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Hyperbolic volume, Heegaard genus and ranks of groups

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Some conjectures about Heegaard genera and ranks of fundamental groups of 3–manifolds are formulated, and it is shown that they imply new statements about hyperbolic volume.

57M05, 57M50

1 Preface

In my talk at the conference on Heegaard splittings at the Technion, I formulated some topological conjectures and discussed how, by adapting some of my work with Marc Culler and others on volumes of hyperbolic 3–manifolds, to use these conjectures—if true—to the problem of relating hyperbolic volume to Heegaard genus.

One of the conjectures that I stated is a modernized version of the antique conjecture that if M is a compact, orientable 3–manifold, the rank of $\pi_1(M)$ is equal to the Heegaard genus of M. The first counterexamples to this old conjecture, in which M is a Seifert fibered manifold, had been given by Boileau and Zieschang in [5]; more general counterexamples, for graph manifolds, were given by Schultens and Weidmann in [17]. Here is the modernized version:

Conjecture 1.1 *If M is a compact, orientable, hyperbolic 3–manifold, the rank of $\pi_1(M)$ is equal to the Heegaard genus of M.*

The main results that I discussed in my talk give connections between Heegaard genus and hyperbolic volume that are conditional on Conjecture 1.1 and another conjecture, which is formulated below as Conjecture 4.3. Those results are incorporated into this article as Corollaries 4.9, 5.3 and 6.8. They are immediate conequences of results, Corollary 4.8 and Propositions 5.2 and 6.2, which involve ranks of groups instead of Heegaard genus and are not conditional on Conjecture 1.1 (although they are conditional on other topological conjectures).

This reorganization of the material may prove valuable if Conjecture 1.1 turns out to be false. The immediate motivation for reorganizing the paper in this way was that

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rumors were circulating to the effect that Conjecture 1.1 had been disproved. These rumors seem to have died down.

In this article I will provide detailed proofs of all the new results, including those that I announce in my talk. I will give less space here than in my talk to summaries of the arguments in Agol–Culler–Shalen [2] and Culler–Shalen [9], on which the proofs are based, as these summaries were not significantly different from the ones given in the introductions to those papers.

All the new results are conditional in the sense that they include various topological conjectures as hypotheses. Some of these conjectures are very new, and I hope that they will motivate new research, whether or not they are true.

I thank Ian Agol for some valuable discussions about Conjecture 1.1. I also thank the referee for pointing out some major errors in the first draft of the article.

2 Introduction

When one studies closed hyperbolic 3–manifolds, the volume is a key invariant, because it is known that up to isometry there exist at most finitely many hyperbolic manifolds of a given finite volume. The volume of a closed hyperbolic 3–manifold M is a topological invariant of M—for example because Mostow rigidity says the hyperbolic structure of M is itself determined by the topology—but the precise connection between volume and more classical topological invariants is far from being well understood.

In this article I'll be concerned with the connection between volume of a hyperbolic 3–manifold and the rank of its fundamental group. An upper bound on rank—or even the Heegaard genus—of a closed hyperbolic 3–manifold does not give an upper bound on the volume. For example, if M_0 is a closed hyperbolic 3–manifold that fibres over the circle, and g is the genus of the fiber, then for every positive integer n there is an n–fold cyclic cover M_n of M which itself fibers over S^1 with genus–g fiber. In particular the Heegaard genus of each M_n is at most $2g+1$, but the volume of M_n is nv_0, where v_0 denotes the volume of M_0.

On the other hand, an upper bound on the volume of a closed hyperbolic 3–manifold definitely does give an upper bound on the Heegaard genus (and hence on the rank). In fact, there is a universal constant $\mu > 0$, the *Margulis constant,* such that for every closed hyperbolic 3–manifold M, the subset M_{thin} of M, defined to consist of all points through which there pass homotopically non-trivial curves of length $\leq \mu$, is a disjoint union of smooth solid tori. It's not hard to show that the submanifold $M_{\text{thick}} = \overline{M - M_{\text{thin}}}$ has a triangulation with at most $C_0 v$ simplices, where v denotes the volume of M and C_0 is another universal constant that can be computed from μ. This implies that there is a constant C such that every closed 3–manifold M has Heegaard genus at most Cv, where v denotes the volume of M.

The problem is that the constant C that comes from this argument is astronomical, and the estimates obtained in this way don't get us anywhere near what we expect from examples. The goal of this article will be to suggest a way of getting good explicit upper bounds for the rank of $\pi_1(M)$, where M is a hyperbolic 3–manifold, in terms of the volume of M.

