

## String topology of Poincaré duality groups

HOSSEIN ABBASPOUR

RALPH COHEN

KATE GRUHER

Let  $G$  be a Poincaré duality group of dimension  $n$ . For a given element  $g \in G$ , let  $C_g$  denote its centralizer subgroup. Let  $\mathcal{L}_G$  be the graded abelian group defined by

$$(\mathcal{L}_G)_p = \bigoplus_{[g]} H_{p+n}(C_g)$$

where the sum is taken over conjugacy classes of elements in  $G$ . In this paper we construct a multiplication on  $\mathcal{L}_G$  directly in terms of intersection products on the centralizers. This multiplication makes  $\mathcal{L}_G$  a graded, associative, commutative algebra. When  $G$  is the fundamental group of an aspherical, closed oriented  $n$ -manifold  $M$ , then  $(\mathcal{L}_G)_* = H_{*+n}(LM)$ , where  $LM$  is the free loop space of  $M$ . We show that the product on  $\mathcal{L}_G$  corresponds to the string topology loop product on  $H_*(LM)$  defined by Chas and Sullivan.

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*Dedicated to Fred Cohen on the occasion of his 60th birthday*

### Introduction

In [2] Chas and Sullivan defined an associative, commutative algebra structure on  $H_*(LM)$ , the homology of the free loop space of a closed, oriented  $n$ -manifold  $M$ . The Chas–Sullivan string topology product has total degree  $-n$ ,  $H_p(LM) \otimes H_q(LM) \rightarrow H_{p+q-n}(LM)$ .

On the other hand it is well known that if  $G$  is any topological group, and  $BG$  is its classifying space, then its loop space is naturally homotopy equivalent to the homotopy orbit space of the adjoint action of the group on itself:

$$L(BG) \simeq EG \times_G G$$

where  $EG$  is a contractible space with a free  $G$  action, and  $G$  acts on itself by conjugation. The space  $EG \times_G G$  is the orbit space of the corresponding diagonal  $G$ -action on  $EG \times G$ . When  $G$  is discrete, we therefore have an isomorphism,

$$H_*(L(BG)) \cong H_*(G; \text{Ad})$$

where  $\text{Ad}$  is the  $G$ -module given by  $\mathbb{Z}[G]$  additively, and the action is given by conjugation. It is an elementary fact that this homology group splits as

$$H_*(G; \text{Ad}) = \bigoplus_{[g]} H_*(C_g)$$

where the direct sum is taken over conjugacy classes of elements in  $G$ , and  $C_g$  is the centralizer of  $g \in G$ .

Combining this with Chas and Sullivan's string topology construction, when  $G$  is the fundamental group of a closed, oriented, aspherical manifold  $M$ , then

$$(\mathcal{L}_G)_* := \bigoplus_{[g]} H_{*+n}(C_g)$$

has the structure of a graded commutative algebra.

It is the goal of this note to describe this product structure directly in terms of homology of groups. Our construction will not involve any differential topology, and so will work for any *Poincaré duality group*: that is a group  $G$  together with a fundamental class  $z \in H_n(G; \mathbb{Z})$  such that capping with this class is an isomorphism

$$(1) \quad \cap z: H^p(G, A) \xrightarrow{\cong} H_{n-p}(G, A)$$

where  $A$  is an arbitrary  $\mathbb{Z}[G]$ -module. Such a structure is called an oriented Poincaré duality group in Brown [1].

Our construction will be defined in terms of the *intersection product* in the homology of subgroups of a Poincaré duality group. Given two such subgroups,  $K, H < G$ , this product is a pairing,

$$H_p(K; \mathbb{Z}) \otimes H_q(H; \mathbb{Z}) \rightarrow \bigoplus_{[g]} H_{p+q-n}(K \cap gHg^{-1}).$$

The target of this map is the sum of the homologies of the intersections of  $K$  with conjugates of  $H$ , taken over all double cosets  $[g] = KgH \subset G$ . This construction is defined directly in terms of the Poincaré duality isomorphism (1). In the case when  $K$  and  $H$  are centralizers, we will be able to assemble the intersection maps into a product on  $\mathcal{L}_G$ . After making this construction we will show that this operation corresponds to the Chas–Sullivan string topology loop product in the case when  $G$  is the fundamental group of an aspherical manifold (see [Theorem 6](#) below).

