Problems

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These are problems on Heegaard splittings, that were raised at the Workshop, listed according to their contributors: David Bachman, Mario Eudave-Muñoz, John Hempel, Tao Li, Yair Minsky, Yoav Moriah and Richard Weidmann. In [34] Hyam Rubinstein gives a personal collection of problems on 3–manifolds.

57M27, 57M25; 57M07

1 David Bachman

We analyze what is known, and what is not known, about the following question:

Question 1.1 *Which manifolds have infinitely many Heegaard splittings of the same genus, and how are the splittings constructed?*

The only known examples of 3–manifolds that admit infinitely many Heegaard splittings of the same genus are given in Sakuma [35], Morimoto and Sakuma [28] and Bachman and Derby-Talbot [3]. By Li's proof of the Waldhausen conjecture [19], any such manifold must have an essential torus T. In the presence of such a torus, it is easy to see how an infinite number of Heegaard splittings of the same genus might arise. Simply take any splitting that intersects T, and Dehn twist the splitting about T. But is this the only way such an infinite collection might arise? What can be said about the manifold if this construction does *not* produce an infinite collection of non-isotopic splittings?

We analyze what is known when the torus is separating. (Some of the results described below hold in the non-separating case as well.) Suppose then T is an essential torus which separates a closed, orientable, irreducible 3–manifold M into X and Y. Then we may think of M as being constructed from X and Y by gluing by some homeomorphism, $\phi\colon \partial X \to \partial Y$.

Fix separate triangulations of X and Y (these do not have to agree in any way on T). By a result of Jaco and Sedgwick [13] the sets Δ_X and Δ_Y of slopes on ∂X and ∂Y that bound normal or almost normal surfaces in X and Y are finite (see Bachman [2]

for a further discussion of the almost normal case). We may thus define the *distance* $d(T)$ *of* T to be the distance between the sets $\phi(\Delta_X)$ and Δ_Y, as measured by the path metric in the Farey graph of ∂Y.

Theorem 1.1   (Bachman, Schleimer, Sedgwick [4])   *If $d(T) \geq 2$ then every Heegaard splitting of M is an amalgamation of Heegaard splittings of X and Y.*

The relevance of this theorem is that Dehn twisting an amalgamation about the torus T will not produce non-isotopic Heegaard splittings. Hence, when $d(T) \geq 2$ the manifold M can only admit an infinite collection of non-isotopic Heegaard splittings if X or Y does. The converse, however, is more subtle. That is, if we know X or Y has infinitely many non-isotopic Heegaard splittings of the same genus, then does it follow that M does as well? We conjecture the following:

Conjecture 1.2   *If $d(T) \geq 3$ and X or Y has infinitely many Heegaard splittings of the same genus then M does as well.*

When $d(T) = 0$ or 1, there is the possibility that there is a strongly irreducible Heegaard splitting of M. Let H denote such a splitting surface. By a classic result of Kobayashi, H can be isotoped to meet T in a non-empty collection of loops that are essential on both surfaces.

One way Dehn twisting H about T can fail to produce a non-isotopic Heegaard splitting is if $H \cap X$ is a fiber of a fibration of X. Then the effect of the Dehn twist can be "undone" by pushing $H \cap X$ around the fibration. This is precisely what happens when T is a separating vertical torus in a Seifert Fibered space. (For a complete resolution of Question 1.1 in the case of Seifert Fibered spaces, see [3].) A second thing that may happen is that $H \cap X$ is the union of two pages of an open book decomposition of X. Then the effect of the Dehn twist can be undone by spinning $H \cup X$ about the open book decomposition. We conjecture that these are the only ways that Dehn twisting H about T can fail to produce non-isotopic Heegaard splittings:

Conjecture 1.3   *If M admits an essential torus T, separating M into X and Y, and a strongly irreducible Heegaard splitting H, then either Dehn twisting H about T produces an infinite collection of non-isotopic Heegaard splittings, or H can be isotoped so that $H \cap X$ or $H \cap Y$ is either a fiber of a fibration of X or Y or two pages of an open book decomposition of X or Y.*