By contrast, if one is willing to settle for bounds on the homological complexity of a manifold M instead of the rank of $\pi_1(M)$, the results that appear in my papers with Culler and others are in the realm of reality and are sometimes sharp. For example, [2, Theorem 1.1] states among other things that if M is a closed, orientable hyperbolic 3–manifold with volume at most 1.22, then $H_1(M;\mathbb{Z}_p)$ has dimension at most 2 for every prime $p \neq 2, 7$. This result is sharp for $p = 3$ and for $p = 5$: the manifolds referred to in the Weeks–Hodgson census [19] as `m003(-3,1)` and `m007(3,1)` have respective volumes $0.94\ldots$ and $1.01\ldots$, while their integer homology groups are respectively isomorphic to $\mathbb{Z}_5 \oplus \mathbb{Z}_5$ and $\mathbb{Z}_3 \oplus \mathbb{Z}_6$.

One would like to obtain bounds for Heegaard genus, or rank of fundamental group, in terms of volume, similar to the bounds on homology that I have described. This will require more knowledge on the topological side. In Section 4 I'll formulate and discuss a pair of topological conjectures, one about rank and one about Heegaard genus; the former will be shown to imply a bound on the rank of the fundamental group in terms of hyperbolic volume. In Section 5 I'll show how to get a stronger bound by combining the conjecture of Section 4 with a conjectured analogue for rank of Moriah and Rubinstein's result in [12] about the behavior of Heegaard genus under Dehn filling. In Section 6 I'll give another consequence of the conjecture of Section 4, partially analogous to the results of [9] and [7].

3 A little background

The following result provides the simplest motivation (from the point of view of studying volumes) for the conjectures on ranks of 3–manifold groups that I'll state in the next section.

Theorem 3.1 *Suppose that M is a closed, orientable hyperbolic 3–manifold with volume less than 1.015. Then $\pi_1(M)$ has a 2–generator subgroup of finite index.*

Proof Let M be any closed, orientable hyperbolic 3–manifold. Let us write $M = \mathbb{H}^3/\Gamma$, where $\Gamma \le \mathrm{Isom}^+(\mathbb{H}^3)$ is discrete, cocompact and torsion-free. If $\Gamma \cong \pi_1(M)$ has no 2–generator subgroup of finite index, it follows from Jaco and Shalen [10, Theorem VI.4.1] that Γ is 2–free in the sense that all its 2–generator subgroups are free.

When Γ is 2–free, the number $\log 3$ is a strong Margulis number for M in the sense of Anderson, Canary, Culler and Shalen [4, Section 8]. Indeed, the case $k = 2$ of [4, Proposition 8.1] asserts that $\log 3$ is a strong Margulis number for M as long as every 2–generator subgroup of Γ is free and topologically tame. But according to the main results of Agol [1] and Calegari–Gabai [6], every finitely generated Kleinian group is topologically tame.

Przeworski [13, Theorem 3] asserts that if the first Betti number $\beta_1(M)$ is at least 3, then the volume of M is at least 1.015. The hypothesis $\beta_1(M) \ge 3$ is used only in order to quote [13, Theorem 1], which is included in [4, Corollary 10.6], and gives a lower bound $V(\lambda)$ on the volume of M in terms of the length λ of a shortest geodesic in M, under the assumption that $\beta_1(M) \ge 3$. Now, according to [4, Corollary 10.5], the number $V(\lambda)$ is a lower bound for the volume of M as long as $\log 3$ is a strong Margulis number for M. Hence the lower bound of 1.015 for the volume of M still holds if one replaces the assumption $\beta_1(M) \ge 3$ by the assumption that $\log 3$ is a strong Margulis number for M; by the discussion above, this holds in particular if $\pi_1(M)$ has no finite-index subgroup of rank 2. □

4 A conjecture on rank and finite covering spaces, and a consequence

Theorem 3.1 raises the following question (most immediately for the case $k = 2$):

Question 4.1 *What restriction does the existence of a k–generator subgroup of finite index in $\pi_1(M)$ place on M?*

There is a *homological* condition which is necessary for the existence of such a subgroup: $H_1(M; \mathbf{Z}_p)$ must have rank $\le k+1$ for every prime p. The necessity of this condition follows from Shalen–Wagreich [18, Proposition 1.1]. This was the starting point for the results relating volume to homology which I described in the introduction.

The following conjecture, which I first formulated in my talk at the Technion, would provide a bound of the rank of $\pi_1(M)$ when $\pi_1(M)$ has a k–generator subgroup of finite index.

Conjecture 4.2 *If M is a compact, orientable hyperbolic 3–manifold such that* $\operatorname{rank}\pi_1(M) = r$*, then for any finite-sheeted covering space $\widetilde{M}$ of M we have* $\operatorname{rank}\pi_1(\widetilde{M}) \geq r-1$.

