

Cohomology of Artin groups of type \tilde{A}_n , B_n and applications

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We consider two natural embeddings between Artin groups: the group $G_{\tilde{A}_{n-1}}$ of type \tilde{A}_{n-1} embeds into the group G_{B_n} of type B_n ; G_{B_n} in turn embeds into the classical braid group $Br_{n+1} := G_{A_n}$ of type A_n . The cohomologies of these groups are related, by standard results, in a precise way. By using techniques developed in previous papers, we give precise formulas (sketching the proofs) for the cohomology of G_{B_n} with coefficients over the module $\mathbb{Q}[q^{\pm 1}, t^{\pm 1}]$, where the action is $(-q)$ -multiplication for the standard generators associated to the first $n - 1$ nodes of the Dynkin diagram, while is $(-t)$ -multiplication for the generator associated to the last node.

As a corollary we obtain the rational cohomology for $G_{\tilde{A}_n}$ as well as the cohomology of Br_{n+1} with coefficients in the $(n + 1)$ -dimensional representation obtained by Tong, Yang and Ma [38].

We stress the topological significance, recalling some constructions of explicit finite CW-complexes for orbit spaces of Artin groups. In case of groups of infinite type, we indicate the (few) variations to be done with respect to the finite type case (see Salvetti [32]). For affine groups, some of these orbit spaces are known to be $K(\pi, 1)$ spaces (in particular, for type \tilde{A}_n).

We point out that the above cohomology of G_{B_n} gives (as a module over the monodromy operator) the rational cohomology of the fibre (analog to a Milnor fibre) of the natural fibration of $K(G_{B_n}, 1)$ onto the 2-torus.

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1 Introduction

The cohomology of classical *braid groups* with trivial coefficients was computed in the seventies by F Cohen [13], and independently by A Vaňštejn [39] (see also Arnol'd [2], Brieskorn and Saito [4; 5] and Fuks [22]). For Artin groups of type C_n , D_n it was computed by Gorjunov [23], and for exceptional cases by Salvetti [32] it was given as a \mathbb{Z} -module, while the ring structure was computed by Landi [27]. Other

cohomologies with twisted coefficients were later considered: an interesting case is over the module of Laurent polynomials $\mathbb{Q}[q^{\pm 1}]$, which gives the \mathbb{Q} -cohomology of the Milnor fibre of the naturally associated bundle. For the case of classical braids many people made computations, independently and using different methods (Frenkel [21], Markaryan [28], Callegaro and Salvetti [9] and De Concini, Procesi and Salvetti [15]), while for cases C_n , D_n see De Concini, Procesi, Salvetti and Stumbo [16] (here the authors use the resolution coming from topological considerations discovered by De Concini and Salvetti [32; 17]; an equivalent resolution was independently discovered by using purely algebraic methods by Squier [34]). Over the integral Laurent polynomials $\mathbb{Z}[q^{\pm 1}]$ not many computations exist: see D Cohen and Suciu [12] for the exceptional cases and recently Callegaro [8] for the case of braid groups, and De Concini, Salvetti and Stumbo [18] for the top cohomologies in all cases.

As regards Artin groups of non-finite type, some computations were done by Salvetti and Stumbo [33] and Charney and Davis [10].

In this paper we give a complete computation of the above cohomologies over $\mathbb{Q}[q^{\pm 1}]$ for the Artin groups $G_{\tilde{A}_n}$ of affine type \tilde{A}_n . By using a natural embedding of $G_{\tilde{A}_{n-1}}$ into the Artin group G_{B_n} of type B_n , we reduce to the equivalent computation of the cohomology of G_{B_n} over the module $\mathbb{Q}[q^{\pm 1}, t^{\pm 1}]$, where the action is $(-q)$ -multiplication for the standard generators associated to the first $n - 1$ nodes of the Dynkin diagram, while is $(-t)$ -multiplication for the generator associated to the last node.

The proof uses techniques similar to [15]: a natural filtration of the complex given in [32] and the associated spectral sequence. We sketch the argument here: details will appear elsewhere.

As a corollary we derive the trivial \mathbb{Q} -cohomology of $G_{\tilde{A}_{n-1}}$.

By using another natural inclusion, a map of $G_{\tilde{B}_n}$ into the classical braid group $Br_{n+1} := G_{A_n}$, we also find an isomorphism with the cohomology of Br_{n+1} over a certain interesting representation, namely the irreducible $(n + 1)$ -dimensional representation of Br_{n+1} found by Tong, Yang and Ma [38], twisted by an abelian representation.

