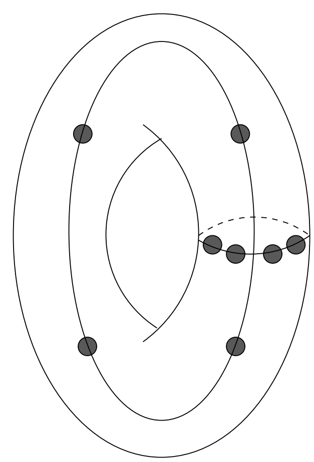

Figure 4: The boundaries of D_1 and D_2 as they intersect

Theorem 19 *The Heegaard splitting (X, Y) of M is stabilized.*

Proof For $i = 1, 2$, choose arcs a^i_1, a^2_1 in $T_i \subset M_i$ as in Definition 11. Then $T_i - (N(a^i_1) \cup N(a^i_2))$ is a disk D_i. It's boundary meets ∂M_i as in Figure 3. After the amalgamation, a copy of D_i survives in $M_i \subset M$, for $i = 1, 2$. How ∂D_1 and ∂D_2 intersect is pictured in Figure 4. Thus (D_1, D_2) are a stabilizing pair of disks. $\square$

Corollary 20 *The Heegaard splitting (X, Y) of M can be destabilized exactly once.*

Exercise 21 Show that destabilizing the Heegaard splitting in Example 17 yields the Heegaard splitting in Example 16.

4 Arbitrarily many destabilizations

We now construct a sequence of pairs of 3–manifolds that exhibit a more general phenomenon. More specifically, for each n, we construct a pair (M_1^n, M_2^n) of 3–manifolds as follows: Given n, take M_1^n to be a Seifert fibered space with base orbifold a disk with $n+1$ cone points. We denote the natural quotient map from M_1^n to the base orbifold by p_n. Take K^n to be a knot that has bridge number n and tunnel number $n-1$. The existence of such knots is guaranteed by [10, Theorem 0.1]. Indeed, in [10], M Lustig and Y Moriah define the class of generalized Montesinos knots. The referenced theorem provides very technical but nevertheless achievable sufficient conditions under which such a knot has bridge number n and tunnel number $n-1$. Take M_2^n to be the complement of K^n in $\mathbb{S}^3$.

Glue M_1^n to M_2^n in such a way that a fiber of M_1^n is identified with a meridian of M_2^n. Denote the 3–manifold obtained in this way by M^n. Consider the following Heegaard splittings of M^n:

Example 22 Let $b_1, \dots, b_n$ be a collection of arcs that cut the base orbifold of M_1^n into disks each with exactly one cone point. Bicolor these disks red and blue, that is, color these disks in such a way that disks abutting along an arc are given distinct colors. The preimage of these arcs in M_1^n is a collection of annuli that cut M_1^n into solid tori. These tori inherit colors from the bicoloring of the disks to which they project. Take V_1^n to be the union of the red tori and W_1^n to be the union of the blue tori.

Let P be a bridge sphere for K^n. Then P divides M_2^n into two components that we label V_2^n and W_2^n. We can clearly assume that the $2n$ meridional boundary curves of $P \cap M_2^n$ match up with the boundary curves of the annuli $b_1, \dots, b_n$ Now set $V^n = V_1^n \cup V_2^n$ and $W^n = W_1^n \cup W_2^n$.

Lemma 23 *The decomposition (V^n, W^n) is a Heegaard splitting of M^n.*

We first prove an auxilliary lemma. It is well known, but we include it here for completeness.

Lemma 24 *Suppose X and Y are handlebodies. Let A be a collection of k essential annuli in ∂X and let B be a primitive collection of k annuli in ∂Y. Glue X to Y by identifying A and B. Denote the result by E. Then E is a handlebody.*

Proof Since B is a primitive collection of k annuli in ∂Y, there is a collection $\mathcal{Y}$ of k disjoint essential disks such that each annulus meets one of the disks in exactly one arc and is disjoint from the other disks. Cutting Y along $\mathcal{Y}$ yields a handlebody Y' and cuts each component of B into a disk. The remnants of $\mathcal{Y} \cup B$ on $\partial Y'$ are disks. Thus a set of defining disks for Y' can be isotoped to be disjoint from the remnants of $\mathcal{Y} \cup B$ on $\partial Y'$. Hence they can be used to augment $\mathcal{Y}$ to a set of defining disks $\mathcal{Y}'$ of Y.