Equivalently, Conjecture 4.2 asserts that if $\pi_1(M)$ has a k–generator subgroup of finite index then $\operatorname{rank}\pi_1(M) \leq k+1$.

In my talk, I also formulated a parallel conjecture about Heegaard genus:

Conjecture 4.3 *If M is a compact, orientable hyperbolic 3–manifold with Heegaard genus g, then any finite-sheeted covering space $\widetilde{M}$ of M has Heegaard genus at least $g-1$.*

Of course, Conjectures 4.2 and 4.3 are equivalent modulo the "rank equals genus" Conjecture 1.1.

4.4 It follows from Agol, Culler and Shalen [2, Corollary 7.3] (which is in turn a refinement of Shalen and Wagreich [18, Proposition 1.1], a result which I mentioned above) that if M is a closed, orientable hyperbolic 3–manifold such that $H_1(M;\mathbb{Z}_p)$ has rank r for a given prime p, then for any finite-sheeted covering space $\widetilde{M}$ of M, the rank of $H_1(\widetilde{M};\mathbb{Z}_p)$ is at least $r-1$. We may regard Conjectures 4.2 and 4.3 as analogues, for the rank of the fundamental group and the Heegaard genus, of this result about the rank of the mod–p homology.

4.5 There appears to be a huge class of examples, for every $g \geq 3$, in which a compact, orientable hyperbolic 3–manifold M with Heegaard genus g has a finite-sheeted covering space $\widetilde{M}$ of Heegaard genus exactly $g-1$. Alan Reid gave the first such example, for $g=3$, in his paper [14], which was inspired by an earlier, weaker version of Theorem 3.1 which appeared in Culler and Shalen [8].

During the conference, Hyam Rubinstein pointed out a systematic way of constructing such examples. Suppose that M is a closed, orientable hyperbolic 3–manifold containing a closed, non-orientable surface F, and that the complement in M of the interior of a regular neighborhood N of F is a handlebody J of genus $g-1$. Since N is a twisted I–bundle over F, it's easy to construct a 2–sheeted covering space $q\colon \widetilde{M} \to M$ to which J lifts, and in which $q^{-1}(N)$ is a trivial I–bundle. It follows that $\widetilde{M}$ has a Heegaard splitting in which the handlebodies are isotopic to the two lifts of J; in particular this splitting has genus $g-1$. On the other hand, if A is a vertical arc in the twisted I–bundle N, and T is a regular neighborhood of A relative to N, then $J \cup T$ and $\overline{N-T}$ are genus–g handlebodies, which define a genus–g Heegaard splitting of M. It appears that for $g \geq 3$ the "generic" situation is that both the genus–g Heegaard splitting of M and the genus–$(g-1)$ Heegaard splitting of $\widetilde{M}$ are minimal.

These examples help show why the lower bound $g-1$ is natural in Conjecture 4.3. I would guess that in many of these examples one can also show that $\operatorname{rank} \pi_1(M) = g$ and $\operatorname{rank} \pi_1(M) = g-1$, which would help show why the lower bound $r-1$ is natural in Conjecture 4.2.

4.6 During the conference, Andrew Casson pointed out that Conjecture 4.2 is trivial for a cyclic regular covering. This is because if any group G has a normal subgroup N with G/N cyclic, the rank of G can obviously exceed the rank of N by at most 1.

4.7 The best known result in the direction of Conjecture 4.3 seems to be the result of Rieck and Rubinstein [15], which gives a lower bound for the Heegaard genus of a two-sheeted covering of M in terms of the Heegaard genus of M.

Theorem 3.1 now has the following immediate consequence:

Corollary 4.8 *If Conjecture 4.2 is true, then for every closed, orientable hyperbolic 3–manifold with volume at most 1.015 we have $\operatorname{rank} \pi_1(M) \leq 3$.* □

Since Conjectures 4.2 and 4.3 are equivalent modulo the "rank equals genus" Conjecture 1.1, we also get:

Corollary 4.9 *If Conjectures 1.1 and 4.3 are true, then for every closed, orientable hyperbolic 3–manifold with volume at most 1.015 we have $\operatorname{rank} \pi_1(M) \leq 3$.* □

5 A conjecture on rank and Dehn filling, and a consequence

It is a consequence of the result proved by Moriah and Rubinstein in [12] that if N is a hyperbolic 3–manifold of finite volume with exactly one cusp, and if g denotes the Heegaard genus of the compact core $\widehat{N}$ of N, then infinitely many Dehn fillings of $\widehat{N}$ yield closed manifolds of Heegaard genus exactly g. (This was re-proved by a purely topological argument in Rieck and Sedgwick [16].)

