Geometry, Heegaard splittings and rank of the fundamental group of hyperbolic 3–manifolds

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In this survey we discuss how geometric methods can be used to study topological properties of 3–manifolds such as their Heegaard genus or the rank of their fundamental group. On the other hand, we also discuss briefly some results relating combinatorial descriptions and geometric properties of hyperbolic 3–manifolds.

57M50; 57M07

A closed, and say orientable, Riemannian 3–manifold (M,ρ) is *hyperbolic* if the metric ρ has constant sectional curvature $\kappa_\rho = -1$. Equivalently, there is a discrete and torsion free group Γ of isometries of hyperbolic 3–space $\mathbb{H}^3$ such that the manifolds (M,ρ) and $\mathbb{H}^3/\Gamma$ are isometric. It is well-known that the fundamental group $\pi_1(M)$ of every closed 3–manifold which admits a hyperbolic metric is a non-elementary Gromov hyperbolic group and hence that it is is infinite and does not contain free abelian subgroups of rank 2. A 3–manifold M whose fundamental group does not have subgroups isomorphic to $\mathbb{Z}^2$ is said to be *atoroidal*. Another well-known property of those 3–manifolds which admit a hyperbolic metric is that they are *irreducible*, ie every embedded sphere bounds a ball. Surprisingly, these conditions suffice to ensure that a closed 3–manifold M admits a hyperbolic metric.

Hyperbolization Theorem (Perelman) *A closed orientable 3–manifold M admits a hyperbolic metric if and only if it is irreducible, atoroidal and has infinite fundamental group.*

Thurston proved the Hyperbolization Theorem in many cases, for instance if M has positive first Betti-number (see Otal [36; 37]). The Hyperbolization Theorem is a particular case of Thurston's Geometrization conjecture recently proved by Perelman [38; 39; 40] (see also Cao–Zhu [14]).

From our point of view, the Hyperbolization theorem is only one half of the coin, the other half being Mostow's rigidity theorem.

Mostow's Rigidity Theorem *Any two closed hyperbolic 3–manifolds which are homotopy equivalent are isometric.*

Published: 1 December 2007 DOI: 10.2140/gtm.2007.12.351

The goal of this note is to describe how the existence and uniqueness of hyperbolic metrics can be used to obtain results about quantities which have been classically studied in 3–dimensional topology. More precisely, we are interested in the *Heegaard genus* $g(M)$ of a 3–manifold M and in the *rank* of its fundamental group. Recall that a *Heegaard splitting* of a closed 3–manifold is a decomposition of the manifold into two handlebodies with disjoint interior. The surface separating both handlebodies is said to be the *Heegaard surface* and its genus is the genus of the Heegaard splitting. Moise [31] proved that every topological 3–manifold admits a Heegaard splitting. The Heegaard genus $g(M)$ of M is the minimal genus of a Heegaard splitting of M. The *rank* of the fundamental group of M is the minimal number of elements needed to generate it.

Unfortunately, the Hyperbolization Theorem only guarantees that a hyperbolic metric exists, but it does not provide any further information about this metric. This is why most results we discuss below are about concrete families of 3–manifolds for which there is enough geometric information available. But this is also why we discuss which geometric information, such as the volume of the hyperbolic metric, can be read from combinatorial information about for example Heegaard splittings.

The paper is organized as follows. In Section 1 and Section 2 we recall some well-known facts about 3–manifolds, Heegaard splittings and hyperbolic geometry. In particular we focus on the consequences of tameness and of Thurston's covering theorem.

In Section 3 we describe different constructions of minimal surfaces in 3–manifolds, in particular the relation between Heegaard splittings and minimal surfaces. The so obtained minimal surfaces are used for example in Section 4 to give a proof of the fact that the mapping torus of a sufficiently high power of a pseudo-Anosov mapping class of a closed surface of genus g has Heegaard genus $2g+1$.

In Section 5 we introduce carrier graphs and discuss some of their most basic properties. Carrier graphs are the way to translate questions about generating sets of the fundamental group of hyperbolic 3–manifolds into a geometric framework. They are used in Section 6 to prove that the fundamental group of the mapping torus of a sufficiently high power of a pseudo-Anosov mapping class of a closed surface of genus g has rank $2g+1$.

In Section 7 we determine the rank of the fundamental group and the Heegaard genus of those 3–manifolds obtained by gluing two handlebodies by a sufficiently large power of a *generic* pseudo-Anosov mapping class. When reading this last sentence, it may have crossed through the mind of the reader that this must somehow be the same situation as for the mapping torus. And in fact, it almost is. However, there is a crucial difference. It follows from the full strength of the geometrization conjecture, not just the Hyperbolization Theorem, that the manifolds in question are hyperbolic; however, a priori nothing is known about these hyperbolic metrics. Instead of using the existence of the hyperbolic metric, in Namazi–Souto [34] we follow a different approach, described below. We construct out of known hyperbolic manifolds a negatively curved metric on the manifolds in question. In particular, we have full control of the metric and our previous strategies can be applied. In principle, our metric and the actual hyperbolic metric are unrelated.

So far, we have considered families of 3–manifolds for which we had a certain degree of geometric control and we have theorems asserting that for most members of these families something happens. In Section 8 we shift our focus to a different situation: we describe a result due to Brock and the author relating the volume of a hyperbolic 3–manifold with a certain combinatorial distance of one of its Heegaard splittings.

Finally, in Section 9 we describe the geometry of those thick hyperbolic 3–manifolds whose fundamental group has rank 2 or 3; in this setting we cannot even describe a conjectural model but the results hint towards the existence of such a construction.

We conclude with a collection of questions and open problems in Section 10.

This note is intended to be a survey and hence most proofs are only sketched, and this only in the simplest cases. However, we hope that these sketches make the underlying principles apparent. It has to be said that this survey is certainly everything but all-inclusive, and that the same holds for the bibliography. We refer mostly to papers read by the author, and not even to all of them. Apart of the fact that many important references are missing, the ones we give are not well distributed. For example Yair Minsky and Dick Canary do not get the credit that they deserve since their work is in the core of almost every result presented here. It also has to be said, that this survey is probably superfluous for those readers who have certain familiarity with (1) the work of Canary and Minsky, (2) the papers [16; 24; 33; 44] by Tobias Colding and Camillo de Lellis, Marc Lackenby, Hossein Namazi and Hyam Rubinstein, and (3) have had a couple of conversations, about math, with Ian Agol, Michel Boileau and Jean-Pierre Otal. Having collaborators such as Jeff Brock also helps.

1 Some 3–dimensional topology

From now on we will only consider orientable 3–manifolds M which are *irreducible*, meaning that every embedded sphere bounds a ball. We will also assume that our manifolds do not contain surfaces homeomorphic to the real projective plane $\mathbb{R}P^2$. This is not much of a restriction because $\mathbb{R}P^3$ is the only orientable, irreducible 3–manifold which contains a copy of $\mathbb{R}P^2$.

A surface S in M with non-positive Euler characteristic $\chi(S) \leq 0$ is said to be π_1*–injective* if the induced homomorphism $\pi_1(S) \to \pi_1(M)$ is injective. A surface is *incompressible* if it is embedded and π_1–injective. An embedded surface which fails to be incompressible is said to be *compressible*. The surface S is said to be *geometrically compressible* if it contains an essential simple closed curve which bounds a disk D in M with $D \cap S = \partial D$; D is said to be a *compressing disk*. Obviously, a geometrically compressible surface is compressible. On the other hand there are geometrically incompressible surfaces which fail to be incompressible. However, the Loop theorem asserts that any such surface must be one-sided. Summing up we have the following proposition.

Proposition 1.1 *A two-sided surface S in M is compressible if and only if it is geometrically compressible.*

If S is geometrically compressible and D is a compressing disk then we can obtain a new surface S' as follows: we cut open S along ∂D and glue to the obtained boundary curves two copies of D. We say that S' arises from S by suturing along D. A surface is obtained from S by *suturing along disks* if it is obtained by repeating this process as often as necessary.

Given two embedded surfaces S and S' in M, we say that S' arises from S by *collapsing along the normal bundle of S'* if there is a regular neighborhood of S' diffeomorphic to the total space of the normal bundle $\pi\colon N(S') \to S'$ containing S and such that the restriction of π to S is a covering of S'.

Definition *Let S and S' be two embedded, possibly empty, surfaces in M. The surface S' arises from S by* surgery *if it does by a combination of isotopies, suturing along disks, discarding inessential spheres and collapsing along the normal bundle of S'.*

Observe that if the surface S bounds a handlebody in M then $\emptyset$ arises from S by surgery. Similarly, the interior boundary of a compression body arises from the exterior boundary by surgery. Recall that a *compression body* C is a compact orientable and irreducible 3–manifold which has a boundary component called the *exterior boundary* $\partial_e C$ such that the homomorphism $\pi_1(\partial_e C) \to \pi_1(C)$ is surjective; the interior boundary is the union of all the remaining boundary components. The genus of a compression body is the genus of its exterior boundary.

A *Heegaard splitting* of the compact 3–manifold M is a decomposition $M = U \cup V$ into two compression bodies with disjoint interior and separated by the corresponding exterior boundary, the so-called *Heegaard surface* $\partial_e U = \partial_e V$. Heegaard splittings can be obtained for example from Morse functions on M. Moise [31] proved that every topological 3–manifold admits a unique smooth structure; in particular, every 3–manifold has a Heegaard splitting. The *Heegaard genus* $g(M)$ of M is the minimal genus of a Heegaard splitting of M. It is a well-known fact that if M is closed then $2g(M)+2$ is equal to the minimal number of critical points of a Morse function on M.

As Morse functions can be perturbed to introduce new critical points new Heegaard splittings can be obtained from other Heegaard splittings by attaching new handles. A Heegaard splitting which is obtained from another one by this process is said to be obtained by *stabilization*. It is well-known that a Heegaard splitting $M = U \cup V$ arises by stabilization if and only if there are two essential properly embedded disks $D_U \subset U$ and $D_V \subset V$ whose boundaries $\partial D_U = \partial D_V$ intersect in a single point. A Heegaard splitting is *reducible* if there are two essential properly embedded disks $D_U \subset U$ and $D_V \subset V$ with $\partial D_U = \partial D_V$. Every reducible Heegaard splitting of an irreducible 3–manifold is stabilized.

Let $\Sigma \subset M$ be a (say) connected surface separating M into two components N_1 and N_2 and f_1 and f_2 be Morse functions on N_1 and N_2 whose values and derivatives of first and second order coincide along Σ. Then the function $f\colon M \to \mathbb{R}$ given by $f(x) = f_i(x)$ if $x \in N_i$ is a Morse function. Similarly, if a 3–manifold M is decomposed into codimension 0 submanifolds $N_1, \dots, N_k$ with disjoint interior then Heegaard splittings of $N_1, \dots, N_k$, fulfilling again some normalization, can be merged to obtain a Heegaard splitting of M. See Schultens [53] for a precise description of this process which is called *amalgamation*. As in the case of stabilization, there is a criterium to determine if a Heegaard splitting arises by amalgamation. One has namely that this is the case for the Heegaard splitting $M = U \cup V$ if and only if there are two essential properly embedded disks $D_U \subset U$ and $D_V \subset V$ whose boundaries are disjoint. A Heegaard splitting $M = U \cup V$ which is not reducible but such that there are two essential properly embedded disks $D_U \subset U$ and $D_V \subset V$ with disjoint boundary is said to be *weakly reducible*. A Heegaard splitting is *strongly irreducible* if it is not reducible or weakly reducible. With this terminology, we can summarize the above discussion as follows:

> Every Heegaard splitting that is not strongly irreducible can be obtained by amalgamation and stabilization from other splittings.

Before stating a more precise version of this claim we need a last definition.

Definition *A* generalized Heegaard splitting *of M is a pair of disjoint embedded possibly disconnected surfaces (Σ_I, Σ_H) such that the following hold.*

(1) Σ_I *divides* M *into two possibly disconnected manifolds* N_1 *and* N_2.

(2) *The surfaces* $\Sigma_1 = \Sigma_H \cap N_1$ *and* $\Sigma_2 = \Sigma_H \cap N_2$ *determine Heegaard splittings of* N_1 *and* N_2 *which can be amalgamated to obtain a Heegaard splitting of* M.

A generalized Heegaard splitting (Σ_I, Σ_H) *of* M *is* strongly irreducible *if* Σ_I *is incompressible and if the Heegaard surface* Σ_H *of* $M \setminus \Sigma_I$ *is strongly irreducible.*

We have now the following crucial theorem,

Theorem 1.2 (Scharlemann–Thompson [49]) *Every genus* g *Heegaard splitting arises from first (1) amalgamating a strongly irreducible generalized Heegaard splitting such that the involved surfaces have at most genus* g *and then (2) stabilizing the obtained Heegaard splitting.*

This theorem is in some way a more precise version of the following result of Casson and Gordon [15].

Theorem 1.3 (Casson–Gordon) *If an irreducible 3–manifold* M *admits a weakly reducible Heegaard splitting then* M *contains an incompressible surface.*

From our point of view, Theorem 1.2 asserts that most of the time it suffices to study strongly irreducible splittings. However, the use of Theorem 1.2 can be quite cumbersome because of the amount of notation needed: for the sake of simplicity we will often just prove claims for strongly irreducible splittings and then claim that the general case follows using Theorem 1.2.

As we just said, questions about Heegaard splittings can be often reduced to questions about strongly irreducible splittings. And this is a lucky fact since, while Heegaard splittings can be quite random, strongly irreducible splittings show an astonishing degree of rigidity as shown for example by the following lemma.

Lemma 1.4 (Scharlemann [47, Lemma 2.2]) *Suppose that an embedded surface* S *determines a strongly irreducible Heegaard splitting* $M = U \cup V$ *of a 3–manifold* M *and that* D *is an embedded disk in* M *transverse to* S *and with* $\partial D \subset S$ *then* ∂D *also bounds a disk in either* U *or* V.

From Lemma 1.4, Scharlemann [47] derived the following useful description of the intersections of a strongly irreducible Heegaard surface with a ball.