Choose a set of defining disks $\mathcal{X}$ for X. We may assume that each component of $\mathcal{X}$ meets each component of A in spanning arcs. (Note that each component of A is met by a non zero number of such arcs, because it is essential.) In E we can place a copy of the appropriate element of $\mathcal{Y}$ along each such spanning arc. Thus in E, the components of $\mathcal{X}$ can be extended into $Y \subset E$ by parallel copies of elements of $\mathcal{Y}$ to an embedded disk. Denote the set of disks resulting from $\mathcal{X}$ via these extensions along with a set of defining disks for Y' by $\mathcal{E}$.

Now the result of cutting E along $\mathcal{E}$ is a 3–ball since it can also be obtained by glueing 3–balls (the result of cutting Y along $\mathcal{E} \cap Y$) to a 3–ball (the result of cutting X along $\mathcal{X}$) along disks. It follows that E is a handlebody. $\square$

We now prove Lemma 23. Fortunately, the hard work has already been accomplished.

Proof of Lemma 23 To see that (V^n, W^n) is a Heegaard splitting, consider the following: Each component of V_1^n and W_1^n is a solid torus. In particular, it is a handlebody. Furthermore, both V_2^n and W_2^n are genus n handlebodies each meeting ∂M_2^n in a primitive collection of n annuli. More specifically, we can take a complete set of strict upper disks or a complete set of strict lower disks, respectively, to be the required collection of disks. See Figure 5.

It thus follows from Lemma 24 that V^n and W^n are handlebodies. Thus (V^n, W^n) is a Heegaard splitting. $\square$

Example 25 Take (X_1^n, Y_1^n) to be a vertical Heegaard splitting of M_1^n. Take $t_1, \dots, t_{n-1}$ to be an upper tunnel system of M_2^n and take (X_2^n, Y_2^n) to be the Heegaard splitting corresponding to $t_1, \dots, t_{n-1}$. Now take (X^n, Y^n) to be the Heegaard splitting of M^n resulting from the amalgamation of (X_1^n, Y_1^n) and (Y_1^n, Y_2^n).

Theorem 26 *For M_1^n, M_2^n, M^n as above,*

$$\operatorname{genus}(M_1^n) + \operatorname{genus}(M_2^n) - \operatorname{genus}(M^n) > n$$

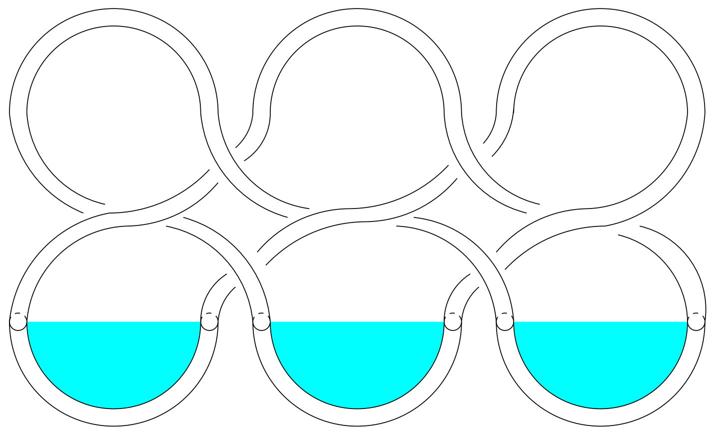

Figure 5: The submanifold V_2^1 or W_2^1 of M_2^1 with a collection of disks meeting primitive annuli as required

and

$$\frac{\text{genus}(M^n)}{\text{genus}(M_1^n)+\text{genus}(M_2^n)} < \frac{1}{2}$$

Proof Note first that $\pi_1(M_1^n)$ maps onto the fundamental group of the base orbifold which is a free product of $n+1$ cyclic groups. Thus $\pi_1(M_1^n)$ is of rank $n+1$ by Grushko's theorem. It follows that the genus of M_1^n is $n+1$. Furthermore, since the tunnel number of K^n is $n-1$, the genus of M_2^n is n. The Heegaard splitting constructed in Example 22 bears witness to the fact that the Heegaard genus of M^n is at most n. $\square$

Again, the manifold pairs M_1^n, M_2^n exhibit the phenomenon of "degeneration of Heegaard genus" under glueing.