## 1 Intersection pairings in Poincaré duality groups and the string product

Throughout this section  $G$  will be a fixed Poincaré duality group of dimension  $n$ . Let  $z \in H_n(G)$  be a fundamental class. We let  $D: H^q(G; M) \xrightarrow{\cong} H_{n-q}(G; M)$  be the Poincaré duality isomorphism given by capping on the left with  $z$ .

We begin by using the Poincaré duality isomorphism to define an *intersection pairing*,

$$\cap: H_p(K; \mathbb{Z}) \otimes H_q(H; \mathbb{Z}) \rightarrow \bigoplus_{[g] \in I} H_{p+q-n}(K \cap gHg^{-1}),$$

where  $K$  and  $H$  are any two subgroups of  $G$ , and  $I$  is an indexing set for the double cosets  $K \backslash G / H$ .

First observe that for any subgroup  $K$  of  $G$  we have

$$H_*(K, \mathbb{Z}) = H_*(G, \mathbb{Z}[G/K])$$

where  $\mathbb{Z}[G/K]$  is the free  $G$ -module generated by the left cosets, equipped with the standard left action of  $G$ .

We next need the following double coset formula.

**Lemma 1** *Suppose  $I$  is a set of representatives for the double cosets  $K \backslash G / H$ . Then there is an isomorphism of  $\mathbb{Z}[G]$ -modules,*

$$(2) \quad \mathbb{Z}[G/K] \otimes \mathbb{Z}[G/H] \cong \bigoplus_{g \in I} \mathbb{Z}[G/K \cap gHg^{-1}],$$

where  $G$  acts diagonally on  $\mathbb{Z}[G/K] \otimes \mathbb{Z}[G/H]$  by the standard left translation.

**Proof** Given  $g_1 K \otimes g_2 H$ , double coset  $K(g_1^{-1} g_2)H$  is well defined and does not depend on the choice of the representatives  $g_1$  and  $g_2$  for the cosets  $g_1 K$  and  $g_2 H$ , so let  $g \in I$  be representative for the double  $K(g_1^{-1} g_2)H$ . If we write  $g_1 K \otimes g_2 H = a(K \otimes gH)$  for some  $a \in G$ , the isomorphism is defined by sending  $g_1 K \otimes g_2 H$  to  $a(K \cap gHg^{-1})$ . This map is well-defined as  $a(K \otimes gH) = b(K \otimes gH)$  implies that  $b^{-1}a \in K \cap gHg^{-1}$ . It is obvious that the map described above is a map of  $G$ -modules.  $\square$

With this we can define the intersection product.

**Definition 2** The intersection pairing

$$\cap: H_i(K; \mathbb{Z}) \otimes H_j(H; \mathbb{Z}) \rightarrow \bigoplus_{[g]} H_{i+j-n}(K \cap gHg^{-1})$$

is given by the composition

$$\begin{aligned} \cap: H_i(K) \otimes H_j(H) &\xrightarrow{\cong} H_i(G, \mathbb{Z}[G/K]) \otimes H_j(G, \mathbb{Z}[G/H]) \xrightarrow{D^{-1} \otimes D^{-1}} \\ &H^{n-i}(G, \mathbb{Z}[G/K]) \otimes H^{n-j}(G, \mathbb{Z}[G/H]) \xrightarrow{\text{cup prod.}} H^{2n-i-j}(G, \mathbb{Z}[G/K] \otimes \mathbb{Z}[G/H]) \\ &\xrightarrow{D} H_{i+j-n}(G, \mathbb{Z}[G/K] \otimes \mathbb{Z}[G/H]) \xrightarrow{\psi_*} \bigoplus_{[g] \in I} H_{i+j-n}(G; \mathbb{Z}[G/K \cap gHg^{-1}]) \\ &\xrightarrow{\cong} \bigoplus_{[g] \in I} H_{i+j-n}(K \cap gHg^{-1}) \end{aligned}$$

where the map  $\psi$  is the isomorphism given by [Lemma 1](#).