Theorem 1.5 (Scharlemann) *Let $M = U \cup V$ be a strongly irreducible Heegaard splitting with Heegaard surface S. Let B be a ball with ∂B transversal to S and such that the two surfaces $\partial B \cap U$ and $\partial B \cap V$ are incompressible in U and V respectively. Then, $S \cap B$ is a connected planar surface properly isotopic in B to one of $U \cap \partial B$ and $V \cap \partial B$.*

In the same paper, Scharlemann also determined how a strongly irreducible Heegaard splitting can intersect a solid torus.

Another instance in which the rigidity of strongly irreducible splittings becomes apparent is the following lemma restricting which surfaces can arise from a strongly irreducible surface by surgery.

Lemma 1.6 (Suoto [57]) *Let S and S' be closed embedded surface in M. If S is a strongly irreducible Heegaard surface and S' is obtained from S by surgery and has no parallel components then one of the following holds.*

(1) *S' is isotopic to S.*

(2) *S' is non-separating and there is a surface $\widehat{S}$ obtained from S by surgery at a single disk and such that $\widehat{S}$ is isotopic to the boundary of a regular neighborhood of S'. In particular, S' is connected and $M \setminus S'$ is a compression body.*

(3) *S' is separating and S is, up to isotopy, disjoint of S'. Moreover, S' is incompressible in the component U of $M \setminus S'$ containing S, S is a strongly irreducible Heegaard surface in U and $M \setminus U$ is a collection of compression bodies.*

Lemma 1.6 is well-known to many experts. In spite of that we include a proof.

Proof Assume that S' is not isotopic to S. We claim that in the process of obtaining S' from S, some surgery along disks must have been made. Otherwise S is, up to isotopy, contained in regular neighborhood $\mathcal{N}(S')$ of S' such that the restriction of the projection $\mathcal{N}(S') \to S'$ is a covering. Since S is connected, this implies that S' is one-sided and that S bounds a regular neighborhood of S'. However, no regular neighborhood of an one-sided surface in an orientable 3–manifold is homeomorphic to a compression body. This contradicts the assumption that S is a Heegaard splitting.

We have proved that there is a surface $\widehat{S}$ obtained from S by surgery along disks and discarding inessential spheres which is, up to isotopy, contained in $\mathcal{N}(S')$ and such that the restriction of the projection $\mathcal{N}(S') \to S'$ to $\widehat{S}$ is a covering. Since S determines a strongly irreducible Heegaard splitting it follows directly from the definition that all surgeries needed to obtain $\widehat{S}$ from S occur to the same side. In particular, $\widehat{S}$ divides M into two components U and V such that U is a compression body with exterior boundary $\widehat{S}$ and such that $\widehat{S}$ is incompressible in V. Moreover, V is connected and S determines a strongly irreducible Heegaard splitting of V.

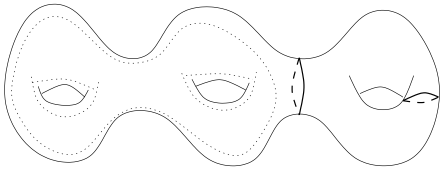

Figure 1: The surface S has genus 3; the thick lines are the boundaries of compressing disks; the surface $\widehat{S}$ obtained from S by surgery along these disks is dotted.

Assume that every component of S' is two-sided. In particular, every component of $\widehat{S}$ is isotopic to a component of S'. If for every component of S' there is a single component of $\widehat{S}$ isotopic to it, then we are in case (3). Assume that this is not the case. Then there are two components $\widehat{S}_1$ and $\widehat{S}_2$ which are isotopic to the same component S'_0 and which bound a trivial interval bundle W homeomorphic to $S'_0 \times [0,1]$ which does not contain any further component of $\widehat{S}$. In particular W is either contained in U or in V. Since the exterior boundary of a compression body is connected we obtain that W must be contained in V and since V is connected we have $W = V$. The assumption that S' does not have parallel components implies that S' is connected. It remains to prove that S' arises from S by surgery along a single disk. The surface S determines a Heegaard splitting of V and hence it is isotopic to the boundary of a regular neighborhood of $\widehat{S} \cup \Gamma$ where Γ is a graph in V whose endpoints are contained in ∂V. If Γ is not a segment then it is easy to find two compressible simple curves on S which intersect only once (see Figure 2).

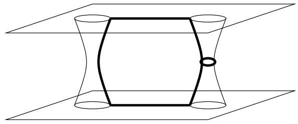

Figure 2: Two disks intersecting once

By Lemma 1.4 each one of these curves bounds a disk in one of the components of $V \setminus S$ contradicting the assumption that S is strongly irreducible. This proves that $\widehat{S}$

arises from S by surgery along a single disk. We are done if every component of $\widehat{S}$ is two-sided.

The argument in the case that S' has at least a one-sided component is similar. $\square$

We conclude this section with some remarks about the curve complex $\mathcal{C}(S)$ of a closed surface S of genus $g \geq 2$ and its relation to Heegaard splittings. The curve complex is the graph whose vertices are isotopy classes of essential simple closed curves in S. The edges correspond to pairs of isotopy classes that can be represented by disjoint curves. Declaring every edge to have unit length we obtain a connected metric graph on which the mapping class group $\mathrm{Map}(S)$ acts by isometries. If S is the a Heegaard surface then we define the *distance in the curve complex* of the associated Heegaard splitting $M = U \cup V$ to be the minimal distance between curves bounding essential disks in U and in V. Using this terminology, $M = U \cup V$ is reducible if the distance is 0, it is weakly reducible if the distance is 1 and it is strongly irreducible if the distance is at least 2. The following result due to Hempel [19] asserts that a manifold admitting a Heegaard splittings with at least distance 3 is irreducible and atoroidal.

Theorem 1.7 (Hempel) *If a 3–manifold admits a Heegaard splitting with at least distance 3 then it is irreducible and atoroidal.*

For more on Heegaard splittings see Scharlemann [48] and for generalized Heegaard splittings Saito–Scharlemann–Schultens [46].

2 Hyperbolic 3–manifolds and Kleinian groups

A *hyperbolic structure* on a compact 3–manifold M is the conjugacy class of a discrete and faithful representation $\rho\colon \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$, such that $N_\rho = \mathbb{H}^3/\rho(\pi_1(M))$ is homeomorphic to the interior of M by a homeomorphism inducing ρ. From this point of view, Mostow's rIgidity Theorem asserts that whenever M is closed then there is at most one hyperbolic structure.

It is never to early to remark that most results concerning hyperbolic 3–manifolds are still valid in the setting of manifolds of pinched negative curvature (see for instance Canary [12] or Agol [3]). In fact, negatively curved metrics have a much greater degree of flexibility than hyperbolic metrics and hence allow certain extremely useful constructions. Further generalizations of hyperbolic metrics such as CAT(-1) metrics are also ubiquitous; again because they are even more flexible than negatively curved metrics. Recall that a geodesic metric space is CAT(-1) if, from the point of view of comparison geometry it is at least as curved as hyperbolic space (Bridson–Haefliger [8]).

The hyperbolic structure N_ρ is *convex-cocompact* if there is a convex $\rho(\pi_1(M))$–invariant subset $K \subset \mathbb{H}^3$ with $K/\rho(\pi_1(M))$ compact. Equivalently, the manifold N_ρ contains a compact convex submanifold C such that $N_\rho \setminus C$ is homeomorphic to $\partial C \times \mathbb{R}$.

If S is a boundary component of M, then the S–*end* of N_ρ is the end corresponding to S under this homeomorphism. The end corresponding to a component S of ∂M is *convex-cocompact* if N_ρ contains a convex-submanifold C such that $N_\rho \setminus C$ is a neighborhood of the S–end of N_ρ homeomorphic to $S \times \mathbb{R}$. Before going further we remind the reader of the following characterization of the convex-cocompact structures.

Lemma 2.1 *A hyperbolic structure N_ρ is convex-cocompact if and only if for some choice, and hence for all, of $p_{\mathbb{H}^3} \in \mathbb{H}^3$ the map*

$$\pi_1(M) \to \mathbb{H}^3, \quad \gamma \mapsto (\rho(\gamma))(p_{\mathbb{H}^3})$$

is a quasi-isometric embedding. Here we endow $\pi_1(M)$ with the left-invariant word-metric corresponding to some finite generating set.

Recall that a map $\phi\colon X_1 \to X_2$ between two metric spaces is an (L, A)–quasi-isometric embedding if

$$\frac{1}{L} d_{X_1}(x, y) - A \le d_{X_2}(\phi(x), \phi(y)) \le L d_{X_1}(x, y) + A$$

for all $x, y \in X_1$. An (L, A)–quasi-isometric embedding $\phi\colon \mathbb{R} \to X$ is said to be a quasi-geodesic.

Through out this note we are mostly interested in hyperbolic manifolds without cusps. If there are cusps, ie if there are elements $\gamma \in \pi_1(M)$ with $\rho(\gamma)$ parabolic, then there is an analogous of the convex-cocompact ends: the geometrically finite ends. Results about hyperbolic 3–manifolds without cusps can be often extended, under suitable conditions and with lots of work, to allow general hyperbolic 3–manifolds. We state this golden rule here.

> Every result mentioned without cusps has an analogous result in the presence of cusps.

The geometry of convex-cocompact ends of N_ρ is well-understood using Ahlfors–Bers theory. Building on the work of Thurston, Canary [12] described a different sort of end.

Definition *An end $\mathcal{E}$ of N_ρ is* simply degenerate *if there is a sequence of surfaces $(S_i) \subset N_\rho$ with the following properties.*

- *Every neighborhood of $\mathcal{E}$ contains all but finitely many of the surfaces S_i.*
- *With respect to the induced path distance, the surface S_i is $CAT(-1)$ for all i.*
- *If $C_1 \subset N_\rho$ is a compact submanifold with $N_\rho \setminus C_1$ homeomorphic to a trivial interval bundle, then S_i is homotopic to ∂C_1 within $N_\rho \setminus C_1$.*

The best understood example of manifolds with simply degenerate ends are obtained as follows. A theorem of Thurston (see for example Otal [36]) asserts that whenever S is a closed surface and $f \in \mathrm{Map}(S)$ is a pseudo-Anosov mapping class on S, then the mapping torus

$$M_f = (S \times [0,1])/((x,1) \simeq (f(x),0) \tag{2.1}$$

admits a hyperbolic metric. The fundamental group of the fiber $\pi_1(S)$ induces an infinite cyclic cover $M'_f \to M_f$ homeomorphic to $S \times \mathbb{R}$. The surface $S \times \{0\}$ lifts to a surface, again denoted by S, in M'_f and there are many known ways to construct $CAT(-1)$ surfaces in M'_f homotopic to S; for instance one can use simplicial hyperbolic surfaces, minimal surfaces or pleated surfaces. For the sake of concreteness, let X be such a surface and F be a generator of the deck transformation group of the covering $M'_f \to M_f$. Then the sequences $(F^i(X))_{i\in\mathbb{N}}$ and $(F^{-i}(X))_{i\in\mathbb{N}}$ fulfill the conditions in the definition above proving that both ends of M'_f are singly degenerate.

Before going further recall that homotopic, π_1–injective simplicial hyperbolic surfaces can be interpolated by simplicial hyperbolic surfaces. In particular, we obtain that every point in M'_f is contained in a $CAT(-1)$ surface. The same holds for every point in a sufficiently small neighborhood of a singly degenerated end. This is the key observation leading to the proof of Thurston and Canary's [13] covering theorem.

Covering Theorem *Let M and N be infinite volume hyperbolic 3–manifolds with finitely generated fundamental group and $\pi\colon M \to N$ be a Riemannian covering. Every simply degenerate end $\mathcal{E}$ of M has a neighborhood homeomorphic to $E = S \times [0,\infty)$ such that $\pi(E) = R \times [0,\infty)$ where R is a closed surface and $\pi|_E\colon E \to \pi(E)$ is a finite-to-one covering.*

The Covering Theorem would be of limited use if there were a third, call it wild, kind of ends. However, the positive solution of the tameness conjecture by Agol [3] and Calegari–Gabai [11], together with an older but amazingly nice result of Canary [12], implies that that every end of a hyperbolic 3–manifold without cusps is either convex-cocompact or simply degenerate; there is an analogous statement in the presence of cusps.

Theorem 2.2 *If M is a hyperbolic 3–manifold with finitely generated fundamental group then M is homeomorphic to the interior of a compact 3–manifold. In particular, in the absence of cusps, every end of M is either convex-cocompact or simply degenerate.*

Theorem 2.2 asserts that in order to prove that a manifold without cusps is convex-cocompact it suffices to prove that it does not have any degenerate ends. On the other hand, the Covering Theorem asserts that degenerate ends cannot cover infinite volume 3–manifolds in interesting ways. We obtain for example the following corollary.

Corollary 2.3 *Let M be trivial interval bundle or a handlebody and let $\rho\colon \pi_1(M) \to \mathrm{PSL}_2\,\mathbb{C}$ be a hyperbolic structure on M such that $N_\rho = \mathbb{H}^3/\rho(\pi_1(M))$ has no cusps. If $\Gamma \subset \pi_1(M)$ is a finitely generated subgroup of infinite index then $\mathbb{H}^3/\rho(\Gamma)$ is convex-cocompact.*

Given a sequence of pointed hyperbolic 3–manifolds (M_i, p_i) such that the injectivity radius $\mathrm{inj}(M_i, p_i)$ of M_i at p_i is uniformly bounded from below, then it is well-known that we may extract a geometrically convergent subsequence; say that it is convergent itself. More precisely, this means that there is some pointed 3–manifold (M, p) such that for every large R and small ϵ, there is some i_0 such that for all $i \geq i_0$, there are $(1+\epsilon)$–bi-Lipschitz, base points preserving, embeddings $\kappa_i^R\colon (B_R(p, M), p) \to (M_i, p_i)$ of the ball $B_R(M, p)$ in M of radius R and center p. Taking R and ϵ in a suitable way, we obtain better and better embeddings of larger and larger balls and we will refer in the sequence to these maps as the *almost isometric embeddings* provided by geometric convergence. If M_i is a sequence of hyperbolic 3–manifolds and M is isometric to the geometric limit of some subsequence of (M_i, p_i) for some choice of base points $p_i \in M_i$ then we say that M is a geometric limit of the sequence (M_i). We state here the following useful observation.