We also describe the cohomology of the braid group over the irreducible representation in [38].

The topological counterpart is given by some very explicit constructions of finite CW-complexes which are retracts of the orbit spaces associated to Artin groups. Following [32], we show the (few) variations to be done in case of groups of infinite type, explicitly showing the affine case (see also [10] for a different construction). From such constructions the standard presentation of the fundamental group comes quite

easily (see Dřng [20]). We also easily deduce a formula for the Euler characteristic of the orbit space in the affine case. It is conjectured that such orbit spaces are always $K(\pi, 1)$ spaces; for the affine groups, this is known in case \tilde{A}_n , \tilde{C}_n (see Okonek [29] and Charney and Peifer [11]; see also [10] for a different class of Artin groups of infinite type).

It is interesting to notice the geometrical meaning of the two-parameters cohomology of G_{B_n} : similar to the one-parameter case, it gives the trivial cohomology of the ‘‘Milnor fibre’’ associated to the natural map of the orbit space onto a two-dimensional torus.

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2 Preliminary results

In this section we briefly fix the notation and recall some preliminary results.

2.1 Coxeter groups and Artin groups

A *Coxeter graph* is a finite undirected graph, whose edges are labelled with integers ≥ 3 or with the symbol ∞ .

Let S be the vertex set of a Coxeter graph. For every pair of vertices $s, t \in S$ ($s \neq t$) joined by an edge, define $m(s, t)$ to be the label of the edge joining them. If s, t are not joined by an edge, set by convention $m(s, t) = 2$. Let also $m(s, s) = 1$ (see Bourbaki [3] and Humphreys [24]).

Two groups are associated to a Coxeter graph: the *Coxeter group* W defined by

$$W = \langle s \in S \mid (st)^{m(s,t)} = 1 \ \forall s, t \in S \text{ such that } m(s, t) \neq \infty \rangle$$

and the *Artin group* G defined by (see Brieskorn and Saito [5] and Deligne [19]):

$$G = \langle s \in S \mid \underbrace{stst\dots}_{m(s,t)\text{-terms}} = \underbrace{tsts\dots}_{m(s,t)\text{-terms}} \ \forall s, t \in S \text{ such that } m(s, t) \neq \infty \rangle.$$

Loosely speaking, G is the group obtained by dropping the relations $s^2 = 1$ ($s \in S$) in the presentation for W .

In this paper, we are primarily interested in Artin groups associated to Coxeter graphs of type A_n , B_n and \tilde{A}_{n-1} (see Figure 1).

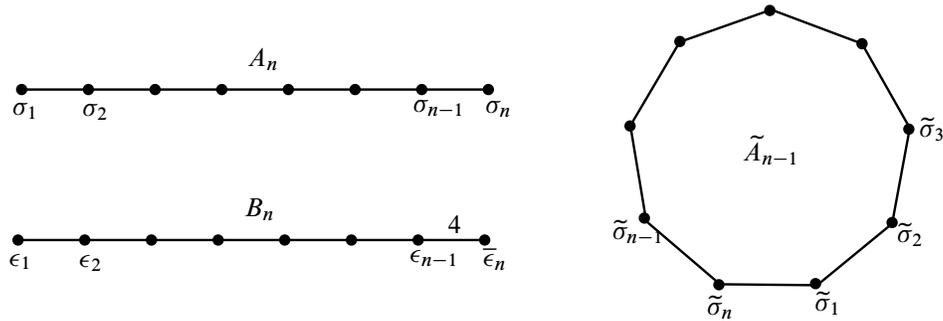


Figure 1: Coxeter graph of type A_n , B_n ($n \geq 2$) and \tilde{A}_{n-1} ($n \geq 3$). Labels equal to 3, as usual, are not shown. Moreover, to fix notation, every vertex is labelled with the corresponding generator in the Artin group.

2.2 Inclusions of Artin groups

Let $\text{Br}_{n+1} := G_{A_n}$ be the braid group on $n + 1$ strands and $\text{Br}_{n+1}^{n+1} < \text{Br}_{n+1}$ be the subgroup of braids fixing the $(n + 1)$ -st strand. The group Br_{n+1}^{n+1} is called the annular braid group, since it can be regarded as the group of braids on n strands on the annulus (see Figure 2).