In view of Moriah and Rubinstein's result, the following conjecture would follow immediately from the "rank equals genus" Conjecture 1.1.

Conjecture 5.1 *Suppose that N is a hyperbolic 3–manifold of finite volume with exactly one cusp, and set $r = \operatorname{rank} \pi_1(N)$. Let $\widehat{N}$ denote the compact core of N. Then there is an infinite sequence (M_i) of manifolds obtained by distinct Dehn fillings of $\widehat{N}$ such that each $\operatorname{rank} \pi_1(M_i) = r$ for each i.*

The main result of [2], Theorem 1.1, states that if M is a closed, orientable hyperbolic 3–manifold with volume at most 1.22, then $H_1(M;\mathbb{Z}_p)$ has dimension at most 2 for every prime $p \neq 2, 7$, and that $H_1(M;\mathbb{Z}_2)$ and $H_1(M;\mathbb{Z}_7)$ have dimension at most 3. The result had originally been proved in a weaker form, which states that if M has volume at most 1.22, then $H_1(M;\mathbb{Z}_p)$ has dimension at most 3 for every prime p. In this section I'll show how to adapt the proof of this weaker theorem to prove an analogous result about the rank of the fundamental group, modulo Conjectures 4.2 and 5.1. This is Proposition 5.2 below. I have not thought about whether one can prove a Heegaard-genus analogue of the strong form of [2, Theorem 1.1] modulo Conjectures 4.2 and 5.1.

Proposition 5.2 *If Conjectures 4.2 and 5.1 are true, then for every closed, orientable hyperbolic 3–manifold M with volume at most 1.22, we have* $\operatorname{rank}\pi_1(M) \leq 3$.

You'll notice that the information given by Proposition 5.2 modulo Conjectures 4.2 and 5.1 is strictly stronger than the information given by Corollary 4.8 modulo Conjecture 4.2 alone. The proof of Proposition 5.2 uses a lot more mathematics than that of Corollary 4.8. In particular, [2, Lemma 3.2], which is quoted in the proof below, depends on a result from Agol, Dunfield, Storm and Thurston [3] which in turn relies on Perelman's estimates for the Ricci flow with surgeries.

Proof of Proposition 5.2 As in [2], we shall say that a hyperbolic manifold M is *exceptional* if every shortest geodesic in M has tube radius at most $(\log 3)/2$.

We first prove the proposition in the case where M is non-exceptional. In this case, by definition, there is a shortest geodesic C in M with $R = \operatorname{tuberad}(C) > (\log 3)/2$. We set $N = \operatorname{drill}_C(M)$. Let $\mathcal{H}$ denote the maximal cusp neighborhood in N. Since $R > (\log 3)/2$, [2, Lemma 3.2] implies that $\operatorname{vol}\mathcal{H} < \pi$.

Now assume that $\operatorname{rank}\pi_1(M) \geq 4$. Set $r = \operatorname{rank}\pi_1(\widehat{N})$. It is obvious that $r \geq \operatorname{rank}\pi_1(M)$, so in particular $r \geq 4$. Conjecture 5.1 implies that there is an infinite sequence (M_i) of manifolds obtained by distinct Dehn fillings of $\widehat{N}$ such that $\operatorname{rank}\pi_1(M_i) = r$ for each i. Since $r \geq 4$, Conjecture 4.2 implies that for each i, every finite-index subgroup of $\pi_1(M_i)$ has rank at least 3. Since $\pi_1(M_i)$ has no 2–generator subgroup of finite index, it follows from Jaco and Shalen [10, Theorem VI.4.1] that $\pi_1(M_i)$ is 2–free in the sense that all its 2–generator subgroups are free. [2, Lemma 4.3] then implies that $\operatorname{vol}\mathcal{H} \geq \pi$, a contradiction. This completes the proof in the non-exceptional case.

We now turn to the case where M is exceptional. If M is isometric to the manifold vol 3 discussed by Jones and Reid [11], then M can be obtained by Dehn filling from a once-punctured torus bundle over S^1, and hence rank $\pi_1(M) \le 3$. If M is not isometric to vol 3, then according to [2, Proposition 7.1], the group $\pi_1(M)$ has a finite-index subgroup of rank at most 2. But if rank $\pi_1(M) \ge 4$, Conjecture 4.2 implies that every finite-index subgroup of $\pi_1(M)$ has rank at least 3. Hence rank $\pi_1(M) \le 3$ in this case as well. □

I pointed out in Section 4 that Conjectures 4.2 and 4.3 are equivalent modulo the "rank equals genus" Conjecture 1.1. I pointed out at the beginning of the present section that Conjecture 5.1 would follow immediately from Conjecture 1.1. Hence the following corollary follows from Proposition 5.2.