Lemma 2.4 *If M is a geometric limit of a sequence (M_i), $K \subset M$ is a compact subset such that the image $[\pi_1(K)]$ of $\pi_1(K)$ in $\pi_1(M)$ is convex-cocompact, and $\kappa_i : K \to M_i$ are the almost isometric maps provided by geometric convergence, then, for all i large enough, the induced homomorphism*

$$(\kappa_i)_*\colon [\pi_1(K)] \to \pi_1(M_i)$$

is injective and has convex-cocompact image.

Proof Let $\widetilde{C}$ be a convex, $[\pi_1(K)]$–invariant subset of $\mathbb{H}^3$ with $\widetilde{C}/[\pi_1(K)]$ compact and let C be the image of $\widetilde{C}$ in M. We find i_0 such that for all $i \geq i_0$ the almost isometric embeddings given by geometric convergence are defined on C and, moreover, the lift $\widetilde{C} \to \mathbb{H}^3$ of the composition of (1) projecting from $\widetilde{C}$ to C, and (2) mapping C into M_i is a quasi-isometry. By Lemma 2.1, $\widetilde{C}$ is quasi-isometric to $[\pi_1(K)]$. The claims follows now applying again Lemma 2.1. $\square$

It is well-known that a pseudo-Anosov mapping class $f \in \mathrm{Map}(S)$ of a surface fixes two projective classes of laminations $\lambda^+, \lambda^- \in \mathcal{PML}(S)$; λ^+ is the attracting fix point and λ^- the repelling. More precisely, for any essential simple closed curve γ in S we have

$$\lim_{n\to\infty} f^n(\gamma) = \lambda^+, \quad \lim_{n\to\infty} f^{-n}(\gamma) = \lambda^-$$

where the limits are taken in $\mathcal{PML}(S)$. If M'_f denotes again the infinite cyclic cover of the mapping torus M_f then when $|n|$ becomes large then the geodesic representatives in M'_f of $f^n(\gamma)$ leave every compact set. This implies that the laminations λ^+ and λ^- are in fact the *ending laminations* of M'_f. In general, every singly degenerate end has an associated ending lamination $\lambda_{\mathcal{E}}$. More precisely, if N_ρ is a hyperbolic structure on M, S is a component of ∂M and $\mathcal{E}$ is the S–end of N_ρ, then the ending lamination $\lambda_{\mathcal{E}}$ is defined as the limit in the space of laminations on S of any sequence of simple closed curves γ_i, on S, whose geodesics representatives γ_i^* tend to the end $\mathcal{E}$ and are homotopic to γ_i within $\mathcal{E}$. Thurston's ending lamination conjecture asserts that every hyperbolic structure on a 3–manifold, say for simplicity without cusps, is fully determined by its *ending invariants*: the conformal structures associated to the convex-cocompact ends and the ending lamination associated to the singly degenerate ends. The ending lamination conjecture has been recently proved by Minsky [30] and Brock–Canary–Minsky [10]. However, from our point of view, the method of proof is much more relevant than the statement itself: the authors prove that given a manifold and ending invariants, satisfying some necessary conditions, then it is possible to construct a metric on the manifold, the *model*, which is bi-Lipschitz equivalent to any hyperbolic metric on the manifold with the given ending invariants.

3 Minimal surfaces and Heegaard splittings

In this section let $M = (M, \rho)$ be a closed Riemannian 3–manifold, not necessarily hyperbolic. We will however assume that M is irreducible. Recall that a surface $F \subset M$ is a minimal surface if it is a critical point for the area functional. More precisely, if $\mathcal{H}^2_M(F)$ is the area, or in other words the two dimensional Hausdorff measure of the surface F in M, then F is *minimal* if for every smooth variation $(F_t)_t$

with $F_0 = F$ one has

$$\frac{d}{dt}\mathcal{H}^2_M(S_t)|_{t=0} = 0. \tag{3.1}$$

The minimal surface F is said to be *stable* if again for every smooth variation the second derivative of the area is positive:

$$\frac{d^2}{dt^2}\mathcal{H}^2_M(S_t)|_{t=0} > 0. \tag{3.2}$$

From a more intrinsic point of view, it is well-known that a surface F in M is minimal if and only if its mean curvature vanishes.

Schoen–Yau [52] and Sacks–Uhlenbeck [45] proved that every geometrically incompressible surface S in M is homotopic to a stable minimal surface F. Later, Freedman–Hass–Scott [18] proved that in fact F is embedded and hence that, by a result of Waldhausen [60] S is isotopic to a connected component of the boundary of a regular neighborhood of F. Summing up one has the following theorem.

Theorem 3.1 *Let S be a geometrically incompressible surface in M. Then there is a stable minimal surface F such that S is either isotopic to F or to the boundary $\partial\mathcal{N}(F)$ of a regular neighborhood of F.*

Theorem 3.1 concludes the discussion about existence of minimal surfaces as long as one is only interested into those surfaces which are geometrically incompressible. We turn now our attention to surfaces which are geometrically compressible.

Not every compressible surface in M needs to be isotopic to a minimal surface. In fact, the following beautiful theorem of Lawson [25] asserts that for example every minimal surface F in the round 3–sphere $\mathbb{S}^3$ is a Heegaard surface.

Theorem 3.2 *Assume that M has positive Ricci-curvature $\mathrm{Ric}(M) > 0$ and let F be a closed embedded minimal surface. Then, $M \setminus F$ consists of one or two handlebodies.*

However, there is a way to associate to every surface in M a (possibly empty) minimal surface. The idea is to consider the set $\mathfrak{Is}(S)$ of all possible surfaces in M isotopic to S and try to minimize area. If S_i is a sequence in $\mathfrak{Is}(S)$ such that

$$\lim_i \mathcal{H}^2_M(S_i) = \inf\{\mathcal{H}^2_M(S') | S' \in \mathfrak{Is}(S)\}$$

then one can try to extract a limit of the surfaces S_i hoping that it will be a minimal surface. However, it is unclear which topology should one consider. The usual approach is to consider S_i as a varifold. A *varifold* is a Radon measure on the Grassmannian

$G^2(M)$ of two-dimensional planes in TM. For all i, the inclusion of the surface S_i lifts to an inclusion

$$S_i \to G^2(M)$$

obtained by sending $x \in S_i$ to the plane $T_x S_i \in G^2(M)$. We obtain now a measure on $G^2(M)$ by pushing forward the Hausdorff measure, ie the area, of S_i. Observe that the total measure of the obtained varifold coincides with the area $\mathcal{H}^2_M(S_i)$ of S_i. In particular, the sequence (S_i), having area uniformly bounded from above, has a convergent subsequence, say the whole sequence, in the space of varifolds. Let $F = \lim_i S_i$ be its limit. The hope now is that F is a varifold induced by an embedded minimal surface. Again in the language of varifolds, F is a so-called *stationary varifold* and Allard's [5] regularity theory asserts that it is induced by a countable collection of minimal surfaces. In fact, using the approach that we just sketched, Meeks–Simon–Yau proved the following theorem.

Theorem 3.3 (Meeks–Simon–Yau [28]) *Let S an embedded surface in M and assume that*

$$\inf\{\mathcal{H}^2(S')|S' \in \mathfrak{I}s(S)\} > 0.$$

Then there is a minimizing sequence in $\mathfrak{I}s(S)$ converging to a varifold V, a properly embedded minimal surface F in M with components $F_1, \dots, F_k$ and a collection of positive integers $m_1, \dots, m_k$ such that $V = \sum m_i F_i$.

Theorem 3.3 applies also if S is a properly embedded surface in a manifold with mean-convex, for instance minimal, boundary.

Theorem 3.2 implies that the minimal surface F provided by Theorem 3.3 is, in general, not isotopic to the surface S we started with. Moreover, since the notion of convergence is quite weak, it seems hopeless to try to relate the topology of both surfaces. However, Meeks–Simon–Yau [28] show, during the proof of Theorem 3.3, that F arises from S through surgery.

Remark Meeks–Simon–Yau say that the minimal surface F arises from S by *γ–convergence* but this is exactly what we call surgery.

Theorem 3.3, being beautiful as it is, can unfortunately not be used if S is a Heegaard surface. Namely, if S is a Heegaard surface in M then there is a sequence of surfaces (S_i) isotopic to S such that $\lim_i \mathcal{H}^2_M(S_i) = 0$. For the sake of comparison, every simple closed curve in the round sphere $\mathbb{S}^2$ is isotopic to curves with arbitrarily short length. The comparison with curves in the sphere is not as far-fetched as it may seem. From this point of view, searching from minimal surfaces amounts to prove that the sphere has a closed geodesic for every Riemannian metric. That this is the case is an old result due to Birkhoff.

Theorem 3.4 (Birkhoff) *If ρ is a Riemannian metric on $\mathbb{S}^2$, then there is a closed non-constant geodesic in $(\mathbb{S}^2, \rho)$.*

The idea of the proof of Birkhoff's theorem is as follows. Fix ρ a Riemannian metric on $\mathbb{S}^2$ and let

$$f\colon \mathbb{S}^1 \times [0,1] \to \mathbb{S}^2, \quad f(\theta, t) = f_t(\theta)$$

be a smooth map with f_0 and f_1 constant and such that f represents a non-trivial element in $\pi_2(\mathbb{S}^2)$. For any g homotopic to f let

$$E(g) = \max\{l_\rho(g_t) | t \in [0,1]\}$$

be the length of the longest of the curves g_t. Observe that since g is not homotopically trivial $E(g)$ is bounded from below by the injecvity radius of $(\mathbb{S}^2, \rho)$. Choose then a sequence (g^i) for maps homotopic to f such that

$$\lim_i E(g^i) = \inf\{E(g) | g \text{ homotopic to } f\}.$$

One proves that there is a *minimax* sequence (t^i) with $t^i \in [0,1]$ such that $E(g_i) = l_\rho(g^i_{t^i})$ and such that the curves $g^i_{t^i}$ converge, when parametrized by arc-length to a non-constant geodesic in $(\mathbb{S}^2, \rho)$.

The strategy of the proof of Birkhoff's theorem was used in the late 70s by Pitts [41] who proved that every closed n–manifold with $n \leq 6$ contains an embedded minimal submanifold of codimension 1 (see also Schoen–Simon [51] for $n = 7$). We describe briefly his proof in the setting of 3–manifolds. The starting point is to consider a Heegaard surface S in M. By definition, the surface S divides M into two handlebodies. In particular, there is a map

$$f\colon (S \times [0,1], S \times \{0,1\}) \to (M, f(S \times \{0,1\})), \quad f(x,t) = f^t(x) \tag{3.3}$$

with positive relative degree, such that for $t \in (0,1)$ the map $f^t\colon S \to M$ is an embedding isotopic to the original embedding $S \hookrightarrow M$ and such that $f^0(S)$ and $f^1(S)$ are graphs. Such a map as in (3.3) is said to be a *sweep-out* of M. Given a sweep-out f one considers $E(f)$ to be the maximal area of the surfaces $f^t(S)$. Pitts proves that there is a minimizing sequences (f_i) of sweep-outs and an associated *minimax* sequence t^i with $E(f^i) = \mathcal{H}^2_M(f_{t^i}(S))$ and such that the surfaces $f_{t^i}(S)$ converge as varifolds to an embedded minimal surface F; perhaps with multiplicity.

Theorem 3.5 (Pitts) *Every closed Riemannian 3–manifold contains an embedded minimal surface.*

Pitts' proof is, at least for non-experts like the author of this note, difficult to read. However, there is an amazingly readable proof due to Colding–de Lellis [16]. In fact, the main technical difficulties can be by-passed, and this is what these authors do, by using Meeks–Simon–Yau's Theorem 3.3. In fact, as it is the case with the Meeks–Simon–Yau theorem, Theorem 3.5 remains true for compact 3–manifolds with mean-convex boundary.

Theorem 3.5 settles the question of existence of minimal surfaces in 3–manifolds. Unfortunately, it does not say anything about the relation between the Heegaard surface S we started with and the obtained minimal surface F. In fact, Colding–de Lellis announce in their paper that in a following paper they are going to prove that the genus does not increase. The concept of convergence of varifolds is so weak that this could well happen. However, in the early 80s, Pitts and Rubinstein affirmed something much stronger: they claimed that F is not stable and arises from S by surgery. This was of the greatest importance in the particular case that the Heegaard surface S is assumed to be strongly irreducible.

By Lemma 1.6, the assumption that the Heegaard surface S is strongly irreducible implies that every surface S' which arises from S by surgery is either isotopic to S or of one of the following two kinds:

(A) Either S' is obtained from S by suturing along disks which are all at the same side, or

(B) S is isotopic to the surface obtained from the boundary of a regular neighborhood of S' by attaching a vertical handle.

In particular, if in the setting of Pitts' theorem we assume that S is strongly irreducible we obtain that this alternative holds for the minimal surface F. In fact, more can be said. If we are in case (A) then F bounds a handlebody H in M such that the surface S is isotopic to a strongly irreducible Heegaard surface in the manifold with boundary $M \setminus H$. The boundary of $M \setminus H$ is minimal an incompressible. In particular, $F = \partial M \setminus H$ is isotopic to some stable minimal surface F' in $M \setminus H$ parallel to ∂M. Observe that $F \neq F'$ because one of them is stable and the other isn't. The stable minimal surface F' bounds in M some submanifold M_1 isotopic in M to $M \setminus H$, in particular the original Heegaard surface S induces a Heegaard splitting of M_1. The boundary of M_1 is minimal and hence mean-convex. In particular, the method of proof of Theorem 3.5 applies and yields an unstable minimal surface F_1 in M_1 obtained

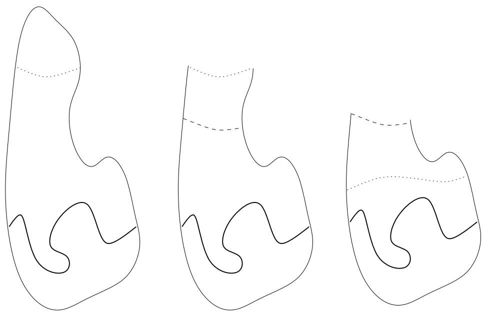

Figure 3: Proof of Theorem 3.6: The thick line is the original surface; the short dotted line is the first minimax surface; the dashed line is the least area surface obtained from the first minimax surface; the long dotted line is the second minimax surface.

from S by surgery. Again we are either in case (B) above, or F_1 is isotopic to S within M_1 and hence within M, or we can repeat this process.