**Remark 3** When  $K = H = G$ , the intersection pairing

$$\cap: H_i(G) \otimes H_j(G) \rightarrow H_{i+j-n}(G)$$

is simply the homomorphism that is Poincaré dual to the cup product in cohomology. In the case when  $G$  is the fundamental group of a closed, oriented, aspherical  $n$ -manifold  $M$ , this is the usual definition (up to sign) of the intersection product on  $H_*(M)$ .

The case of interest for us is when the subgroups are the centralizers of the elements of  $G$ . Suppose  $C_\alpha$  and  $C_\beta$  are the centralizers of  $\alpha$  and  $\beta$  in  $G$ . Then we have

$$(3) \quad \cap: H_i(C_\alpha) \otimes H_j(C_\beta) \rightarrow \bigoplus_{g \in I} H_{n-i-j}(G, \mathbb{Z}[G/C_\alpha \cap gC_\beta g^{-1}]).$$

Now the conjugate of a centralizer subgroup is the centralizer of the conjugate,

$$gC_\beta g^{-1} = C_{g\beta g^{-1}}.$$

Furthermore we have inclusions of subgroups,

$$j: C_\alpha \cap gC_\beta g^{-1} = C_\alpha \cap C_{g\beta g^{-1}} \leq C_{\alpha g\beta g^{-1}}.$$

This will allow us to make the following definition.

**Definition 4** Define the string product,

$$\begin{aligned} \mu: (\mathcal{L}_G)_p \otimes (\mathcal{L}_G)_q &\longrightarrow (\mathcal{L}_G)_{p+q} \\ \left( \bigoplus_{\alpha} H_{p+n}(C_{\alpha}) \right) \otimes \left( \bigoplus_{\beta} H_{q+n}(C_{\beta}) \right) &\longmapsto \left( \bigoplus_{\gamma} H_{p+q+n}(C_{\gamma}) \right) \end{aligned}$$

to be the composition

$$\begin{aligned} \mu = j_* \circ \cap: H_{p+n}(C_{\alpha}) \otimes H_{q+n}(C_{\beta}) &\longrightarrow \\ \bigoplus_{g \in I} H_{p+q+n}(C_{\alpha} \cap g C_{\beta} g^{-1}) &\longrightarrow \bigoplus_{g \in I} H_{p+q+n}(C_{\alpha g \beta g^{-1}}). \end{aligned}$$

**Theorem 5** The product  $\mu$  makes  $\mathcal{L}_G$  a graded, associative, commutative algebra with unit.

**Proof** The associativity is a direct check of the definitions. The unit in the algebra is given by the element  $z \in H_n(G) \subset (\mathcal{L}_G)_0$ . Again, it is an immediate check of definitions to verify that this class acts as the identity.

To see commutativity, consider the map

$$\bigoplus_{\alpha, \beta} \bigoplus_{g \in I} H_{*+n}(C_{\alpha} \cap C_{g \beta g^{-1}}) \xrightarrow{j_*} \bigoplus_{\alpha, \beta} \bigoplus_{g \in I} H_{*+n}(C_{\alpha g \beta g^{-1}}) \rightarrow (\mathcal{L}_G)_* = H_{*+n}(G, \text{Ad}).$$

Notice that  $\bigoplus_{\alpha, \beta} \bigoplus_{g \in I} H_{*+n}(C_{\alpha} \cap C_{g \beta g^{-1}})$  has an involution on it, by interchanging the roles of  $\alpha$  and  $\beta$ , and mapping  $g$  to  $g^{-1}$ . We call that involution  $\tau$ . Notice that for  $x \in H_{p+n}(C_{\alpha})$ , and  $y \in H_{q+n}(C_{\beta})$ , we have that

$$\mu(y, x) = j_* \circ \cap(y, x) = (1)^{pq} j_* \circ \tau \circ \cap(x, y).$$

Now notice the obvious conjugation relation,

$$g^{-1} \alpha^{-1} (C_{\alpha g \beta g^{-1}}) \alpha g = C_{\beta g^{-1} \alpha g},$$