Corollary 5.3 *If Conjectures 1.1 and 4.3 are true, then for every closed, orientable hyperbolic 3–manifold M with volume at most 1.22, the Heegaard genus of M is at most 3.*

6 A hybrid consequence of the conjecture on covering spaces

In [7], Culler and I prove:

Theorem 6.1 *If M is a closed, orientable hyperbolic 3–manifold with volume at most 3.08, then $H_1(M;\mathbf{Z}_2)$ has rank at most 7.*

The weaker version that the rank is at most 10 is somewhat easier to prove, and is established in [9].

In this section I'll show how to adapt the proof of this weaker result to get the following result. The information given by this result modulo Conjecture 4.2 is not exactly an analogue of the result of [9] involving the rank of the fundamental group, but rather a hybrid result involving both the rank of the fundamental group and the rank of the mod–2 first homology.

Proposition 6.2 *If Conjecture 4.2 is true, then for every closed, orientable hyperbolic 3–manifold M with volume at most 3.08, either rank $\pi_1(M) \le 10$, or dim $H_1(M;\mathbb{Z}_2) \le 4$.*

The proof of Proposition 6.2 will depend on refining a number of the results proved in [9]. For the rest of this section I shall use the notation and definitions of [9], including the definition of a book of I–bundles. As in [9], I shall write $\overline{\chi}(X) = -\chi(X)$, where X is any space homeomorphic to a finite polyhedron and $\chi(X)$ denotes its Euler characteristic.

Lemma 6.3 *Let X be a compact, connected 3–manifold, and let $P \neq X$ be a submanifold of X. Suppose that P is an I–bundle over a compact, orientable surface with non-empty boundary, that the frontier F of P in X is the vertical boundary of P, and that F is properly embedded in X. Let Y be a component of $\overline{X-P}$, and assume that every component of $\overline{X-(P\cup Y)}$ is a solid torus. (The last condition holds vacuously if $\overline{X-P}$ is connected.) Then*

$$\operatorname{rank}\pi_1(X) \leq \operatorname{rank}\pi_1(Y) + \operatorname{rank}\pi_1(P).$$

Proof Let S denote the base of the I–bundle P. Let $m+1$ denote the number of components of F. If $m=0$ then F is connected, and the assertion of the lemma is immediate from the Seifert-van Kampen theorem. Hence we may assume that $m \geq 1$.

Set $r = \operatorname{rank}\pi_1(Y)$ and $s = \operatorname{rank}\pi_1(P) = \operatorname{rank}\pi_1(S)$. We have $s \geq m$, with equality if and only if S is planar.

Let us denote the components of $\overline{X-P}$ by $Z_0,\dots,Z_n$, where $n \geq 0$, $Z_0 = Y$, and Z_j is a solid torus for each j with $0 < j \leq n$. Let $A_0,\dots,A_m$ denote the components of F, which we index in such a way that $A_j \subset Z_j$ for $0 \leq j \leq n$. For each $i \in \{0,\dots,m\}$, the annulus A_i is contained in a unique component $Z_{q(i)}$ of $\overline{X-P}$; thus $q : \{0,\dots,m\} \to \{0,\dots,n\}$ is a well-defined surjection, and our indexing of the A_i implies that $q(i) = i$ for $i = 0,\dots,n$. (In particular $n \leq m$.)

For $i = 0,\dots,m$, fix a point $a_i \in A_i$, and fix a loop α_i in A_i, based at a_i, which represents a generator of $\pi_1(A_i,a_i)$. Let β_0 denote the constant path at a_0, and for each i with $0 < i \leq m$ let β_i denote a path in P from a_0 to a_i, which projects to an embedded arc B_i in S. We may suppose the β_i to be chosen so that the arcs B_i meet only at the point a_0. For $i = 0,\dots,m$ let c_i denote the element of $\pi_1(P,a_0)$ represented by the loop $\beta_i * \alpha_i * \overline{\beta_i}$. Then $\pi_1(P,a_0)$ has a minimal generating set $\{x_0,\dots,x_{s-1}\}$ such that $x_i = c_i$ for $i = 0,\dots,m-1$.