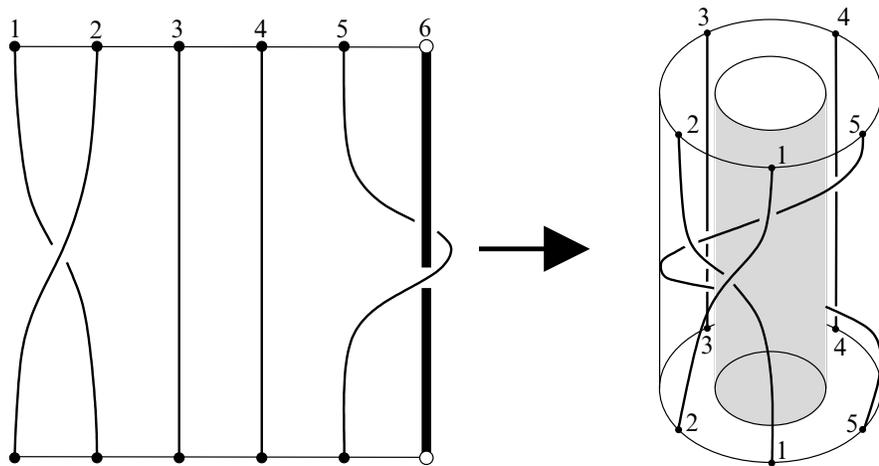


Figure 2: A braid in Br_6^6 represented as an annular braid on 5 strands.

It is well known that the annular braid group is indeed isomorphic to the Artin group G_{B_n} of type B_n . For a proof of the following Theorem see Lambropoulou [26] or Crisp [14].

Theorem 2.1 *Let $\sigma_1, \dots, \sigma_n$ be the standard generators for G_{A_n} and let $\epsilon_1, \dots, \epsilon_{n-1}, \bar{\epsilon}_n$ be the generators for G_{B_n} . The map*

$$\begin{aligned} G_{B_n} &\rightarrow \text{Br}_{n+1}^{n+1} < \text{Br}_{n+1} \\ \epsilon_i &\mapsto \sigma_i \quad \text{for } 1 \leq i \leq n-1 \\ \bar{\epsilon}_n &\mapsto \sigma_n^2 \end{aligned}$$

is an isomorphism.

Using the suggestion given by the identification with the annular braid group, a new interesting presentation for G_{B_n} can be worked out. Let $\tau = \bar{\epsilon}_n \epsilon_{n-1} \cdots \epsilon_2 \epsilon_1$. See Figure 3.

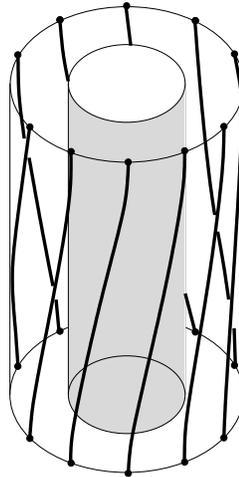


Figure 3: As an annular braid the element τ is obtained turning the bottom annulus by a rotation of $2\pi/n$.

It is easy to verify that:

$$\tau^{-1} \epsilon_i \tau = \epsilon_{i+1} \quad \text{for } 1 \leq i < n-1$$

ie, conjugation by τ shifts forward the first $n-2$ standard generators. By analogy, let $\epsilon_n = \tau^{-1} \epsilon_{n-1} \tau$.

We have the following theorem:

Theorem 2.2 (Kent and Peifer [25]) *The group G_{B_n} has presentation $\langle \mathcal{G} | \mathcal{R} \rangle$ where*

$$\begin{aligned} \mathcal{G} &= \{\tau, \epsilon_1, \epsilon_2, \dots, \epsilon_n\} \\ \mathcal{R} &= \{\epsilon_i \epsilon_j = \epsilon_j \epsilon_i \text{ for } i \neq j-1, j+1\} \cup \\ &\quad \{\epsilon_i \epsilon_{i+1} \epsilon_i = \epsilon_{i+1} \epsilon_i \epsilon_{i+1}\} \cup \\ &\quad \{\tau^{-1} \epsilon_i \tau = \epsilon_{i+1}\} \end{aligned}$$

where all indexes have to be taken modulo n .