If this process goes for ever, we obtain a sequences of disjoint embedded minimal surfaces in M with genus less than that of S. This means that the metric in M is not *bumpy*. However, if M is not bumpy, then we can use a result of White [62] and perturb it slightly so that it becomes bumpy. It follows from the above that for any such perturbation the process ends and we obtain a minimal surface which is either as in (B) or actually isotopic to the Heegaard surface S. Taking a sequence of smaller and smaller perturbations and passing to a limit we obtain a minimal surface F in M, with respect to the original metric, which is either isotopic to S or such that we are in case (B) above. In other words we have the following theorem.

Theorem 3.6 (Pitts-Rubinstein) *If S is a strongly irreducible Heegaard surface in a closed 3–manifold then there is a minimal surface F such that S is either isotopic to F or to the surface obtained from the boundary of a regular neighborhood of F by attaching a vertical 1–handle.*

Unfortunately, Pitts and Rubinstein never wrote the proof of Theorem 3.6 above and it seems unlikely that they are ever going to do so. The most precise version known to the author is a sketch of the proof due to Rubinstein [44]. This lack of written proof has made doubtful if one could use Theorem 3.6 safely or not. However, all that is left is to prove that the minimal surface provided in the proof of Pitts' Theorem 3.5 is unstable and obtained from S by surgery. The author of this note has written a proof [57] and is working on a longer text, perhaps a book, explaining it and some applications of the Pitts–Rubinstein theorem.

Before concluding this section we should remember that one can combine Theorem 1.2 and Theorem 3.6 as follows: Given a Heegaard surface we first destabilize as far as possible, then we obtain using Theorem 1.2 a generalized Heegaard surface (Σ_I, Σ_H). The surface Σ_I is incompressible and hence can be made minimal by Theorem 3.1; now the surface Σ_H can be made minimal using Theorem 3.6.

4 Using geometric means to determine the Heegaard genus

In this section we will show how minimal surfaces can be used to compute the Heegaard genus of some manifolds. Most, if not all, of the results we discuss here can be proved using purely topological arguments but, in the opinion of the author, the geometric proofs are beautiful.

We start considering the mapping torus M_ϕ of a pseudo-Anosov mapping class $\phi \in \mathrm{Map}(\Sigma_g)$ on a closed surface of genus g; compare with (2.1). It is well-known that M_ϕ admits a weakly reducible Heegaard splitting of genus $2g+1$. In particular we have the following bound for the Heegaard genus

$$g(M_\phi) \leq 2g+1.$$

There are manifolds which admit different descriptions as a mapping torus. In particular, we cannot expect that equality always holds. However, equality is to be expected if monodromy map ϕ is complicated enough.

Theorem 4.1 *Let Σ_g be a closed surface of genus g and $\phi \in \mathrm{Map}(\Sigma_g)$ a pseudo-Anosov mapping class. Then there is $n_\phi > 0$ such that for all $n \geq n_\phi$ one has $g(M_{\phi^n}) = 2g+1$. Moreover, for every such n there is, up to isotopy a unique Heegaard splitting of M_{ϕ^n} of genus $2g+1$.*

We sketch now the proof of Theorem 4.1. More precisely, we will prove that, for large n, there is no strongly irreducible Heegaard splitting of M_{ϕ^n} of genus at most $2g+1$. The general case follows, after some work, using Theorem 1.2.

Seeking a contradiction, assume that M_{ϕ^n} admits a strongly irreducible splitting with at most genus $2g+1$. Then, endowing M_{ϕ^n} with its hyperbolic metric, we obtain from Theorem 3.6 that M_{ϕ^n} contains a minimal surface F of at most genus $2g+1$ and such that every component of $M_{\phi^n} \setminus F$ is a handlebody. In particular, F intersects every copy of the fiber Σ_g since the later is incompressible and a handlebody does not contain any incompressible surfaces. For all n the manifold M_{ϕ^n} covers the manifold M_ϕ. In particular, we have first the following lower bound for the injectivity radius

$$\mathrm{inj}(M_{\phi^n}) \geq \mathrm{inj}(M_\phi)$$

and secondly that increasing n we can find two copies of the fiber which are at arbitrary large distances. On the other hand, the bounded diameter lemma for minimal surfaces below shows that the diameter of a minimal surface in a hyperbolic 3–manifold is bounded from above only in terms of its genus and of the injectivity radius of the manifold. This shows that if n is large the minimal surface F cannot exist.

Bounded diameter lemma for minimal surfaces (first version) *Let F be a connected minimal surface in a hyperbolic 3–manifold M with at least injectivity radius ϵ. Then we have*

$$\mathrm{diam}(F) \leq \frac{4|\chi(F)|}{\epsilon} + 2\epsilon$$

where $\mathrm{diam}(F)$ *is the diameter of F in M.*

Proof The motonicity formula (Colding–Minicozzi [17]) asserts that for every point $x \in F$ we have $\mathcal{H}^2_M(F \cap B_x(M,\epsilon)) \geq \pi\epsilon^2$ where $B_x(M,\epsilon)$ is the ball in M centered at x and with radius ϵ. If F has diameter D in M we can find at least $\frac{D}{2\epsilon} - 1$ points which are at distance at least ϵ from each other. On the other hand, the curvature of F is bounded from above by -1 and hence the total area is bounded by $\mathcal{H}^2_M(F) \leq 2\pi|\chi(F)|$. In particular we obtain that

$$\left(\frac{D}{2\epsilon} - 1\right)\pi\epsilon^2 \leq 2\pi|\chi(F)|.$$

This concludes the proof. □

We stated this as a *first version* because in some sense the role of the injectivity radius of M is disappointing. However, it is not difficult to construct hyperbolic 3–manifolds containing minimal surfaces of say genus 2 with arbitrarily large diameter. In order to by-pass this difficulty we define, following Thurston, for some ϵ positive the length of a curve γ relative to the ϵ–thin part $M^{<\epsilon}$ of M to be the length of the intersection of γ with the set of points in M with injectivity radius at least ϵ. Then, the distance $d_{\mathrm{rel}\, M^{<\epsilon}}(x,y)$ of two points $x, y \in M$ relative to the the ϵ-thin part is the infimum of the lengths relative to the ϵ-thin part of paths joining x and y. Using this pseudo-distance we obtain with essentially the same proof the following final version of the bounded diameter lemma.

Bounded diameter lemma for minimal surfaces *Let F be a connected minimal surface in a hyperbolic 3–manifold M and let $\mu > 0$ be the Margulis constant. Then we have*

$$\mathrm{diam}_{\mathrm{rel}\, M^{<\mu}}(F) \leq \frac{8|\chi(F)|}{\mu}$$

where $\mathrm{diam}_{\mathrm{rel}\, M^{<\mu}}(F)$ *is the diameter of F in M with respect to $d_{\mathrm{rel}\, M^{<\mu}}$.*

The bounded diameter lemma, together with the argument used in the proof of Theorem 4.1 shows now that whenever M_ϕ is the mapping torus of a pseudo-Anosov mapping class on the surface Σ_g and such that M_ϕ contains two fibers which are at least at distance $\frac{8(2g+4)}{\mu}$ with respect to $d_{\mathrm{rel}\, M^{<\mu}}$ then M_ϕ does not have strongly irreducible Heegaard splittings of genus less than $2g+1$. Moreover, one gets as above that M_ϕ has genus $2g+1$ and, up to isotopy, a single Heegaard splitting of minimal genus. In particular, in order to generalize Theorem 4.1, it suffices to give conditions on the monodromy ϕ ensuring that the mapping torus M_ϕ contains fibers at large distance. It follows for example from the work of Minsky [29] that this is the case if the translation length of ϕ in the curve complex $\mathcal{C}(\Sigma_g)$ is large enough. In particular we have the following theorem.

Theorem 4.2 *For every g there is $D_g > 0$ such that the following holds: If ϕ is a pseudo-Anosov mapping class on Σ_g with at least translation length D_g in the curve complex $\mathcal{C}(\Sigma_g)$ of Σ_g, then the mapping torus M_ϕ has Heegaard genus $g(M_\phi) = 2g+1$ and there is, up to isotopy, a unique minimal genus Heegaard splitting.*

Observe that whenever ϕ is pseudo-Anosov, the translation lengths of ϕ^n in $\mathcal{C}(\Sigma_g)$ tends to ∞ when n becomes large. In other word, Theorem 4.1 follows from Theorem 4.2.

An other result in the same spirit is due to Lackenby [24], who proved that whenever M_1 and M_2 are compact, irreducible, atoroidal 3–manifolds with incompressible and acylindrical homeomorphic connected boundaries $\partial M_1 = \partial M_2$ of at least genus 2 and $\phi \in \mathrm{Map}(\partial M_1)$ is a pseudo-Anosov mapping class then the manifold $M_1 \cup_{\phi^n} M_2$ obtained by gluing M_1 and M_2 via ϕ^n has Heegaard genus $g(M_1)+g(M_2)-g(\partial M_1)$. In this setting the key point is again that if n is large then the manifold $M_1 \cup_{\phi^n} M_2$ contains two surfaces isopic to the gluing surface and which are at large distance. And again there is a generalization involving the curve complex.

Theorem 4.3 (Souto [59]) *Let M_1 and M_2 be compact, irreducible, atoroidal 3–manifolds with incompressible and acylindrical homeomorphic connected boundaries $\partial M_1 = \partial M_2$ of genus at least two and fix an essential simple closed curve α in*

∂M_1. Then there is a constant D such that every minimal genus Heegaard splitting of $M_1 \cup_\phi M_2$ is constructed amalgamating splittings of M_1 and M_2 and hence

$$g(M_1 \cup_\phi M_2) = g(M_1) + g(M_2) - g(\partial M_1)$$

for every diffeomorphism $\phi\colon \partial M_1 \to \partial M_1$ with $d_{\mathcal{C}(\partial M_1)}(\phi(\alpha), \alpha) \geq D$.

Lackenby's argument in fact uses minimal surfaces and it was reading his paper when the author of this note became interested in the relation between minimal surfaces and Heegaard splittings. Lackenby's paper is also beautifully written.

A different result, the oldest of the ones presented in this section, and also proved using minimal surfaces, involves the Heegaard genus of those hyperbolic 3–manifolds obtained by Dehn-filling a finite volume manifold with cusps. Recall that every complete, non-compact, orientable complete hyperbolic manifold M with finite volume is homeomorphic to the interior of a compact manifold $\bar{M}$ with torus boundary. For simplicity we will assume that $\partial \bar{M}$ has only one component; we say that M has a single cusp. Identifying $\partial \bar{M}$ with the boundary of a solid torus $\mathbb{D}^2 \times \mathbb{S}^1$ via some map ϕ and gluing both $\bar{M}$ and $\mathbb{D}^2 \times \mathbb{S}^1$ via this identification we obtain a closed 3–manifold M_ϕ which is said to arise from M by Dehn-filling. In fact, the homeomorphism type of M_ϕ depends only on the homotopy class in $\partial \bar{M}$ of the meridian of the attached solid torus. In other words, for every essential simple closed curve γ in ∂M there is a unique 3–manifold M_γ obtained by Dehn-filling along γ. Thurston's beautiful Dehn filling theorem asserts that for all but finitely many γ the manifold M_γ admits a hyperbolic metric. An extension due to Hodgson–Kerckhoff [20] of this result asserts that the number of exceptions is in fact bounded independently of the manifold M.

By construction, there is a natural embedding $M \hookrightarrow M_\gamma$ and it is easy to see that every Heegaard surface of M is, via this embedding, also a Heegaard surface of M_γ for every Dehn-filling of M. This proves that

$$g(M_\gamma) \leq g(M)$$

for all M_γ. It is not difficult to construct examples which show that there are finite volume hyperbolic manifolds admitting infinitely many Dehn-fillings for which equality does not hold. However, in some sense, equality holds for most Dehn-fillings of M. More precisely, identifying the set $\mathcal{S}(\partial \bar{M})$ of homotopy class of essential simple curves in $\partial \bar{M}$ with the set of vertices of the Farey graph one obtains a distance on $\mathcal{S}(\partial \bar{M})$. Using this distance one has the following theorem.

Theorem 4.4 (Moriah–Rubinstein [32]) *Let M be a complete, oriented finite volume hyperbolic manifold with a cusp. Then there is a bounded set K in $\mathcal{S}(\partial \bar{M})$ with $g(M_\gamma) = g(M)$ for every $\gamma \notin K$.*

Before concluding this section observe that the Farey graph is not locally compact and hence bounded sets may be infinite. Observe also that a purely topological proof of Theorem 4.4 is due to Rieck–Sedgwick [42].

5 Generators of the fundamental group and carrier graphs

Let M be a closed hyperbolic, or more generally, negatively curved 3–manifold. In this section we relate generating sets, or more precisely Nielsen equivalence classes of generating sets, of $\pi_1(M)$ to some graphs in M with nice geometric properties.

Recall that two (ordered) generating sets $\mathcal{S} = (g_1, \ldots, g_r)$ and $\mathcal{S}' = (g'_1, \ldots, g'_r)$ of a group are *Nielsen equivalent* if they belong to the same class of the equivalence relation generated by the following three moves:

$$\text{Inversion of } g_i \qquad \begin{cases} g'_i = g_i^{-1} \\ g'_k = g_k \quad k \neq i. \end{cases}$$

$$\text{Permutation of } g_i \text{ and } g_j \text{ with } i \neq j \qquad \begin{cases} g'_i = g_j \\ g'_j = g_i \\ g'_k = g_k \quad k \neq i, j. \end{cases}$$

$$\text{Twist of } g_i \text{ by } g_j \text{ with } i \neq j \qquad \begin{cases} g'_i = g_i g_j \\ g'_k = g_k \quad k \neq i. \end{cases}$$

To every Nielsen equivalence class of generators of $\pi_1(M)$ one can associate an equivalence class of carrier graphs.