Note that the genus of a Heegaard splitting of M^n resulting from an amalgamation of minimal genus Heegaard splittings is $2n$. In particular, the genus of (X^n, Y^n) is $2n$.

Theorem 27 *There are n disjoint pairs of stabilizing disks for (X^n, Y^n). In other words, the Heegaard splitting (X^n, Y^n) of M^n can be destabilized successively at least n times. Specifically, the Heegaard splitting obtained from (X^n, Y^n) is the result of stabilizing (V^n, W^n) n times.*

Proof Recall that M_2^n is the complement of K^n and that Y_2^n is the complement of K^n together with an upper tunnel system. See Figure 6. Recall also that after amalgamation, (a collar of) Y_2^n is a subset of X^n.

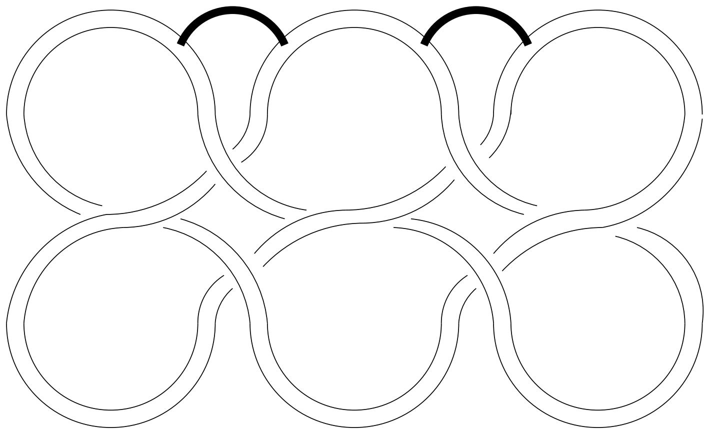

Figure 6: K^3 with an upper tunnel system

Denote the torus resulting from the identification of ∂M_1^n and ∂M_2^n by T. Recall that after the amalgamation, the torus T minus the attaching disks for the 1–handles with cores $b_1, \dots, b_n$ to one side and the upper tunnel system to the other side lies in the splitting surface F^n of (X^n, Y^n). We isotope n essential subannuli of T into M_1^n and denote the resulting annuli by $U_1, \dots, U_n$. We isotope the other n subannuli of T into M_2^n and denote the result by $A_1, \dots, A_n$. We subdivide T into these subannuli in such a way that $U_1, \dots, U_n$ are vertical in M_1^n and $A_1, \dots, A_n$ are meridional in M_2^n. Furthermore, we subdivide T into these subannuli in such a way that U_i meets the endpoints of exactly two distinct components of $b_1, \dots, b_n$. See Figures 7 and 8.

Consider the portion of F^n lying in M_2^n. See Figure 8. It is a punctured sphere. Moreover, it is isotopic to a punctured sphere that consists of a level disk with $2n$ punctures and an upper hemisphere. See Figure 9. Now note that the portion of $\mathbb{S}^3$ above a bridge sphere that coincides with this level punctured disk and above the upper hemisphere is a 3–ball. (Replacing the upper hemisphere of this sphere with a level disk is equivalent to isotoping the upper hemisphere of this sphere through infinity. For details, see [15, Lemma 1].) Thus the portion of F^n lying in M_2^n is isotopic to a bridge sphere. It is hence as required in M_2^n.

It now suffices to verify that the portion of F^n lying in M_1^n admits the required pairs of disks. After a small isotopy, $b_1, \dots, b_n$ lie in the interior of M_1^n. We then see that the portion of F^n lying in M_1^n may be reconstructed from n vertical annuli and one torus by ambient 1–surgery along arcs dual to $b_1, \dots, b_n$. See Figure 10. (Compare to Figure 7.)