For each i with $n < i \leq m$, we fix a path γ_i in $Z_{q(i)}$ from $a_{q(i)}$ to a_i, and define an element $t_i \in \pi_1(X,a_0)$ by $t_i = [\beta_{q(i)} * \gamma_i * \overline{\beta_i}]$, where brackets denote the based homotopy class of a loop in X. We fix a generating set $\{y_1,\dots,y_r\}$ for $\pi_1(Y,a_0)$. For $j = 1,\dots,n$ we fix a loop ζ_j in Z_j based at a_j which represents a generator for the cyclic group $\pi_1(Z_j,a_j)$, and we set $z_j = [\beta_j * \zeta_j * \overline{\beta_j}] \in \pi_1(X,a_0)$. Then $\pi_1(X,a_0)$ is generated by the set

$$\{\widehat{x}_i : 0 \leq i < s\} \cup \{\widehat{y}_k : 1 \leq k \leq r\} \cup \{z_j : 1 \leq j \leq n\} \cup \{t_i : n < i \leq m\},$$

where $\widehat{x}_i$ and $\widehat{y}_k$ denote the images of x_i and y_k under the inclusion homomorphisms from $\pi_1(P,a_0)$ and $\pi_1(Y,a_0)$ to $\pi_1(X,a_0)$.

For $i = 0, \dots, m$, let $\widehat{c}_i$ denote the image of c_i under the inclusion homomorphism $\pi_1(P, a_0) \to \pi_1(X, a_0)$. Since α_0 is a loop in $A_0 \subset Y$, and β_0 is the constant path, $\widehat{c}_0$ is a word in the $\widehat{y}_k$. Likewise, for $0 < i \le n$, since α_i is a loop in $A_i \subset Z_i$, the element $\widehat{c}_i = [\beta_i * \alpha_i * \overline{\beta_i}]$ is a power of z_i. Furthermore, if $n < i \le m$, and if we set $j = q(i)$, then $\widehat{c}_i = t_i^{-1} u_i t_i$, where $u_i = [\beta_j * \gamma_i * \alpha_i * \overline{\gamma_i} * \overline{\beta_j}]$. Since $\gamma_i * \alpha_i * \overline{\gamma_i}$ is a loop in Z_j based at a_i, it follows that u_i is a word in the $\widehat{y}_k$ if $q(i) = 0$, and is a power of z_j if $q(i) > 0$.

In particular, the elements $\widehat{c}_0, \dots, \widehat{c}_m$ all lie in the subgroup of $\pi_1(X, a_0)$ generated by the $\widehat{y}_i$, the z_j and the t_i. Since we have $\widehat{x}_i = \widehat{c}_i$ for $i = 0, \dots, m-1$, we deduce that $\pi_1(X, a_0)$ is generated by the set

$$\{\widehat{x}_i : m \le i < s\} \cup \{\widehat{y}_k : 1 \le k \le r\} \cup \{z_j : 1 \le j \le n\} \cup \{t_i : n < i \le m\}.$$

Hence

$$\operatorname{rank} \pi_1(X) \le (s - m) + r + n + (m - n) = r + s,$$

which is the conclusion of the lemma. □

The following lemma is a refined version of [9, Lemma 2.21].

Lemma 6.4 *If $\mathcal{W}$ is a connected normal book of I–bundles, the rank of $\pi_1(|\mathcal{W}|)$ is at most $2\overline{\chi}(|\mathcal{W}|) + 1$.*

Proof Set $W = |\mathcal{W}|$. If W is an I–bundle over a closed surface, we have

$$\operatorname{rank} \pi_1(W) = \overline{\chi}(W) + 2;$$

by normality we have $\overline{\chi}(W) > 0$, so the conclusion holds in this case. Hence we may assume that every page of $\mathcal{W}$ meets at least one binding.

Let p denote the number of pages of $\mathcal{W}$. We shall recursively construct a finite sequence of connected sub-books $\mathcal{W}_1, \dots, \mathcal{W}_p$ of $\mathcal{W}$, where $\mathcal{W}_i$ has exactly i pages. To begin the recursion, we choose an arbitrary page P_1 of $\mathcal{W}$ and define $\mathcal{W}_1$ to consist of P_1 and the bindings that meet it. Now assume that $\mathcal{W}_i$ has been constructed for a given $i < p$. Since $\mathcal{W}$ has p pages, $\mathcal{W}_i$ is a proper sub-book of $\mathcal{W}$. Since $\mathcal{W}$ is connected, $|\mathcal{W}_i|$ must meet some page P_{i+1} not contained in $|\mathcal{W}_i|$. We define $\mathcal{W}_{i+1}$ to consist of the pages and bindings of $\mathcal{W}_i$, the page P_{i+1}, and all bindings of $\mathcal{W}$ that meet P_{i+1}.

We set $W_i = |\mathcal{W}_i|$ for $i = 1, \dots, p$, and we let W_0 denote some (arbitrarily chosen) binding of $\mathcal{W}_1$. We shall show that for $i = 0, \dots, p$ the rank of $\pi_1(W_i)$ is at most $2\overline{\chi}(W_i) + 1$. For $i = 0$ this is obvious, and for $i = p$ it is the conclusion of the lemma. It therefore suffices to show that if $0 < k \le p$ and $\pi_1(W_{k-1})$ has rank at most $2\overline{\chi}(W_{k-1}) + 1$, then $\pi_1(W_k)$ has rank at most $2\overline{\chi}(W_k) + 1$.