Definition *A continuous map $f\colon X \to M$ of a connected graph X into a hyperbolic 3–manifold M is a* carrier graph *if the induced homomorphism $f_*\colon \pi_1(X) \to \pi_1(M)$ is surjective. Two carrier graphs $f\colon X \to M$ and $g\colon Y \to M$ are* equivalent *if there is a homotopy equivalence $h\colon X \to Y$ such that f and $g \circ h$ are free homotopic.*

Given a generating set $\mathcal{S} = (g_1, \ldots, g_r)$ of $\pi_1(M)$ let $\mathbb{F}_{\mathcal{S}}$ be the free non-abelian group generated by the set $\mathcal{S}$, $\phi_{\mathcal{S}}\colon \mathbb{F}_{\mathcal{S}} \to \pi_1(M)$ the homomorphism given by mapping the free basis $\mathcal{S} \subset \mathbb{F}_{\mathcal{S}}$ to the generating set $\mathcal{S} \subset \pi_1(M)$ and $X_{\mathcal{S}}$ a graph with $\pi_1(X_{\mathcal{S}}) = \mathbb{F}_{\mathcal{S}}$. The surjective homomorphism $\phi_{\mathcal{S}}\colon \mathbb{F}_{\mathcal{S}} \to \pi_1(M)$ determines a free homotopy class of maps $f_{\mathcal{S}}\colon X_{\mathcal{S}} \to M$, ie a carrier graph, and any two carrier graphs obtained in this way are equivalent. The so determined equivalence class is said to be the *equivalence class of carrier graphs associated to $\mathcal{S}$*.

Lemma 5.1 *Let $\mathcal{S}$ and $\mathcal{S}'$ be finite generating sets of $\pi_1(M)$ with the same cardinality. Then the following are equivalent.*

(1) *$\mathcal{S}$ and $\mathcal{S}'$ are Nielsen equivalent.*

(2) *There is a free basis $\bar{\mathcal{S}}$ of $\mathbb{F}_{\mathcal{S}'}$ with $\mathcal{S} = \phi_{\mathcal{S}'}(\bar{\mathcal{S}})$.*

(3) *There is an isomorphism $\psi\colon \mathbb{F}_{\mathcal{S}} \to \mathbb{F}_{\mathcal{S}'}$ with $\phi_{\mathcal{S}} = \phi_{\mathcal{S}'} \circ \psi$.*

(4) *$\mathcal{S}$ and $\mathcal{S}'$ have the same associated equivalence classes of carrier graphs.*

We will only consider carrier graphs $f\colon X \to M$ with $\operatorname{rank}(\pi_1(X)) = \operatorname{rank}(\pi_1(M))$. Equivalently we only consider generating sets with minimal cardinality.

If $f\colon X \to M$ is a carrier then let $X^{(0)}$ be the set of vertices of X and $X^{(1)}$ that of edges. The *length* of a carrier graph $f\colon X \to M$ is defined as the sum of the lengths of the images of the edges

$$l_{f\colon X\to M}(X) = \sum_{e\in X^{(1)}} l_M(f(e)).$$

A *minimal length* carrier graph is a carrier graph $f\colon X \to M$ with

$$l_{f\colon X\to M}(X) \leq l_{f'\colon X'\to M}(X')$$

for every other equivalent carrier graph $f'\colon X' \to M$. The existence of minimal length carrier graphs follows from the Arzela–Ascoli theorem if M is closed and in fact one has the following lemma.

Lemma 5.2 White [63, Section 2] *If M is a closed hyperbolic 3–manifold, then there is a minimal length carrier graph $f\colon X \to M$. Moreover, every such minimal length carrier graph is trivalent, hence it has $3(\operatorname{rank}(M)-1)$ edges, the image in M of its edges are geodesic segments, the angle between any two adjacent edges is $\frac{2\pi}{3}$ and every simple closed path in X represents a non-trivial element in $\pi_1(M)$.*

It is not difficult to see that if M_{ϕ^n} is the mapping torus of a high power of a pseudo-Anosov mapping class then every minimal length carrier graph, with minimal cardinality, has huge total length; in particular it contains some large edge. However, the following simple lemma asserts that there is a universal upper bound for the length of the shortest edge in a carrier graph in closed hyperbolic 3–manifold.

Lemma 5.3 *There is some positive L such that every minimal length carrier graph in a closed hyperbolic 3–manifold has an edge shorter than L.*

Proof Let $M = \mathbb{H}^3/\Gamma$ be a hyperbolic 3–manifold and assume that there is a minimal carrier graph $f\colon X \to M$ consist of only extremely long edges. Denote by $\tilde{f}\colon \tilde{X} \to \mathbb{H}^3$ the lift of f to a map between the universal covers. The image under $\tilde{f}$ of every monotonous bi-infinite path in $\widetilde{X}$ consists of extremely long geodesic segments joined at corners with angle $\frac{2\pi}{3}$. In particular, every such path is a quasi-geodesic and $\widetilde{f}$ is a quasi-geodesic embedding, implying that the homomorphism $f_*\colon \pi_1(X) \to \pi_1(M)$ is injective. Hence $\pi_1(M)$ is free contradicting the assumption that M is closed. $\square$

Lemma 5.3 for itself is of little use; the point is that it has some grown up relatives which, in some sense made precise below, allow to decompose carrier graphs into short pieces. For instance, White [63] proved that again there is some positive constant L such that every carrier graph in a closed hyperbolic 3–manifold admits a circuit shorter than L. In particular he obtained the following beautiful result.

Theorem 5.4 White [63] *For every integer r there is a positive constant L with* $\mathrm{inj}(M) \leq L$ *for every closed hyperbolic 3–manifold whose fundamental group has at most rank r.*

Unfortunately, White's observation is the end if one takes the most naive point of view: there are examples showing that every subgraph with non-abelian fundamental group can be made as long as one wishes or fears. However, the idea is still to obtain a exhaustion of every carrier graph by subgraphs which in some sense are short. The solution is to consider the length of a carrier graph relative to the convex-hull of a subgraph.

However, before making this more precise we need some more notation. If $f\colon X \to M$ is a carrier graph and $Y \subset X$ is a subgraph then let $Y^{(0)}$ be again the set of vertices and $Y^{(1)}$ that of edges; let also $\pi_X\colon \widetilde{X} \to X$ be the universal covering of X and $\widetilde{f}\colon \widetilde{X} \to \mathbb{H}^3$ a fixed lift of f to a map between the universal coverings of X and M. If $Y \subset X$ is a connected subgraph of X then every connected component of $\pi_X^{-1}(Y)$ can be identified with the universal cover of Y. Given such a component $\widetilde{Y}$ of $\pi_X^{-1}(Y)$ let $G(\widetilde{Y}) \subset \pi_1(X)$ be the group of all covering transformations of $\pi_X\colon \widetilde{X} \to X$ preserving $\widetilde{Y}$; $G(\widetilde{Y})$ is isomorphic to $\pi_1(Y)$. Denote by $\Gamma_{\widetilde{Y}}$ the image of $G(\widetilde{Y})$ under the homomorphism $f_*\colon \pi_1(X) \to \pi_1(M)$.

If Y is a connected subgraph of a carrier graph $f\colon X \to M$ and $\widetilde{Y}$ is a component of $\pi_X^{-1}(Y)$ we define the *thick convex-hull* $TCH(\widetilde{Y})$ as follows.

Definition *The* thick convex-hull $TCH(\widetilde{Y})$ *of a component* $\widetilde{Y}$ *of* $\pi_X^{-1}(Y)$ *is the smallest closed convex subset of* $\mathbb{H}^3$ *containing* $\widetilde{f}(\widetilde{Y})$ *and with*

$$d_{\mathbb{H}^3}(x, \gamma x) \geq 1$$

for all non-trivial $\gamma \in \Gamma_{\widetilde{Y}}$ *and* $x \notin TCH(\widetilde{Y})$.

The thick convex-hull is unique because intersection of convex subsets is convex and uniqueness implies that $TCH(\widetilde{Y})$ is invariant under $\Gamma_{\widetilde{Y}}$ and in particular it contains the convex-hull of the limit set of $\Gamma_{\widetilde{Y}}$. However, there are several reasons for introducing the thick convex-hull $TCH(\widetilde{Y})$ instead of working directly with the convex-hull of the limit set of $\Gamma_{\widetilde{Y}}$. For example we want to avoid treating differently the case that Y is a tree.

We are now ready to formally define the length of a carrier graph $f\colon X \to M$ relative to a subgraph Y with $X^{(0)} \subset Y$. If $e \in X^{(1)} \setminus Y^{(1)}$ is an edge which is not contained in Y and $\widetilde{e}$ is a lift of e to the universal cover $\widetilde{X}$ of X then the vertices of $\widetilde{e}$ are contained in two different components $\widetilde{Y}_1$ and $\widetilde{Y}_2$ of $\pi_X^{-1}(Y)$. We define the length of $\widetilde{e}$ relative to $\pi_X^{-1}(Y)$ to be the length of the part of $\widetilde{f}(\widetilde{e})$ which is disjoint of the union

$$TCH(\widetilde{Y}_1) \cup TCH(\widetilde{Y}_2)$$

of the thick convex-hulls of $\widetilde{Y}_1$ and $\widetilde{Y}_2$. If $\widetilde{e}'$ is a second lift of e to $\widetilde{X}$ then both $\widetilde{e}$ and $\widetilde{e}'$ have the same length relative to $\pi_X^{-1}(Y)$. In particular, the relative length with respect to Y

$$l_{f\colon X\to M,\mathrm{rel}(Y)}(e)$$

of the edge e is well-defined.

If $Z \subset X$ is a second subgraph with $Y \subset Z$ then we define the length of Z relative to Y to be the sum of the relative lengths of all the edges contained in Z but not in Y:

$$l_{f\colon X\to M,\mathrm{rel}(Y)}(Z) = \sum_{e\in Z^{(1)}\setminus Y^{(1)}} l_{f\colon X\to M,\mathrm{rel}(Y)}(e).$$

Observe that $l_{f\colon X\to M,\mathrm{rel}(X^{(0)})}(X) = l_{f\colon X\to M}(X)$.

The most important observation is the following proposition.

Proposition 5.5 *There is L such that whenever M is a closed hyperbolic 3–manifold and $f\colon X \to M$ is a minimal length carrier graph then there is a chain of subgraphs*

$$X^{(0)} = Y_0 \subset Y_1 \subset \cdots \subset Y_k = X$$

with $l_{f\colon X\to M,\mathrm{rel}(Y_{i-1})}(Y_i) \le L$ for all $i = 1,\ldots,k$.

The idea behind Proposition 5.5 is that in the proof of Lemma 5.3 one can replace the vertices of $\widetilde{X}$ by convex subsets. See Souto [56] for details.

6 The rank of the fundamental group of simple complicated mapping tori

Recall that the *rank* of a finitely generated group G is the minimal number of elements needed to generate G. While in general the rank of even a hyperbolic group is not computable (Rips [43]), the situation changes if one is interested in those groups arising as the fundamental group of a closed 3–manifold.

Theorem 6.1 Kapovich–Weidmann [21] *There exists an algorithm which, given a finite presentation of the fundamental group of a hyperbolic 3–manifold, finds the rank of G.*

However, it is not possible to give a priori bounds on the complexity of the algorithm provided by the Kapovich–Weidmann theorem and hence it seems difficult to use it directly to obtain precise results in concrete situations.

Here, we show how to derive from the results in the previous section the following theorem analogous to Theorem 4.1.

Theorem 6.2 (Souto [55]) *Let Σ_g be the closed surface of genus $g \geq 2$, $\phi \in \mathrm{Map}(\Sigma_g)$ a pseudo-Anosov mapping class and M_{ϕ^n} the mapping torus of ϕ^n. There is n_ϕ such that for all $n \geq n_\phi$*

$$\mathrm{rank}(\pi_1(M_{\phi^n})) = 2g + 1.$$

Moreover for any such n any generating set of $\pi_1(M_{\phi^n})$ with minimal cardinality is Nielsen equivalent to an standard generating set.

Observe that by construction we have $\pi_1(M_\phi) = \pi_1(\Sigma_g)*_{\mathbb{Z}}$ and hence, considering $2g$ generators of $\pi_1(\Sigma_g)$ and adding a further element corresponding to the HNN-extension we obtain generating sets of $\pi_1(M_\phi)$ with $2g+1$ elements. These are the so-called *standard* generating sets.

Before going further, we would like to remark that recently an extension of Theorem 6.2 has been obtained by Ian Biringer.

Theorem 6.3 (Biringer) *For every g and ϵ there are at most finitely many hyperbolic 3–manifolds M with $\mathrm{inj}(M) \geq \epsilon$, fibering over the circle with fiber Σ_g and with $\mathrm{rank}(M) \neq 2g+1$.*

Proof Sketch proof of Theorem 6.2 Given some ϕ pseudo-Anosov, let $M_n = M_{\phi^n}$ be the mapping torus of ϕ^n. Let also $f_n\colon X_n \to M_n$ be a minimal length carrier graph for each n. Moreover, let $Y_n \subset X_n$ be a sequence of subgraphs of X_n with the following properties.

- There is some constant C with $l_{f_n\colon X_n \to M_n}(Y_n) \leq C$ for all n.
- If $Y'_n \subset X_n$ is a sequence of subgraphs of X_n properly containing Y_n for all n then $\lim_n l_{f_n\colon X_n \to M_n}(Y'_n) = \infty$.

We claim that for all sufficiently large n the graph Y_n has a connected component $\widehat{Y}_n$ such that the image of $\pi_1(\widehat{Y}_n)$ in $\pi_1(M_n)$ generates the fundamental group of the fiber. In particular we have

$$2g \leq \mathrm{rank}(\pi_1(\widehat{Y}_n)) < \mathrm{rank}(\pi_1(M_n)) \leq 2g+1.$$

The claim of Theorem 6.2 follows.

Seeking a contradiction assume that the subgraphs $\widehat{Y}_n$ don't exist for some subsequence (n_i) and let $\overline{Y}_{n_i}$ be a sequence of connected components in Y_{n_i}. Since the graph Y_{n_i} has at most length C we may assume, up to forgetting finitely many, that for all i the graph $\overline{Y}_{n_i}$ lifts to the infinite cyclic cover M' of M_{n_i}. Moreover, the space of graphs in M' with bounded length is, up to the natural $\mathbb{Z}$ action, compact. In particular, we may assume, up to taking a further subsequence and perhaps shifting our lift by a deck-transformation, that the images of $\pi_1(\overline{Y}_{n_i})$ and $\pi_1(\overline{Y}_{n_j})$ coincide for all i and j.