Using the fact that if  $h_1$  is conjugate to  $h_2$  then the images of the homology of the centralizers,  $H_*(C_{h_1}) \hookrightarrow H_*(G; \text{Ad})$  and  $H_*(C_{h_2}) \hookrightarrow H_*(G; \text{Ad})$  are equal, we see that  $j_* \circ \tau = j_*$ , and so  $\mu(x, y) = (-1)^{pq} \mu(y, x)$ .  $\square$

Now as mentioned above, the free loop space,  $LBG$  is homotopy equivalent to the homotopy orbit space of the adjoint action,  $EG \times_{\text{Ad}} G$ . This gives a natural isomorphism of homologies,

$$H_*(LBG) \cong H_*(G; \text{Ad}).$$

Now assume that  $G = \pi_1(M^n)$ , where  $M^n$  is an aspherical, closed, oriented  $n$ -manifold. In particular this says that  $M^n \simeq BG$ , and so there is a homotopy equivalence,

$$LM \simeq EG \times_{\text{Ad}} G$$

and so an isomorphism,

$$H_*(LM) \cong H_*(G; \text{Ad}).$$

Using transversal intersection theory, in [2] Chas and Sullivan described a string topology loop product on the regraded homology groups,  $\mathbb{H}_*(LM^n) = H_{*+n}(LM)$ ,

$$\circ: \mathbb{H}_*(LM^n) \otimes \mathbb{H}_*(LM^n) \rightarrow \mathbb{H}_*(LM^n).$$

Notice with respect to this regrading, we have an isomorphism,  $\Phi: \mathbb{H}_*(LM^n) \cong (\mathcal{L}_G)_*$ . The following says that our algebraically defined product on  $(\mathcal{L}_G)_*$  corresponds to the Chas–Sullivan string topology product on  $\mathbb{H}_*(LM)$ .

**Theorem 6** *The following diagram commutes:*

$$\begin{array}{ccc} \mathbb{H}_*(LM^n) \otimes \mathbb{H}_*(LM^n) & \xrightarrow{\circ} & \mathbb{H}_*(LM^n) \\ \Phi \cong \downarrow & & \cong \downarrow \Phi \\ (\mathcal{L}_G)_* \otimes (\mathcal{L}_G)_* & \xrightarrow{\mu} & (\mathcal{L}_G)_* \end{array}$$

**Proof** Recall that the string topology product was described by Cohen and Jones [3] in the following way. Consider the diagram

$$LM \xleftarrow{\gamma} LM \times_M LM \xrightarrow{\iota} LM \times LM$$

where  $LM \times_M LM$  is the subspace of  $LM \times LM$  consisting of those pairs of loops  $(\alpha, \beta)$ , with  $\alpha(0) = \beta(0)$ .  $\iota: LM \times_M LM \hookrightarrow LM \times LM$  is the natural inclusion.  $\gamma: LM \times_M LM \rightarrow LM$  is the map obtained by thinking of  $LM \times_M LM$  as the maps from the figure 8 to  $M$ ,  $LM \times_M LM \cong \text{Map}(8, M)$ , and then mapping this space to  $LM$  by sending a map from the figure 8 to the loop obtained by starting at the intersection point of the 8, first traversing the upper circle, and then traversing the lower circle of the 8. See Chas–Sullivan [2] and Cohen–Jones [3] for details. Notice that  $LM \times_M LM$  is the fiber product for the square,

$$(4) \quad \begin{array}{ccc} LM \times_M LM & \xrightarrow{\iota} & LM \times LM \\ \downarrow & & \downarrow \text{ev} \times \text{ev} \\ M & \xrightarrow{\Delta} & M \times M \end{array}$$

Here  $\text{ev}: LM \rightarrow M$  evaluates a loop at the starting point,  $0 \in \mathbb{R}/\mathbb{Z} = S^1$ . In [3] a Pontrjagin–Thom map  $\tau_\iota: LM \times LM \rightarrow (LM \times_M LM)^{TM}$  was constructed, where the target is the Thom space of the tangent bundle  $TM \rightarrow M$  pulled back over  $LM \times_M LM$ . This map was defined by finding a tubular neighborhood of the embedding  $\iota$ , and collapsing everything in  $LM \times LM$  outside the tubular neighborhood to a point. When one applies homology and the Thom isomorphism to the map  $\tau_\iota$  one obtains an “umkehr map”,

$$(5) \quad \iota!: H_*(LM \times LM) \rightarrow H_{*-n}(LM \times_M LM)$$

and the basic string topology product is the composition,

$$\circ = \gamma_* \circ \iota!: H_*(LM \times LM) \rightarrow H_{*-n}(LM \times_M LM) \rightarrow H_{*-n}(LM).$$

See [3] for details.