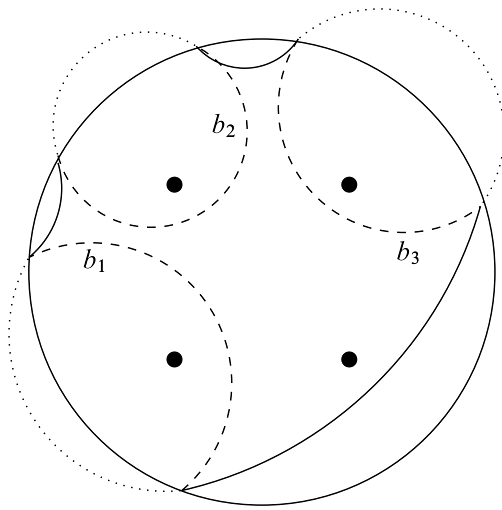

Figure 7: Vertical annuli in M_1^3

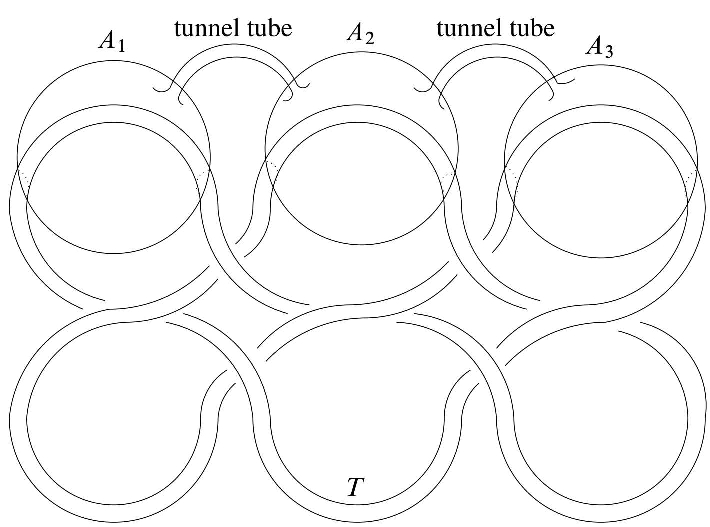

Figure 8: The result of isotoping the annuli A_1, A_2, A_3 into M_2^3

Comparing the decomposition here with (V^n, W^n), we see that the splitting surface F^n is entirely contained in a collar of one of the handlebodies V^n, W^n, say V^n. Furthermore, it induces a Heegaard splitting (X_v^n, Y_v^n) of V^n as follows: Take X_v^n to be $X^n \cap V^n = X^n$ and take Y_v^n to be the collar of ∂V^n together with $Y^n \cap V^n$.