On the other hand, by assumption $\pi_1(\overline{Y}_{n_i})$ does not generate $\pi_1(M') = \pi_1(\Sigma_g)$. Moreover, since every proper finite index subgroup of $\pi_1(M')$ has rank larger than $2g$, we have that the image of $\pi_1(\overline{Y}_{n_i})$ in $\pi_1(M')$ has infinite index and hence is a free group. This implies that the homomorphism $\pi_1(\overline{Y}_{n_i}) \to \pi_1(M')$ is injective for otherwise we would be able to find carrier graphs for M_n with rank less than $\mathrm{rank}(\pi_1(X_n))$ (compare with the remark following Lemma 5.1). Thurston's Covering Theorem (compare with Corollary 2.3) implies now that in fact the image of $\pi_1(\overline{Y}_{n_i})$ is convex-cocompact. In particular, the quotient under $\pi_1(\overline{Y}_{n_i})$ of its thick-convex-hull has bounded diameter. Then, minimality of the graph $f_{n_i}\colon X_{n_i} \to M_{n_i}$ implies that there is some D such that for every edge e of X_{n_i} one has

$$l_{f_{n_i}\colon X_{n_i} \to M_{n_i}}(e) \leq D + l_{f_{n_i}\colon X_{n_i} \to M_{n_i},\mathrm{rel}(\overline{Y}_i)}(e). \tag{6.1}$$

Equation (6.1) applies by assumption to all components $\overline{Y}_i$ of Y_i. In particular we obtain some D' depending on D and the maximal number of components, ie of D and g, such that

$$l_{f_{n_i}\colon X_{n_i} \to M_{n_i}}(e) \leq D' + l_{f_{n_i}\colon X_{n_i} \to M_{n_i},\mathrm{rel}(Y_i)}(e). \tag{6.2}$$

Equation (6.2) contradicts Proposition 5.5. This concludes the (sketch of the) proof of Theorem 6.2. □

7 Another nice family of manifolds

Until now, we have mostly considered 3–manifolds arising as a mapping torus. Mapping tori are particularly nice 3–manifolds whose construction is also particularly simple to describe. But this is not the real reason why we were until now mainly concerned with them. The underlying geometric facts needed in the proofs of Theorem 4.1 and Theorem 6.2 are the following:

- Thurston's theorem asserting that the mapping torus M_ϕ of a pseudo-Anosov mapping class ϕ admits a hyperbolic metric and
- the fact that every geometric limit of the sequence $(M_{\phi^n})_n$ is isometric to the infinite cyclic cover corresponding to the fiber.

In some cases, for instance in the proof of Theorem 4.1, one does not need the full understanding of the possible geometric limits. But still, without understanding enough of the geometry it is not possible, using our methods, to obtain topological facts. In this section we discuss results obtained by Hossein Namazi and the author [34] concerning a different family of 3–manifolds.

Let M^+ and M^- be 3-dimensional handlebodies of genus $g > 1$ with homeomorphic boundary $\partial M^+ = \partial M^-$. Given a mapping class $f \in \text{Map}(\partial M^+)$ we consider the closed, oriented 3–manifold

$$N_f = M^+ \cup_f M^-$$

obtained by identifying the boundaries of M^+ and M^- via f. By construction, the manifold N_f has a *standard* Heegaard splitting of genus g. Also, generating sets of the free groups $\pi_1(M^\pm)$ generate $\pi_1(N_f)$ and hence $\text{rank}(\pi_1(N_f)) \leq g$. We will be interested in those manifolds N_{f^n} where the gluing map is a high power of a sufficiently complicated mapping class hoping that if this is the case, then

$$g(N_{f^n}) = \text{rank}(\pi_1(N_{f^n})) = g.$$

However, we must be a little bit careful with what we mean under "complicated mapping class". The problem is that it does not suffice to assume that f is pseudo-Anosov because there are homeomorphisms F of M^+ which induce pseudo-Anosov mapping classes f on ∂M^+ and for any such map we have $N_f = N_{f^n}$ for all n. For example, N_{f^n} could be the 3–sphere for all n. In particular, we have to rule out that f extends to either M^+ or M^-. In order to give a precise sufficient condition recall that every pseudo-Anosov map has an stable lamination λ^+ and an unstable lamination λ^-. If f extends to a homeomorphism of M^+ then it maps meridians, ie essential simple closed curves on ∂M^+ which are homotopically trivial in M^+, to meridians; in particular, λ^+ is a limit in the space $\mathcal{PML}$ of measured laminations on ∂M^+ of meridians of M^+. With this in mind, we say that a pseudo-Anosov mapping class f on $\partial M^+ = \partial M^-$ is *generic* if the following two conditions hold:

- the stable lamination is not a limit of meridians in ∂M^+ and
- the unstable lamination is not a limit of meridians in ∂M^-.

The term *generic* is justified because Kerckhoff [22] proved that the closure in $\mathcal{PML}$ of the set of meridians of M^+ and M^- have zero measure with respect to the canonical measure class of $\mathcal{PML}$. Moreover, it is not difficult to construct examples of generic pseudo-Anosov maps by hand.

In this section we consider manifolds $N_{f^n} = M^+ \cup_{f^n} M^-$ obtained by gluing M^+ and M^- by a high power of a generic pseudo-Anosov mapping class.

We should point out that the above construction is due to Feng Luo by using an idea of Kobayashi: the manifolds N_{f^n} are also interesting because the standard Heegaard splittings can have arbitrarily large distance in the curve complex. In particular, by Theorem 1.7, the manifold N_{f^n} is irreducible and atoroidal for all sufficiently large n. Hence, it should be hyperbolic. In fact, according to Perelman's Hyperbolization Theorem the manifold N_{f^n} is hyperbolic provided that its fundamental group is not finite. To check that this is the case ought to be easy... one thinks. It is not. The following is the simplest "topological" proof known to the author that N_{f^n} is not simply connected for large n.

Lemma 7.1 *For all sufficiently large n the manifold N_{f^n} is not simply connected.*

Proof By the proof of the Poincaré conjecture, also by Perelman, it suffices to prove that N_{f^n} is not the sphere $\mathbb{S}^3$. However, by Waldhausen's classification of the Heegaard splittings of the sphere, there is only one for each g. And this one is reducible and hence has distance 0 in the curve-complex. The standard Heegaard splitting of N_{f^n} has, for large n, large distance and therefore $N_{f^n} \neq \mathbb{S}^3$. □

Using the classification of the Heegaard splittings of the lens spaces one can also check that for large n the manifold N_{f^n} is not a lens space. Probably something similar can be made to rule out every other spherical manifold. So, using all these classification theorems for Heegaard splittings and the geometrization conjecture one finally obtains that the fundamental group of N_{f^n} is infinite.

The discussion that we just concluded shows how difficult is to prove anything using topological methods for the manifolds N_{f^n}. Even considering the proof of the geometrization conjucture to be topological.

In Namazi–Souto [34] we don't show that N_{f^n} is hyperbolic but we construct, by gluing known hyperbolic metrics on M^+ and M^-, explicit negatively curve metrics on N_{f^n} for large n.

Theorem 7.2 *For arbitrary $\epsilon > 0$ there is n_ϵ such that the manifold N_{f^n} admits a Riemannian metric ρ_n with all sectional curvatures pinched by $-1-\epsilon$ and $-1+\epsilon$ for all $n \geq n_\epsilon$. Moreover, the injectivity radius of the metric ρ_n is bounded from below independently of n and ϵ.*

The idea behind the proof of Theorem 7.2 can be summarized as follows: By the work of Kleineidam and the author [23] there are two hyperbolic manifolds homeomorphic to M^+ and M^- which have respectively ending laminations λ^+ and λ^-. The ends of these two hyperbolic 3–manifolds are asymptotically isometric. In particular, it is possible to construct N_{f^n}, for large n, by gluing compact pieces of both manifolds by maps very close to being an isometry. On the gluing region, a convex-combinations of both hyperbolic metrics yields a negatively curved metric.

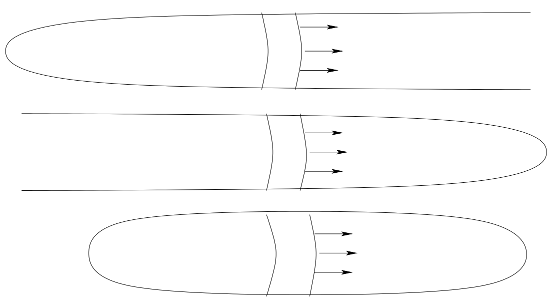

Figure 4: Proof of Theorem 7.2: The two open manifolds M^+ and M^- being glued along the gluing region to obtain the manifold N_{f^n}.

Apart from providing a negatively curved metric, the key point to all further results, is that the metrics ρ_n are *explicits*. In particular, as one sees from the sketch of the construction above, one has automatically a complete understanding of the possible geometric limits of the sequence (N_{f^n}, ρ_n). We obtain for example the following theorem.

Theorem 7.3 *Every geometric limit of the sequence $(N_{f^n})_n$ is hyperbolic and is either homeomorphic to a handlebody of genus g or to a trivial interval bundle over a closed surface of genus g.*

Theorem 7.2 shows that $\pi_1(N_{f^n})$ is infinite and word hyperbolic for all large n. In particular, it has solvable word problem and many other highly desirable algorithmic properties. However, this does not say much about the specific group $\pi_1(N_{f^n})$; it just says as much as the geometrization conjecture does. For instance, it does not explain to which extent the homomorphism $\pi_1(M^+) \to \pi_1(N_{f^n})$ is injective.

Theorem 7.4 *If $\Gamma \subset \pi_1(M^+)$ is a finitely generated subgroup of infinite index, then there is some n_Γ such that for all $n \geq n_\Gamma$ the map $\Gamma \to \pi_1(N_{f^n})$ given by the inclusion $M^+ \hookrightarrow N_{f^n}$ is injective.*

The proof of Theorem 7.4 is based on the fact that M^+ can be canonically identified with one of the possible geometric limits of the sequence (N_{f^n}, ρ_n) and that, by Corollary 2.3, every infinite degree covering of this geometric limit is convex-cocompact; compare with Lemma 2.4.

Another consequence of Theorem 7.2 and Theorem 7.3 is that, again for large n, the fundamental group of the manifold N_{f^n} has rank g. The proof of this fact is almost word-by-word the same as the proof of Theorem 6.2.

Theorem 7.5 *There is n_f with* $\mathrm{rank}(\pi_1(N_{f^n})) = g$ *for all $n \geq n_f$. Moreover, every minimal generating set of $\pi_1(N_{f^n})$ is Nielsen equivalent to a standard generating set for all sufficiently large n. In particular $\pi_1(N_{f^n})$ has at most 2 Nielsen equivalence classes of minimal generating sets.*

In [34] we also use similar arguments as in the proof of Theorem 6.2 and Theorem 7.5 to prove for example that for sufficiently large n, every set of at most $2g-2$ elements in $\pi_1(N_{f^n})$ which generate a proper subgroup does in fact generate a free subgroup. This bound is sharp. Finally, we use the same arguments as outlined in the proof of Theorem 4.1 to prove that N_{f^n} has, again for large n, Heegaard genus g and that the standard Heegaard splitting is, up to isotopy, the unique minimal genus Heegaard splitting.

Theorem 7.6 *There is n_f such that for all $n \geq n_f$ the following holds: every minimal genus Heegaard splitting of N_{f^n} is isotopic to the standard one.*

Observe that the bound on the Heegaard genus follows also from Theorem 7.5. Theorem 7.6 is also due to Scharlemann–Tomova [50] but their methods are completely different.

As mentioned above, Theorem 7.2 and Theorem 7.3 are the key to all the subsequent topological results. While Theorem 7.2 is a consequence of the positive answer to Thurston's geometrization conjecture, the author would like to remark that even assuming the mere existence of negatively curved metric, or even hyperbolic, on N_{f^n} it is not obvious how to derive any of the topological applications outlined above. Recall for example that it was not obvious how to prove, even using the geometrization conjecture, that N_{f^n} is for large n not finitely covered by $\mathbb{S}^3$. It is the control on the geometry of the manifolds N_{f^n} provided by Theorem 7.3 that opens the door to all the subsequent results.

8 Bounding the volume in terms of combinatorial data

Until now we have mainly studied quite particular classes of manifolds; in this section our point of view changes. Here we discuss results due to Brock [9], and Brock and the author, showing that it is possible to give linear upper and lower bounds for the volume of a hyperbolic 3–manifold in terms of combinatorial distances.

Given a closed surface S let $\mathcal{P}(S)$ be its *pants-complex*, ie the graph whose vertices are isotopy classes of pants decomposition and where two pants decompositions P and P' are at distance one if they differ by an elementary move. Here, two pants decompositions $P = \{\gamma_1, \dots, \gamma_{3g-3}\}$ and $P' = \{\gamma'_1, \dots, \gamma'_{3g-3}\}$ differ by an elementary move if there is some j such that

- $\gamma_i = \gamma'_i$ for all $i \neq j$ and
- such that γ_j and γ'_j are different curves with the minimal possible intersection number in $S \setminus \{\gamma_1, \dots, \gamma_{j-1}, \gamma_{j+1}, \dots, \gamma_{3g-3}\}$.

In more mundane terms, in order to change P by an elementary move, one keeps all components of P but one fixed and this component is changed in the simplest possible way.

The pants complex $\mathcal{P}(S)$ is known to be connected. In particular, declaring every edge to have unit length one obtains an interior distance invariant under the natural action of the mapping class group on the pants complex. From our point of view, the pants complex is important because of the following result due to Brock [9] relating volumes of mapping tori to distances in the pants complex.

Theorem 8.1 (Brock [9]) *For every g there is a constant $L_g > 0$ with*

$$L_g^{-1} l_{\mathcal{P}(\Sigma_g)}(f) \leq \mathrm{vol}(M_f) \leq L_g l_{\mathcal{P}(\Sigma_g)}(f)$$

for every pseudo-Anosov mapping class f of the surface Σ_g of genus g. Here M_f is the mapping torus of f and

$$l_{\mathcal{P}(\Sigma_g)}(f) = \inf\{d_{\mathcal{P}(\Sigma_g)}(P, f(P)) | P \in \mathcal{P}(\Sigma_g)\}$$

is the translation length of f in the pants complex of Σ_g.