The hypotheses of Lemma 6.3 hold if we set $X = W_k$, $P = P_k$, and $Y = W_{k-1}$. Hence we have

$$\begin{aligned}\operatorname{rank}\pi_1(W_k) &\le \operatorname{rank}\pi_1(W_{k-1}) + \operatorname{rank}\pi_1(P_k)\\ &\le (2\overline{\chi}(W_{k-1}) + 1) + \operatorname{rank}\pi_1(P_k).\end{aligned}$$

Now since P_k is an I–bundle over a compact, connected surface with non-empty boundary, we have $\operatorname{rank}\pi_1(P_k) = 1 + \overline{\chi}(P_k)$. But by the definition of a normal book of I–bundles we have $\overline{\chi}(P_k) \ge 1$, and hence

$$\operatorname{rank}\pi_1(P_k) \le 2\overline{\chi}(P_k).$$

It follows that

$$\operatorname{rank}\pi_1(W_k) \le (2\overline{\chi}(W_{k-1}) + 1) + 2\overline{\chi}(P_k) = 2\overline{\chi}(W_k) + 1,$$

as required. □

Our next result is a analogue, in the context of the present section, of [9, Theorem 9.13].

Lemma 6.5 *Assume that Conjecture 4.2 is true. Let M be a closed simple 3–manifold with $\operatorname{rank}\pi_1(M) \ge 11$. Suppose that $\dim H_1(M;\mathbb{Z}_2) \ge 5$, and that $\pi_1(M)$ has a subgroup isomorphic to a genus–2 surface group. Then there is a connected, normal book of I–bundles $\mathcal{W}$ with $W = |\mathcal{W}| \subset M$ such that ∂W is incompressible in M and $\overline{\chi}(W) = 2$.*

Proof We shall adapt the proof of [9, Theorem 9.13]. The latter result has the same conclusion as the present lemma, but in place of the hypothesis $\operatorname{rank}\pi_1(M) \ge 11$, that $\dim H_1(M;\mathbb{Z}_2) \ge 5$, and that Conjecture 4.2 is true, it has the hypothesis that $\dim H_1(M;\mathbb{Z}_2)$ has rank at least 11. This hypothesis is used twice in the proof of [9, Theorem 9.13]: once in the first sentence to allow the application of [9, Corollary 9.11], and again in the fifth sentence of the fifth and final paragraph of the proof. The application of [9, Corollary 9.11] requires only the lower bound of 5 for $\dim H_1(M;\mathbb{Z}_2)$. Hence, under the hypotheses of the present lemma, the first four paragraphs of the proof of [9, Theorem 9.13], and the first sentence of the fifth paragraph, go through without change, and show that either (a) there is a connected, normal book of I–bundles $\mathcal{W}$ with $W = |\mathcal{W}| \subset M$ such that ∂W is incompressible in M and $\overline{\chi}(W) = 2$ (this is the case $m = 0$ in the notation of the proof in [9]) or (b) there exist a finite-sheeted covering space $\widetilde{M}$ of M and a connected normal book of I–bundles $\mathcal{W}$ with $W = |\mathcal{W}| \subset \widetilde{M}$ such that the inclusion homomorphism $\pi_1(W) \to \pi_1(\widetilde{M})$ is surjective and $\overline{\chi}(W) \le 4$. (Alternative (b) corresponds to the case $m > 0$ in the notation of the proof in [9], and we take $\widetilde{M} = N_{m-1}$ in the notation of that proof. Since N_{m-1} is closed, it is a finite-sheeted covering space of M according to [9, Section 9.3].)

Now by Lemma 6.4 we have $\operatorname{rank} \pi_1(|\mathcal{W}|) \le 2\overline{\chi}(|\mathcal{W}|) + 1 \le 9$. The surjectivity of $\pi_1(W) \to \pi_1(\widetilde{M})$ therefore implies that $\operatorname{rank} \pi_1(\widetilde{M}) \le 9$. On the other hand, since by hypothesis we have $\operatorname{rank} \pi_1(M) \ge 11$, Conjecture 4.2 implies that $\operatorname{rank} \pi_1(\widetilde{M}) \ge 10$. This is a contradiction, and the proof is complete. □

The following result follows from Lemma 6.5 above in exactly the same way that [9, Corollary 9.14] follows from [9, Theorem 9.13].