Letting $\tilde{\sigma}_1, \tilde{\sigma}_2, \dots, \tilde{\sigma}_n$ be the standard generators of the Artin group of type \tilde{A}_{n-1} , we have the following immediate corollary:

Corollary 2.3 (Kent and Peifer [25], see also tom Dieck [37] and Allcock [1]) *The map*

$$G_{\tilde{A}_{n-1}} \ni \tilde{\sigma}_i \mapsto \epsilon_i \in G_{B_n}$$

gives an isomorphism between $G_{\tilde{A}_{n-1}}$ and the subgroup of G_{B_n} generated by $\epsilon_1, \dots, \epsilon_n$. Moreover, we have a semidirect product decomposition $G_{B_n} \cong G_{\tilde{A}_{n-1}} \rtimes \langle \tau \rangle$.

We have thus a ‘curious’ inclusion of the Artin group of infinite type \tilde{A}_{n-1} into the Artin group of finite type B_n .

Remark The proof of Theorem 2.2 presented in [25] is algebraic and based on Tietze moves; a somewhat more concise proof is obtained by standard topological constructions. Indeed, one can exhibit an explicit infinite cyclic covering $K(G_{\tilde{A}_{n-1}}, 1) \rightarrow K(G_{B_n}, 1)$ (see [1]).

2.3 Local coefficients and induced representations

The previous inclusions allow to relate the homology of the involved groups by means of Shapiro’s lemma (see for instance Brown [6]), of which we explore some consequences.

Let $M := \mathbb{Q}[q^{\pm 1}]$. We indicate by M_q the $G_{\tilde{A}_{n-1}}$ -module where the action of the standard generators is $(-q)$ -multiplication.

Proposition 2.4 *We have*

$$H_*(G_{\tilde{A}_{n-1}}, M_q) \cong H_*(G_{B_n}, M[t^{\pm 1}]_{q,t})$$

$$H^*(G_{\tilde{A}_{n-1}}, M_q) \cong H^*(G_{B_n}, M[[t^{\pm 1}]]_{q,t})$$

where the action of G_{B_n} on $M[t^{\pm 1}]_{q,t}$ (and on $M[[t^{\pm 1}]]_{q,t}$) is given by $(-q)$ -multiplication for the generators $\epsilon_1, \dots, \epsilon_{n-1}$ and $(-t)$ -multiplication for the last generator $\bar{\epsilon}_n$.

Proof Applying Shapiro’s lemma to the inclusion $\tilde{A}_{n-1} < G_{B_n}$, one obtains:

$$H_*(G_{\tilde{A}_{n-1}}, M_q) \cong H_*(G_{B_n}, \text{Ind}_{G_{\tilde{A}_{n-1}}}^{G_{B_n}} M_q)$$

$$H^*(G_{\tilde{A}_{n-1}}, M_q) \cong H^*(G_{B_n}, \text{Coind}_{G_{\tilde{A}_{n-1}}}^{G_{B_n}} M_q).$$

By Corollary 2.3, any element of $\text{Ind}_{G_{\tilde{A}_{n-1}}}^{G_{B_n}} M_q := \mathbb{Z}[G_{B_n}] \otimes_{G_{\tilde{A}_{n-1}}} M_q$ can be represented as a sum of elements of the form $\tau^\alpha \otimes q^m$. Now, we have an isomorphism of $\mathbb{Z}[G_{B_n}]$ -modules

$$\mathbb{Z}[G_{B_n}] \otimes_{G_{\tilde{A}_{n-1}}} M_q \rightarrow M[t^{\pm 1}]_{q,t}$$

defined by sending $\tau^\alpha \otimes q^m \mapsto (-1)^{n\alpha} t^\alpha q^{(n-1)\alpha+m}$ and the result follows.

In cohomology we have similarly:

$$\text{Coind}_{G_{\tilde{A}_{n-1}}}^{G_{B_n}} M_q := \text{Hom}_{G_{\tilde{A}_{n-1}}}(\mathbb{Z}[G_{B_n}], M_q) \cong M[[t^{\pm 1}]]_{q,t}. \quad \square$$

It is interesting to note what happens inducing again via the inclusion $G_{B_n} < G_{A_n}$.

Let $V = \bigoplus_{i=1}^{n+1} \mathbb{Q}[u^{\pm 1}]e_i$ be an $(n + 1)$ -dimensional free $\mathbb{Q}[u^{\pm 1}]$ module.

Definition 2.5 The Tong–Yang–Ma representation [38] is the representation

$$\rho : G_{A_n} \rightarrow \text{Gl}_{\mathbb{Q}[u^{\pm 1}]}(V)$$

defined w.r.t. the basis e_1, \dots, e_{n+1} by:

$$\rho(\sigma_i) = \begin{pmatrix} I_{i-1} & & & \\ & 0 & 1 & \\ & u & 0 & \\ & & & I_{n-i} \end{pmatrix}$$

where I_j denote the j -dimensional identity matrix and all other entries are zero.