It is a beautiful observation due to Brock [9] that the pants complex is quasi-isometric to the Teichmüller metric when endowed with the Weil-Petersson metric. In particular, Theorem 8.1 is stated in [9] in terms of translation distances in the Teichmüller space.

In the setting of Theorem 8.1, getting upper bounds for the volume is not difficult. The idea is that if $(P_1, P_2, \ldots, P_d)$ is a path in the pants complex with $P_d = f(P_1)$ then one obtains an ideal triangulation of the mapping torus M_f with some controlled number of simplices. In particular, the translation length bounds the simplicial volume of M_f, and hence bounds the volume itself. Agol [2, Cor 2.4] obtained sharp upper bounds for the volume of M_f in terms of $l_{\mathcal{P}(\Sigma_g)}(f)$ from which it follows for example that

$$\mathrm{vol}(M_f) \leq 2V_{oct} l_{\mathcal{P}(\Sigma_g)}(f)$$

where V_{oct} is the volume of a regular ideal octahedron.

Proof The bulk of the proof of Theorem 8.1 is to show that the volume can be bounded from below in terms of the translation distance in the pants-complex. In [9], this lower bound is derived using deep difficult results due to Masur–Minsky [26; 27] about the geometry of the curve-complex and exploiting the relation between the curve-complex and the geometry of hyperbolic manifolds discovered by Minsky. However, a simpler proof of the lower bound was found by Brock and the author.

The first observation is that the volume of M_f is roughly the same as the volume of its thick part. The idea of the proof is to decompose the thick part of M_f in a collection of disjoint pieces, called pockets, and then use geometric limit arguments to estimate the volume of each single pocket.

Consider for the time being only mapping tori M_f with injectivity radius at least ϵ. This assumption has the following consequence:

(*) If p_i is a sequence of points in different mapping tori M_{f_i} with $\mathrm{inj}(M_{f_i}) \geq \epsilon$ for all i then, up to passing to a subsequence, the sequence of pointed manifolds (M_{f_i}, p_i) converges geometrically to a pointed hyperbolic manifold (M_∞, p_∞) homeomorphic to $\Sigma_g \times \mathbb{R}$ and with at least injectivity radius ϵ.

In some way (*) gives full geometric control. For instance, it is known that every point p in a mapping torus M_f is contained in a surface S_p homotopic to the fiber of M_f and with curvature ≤ -1. Thurston's bounded diameter lemma asserts that

$$\mathrm{diam}(S_p) \leq \frac{4g-4}{\epsilon^2}$$

and hence it follows directly from (*) that there is a constant c_1 depending of ϵ and g such that if M_f has at least injectivity radius ϵ then for all $p \in M_f$ the surface S_p is in fact homotopic to an embedded surface S'_p by a homotopy whose tracks have at most length c_1 (see footnote[1]). For every p we choose a minimal length pants decomposition P_p in the surface S_p.

Choose D to be a sufficiently large constant, depending on g and ϵ. Given a mapping torus M_f with injectivity radius $\mathrm{inj}(M_f) \geq \epsilon$ let $\{p_1, \dots, p_k\}$ be a maximal collection of points in M_f which are at at least distance D from each other. Observe that for every i there is $j \neq i$ such that p_i and p_j are at at most distance $2D$. For each $i = 1, \dots, k$ we consider the negatively curved surface S_{p_i} passing trough p_i and the associated embedded surface S'_{p_i}. Then, since D is large and the diameter of the surfaces S_{p_i} is bounded we have that the surfaces S_{p_i} and S_{p_j} are disjoint for all $i \neq j$; the same also holds for the associated embedded surfaces S'_{p_i} and S'_{p_j} by the bound on the length of the tracks of the homotopies. In particular, the surfaces S'_{p_i} and S'_{p_j} are disjoint, embedded and homotopic to the fiber. We obtain from Waldhausen's cobordism theorem that they bound in M_f a product region homeomorphic to $\Sigma_g \times [0,1]$. In particular, the set $\{p_1, \dots, p_k\}$ is cyclically ordered, say as $[p_1, \dots, p_k]$.

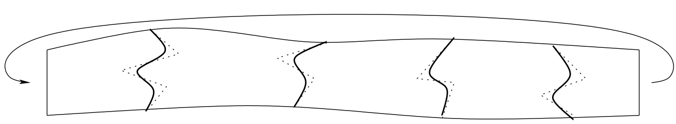

Figure 5: Proof of Theorem 8.1 in the thick case: The dotted lines are simplicial hyperbolic surfaces and the thick lines are close-by embedded surfaces dividing the manifold M_f into roughly uniform pieces.

In other words, under the assumption that $\mathrm{inj}(M_f) \geq \epsilon$ and that $\mathrm{diam}(M_g) \geq 10D$, we have that a decomposition of the mapping torus M_f into cyclically ordered product

[1] In order to see that the constant c_1 exists we can proceed as follows: given a surface S in a 3–manifold consider the infimum of all possible lengths of tracks of homotopies between S and an embedded surface; this infimum may be ∞ if such a homotopy does not exist. Then it is easy to see that this function is semi continuous in the geometric topology and hence bounded on compact sets; (*) is a compactness statement.

regions $[U_1, \dots, U_k]$ with

$$\partial U_1 = S'_{p_1} \cup S'_{p_2}, \cdots, \partial U_{k-1} = S'_{p_{k-1}} \cup S'_{p_k}, \partial U_k = S'_{p_k} \cup S'_{p_1}$$

and such that for all $i = 1, \dots, k$ if $p, q \in \partial U_i$ are in different components then

$$D - 2\frac{4g-4}{\epsilon^2} \le d_{M_f}(p,q) \le 2D + 2\frac{4g-4}{\epsilon^2}.$$

These bounds, together with (*) implies that there are constants V and L depending only on g and ϵ such that for every one of the product regions U_i we have

$$\mathrm{vol}(U_i) \ge V, \quad d_{\mathcal{P}(\Sigma_g)}(P_{p_i}, P_{p_{i+1}}) \le L.$$

From the first inequality we obtain that there are at most $\frac{1}{V}\,\mathrm{vol}(M_f)$ product regions. From the second we deduce that

$$d_{\mathcal{P}(\Sigma_g)}(P_{p_1}, f(P_{p_1})) \le \frac{L}{V}\,\mathrm{vol}(M_f).$$

This concludes the proof of Theorem 8.1 under the additional assumption that the mapping torus in question has injectivity radius $\mathrm{inj}(M_f) \ge \epsilon$.

In general the idea is to decompose the thick part of M_f into product regions. The key point is the following well-known result due to Otal [35].

Theorem 8.2 (Otal's unknotting theorem) *For every g there is a constant ϵ_g such that for every $f\colon \Sigma_g \to \Sigma_g$ pseudo-Anosov the following holds: There is a collection of disjoint surfaces $S_1, \dots, S_k$ parallel to $\Sigma_g \times \{0\}$ with the property that every primitive geodesic in M_f shorter than ϵ_g is contained in $\cup S_i$.*

In the sequel denote by Γ the collection of primitive geodesics in M_f that are shorter than ϵ_g. By Thurston's Hyperbolization Theorem the manifold $M_f^* = M_f \setminus \Gamma$ admits a complete finite volume hyperbolic metric. Moreover, it follows from the deformation theory of cone-manifolds due to Hodgson–Kerckhoff [20] that up to assuming that ϵ_g is smaller than some other universal constant the ratio between the volumes of M_f and M_f^* is close to 1, and

(**) every geodesic in M_f^* has at least length $\frac{\epsilon_g}{2}$.

Theorem 3.1 and Otal's theorem imply that M_f^* can be cut open long minimal surfaces into disjoint product regions $V_i = F_i \times (0,1)$ where F_i is a subsurface of Σ_g; the V_i's are the so-called pockets. The volume of M_f^* is the sum of the volumes of the

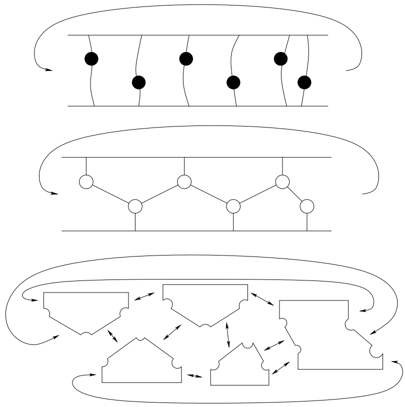

Figure 6: Proof of Theorem 8.1: The first picture represents the manifold M_f where the black dots are short geodesics and the lines are the surfaces provided by Otal's unknotting theorem. In the second picture one has the manifold M_f^* and the straight lines are obtained from Otal's surfaces by (1) isotoping as much as possible into the cusps, (2) removing parallel components, and (3) pulling tight. In the third image one sees the associated pocket decomposition.

regions V_i. Moreover, choosing for each i minimal length pants decompositions P_i^- and P_i^+ of the boundary of V_i it is not difficult to see that

$$l_{\mathcal{P}(\Sigma_g)} \leq \sum_{V_i} d_{\mathcal{P}(F_i)}(P_i^-, P_i^+).$$

In particular, it suffices to prove that the volume of each one of the product regions V_i bounds the distance $d_{\mathcal{P}(F_i)}(P_i^-, P_i^+)$ from above. However, since in the setting we have (**) we are again in the situation that V_i is thick: the same arguments used above yield the desired result in this case.

This concludes the sketch of the proof of Theorem 8.1. □

The following is an analogous of Otal's theorem in the setting of Heegaard splittings.

Theorem 8.3 (Souto) *For every g there is a constant ϵ_g such that for every strongly irreducible genus g Heegaard surface S in a closed hyperbolic 3–manifold M the following holds: There is a collection of disjoint surfaces $S_1, \dots, S_k$ parallel to S with the property that every primitive geodesic shorter than ϵ_g in M is contained in $\cup S_i$.*

Proof The idea of the proof of Theorem 8.3 is the following. By Pitts–Rubinstein's Theorem 3.6, we can associate to the surface S a minimal surface F. For the sake of simplicity assume that F is isotopic to S and contained in the thick part of M. Cutting M along S we obtain two handlebodies; let U be the metric completion of one of these handlebodies and $\widetilde{U}$ its universal cover. Then, by a theorem of Alexander–Berg–Bishop [4], $\widetilde{U}$ is a $\mathrm{CAT}(-1)$-space and hence, from a synthetic point of view, it behaves very much like hyperbolic space. In particular, one can use word-by-word the proof in (Souto [58]) that short curves in hyperbolic handlebodies are parallel to the boundary and obtain that short curves in the handlebody U are parallel to the boundary $\partial U = F \simeq S$. □

Theorem 8.3 opens the door to analogous results to Theorem 8.1 for Heegaard splittings. In some sense, the main difficulty is to decide what has to replace the translation length $l_{\mathcal{P}}(f)$.

Definition *The* handlebody set $\mathcal{H}(H)$ *of a handlebody H is the subset of the pants complex $\mathcal{P}(\partial H)$ of its boundary consisting of those pants decompositions P with the following property: There is a collection $\mathcal{D}$ of properly embedded disks in H with boundary in P and such that $H \setminus \mathcal{D}$ is homeomorphic to a collection of solid tori.*

Given a Heegaard splitting $M = U \cup V$ of a 3–manifold we can now define its distance in the pants complex of the Heegaard surface S as follows:

$$\delta_{\mathcal{P}}(U, V) = \min\{d_{\mathcal{P}(S)}(P_U, P_V) | P_U \in \mathcal{H}(U), P_V \in \mathcal{H}(P_V)\}.$$

Theorem 8.3 and similar geometric limit arguments as the ones used in the proof of Theorem 8.1 yield the following theorem.

Theorem 8.4 (Brock-Souto) *For every g there is a constant $L_g > 0$ with*

$$L_g^{-1}\delta_{\mathcal{P}}(U, V) \leq \mathrm{vol}(M) \leq L_g \delta_{\mathcal{P}}(U, V)$$

for every genus g strongly irreducible Heegaard spitting $M = U \cup V$ of a hyperbolic 3–manifold M.

9 Hyperbolic manifolds with given rank

In the last section we studied the relation between the geometry of hyperbolic 3–manifolds and the combinatorics of Heegaard splittings. In this section we sketch some results about what can be said about the geometry of a hyperbolic 3–manifold with given rank of the fundamental group. We will be mainly interested in the following two conjectures which assert that (A) the radius of the largest embedded ball in a closed hyperbolic 3–manifold and (B) its Heegaard genus are bounded from above in terms of the rank of the fundamental group.

Conjecture (A) (McMullen) *For all k there is some R with*

$$\mathrm{inj}(M,x) \leq R$$

for all x in a closed hyperbolic 3–manifold M with $\mathrm{rank}(\pi_1(M)) = k$.

Observe that $\max_{x\in M} \mathrm{inj}(M,x)$ is the radius of the largest embedded ball in M. McMullen's conjecture admits a suitable generalization to the setting of infinite volume hyperbolic 3–manifolds. In fact, it was in this setting in which McMullen's conjecture was first formulated because of its implications to holomorphic dynamics. However, it follows from the work of Gero Kleineidam and the author of this note that the general case can be reduced to the closed case. Before going further we should recall the injectivity radius itself, ie the minimum of the injectivity radius over all points of the manifold, is bounded from above in terms of the rank by White's Theorem 5.4.

Conjecture (B) (Waldhausen) *For every k there is g such that every closed 3–manifold with* $\mathrm{rank}(\pi_1(M)) = k$ *has Heegaard genus $g(M) \leq g$.*

Waldhausen asked in fact if for every 3–manifold $\mathrm{rank}(\pi_1(M)) = g(M)$. However, the stronger form of the Conjecture B was answered in the negative by Boileau–Zieschang [7] who presented an example of a Seifert-fibered 3–manifold M with $g(M) = 3$ and $\mathrm{rank}(\pi_1(M)) = 2$. Examples of 3–manifolds with $g(M) = 4k$ and $\mathrm{rank}(\pi_1(M)) = 3k$ were constructed by Schultens-Weidmann [54]. Recently Abert–Nikolov [1] have announced that there are also hyperbolic 3–manifolds with larger Heegaard genus than rank.

Conjecture A and Conjecture B are related by a result of Bachmann–Cooper–White [6] who proved the following theorem.