Lemma 6.6 *Assume that Conjecture 4.2 is true. Let M be a closed simple 3–manifold with $\operatorname{rank} \pi_1(M) \ge 11$. Suppose that $\pi_1(M)$ has a subgroup isomorphic to a genus–2 surface group, and that $\dim H_1(M; \mathbb{Z}_2) \ge 5$. Then M contains either a connected incompressible surface of genus 2 or a separating, connected incompressible surface of genus 3.* □

The following result is an analogue of [9, Proposition 10.5] in the context of the present section.

Lemma 6.7 *Assume that Conjecture 4.2 is true. Suppose that M is a closed orientable hyperbolic 3–manifold with $\operatorname{rank} \pi_1(M) \ge 11$. Suppose that $\pi_1(M)$ has a subgroup isomorphic to a genus–2 surface group, and that $\dim H_1(M; \mathbb{Z}_2) \ge 5$. Then $\operatorname{vol} M \ge 3.66$.*

Proof It follows from Lemma 6.6 that either

(i) M contains either a separating incompressible surface of genus 2 or 3, or

(ii) M contains a non-separating incompressible surface of genus 2.

Suppose that (i) holds but that $\operatorname{vol} M < 3.66$. Let X_1 and X_2 denote the closures of the components of $M - S$. According to [9, Theorem 10.4] (a result deduced from the main result of Agol, Dunfield, Storm and Thurston [3]), each X_i has the form $|\mathcal{W}_i|$ for some book of I–bundles $\mathcal{W}_i$. For $i = 1, 2$ we have

$$\overline{\chi}(X_i) = \tfrac{1}{2}\overline{\chi}(S) \le 2.$$

By Lemma 6.4, applied to $\mathcal{W} = \mathcal{W}_i$, it follows that

$$\operatorname{rank} \pi_1(X_i) \le 2\overline{\chi}(X_i) + 1 \le 5$$

for $i = 1, 2$. Hence by the Seifert-van Kampen theorem,

$$\operatorname{rank} \pi_1(M) \le \operatorname{rank} \pi_1(X_1) + \operatorname{rank} \pi_1(X_2) \le 10,$$

a contradiction to the hypothesis.

Now suppose that (ii) holds but that $\operatorname{vol} M < 3.66$. Let X denote the connected manifold obtained by splitting M along S. According to [9, Theorem 10.4] we have $X = |\mathcal{W}|$ for some book of I–bundles $\mathcal{W}$. We have

$$\overline{\chi}(X) = \tfrac{1}{2}\overline{\chi}(\partial X) = \overline{\chi}(S) = 2.$$

By Lemma 6.4 it follows that

$$\operatorname{rank} \pi_1(X) \le 2\overline{\chi}(X) + 1 \le 5.$$

Hence

$$\operatorname{rank} \pi_1(M) \le \operatorname{rank} \pi_1(X) + 1 \le 6,$$

and again we have a contradiction. □

Proof of Proposition 6.2 Assume that the conclusion is false, i.e. that $\operatorname{rank} \pi_1(M) \ge 11$ and that $\dim H_1(M; \mathbb{Z}_2) \ge 5$. If $\pi_1(M)$ has a subgroup isomorphic to a genus–2 surface group, then it follows from Lemma 6.7 that $\operatorname{vol} M \ge 3.66 > 3.08$, a contradiction to the hypothesis.

There remains the possibility that $\pi_1(M)$ has no subgroup isomorphic to a genus–2 surface group. Now since $H_1(M; \mathbb{Z}_2)$ has rank at least 5, it follows from [18, Proposition 1.1] that every subgroup of rank at most 3 in $\pi_1(M)$ has infinite index. But it follows from Anderson, Canary, Culler and Shalen [4, Proposition 7.3, Remark 7.5] that if M is an orientable hyperbolic 3–manifold without cusps such that $\pi_1(M)$ contains no genus–2 surface subgroup and such that every subgroup of rank at most 3 in $\pi_1(M)$ has infinite index, then $\pi_1(M)$ is 3–free, in the sense that each subgroup of rank at most 3 is free. And according to [9, Corollary 10.3], if a closed, orientable hyperbolic 3–manifold has 3–free fundamental group, then its volume exceeds 3.08. Again the hypothesis is contradicted. □

Since Conjectures 4.2 and 4.3 are equivalent modulo the "rank equals genus" Conjecture 1.1, Proposition 6.2 has the following immediate consequence.

Corollary 6.8 *If Conjectures 1.1 and 4.3 are true, then for every closed, orientable hyperbolic* 3*–manifold* M *with volume at most* 3.08*, either* M *has Heegaard genus at most* 10*, or* $\dim H_1(M;\mathbb{Z}_2) \leq 4$*.*

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