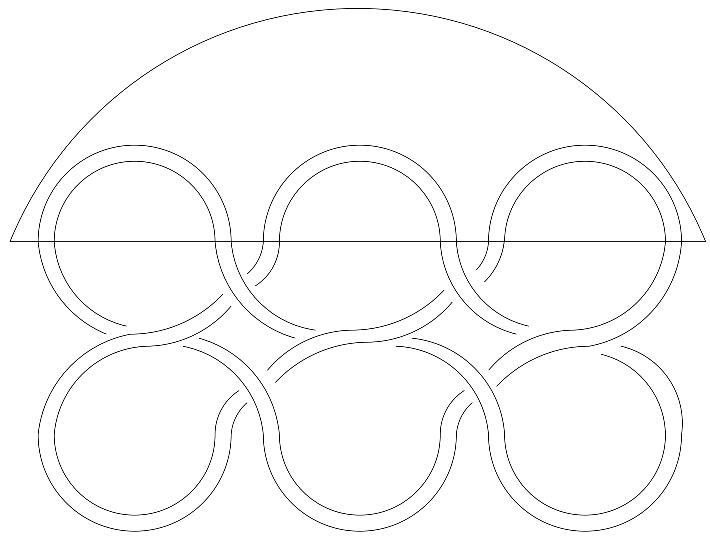

Figure 9: The punctured sphere in M_2^3 that is isotopic to a bridge sphere

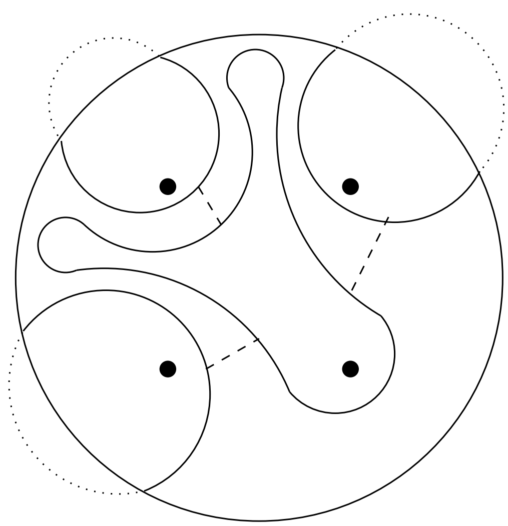

Figure 10: A dual schematic for $F^n \cap M_1^n$

Then $Y_v^n = (\text{collar of } V^n) \cup (\text{solid torus}) \cup (1\text{–handles})$ and hence X_v^n and Y_v^n are both handlebodies.

However, the genus of F^n is $2n$ and the genus of ∂V^n is n. It thus follows from Scharlemann and Thompson [14, Lemma 2.7] that (X_v^n, Y_v^n) and thus (X^n, Y^n) is stabilized. By applying [14, Lemma 2.7] to locate a stabilizing pair of disks and using one of the disks to destabilize n times in succession, we locate the n pairs of stabilizing disks required. □

References

[1] **D Bachman**, *Connected sums of unstabilized Heegaard splittings are unstabilized*, preprint

[2] **D Bachman**, **S Schleimer**, **E Sedgwick**, *Sweepouts of amalgamated 3–manifolds*, Algebr. Geom. Topol. 6 (2006) 171–194 MR2199458

[3] **G Burde**, **H Zieschang**, *Knots*, de Gruyter Studies in Mathematics 5, Walter de Gruyter & Co., Berlin (1985) MR808776

[4] **J Hempel**, 3*-Manifolds*, Ann. of Math. Studies 86, Princeton University Press, Princeton, NJ (1976) MR0415619

[5] **W Jaco**, *Lectures on three-manifold topology*, CBMS Regional Conference Series in Mathematics 43, American Mathematical Society, Providence, R.I. (1980) MR565450

[6] **K Johannson**, *Topology and combinatorics of 3–manifolds*, Lecture Notes in Mathematics 1599, Springer, Berlin (1995) MR1439249

[7] **T Kobayashi**, **R Qiu**, **Y Rieck**, **S Wang**, *Separating incompressible surfaces and stabilizations of Heegaard splittings*, Math. Proc. Cambridge Philos. Soc. 137 (2004) 633–643 MR2103921

[8] **M Lackenby**, *The Heegaard genus of amalgamated 3–manifolds*, Geom. Dedicata 109 (2004) 139–145 MR2113191

[9] **W B R Lickorish**, *An introduction to knot theory*, Graduate Texts in Mathematics 175, Springer, New York (1997) MR1472978

[10] **M Lustig**, **Y Moriah**, *Generalized Montesinos knots, tunnels and $\mathcal{N}$–torsion*, Math. Ann. 295 (1993) 167–189 MR1198847

[11] **R Qiu**, *Stabilizations of reducible Heegaard splittings*, preprint

[12] **D Rolfsen**, *Knots and links*, Mathematics Lecture Series 7, Publish or Perish, Berkeley, CA (1976) MR0515288

[13] **M Scharlemann**, **J Schultens**, *Annuli in generalized Heegaard splittings and degeneration of tunnel number*, Math. Ann. 317 (2000) 783–820 MR1777119

[14] **M Scharlemann**, **A Thompson**, *Heegaard splittings of* (surface)$\times I$ *are standard*, Math. Ann. 295 (1993) 549–564 MR1204837

[15] **J Schultens**, *Additivity of bridge numbers of knots*, Math. Proc. Cambridge Philos. Soc. 135 (2003) 539–544 MR2018265

[16] **J Schultens**, *Heegaard genus formula for Haken manifolds*, Geom. Dedicata 119 (2006) 49–68 MR2247647

Department of Mathematics, University of California, Davis
1 Shields Avenue, Davis CA 95616, USA

Department of Mathematics and Maxwell Institute of Mathematical Sciences
Heriot-Watt University, Riccarton, Edinburgh, EH14 4AS, Scotland

jcs@math.ucdavis.edu, R.Weidmann@ma.hw.ac.uk

Received: 20 January 2007 Revised: 12 March 2007