2 Mario Eudave-Muñoz

Let F be a closed surface of genus 1 standardly embedded in S^3, that is, it bounds a solid torus on each of its sides. We say that a knot K has a $(1, b)$–*presentation* or that it is in a $(1, b)$–*position*, if K has been isotoped to intersect F transversely in $2b$ points that divide K into $2b$ arcs, so that the b arcs in each side can be isotoped, keeping the endpoints fixed, to disjoint arcs on F. The *genus–1–bridge number* of K, $b_1(K)$, is the smallest integer n for which K has a $(1, n)$–presentation. We say that a knot is a $(1, n)$–*knot* if $b_1(K) \leq n$. If K is a $(1, 1)$–knot, then it is easy to see that K has tunnel number one. On the other hand, if K has tunnel number one, it seems to be very difficult to determine $b_1(K)$. It has been of interest to find tunnel number one knots with large genus–1–bridge number, see for example Kobayashi and Rieck [16].

Moriah and Rubinstein [24] showed the existence of tunnel number one knots K with $b_1(K) \geq 2$. Morimoto, Sakuma and Yokota also showed this, and gave explicit examples of knots K, with tunnel number one and $b_1(K) = 2$ [29]. It was shown by Eudave-Muñoz [7], that many of the tunnel number one knots K constructed in [5] are not $(1, 1)$–knots; this was extended in [8], where it is shown that many such knots are not $(1, 2)$–knots. We remark that such knots can be explicitly described, for example the knot K shown in Figure 13 of [8] satisfies $3 \leq b_1(K) \leq 4$. Combining results of [6] and [9], it is also possible to give explicit examples of knots K, with tunnel number one and $b_1(K) = 2$, for example the knot shown in Figure 4 of [6]. Valdez-Sánchez and Ramírez-Losada have also shown explicit examples of tunnel number one knots K with $b_1(K) = 2$ (personal communication). These knots bound punctured Klein bottles but are not contained in the $(1, 1)$–knots bounding Klein bottles determined by the same authors [33].

Johnson and Thompson [14], and independently Minsky, Moriah and Schleimer [22] have shown that for any given n, there exist tunnel number one knots which are not $(1, n)$–knots. The two papers use similar techniques, prove the existence of such knots, but do not give explicit examples. For a tunnel number one knot, let Σ be the Heegaard splitting of the knot exterior determined by the unknotting tunnel, and let $d(\Sigma)$ denote the Hempel distance of the splitting [12]. In [14] and [22] the existence of tunnel number one knots with large Hempel distance is shown, and then results of Scharlemann and Tomova [37], [40] are used to ensure the knots have large genus–1–bridge number.

We propose the following problems:

Problem 2.1 *For a given integer n, give explicit examples of tunnel number one knots K with $b_1(K) \geq n$.*

Problem 2.2 *For a given integer N, give explicit examples of tunnel number one knots with $d(\Sigma) \geq N$.*

Problem 2.3 *For a given integer n, give explicit examples of tunnel number one knots K with $b_1(K) \geq n$, but with bounded Hempel distance, say with $d(\Sigma) = 2$.*

3   John Hempel

Questions on the curve complex

Let S be a surface and $C(S)$ be its curve complex whose vertices are the isotopy classes of essential simple closed curves in S and whose n–simplexes are determined by $n + 1$ distinct vertices with pairwise disjoint representatives. The *distance* between vertices is the number of edges in a minimal edge path joining them. Understanding this distance function is a daunting task – it is hard to tell where one is headed as one moves away from a given vertex:

Question 3.1 *Given vertices x, y of $C(S)$ is there an "easy" way to find a vertex x_1 with $x \cap x_1 = \varnothing$ and $d(x_1, y) = d(x, y) - 1$?*

One answer is given by K Shackleton [38], but it involves extending the curves to multicurves which satisfy an additional "tightness" condition and requires a search space whose size grows very rapidly with the complexity of the problem. Also, the complexity is measured in terms of the intersection numbers of the multicurves involved. This may not be the most natural measure and could be a distraction. I am hoping for something easier. The difficulty will be in making a proof.