Now in the case of interest, where  $M$  is an aspherical manifold with fundamental group  $G$ , the tubular neighborhood of  $\iota: LM \times_M LM \hookrightarrow LM \times LM$ , and the resulting Thom–Pontrjagin construction has the following description.

Let  $\tilde{M} \rightarrow M$  be the universal cover. Since  $M$  is aspherical,  $\tilde{M}$  is a contractible space with a free  $G$ -action. We can therefore write the homotopy equivalence,

$$LM \cong \tilde{M} \times_{\text{Ad}} G,$$

and up to homotopy, the evaluation map  $\text{ev}: LM \rightarrow M$  is given by the projection map  $p: \tilde{M} \times_{\text{Ad}} G \rightarrow M$ . Now as seen before,  $\tilde{M} \times_{\text{Ad}} G$  decomposes as a disjoint union of spaces,

$$\tilde{M} \times_{\text{Ad}} G = \coprod_{[\alpha]} \tilde{M} \times_G (G/C_\alpha)$$

where the disjoint union is taken over conjugacy classes of elements  $\alpha \in G$ . Thus the pullback square (4) is the disjoint union, over pairs of conjugacy classes,  $\alpha, \beta$ , of the pullback squares,

$$(6) \quad \begin{array}{ccc} \tilde{M} \times_G (G/C_\alpha \times G/C_\beta) & \xrightarrow{\tilde{\Delta} \times 1} & (\tilde{M} \times \tilde{M}) \times_{G \times G} (G/C_\alpha \times G/C_\beta) \\ p \downarrow & & \downarrow p \times p \\ M & \xrightarrow{\Delta} & M \times M \end{array}$$

A tubular neighborhood of the diagonal map,  $\Delta: M \rightarrow M \times M$  lifts to a  $G$ -equivariant tubular neighborhood of the diagonal,  $\tilde{\Delta}: \tilde{M} \rightarrow \tilde{M} \times \tilde{M}$ , which has normal bundle

isomorphic to the tangent bundle,  $T\tilde{M} \rightarrow \tilde{M}$ . Therefore the Pontrjagin–Thom collapse map in this setting, is  $G$ –equivariant,

$$\tau: \tilde{M} \times \tilde{M} \rightarrow (\tilde{M})^{T\tilde{M}}.$$

This says that the Pontrjagin–Thom map for the embedding  $\tilde{\Delta} \times 1$  from diagram (6), a disjoint union of which model the Pontrjagin–Thom map of the embedding  $\iota$  from diagram (4), is given by

$$\tau \times 1: (\tilde{M} \times \tilde{M}) \times_{G \times G} (G/C_\alpha \times G/C_\beta) \rightarrow (\tilde{M})_+^{T\tilde{M}} \wedge_{G \times G} (G/C_\alpha \times G/C_\beta).$$

Applying homology and the Thom isomorphism, one gets umkehr maps,

$$(\iota_{\alpha,\beta})!: H_*((\tilde{M} \times \tilde{M}) \times_{G \times G} (G/C_\alpha \times G/C_\beta)) \rightarrow H_{*-n}(\tilde{M} \times_G (G/C_\alpha \times G/C_\beta)),$$