We refer to Sysoeva [35] for a discussion of the relevance of Tong–Yang–Ma representation in braid group representation theory. We recall that the image of the pure braid group in the Tong–Yang–Ma representation is abelian; hence this representation factors through the *extended Coxeter group* presented by Tits [36].

Proposition 2.6 We have

$$H_*(G_{B_n}, M[t^{\pm 1}]_{q,t}) \cong H_*(G_{A_n}, M_q \otimes V)$$

$$H^*(G_{B_n}, M[t^{\pm 1}]_{q,t}) \cong H^*(G_{A_n}, M_q \otimes V)$$

where the action of G_{A_n} on M_q is defined sending the standard generators to $(-q)$ -multiplication.

Sketch of proof For the statement in homology, by Shapiro's lemma, it is enough to show that $\text{Ind}_{G_{B_n}}^{G_{A_n}} M[t^{\pm 1}]_{q,t} \cong M_q \otimes V$. Note that $[G_{A_n} : G_{B_n}] = n + 1$ and let choose as coset representatives for G_{A_n}/G_{B_n} the elements

$$\alpha_i = (\sigma_i \sigma_{i+1} \cdots \sigma_{n-1}) \sigma_n (\sigma_i \sigma_{i+1} \cdots \sigma_{n-1})^{-1}$$

for $1 \leq i \leq n-1$, $\alpha_n = \sigma_n$, $\alpha_{n+1} = e$.

Then, by definition of induced representation,

$$\text{Ind}_{G_{B_n}}^{G_{A_n}} M[t^{\pm 1}]_{q,t} = \bigoplus_{i=1}^{n+1} M[t^{\pm 1}]e_i$$

with the following action. For an element $x \in G_{A_n}$, write $x\alpha_k = \alpha_{k'}x'$ with $x' \in G_{B_n}$. Then x acts on an element $r \cdot e_k \in \bigoplus_{i=1}^{n+1} M[t^{\pm 1}]e_i$ as $x(r \cdot e_k) = (x'r) \cdot e_{k'}$.

After some easy computations, one can write the representation in the following matrix form:

$$\sigma_i \mapsto \begin{pmatrix} -qI_{i-1} & & & \\ & 0 & -q & \\ & q^{-1}t & 0 & \\ & & & -qI_{n-i} \end{pmatrix}$$

for $1 \leq i \leq n-1$, whereas

$$\sigma_n \mapsto \begin{pmatrix} -qI_{n-1} & & \\ & 0 & 1 \\ & -t & 0 \end{pmatrix}.$$

Conjugating by $U = \text{Diag}(1, 1, \dots, 1, -q^{-1})$ and setting $u = -q^{-2}t$, one obtains the desired result.

Finally, since $[G_{A_n} : G_{B_n}] = n + 1 < \infty$, the induced and coinduced representation are isomorphic; so the analogous statement in cohomology holds. \square

Remark Specializing q to 1, we have in particular that homology of $G_{\tilde{A}_{n-1}}$ with trivial coefficients is isomorphic to homology of G_{A_n} with coefficients in the Tong–Yang–Ma representation.

By means of Propositions 2.4 and 2.6, in the following all cohomology computations will be performed in G_{B_n} using the double-weight coefficient system. We conclude remarking that cohomology computations can be even reduced from the ring of Laurent series to the ring of Laurent polynomials by the following result about degree shift, obtained by Callegaro [7] in a slightly weaker form, but which is possible to extend to our case with little effort.

Proposition 2.7 (Degree shift)

$$H^*(G_{B_n}, M[[t^{\pm 1}]]_{q,t}) \cong H^{*+1}(G_{B_n}, M[t^{\pm 1}]_{q,t}).$$

2.4 (q, t) -weighted Poincaré series for B_n

For future use in cohomology computations, we are interested in a (q, t) -analog of the usual Poincaré series for B_n . This result and similar ones are studied in Reiner [30], to which we refer for details. We also use classical results from [3; 24] without further reference.

Consider the Coxeter group W of type B_n with its standard generating reflections s_1, s_2, \dots, s_n .

For $w \in W$, let $n(w)$ be the number of times s_n appears in a reduced expression for w . By standard facts, $n(w)$ is well-defined.