If $d(x, y) > 2$, then $x \cup y$ splits S into contractible regions, each with an even number of edges, alternating between x and y. We assume there are no bigons. Euler characteristic calculations yield that most of these regions are squares. When we look in T, a component of S split along x, we see families of parallel arcs from y successively in the boundaries of these squares whose unions we call y–*stacks*. T deforms to a graph Γ with one vertex in each large region of $S - y$ and one edge crossing each y–stack (in T). Choose a maximal tree Δ in Γ. This determines a free basis for $\pi_1(T)$ whose elements are represented by simple closed curves each of which crosses exactly one of the y–stacks corresponding to an edge of $\Gamma - \Delta$ and crossing it once. We call any such basis a y–determined free basis and its elements y–determined free generators. They are in many ways the most simple, relative to y, curves in $S - x$. It is natural to ask:

Question 3.2 *Is there some y–determined free generator x_1 for $\pi_1(T)$ with $d(x_1, y)$ $< d(x, y)$? Does every y–determined free basis contain such an element?*

Caution: there are examples in which not every y–determined free generator, x_1 satisfies $d(x_1, y) < d(x, y)$. If Question 3.2 can't be answered, then:

Question 3.3 *In terms of word length, in a y–determined free basis, how far must we look in order to find a simple closed curve $x_1 \subset T$ with $d(x_1, y) < d(x, y)$?*

4 Tao Li

Let M_1 and M_2 be two manifolds with connected boundary $\partial M_1 \cong \partial M_2 \cong S$ and $\phi\colon \partial M_1 \to \partial M_2$ a homeomorphism. If one glues M_1 to M_2 by identifying x to $\phi(x)$ for each $x \in \partial M_1$, one obtains a closed manifold M. We say that M is an amalgamation of M_1 and M_2.

Let $\mathcal{D}_i$ $(i = 1, 2)$ be the set of curves in ∂M_i that bound essential disks in M_i. We propose the following conjecture.

Conjecture 4.1 *There is an essential curve $\mathcal{C}_i$ $(i = 1, 2)$ in ∂M_i such that if the distance between $\mathcal{D}_2 \cup \mathcal{C}_2$ and $\phi(\mathcal{D}_1 \cup \mathcal{C}_1)$ in the curve complex $\mathcal{C}(S)$ $(S = \partial M_2)$ is sufficiently large, then either the minimal-genus Heegaard splitting of $M = M_1 \cup_\phi M_2$ is an amalgamation or S itself is a minimal-genus Heegaard surface in which case both M_1 and M_2 are handlebodies.*

Conjecture 4.1 is a generalization of two recent theorems. In the case that M_1 and M_2 are atoroidal and have incompressible boundaries, ie, $\mathcal{D}_1 = \mathcal{D}_2 = \varnothing$, the conjecture is proved by Souto [39] and Li [20]. Note that [20] also gives an algorithm to find $\mathcal{C}_1$ and $\mathcal{C}_2$.

In the case that both M_1 and M_2 are handlebodies, $\mathcal{C}_1$ and $\mathcal{C}_2$ can be chosen to be empty and Conjecture 4.1 follows from a recent theorem of Scharlemann and Tomova [37]. The theorem of Scharlemann and Tomova can be formulated as: if the distance between $\mathcal{D}_2$ and $\phi(\mathcal{D}_1)$ (ie, the Hempel distance) is large, then the genus of any other Heegaard splitting must be large unless it is a stabilized copy of S.

5 Yair Minsky

For a handlebody H we have an inclusion of mapping class groups, $MCG(H) < MCG(\partial H)$. If $M = H_+ \cup_S H_-$ is a Heegaard splitting we denote $\Gamma_\pm = MCG(H_\pm)$

$< MCG(S)$, and moreover let $\Gamma_{\pm}^0$ be the kernel of the map $MCG(H_{\pm}) \to$ Out$(\pi_1(H_{\pm}))$ (ie, $\Gamma_{\pm}^0$ is the group of mapping classes of $H_{\pm}$ that are *homotopic to the identity on $H_{\pm}$*).

Question 5.1 *When is $\Gamma_+ \cap \Gamma_-$ finite? ... finitely generated? ... finitely presented?*

Question 5.2 *When is $\langle \Gamma_+, \Gamma_- \rangle$ equal to the amalgamation $\Gamma_+ *_{\Gamma_+ \cap \Gamma_-} \Gamma_-$?*

Question 5.3 *When is the map $\Gamma_+ \cap \Gamma_- \to MCG(M)$ injective?*

When $M = S^3$ and genus$(S) = 2$, Akbas [1] proved $\Gamma_+ \cap \Gamma_-$ is finitely presented. (Finitely generated has a longer history starting with Goeritz [11] – for details see Scharlemann [36].)