A direct sum of these umkehr maps, (over conjugacy classes  $\alpha$  and  $\beta$ ) gives a model for the umkehr map  $\iota_!$  given in (5) above:

$$(7) \quad \begin{array}{ccc} \bigoplus_{\alpha,\beta} H_*((\tilde{M} \times \tilde{M}) \times_{G \times G} (G/C_\alpha \times G/C_\beta)) & \xrightarrow{\bigoplus (\iota_{\alpha,\beta})!} & \bigoplus_{\alpha,\beta} H_{*-n}(\tilde{M} \times_G (G/C_\alpha \times G/C_\beta)) \\ \cong \downarrow & & \downarrow \cong \\ H_*(LM \times LM) & \xrightarrow{\iota_!} & H_{*-n}(LM \times_M LM) \end{array}$$

In terms of group homology, the top line of this diagram is a sum of terms of the form,

$$(\iota_{\alpha,\beta})!: H_*(G \times G; \mathbb{Z}[G/C_\alpha \times G/C_\beta]) \rightarrow H_{*-n}(G; \mathbb{Z}[G/C_\alpha \times G/C_\beta]).$$

Now the standard relationship between umkehr maps for embeddings of closed manifolds and Poincaré duality says that  $(\iota_{\alpha,\beta})!$  is Poincaré dual to the induced homomorphism in cohomology,

$$(\tilde{\Delta} \times 1)^*: H^*((\tilde{M} \times \tilde{M}) \times_{G \times G} (G/C_\alpha \times G/C_\beta)) \rightarrow H^*(\tilde{M} \times_G (G/C_\alpha \times G/C_\beta))$$

or, in terms of group cohomology,

$$(\tilde{\Delta} \times 1)^*: H^*(G \times G; \mathbb{Z}[G/C_\alpha \times G/C_\beta]) \rightarrow H^*(G; \mathbb{Z}[G/C_\alpha \times G/C_\beta]).$$

But the diagonal map in group cohomology induces the cup product,

$$\cup: H^*(G; \mathbb{Z}[G/C_\alpha]) \otimes H^*(G; \mathbb{Z}[G/C_\beta]) \rightarrow H^*(G; \mathbb{Z}[G/C_\alpha \times G/C_\beta]).$$

Putting these observations together with the definition of the intersection pairing in the homologies of the centralizers (Definition 4), we see that the following diagram



commutes:

$$\begin{array}{ccc}
 \bigoplus_{\alpha,\beta} H_*(C_\alpha) \otimes H_*(C_\beta) & \xrightarrow{\cap} & \bigoplus_{[g] \in I} \bigoplus_{\alpha,\beta} H_{*-n}(C_\alpha \cap g C_\beta g^{-1}) \\
 \cong \downarrow & & \cong \downarrow \\
 H_*(LM) \otimes H_*(LM) & \xrightarrow{\mu} & \bigoplus_{\alpha,\beta} H_{*-n}(G; \mathbb{Z}[G/C_\alpha \times G/C_\beta]) \\
 & & \cong \downarrow \\
 & & H_{*-n}(LM \times_M LM)
 \end{array}$$

On the other hand, it is clear that the following diagram also commutes,

$$\begin{array}{ccc}
 \bigoplus_{\alpha,\beta} \bigoplus_{[g] \in I} H_{*-n}(C_\alpha \cap g C_\beta g^{-1}) & \xrightarrow{j} & \bigoplus_{\alpha,\beta} \bigoplus_{[g] \in I} H_{*-n}(C_{\alpha g \beta g^{-1}}) \\
 \cong \downarrow & & \downarrow \\
 H_{*-n}(LM \times_M LM) & \xrightarrow{\gamma_*} & H_{*-n}(LM) \\
 & & \cong \downarrow \\
 & & (\mathcal{L}G)_*
 \end{array}$$

The composition of the bottom rows of these two diagrams is the string topology loop product. The composition of the top rows of these two diagrams is the map  $\mu$ . Thus the theorem is proven.  $\square$

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*Centre de Mathématiques, École Polytechnique  
Palaiseau, France*

*Department of Mathematics, Stanford University  
Stanford CA 94305, USA*

*Department of Mathematics, Stanford University  
Stanford CA 94305, USA*

[abbaspou@mpim-bonn.mpg.de](mailto:abbaspou@mpim-bonn.mpg.de), [ralph@math.stanford.edu](mailto:ralph@math.stanford.edu),  
[gruher@math.stanford.edu](mailto:gruher@math.stanford.edu)

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