Theorem 9.1 (Bachmann–Cooper–White) *Suppose that M is a closed, orientable, connected Riemannian 3–manifold with all sectional curvatures less than or equal to -1 and with Heegaard genus $g(M)$. Then*

$$g(M) \geq \frac{\cosh(r)+1}{2}$$

where $r = \max_{x\in M} \mathrm{inj}_x(M)$.

The first positive result towards Conjecture A and Conjecture B is due to Agol, who proved the next theorem.

Theorem 9.2 (Agol) *For every $\epsilon > 0$ there is $V > 0$ such that every hyperbolic 3–manifold with injectivity radius $\mathrm{inj}(M) > \epsilon$, volume $\mathrm{vol}(M) > V$ and $\mathrm{rank}(\pi_1(M)) = 2$ has Heegaard genus $g(M) = 2$. In particular, the injectivity radius at every point is bounded from above by $\mathrm{arccosh}(3) - 1$.*

Proof Agol's theorem is unfortunately not available in print. However, the idea of the proof is not so difficult to explain. Assume that (M_i) is a sequence of hyperbolic 3–manifolds with $\mathrm{inj}(M_i) \geq \epsilon$, $\mathrm{rank}(\pi_1(M_i)) = 2$ and $\mathrm{vol}(M_i) \to \infty$. A special case of Proposition 5.5 implies that there is for all i a minimal length carrier graph $f_i\colon X_i \to M_i$ with length $l_{f_i\colon X_i \to M_i}(X_i)$ bounded from above by some universal constant. Passing to a subsequence we may assume that the manifold M_i converge geometrically to a hyperbolic manifold M_∞, the graphs X_i to a graph X_∞ and the maps f_i to a map $f_\infty\colon X_\infty \to M_\infty$.

We claim that M_∞ is homeomorphic to a handlebody of genus 2. In fact, in order to see that this is the case, it suffices to observe that it has infinite volume that that its fundamental group is generated by 2 elements. In other words, it suffices to prove that $f_\infty\colon X_\infty \to M_\infty$ is a carrier graph. In order to do so, we consider the covering of $M'_\infty \to M_\infty$ determined by the image of $\pi_1(X_\infty)$. The proof of the tameness conjecture by Agol [3] and Calegari–Gabai [11] implies that M'_∞ is homeomorphic to a handlebody. Moreover, M'_∞ is not convex-cocompact because otherwise the homomorphism $(f_i)_*\colon \pi_1(X_i) \to \pi_1(M_i)$ would be injective, and hence $\pi_1(M_i)$ free, for large i. Since M'_∞ is not convex-cocompact, Thurston and Canary's Covering Theorem [13] implies that the covering $M'_\infty \to M_\infty$ is finite-to-one; in fact this covering is trivial because M'_∞ is a handlebody of genus 2 and a surface of genus 2 does not cover any other surface. This proves that M_∞ itself is a handlebody.

Choose now $C \subset M_\infty$ a compact core, ie a compact submanifold of M_∞ such that $M_\infty \setminus C$ is homeomorphic to a product. Pushing back the core C to the approximating manifolds M_i we obtain in each M_i a handlebody C_i. In order to conclude the proof of Theorem 9.2 it suffices to show that its complement is a handlebody as well. Furthermore, it suffices to show that ∂C_i is compressible in $M_i \setminus C_i$. However, if this is not the case it is possible to deduce from the Covering Theorem that there is a non-trivial homotopy from ∂C_i to itself supported in $M_i \setminus C_i$. In particular, $M_i \setminus C_i$ is homeomorphic to a twisted interval bundle and hence there is a non-trivial homology class supported in the complement of C_i. This contradicts the assumption that $\pi_1(X_i)$, and hence $\pi_1(C_i)$, surjects onto $\pi_1(M_i)$. Hence ∂C_i is, for large i, compressible in $M_i \setminus C_i$ and the latter is a handlebody. This concludes the sketch of the proof of Theorem 9.2. □

As we saw during the sketch of the proof the case of $\operatorname{rank}(\pi_1(M)) = 2$ is quite particular because of two reasons.

- If M is a thick hyperbolic 3–manifold with $\operatorname{rank}(\pi_1(M)) = 2$ then there is a carrier graph whose length is uniformly bounded from above.
- If M is a thick non-compact complete hyperbolic 3–manifold whose fundamental group is generated by two elements then M is a handlebody.

This two facts were heavily used in the proof of Theorem 9.2. If the rank of $\pi_1(M)$ is higher than two, then both statements fail, However, using a similar strategy as in the proof of Theorem 9.2, together with the facts about carrier graphs explained in Section 5 it is possible to prove the following claim: Given a sequence (M_i) of closed hyperbolic 3–manifolds with $\operatorname{inj}(M_i) > \epsilon$ and $\operatorname{rank}(\pi_1(M)) = 3$ there is a compact, atoroidal and irreducible 3–manifold N and a subsequence (M_{i_j}) such that for all j the manifold M_{i_j} contains a compact submanifold homeomorphic to N whose complement is a union of handlebodies. In particular, one obtains the following structure theorem of those manifolds whose fundamental group has rank 3.

Theorem 9.3 (Souto [56]) *For all positive ϵ there is a finite collection $N_1, \dots, N_k$ of compact, atoroidal and irreducible 3–manifolds such that every closed hyperbolic 3–manifold M with $\operatorname{inj}(M) > \epsilon$ and $\operatorname{rank}(\pi_1(M)) = 3$ contains a compact submanifold N homeomorphic N_i for some i such that $M \setminus N$ is a union of handlebodies.*

From Theorem 9.3 the next theorem follows.

Theorem 9.4 *For every $\epsilon > 0$ there is some g such that every closed hyperbolic 3–manifold M with $\operatorname{inj}(M) > \epsilon$ and $\operatorname{rank}(\pi_1(M)) = 3$ has Heegaard genus $g(M) \le g$.*

Proof Given ϵ positive let $N_1, \dots, N_k$ be the finite collection of manifolds provided by Theorem 9.3 and let

$$g = \max\{g(N_1), \dots, g(N_k)\}.$$

If M is a closed hyperbolic 3–manifold with $\operatorname{inj}(M) \ge \epsilon$ and $\operatorname{rank}(\pi_1(M)) = 3$ then, the Theorem 9.3, the manifold M contains a submanifold N homeomorphic to some N_i such that $M \setminus N$ is a collection of handlebodies. In particular, every Heegaard splitting of N extends to a Heegaard splitting of M; hence $g(M) \le g(N) = g(N_i) \le g$. □

The next theorem follows from Theorem 9.1.

Theorem 9.5 *For all ϵ positive there is R such that for every closed hyperbolic 3-manifold M with $\operatorname{inj}(M) \geq \epsilon$ and $\operatorname{rank}(\pi_1(M)) = 3$ and for every $x \in M$ one has $\operatorname{inj}(M, x) \leq R$.*

In some sense the claim of the Theorem 9.3 may seem redundant once one has Theorem 9.4. However one gets from the proof of Theorem 9.3 some additional information about the geometry of the manifolds. For example, one can use this additional structure together with Pitts–Rubinstein's Theorem 3.6 to prove the next theorem.

Theorem 9.6 *For all ϵ and g there is a number k such that every ϵ-thick hyperbolic 3–manifold M with $\operatorname{rank}(\pi_1(M)) = 3$ admits at most k isotopy classes of Heegaard surfaces of genus g.*

Theorem 9.7 *For all ϵ there is d such that every closed ϵ–thick hyperbolic 3-manifold M which admits a genus 4 Heegaard spitting with at least distance d in the curve complex has $\operatorname{rank}(\pi_1(M)) = 4$.*

10 Open questions

In this section we outline some open questions and problems.

Problem 1 Construct models for hyperbolic 3–manifolds in terms of combinatorial data given by a Heegaard splitting.

A satisfactory answer of Problem 1 would be given by a machine which, when fed with the combinatorial data of a Heegaard splitting of genus g of a 3–manifold M, yields a metric ρ on M such that whenever M admits a hyperbolic metric ρ_0, then (M, ρ) and (M, ρ_0) are L_g-bi-Lipschitz where L_g depends only on g. If such a machine exists, then the obtained metric ρ captures all the coarse information about M.

Partial results have been obtained towards an answer of Problem 1. Let (M, ρ_0) be a hyperbolic manifold and $M = U \cup V$ a Heegaard splitting of genus g. Then it is possible to construct out of combinatorial data determined by the Heegaard splitting $M = U \cup V$ a metric ρ on M such that (M, ρ) and (M, ρ_0) are $L_{g,\operatorname{inj}(M,\rho_0)}$-bi-Lipschitz. In other words, the constant in question depends on the genus of the splitting and the injectivitiy radius of the hyperbolic metric. On the other hand, it is also possible to give lower bounds on the injectivity radius of the hyperbolic metric in terms of combinatorial data of the Heegaard splitting of M. If γ is a simple closed curve on the Heegaard surface such that (U, γ) and (V, γ) has incompressible and acylindrical pared boundary, then it is also possible to determine if the homotopy class of γ is non-trivial and short in (M, ρ_0). All these results, due to Jeff Brock, Yair Minsky, Hossein Namazi and the author, are steps towards a positive answer to Problem 1.

Answering Problem 1 also opens the door to obtain partial proofs of the geometrization conjecture. The idea being that if under the assumption of the existence of a hyperbolic metric one is able to construct metrics close to the hyperbolic metric, then one can perhaps be smarter and use the machine developed to answer Problem 1 to construct hyperbolic, or at least negatively curved, metrics. In some sense, this is the idea behind the results presented in Section 7; compare with Theorem 7.2. The construction used in the proof of Section 7 is a special case of a more general construction due to Hossein Namazi [33]. Recently, Jeff Brock, Yair Minsky, Hossein Namazi and the author have proved that for every g there is a constant D_g such that every closed 3–manifold which admits a genus g Heegaard splitting with at least distance D_g in the curve complex admits also a negatively curved metric.

Problem 2 Show that for every g there are at most finitely many counter-examples to the geometrization conjecture which admits a genus g Heegaard splitting.

After the proof of the geometrization conjecture by Perelman, Problem 2 may seem redundant. However, a satisfactory answer to Problem 2 would consist of presenting a combinatorial, Ricci-flow-free, construction of the hyperbolic metric in question. In fact, after answering Problem 2 one could try to prove that for example there are no counter-examples to the geometrization conjecture which admit a genus 10 Heegaard splitting. Apart from the difficulty of checking if a manifold is hyperbolic or not there is the conceptual problem that unless the constants involved in the answers of Problem 1 and Problem 2 are computable then there can be no a priori bounds on the number of possible exceptions to the geometrization conjecture. So far, all similar constants are obtained using geometric limits and compactness and hence they are not computable.

Problem 3 Obtain explicit constants.

We turn now to questions related to the rank of the fundamental group. If Σ_g is a closed surface of genus g then it is well-known that $\mathrm{rank}(\pi_1(\Sigma_g)) = 2g$. In [64], Zieschang proved that in fact $\pi_1(\Sigma_g)$ has a single Nielsen equivalence class of minimal generating sets. Zieschang's proof is quite combinatorial and difficult to read.

Problem 4 Give a geometric proof of Zieschang's result.

A geometric proof of Zieschang's result would also shed some light on the fact that there are 2–dimensional hyperbolic orbifolds whose fundamental groups admit two different Nielsen equivalence classes of minimal generating sets. Compare with Weidmann [61].

In general, if G is a finitely generated group then every generating set $(g_1, \dots, g_r)$ with r elements can be *stabilized* to a generating set $(g_1, \dots, g_r, e_G)$ with $r+1$ elements by adding the identity to it. It is easy to see that any two generating sets with r elements are Nielsen-equivalent after r stabilizations. Clearly, if two generating sets with the same cardinality are not Nielsen equivalent, then one needs at least a stabilization so that they can be connected by a sequence of Nielsen moves. The author suspects that the two standard generating sets of the manifolds considered in Section 7 are not Nielsen equivalent, and in fact that one need g stabilizations to make them Nielsen equivalent; here g is the genus of the involved handlebodies.

Problem 5 If N_{f^n} is one of the manifolds considered in Section 7 and n is very large, determine how many stabilizations does one need so that the standard Nielsen equivalence classes of generating sets coincide.

Remark As remarked by Richard Weidmann, Problem 5 can be easily solved if the genus is equal to 2. His argument is very special to this situation.

The relation between Heegaard splittings and geometry is much better understood than the relation between rank of the fundamental group and geometry. In some sense, one can take any result relating geometry of hyperbolic manifolds and Heegaard splittings and try to prove the analogous statement for the rank of the fundamental group. For example the following question.

Problem 6 Prove that for every g there is some D_g such that whenever $\phi \in \mathrm{Map}(\Sigma_g)$ is a pseudo-Anosov mapping class with at least translation length D_g in the curve complex, then the fundamental group of the mapping torus M_ϕ has rank $2g+1$ and a single Nielsen equivalence class of generating sets.

More ambitiously, one could remark that by Mostow's rIgidity Theorem a hyperbolic 3–manifold is determined by its fundamental group. In particular, the following questions make sense.

Problem 7 Given a presentation of the fundamental group of a 3–manifold estimate the volume.

Problem 8 Construct models out of the algebraic data provided by a presentation.

Most probably, satisfactory answers to these questions would imply answers to Conjecture A and Conjecture B from Section 9.

Another question related to the relation between Heegaard genus and rank of the fundamental group is the following. As mentioned above, Abert and Nikolov [1] have announced that there are hyperbolic 3–manifolds with larger Heegaard genus than rank. However their proof is not constructive. Essentially they show that there is no equality for all but finitely many elements in a certain sequence of hyperbolic 3–manifolds. It would be interesting to have concrete examples.

Problem 9 Construct concrete examples of hyperbolic 3–manifolds with larger rank than Heegaard genus.

A possible other interesting line of research would be to use minimal surfaces to obtain geometric proofs about Heegaard splittings of non-hyperbolic 3–manifolds. For example, using that the 3–sphere admits a collapse, it is possible to obtain a proof of Waldhausen's classification of the Heegaard splittings of the sphere. This strategy can be probably applied to other Seifert manifolds.

Problem 10 Give a geometric proof of the classification of Heegaard splittings of Seifert manifolds.

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Received: 24 October 2006 Revised: 2 May 2007