Let $\Delta_{\pm} \subset \mathcal{C}(S)$ be the set of (isotopy classes of) simple curves in S which bound disks in $H_{\pm}$. Let $Z \subset \mathcal{C}(S)$ be the simple curves in S that are homotopic to the identity in M. Note that Z contains $\Delta_{\pm}$, and is invariant under $\Gamma_{\pm}^0$.

Namazi [30] showed if the distance of the splitting (dist$_{\mathcal{C}(S)}(\Delta_+, \Delta_-)$) is sufficiently large then $\Gamma_+ \cap \Gamma_-$ is finite. Note also the Geometrization Theorem plus Hempel [12], plus Mostow rigidity plus Gabai–Meyerhoff–Thurston [10], implies that the image of $\Gamma_+ \cap \Gamma_-$ in $MCG(M)$ is finite when the splitting distance is at least 3.

Question 5.4 *When is Z equal to the orbit $\langle \Gamma_+^0, \Gamma_-^0 \rangle (\Delta_+ \cup \Delta_-)$?*

Remarks When $M = S^3$ or a connected sum of $S^2 \times S^1$'s, with the natural splitting, equality holds trivially. If M is a lens space $L_{p,q}$, with $p \geq 2$, and S is the torus, then they are not equal – Z is all of $\mathcal{C}(S) = \mathbb{Q} \cup \{\infty\}$, whereas the orbit of $\Delta_+ \cup \Delta_- = \{\frac{0}{1}, \frac{p}{q}\}$ under $\langle \Gamma_+^0, \Gamma_-^0 \rangle < SL(2, \mathbb{Z})$ is strictly smaller than $\mathbb{Q} \cup \{\infty\}$. One might reasonably ask if there is equality when M is hyperbolic, or if the splitting distance is sufficiently large.

6 Yoav Moriah

Question 6.1 *Give an example of a weakly reducible but non-stabilized non-minimal Heegaard splitting of a closed 3–manifold.*

There are plenty of examples for manifolds with strongly irreducible Heegaard splittings of arbitrarily high genus. However there are no examples of manifolds (closed or with a single boundary component) with a weakly reducible but non-stabilized non-minimal Heegaard splitting.

Question 6.2 *Prove that if a manifold has strongly irreducible Heegaard splittings of arbitrarily high genus then the Heegaard splittings are of the form $H + nK$ where H and K are surfaces with K perhaps not connected. (See Moriah, Schleimer and Sedgwick [25] and Li [19].) Or give a counterexample.*

Question 6.3 *There are examples by Kobayashi and Rieck (see below) of a 3– manifold which has both a weakly reducible and strongly irreducible minimal genus Heegaard splitting. These examples are very particular. Are there other such examples of a different nature?*

Theorem 6.1 (T Kobayashi and Y Rieck [18]) *There are infinitely many 3–manifolds which have both strongly irreducible and weakly reducible Heegaard splittings of minimal genus.*

Proof Let $X = S^3 - N(K_1)$, $Y = S^3 - N(K_2)$ and $Z = S^3 - N(K_3)$, where $K_1 = T_{(2,3)}$ the trefoil knot, $K_2 = L(\alpha, \beta)$ with α even, is any 2–bridge link which is not the Hopf link and $K_3 = K(2, 5)$ is the figure 8 knot. Let μ_1 and μ_2 be meridians of the 2–bridge link $L(\alpha, \beta)$, λ be the longitude of K_3 and γ be the boundary of the annulus in X.

Attach X and Z to Y by gluing their tori boundaries so that γ is mapped to μ_1 and λ to μ_2. We obtain a closed 3–manifold M with incompressible tori. As the genus two toroidal 3–manifolds are classified by Kobayashi [15] this manifold cannot have genus 2. The surface S obtained from the bridge sphere Σ union the annulus in the trefoil complement and the two genus one Seifert surfaces of the figure 8 complement is a closed surface of genus $g = 1 - \frac{\chi(S)}{2} = 1 - ((-2) + (-2) + 0)/2 = 3$. Hence it is a minimal genus Heegaard splitting if it is a Heegaard surface.

Let V_1, V_2 be the two components of $S^3 - N(K_1)$. So on one side of S we have at the first stage the solid torus V_1, say, glued to the genus two handlebody W_1, which is one of the two components W_1, W_2 of $Y - \Sigma$, along a primitive meridional annulus to obtain a genus two handlebody. This handlebody is glued in the second stage to the genus two handlebody which is a regular neighborhood of the Seifert surface along a primitive meridional annulus. So we get a genus three handlebody U_1.

On the other side of S we have V_2 glued to the genus two handlebody W_2 along a primitive meridional annulus to obtain a genus two handlebody which is then glued again along a primitive meridional annulus to a genus two handlebody which is a regular neighborhood of the Seifert surface of the figure 8 knot complement. Thus we get a genus three handlebody U_2 and (U_1, U_2) is a genus three Heegaard splitting for M.

In [15, Proposition 3.1] Kobayashi proves that a Heegaard splitting of the form (U_1, U_2) is always strongly irreducible if the link $L(\alpha, \beta)$ is not trivial or a Hopf link.

Now consider the union along the torus boundary of X and Y. This is a manifold with a minimal genus two Heegaard splitting and with one torus boundary component. Hence when this Heegaard splitting is amalgamated with the genus two Heegaard splitting of Z we obtain a genus three Heegaard splitting for M which is weakly reducible as it is obtained by amalgamation.   □

Conjecture 6.2 Let K_1 and K_2 be *prime knots in* S^3 *then* $t(K_1\#K_2) \leq t(K_1) + t(K_2)$ *if and only if one of* K_i *has a minimal genus Heegaard splitting with primitive meridian. (See Moriah [23].)*

Recently T Kobayashi and Y Rieck disproved the conjecture for knots which are not prime (see [17]). The conjecture is known by work of Morimoto for tunnel number one knots and for knots which are connected sums of two prime knots each of which is also m–small. (See [23] for more references.)

Conjecture 6.3 *Given a knot* $K \subset S^3$ *which is not* γ–*primitive then a boundary stabilization of a minimal genus Heegaard splitting of* $E(K)$ *is non-stabilized.*

Conjecture 6.4 *All twisted torus knots of type* $K = T(p, q, 2, r)$ *which are not* μ–*primitive have a unique (minimal) genus two Heegaard splitting. (See Moriah and Sedgwick [26].)*

Conjecture 6.5 *What are the properties of meridional essential surfaces which ensure that the tunnel number degenerate? Can these surfaces be classified?*

Question 6.4 *Are there knots which are not* $K_1 = K^n(-2, 3, -3, 2)$ *and* 2–*bridge knots so that* $t(K_1\#K_2) < t(K_1) + t(K_2)$? *(See Morimoto [27].)*

Question 6.5 *Does rank equal genus for hyperbolic* 3–*manifolds?*

In [21] Lustig and Moriah defined a condition on complete disks systems in a Heegaard splitting, called the *double rectangle condition*. It was shown that if a manifold has a Heegaard splitting which has some complete disk system which satisfies this condition then there are only finitely many such disk systems with that property. However the double rectangle condition is clearly non-"generic". Is it possible to define some other condition which will be "generic" in some reasonable sense? The intuition is that if the Heegaard distance of the splitting is sufficiently high then some form of this might be possible.

7 Richard Weidmann

Let $(M; V, W)$ be a Heegaard splitting of genus n. Let $g_1, \ldots, g_n \in \pi_1(M)$ be the elements corresponding to a spine of V.

Question 7.1 *When is $\langle g_1, \ldots, g_{n-1} \rangle$ a free group of rank $n - 1$?*

(This would be a kind of "Freiheitsatz".) For instance, what happens with the Casson–Gordon examples of strongly irreducible splittings of a fixed M of arbitrarily high genus?

Note that the freeness holds for the examples exhibited by Namazi [31] and Namazi–Souto [32].

Question 7.2 *Find (3–manifold) groups that have only finitely many irreducible Nielsen equivalence classes of generating tuples.*

Here a n–tuple is called irreducible if it is not Nielsen equivalent to a tuple of type $(x_1, \ldots, x_{n-1}, 1)$. Note that free groups and free Abelian groups have this property, in fact for those groups there is precisely one irreducible Nielsen equivalence class. Note that for Heegaard splittings it has been shown by Tao Li that closed non-Haken 3–manifolds only have finitely many isotopy classes of irreducible Heegaard splittings so those groups might be potential examples.

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