Let also $W(q, t) = \sum_{w \in W} q^{\ell(w) - n(w)} t^{n(w)}$ be the (q, t) -weighted Poincaré series, where ℓ is the length function.

We recall some notation. We write $\varphi_m(q)$ for the m -th cyclotomic polynomial in the variable q and we define the q -analog of the number m by the polynomial

$$[m]_q := 1 + q + \dots + q^{m-1} = \frac{q^m - 1}{q - 1}.$$

It is easy to see that $[m] = \prod_{i|m, i \neq 1} \varphi_i(q)$. Moreover we define the q -factorial $[m]_q!$ as the product

$$\prod_{i=1}^m [i]_q$$

and the q -analog of the binomial $\binom{m}{i}$ as the polynomial

$$\left[\begin{matrix} m \\ i \end{matrix} \right]_q := \frac{[m]_q!}{[i]_q! [m-i]_q!}.$$

We can also define the (q, t) -analog of an even number

$$[2m]_{q,t} := [m]_q(1 + tq^{m-1})$$

and of the double factorial

$$[2m]_{q,t}!! := \prod_{i=1}^m [2i]_{q,t} = [m]_q! \prod_{i=0}^{m-1} (1 + tq^i).$$

Finally, we define the polynomial

$$(1) \quad \left[\begin{matrix} m \\ i \end{matrix} \right]'_{q,t} := \frac{[2m]_{q,t}!!}{[2i]_{q,t}!! [m-i]_q!} = \left[\begin{matrix} m \\ i \end{matrix} \right]_q \prod_{j=i}^{m-1} (1 + tq^j).$$

Proposition 2.8 [30]

$$W(q, t) = [2n]_{q,t}!!.$$

Proof Consider the parabolic subgroup W_I associated to the subset of reflections $I = \{s_1, \dots, s_{n-1}\}$. Notice that W_I is isomorphic to the symmetric group on n letters A_{n-1} and that it has index 2^n in B_n . Let W^I be the set of minimal coset representatives for W/W_I . Then, by multiplicative properties on reduced expressions:

$$(2) \quad \begin{aligned} W(q, t) &= \sum_{w \in W} q^{\ell(w)-n(w)} t^{n(w)} \\ &= \left(\sum_{w' \in W^I} q^{\ell(w')-n(w')} t^{n(w')} \right) \cdot \left(\sum_{w'' \in W_I} q^{\ell(w'')-n(w'')} t^{n(w'')} \right). \end{aligned}$$

Clearly, for elements $w'' \in W_I$, we have $n(w'') = 0$; so the second factor in (2) reduces to the well-known Poincaré series for A_{n-1} :

$$\sum_{w'' \in W_I} q^{\ell(w'')-n(w'')} t^{n(w'')} = [n]_q!.$$

The remaining of the proof uses an explicit computation of minimal coset representatives in W^I . □

3 The cohomology of G_{B_n}

In this section we will compute the cohomology groups $H^*(G_{B_n}, R_{q,t})$, where $R_{q,t}$ is the local system over the ring of Laurent polynomials $R = \mathbb{Q}[q^{\pm 1}, t^{\pm 1}]$ and the action is $(-q)$ -multiplication for the standard generators associated to the first $n - 1$

nodes of the Dynkin diagram, while is $(-t)$ -multiplication for the generator associated to the last node.

In order to state our result we need to define, for $m \geq 2$, R -modules

$$\{m\}_i = R/(\varphi_m(q), q^i t + 1).$$

For $m = 1$ we set:

$$\{1\}_i = R/(q^i t + 1).$$

Notice that the modules $\{m\}_i$ are pairwise non isomorphic as R -modules. $\{m\}_i$ and $\{m'\}_{i'}$ are isomorphic as $\mathbb{Q}[q^{\pm 1}]$ -modules if and only if $m = m'$ and are isomorphic as $\mathbb{Q}[t^{\pm 1}]$ -modules if and only if $\phi(m) = \phi(m')$ and $\frac{m}{(m,i)} = \frac{m'}{(m,i')}$.

Our main result is the following:

Theorem 3.1

$$H^i(G_{B_n}, R_{q,t}) = \begin{cases} \bigoplus_{d|n, 0 \leq k \leq d-2} \{d\}_k \oplus \{1\}_{n-1} & \text{if } i = n \\ \bigoplus_{d|n, 0 \leq k \leq d-2, d \leq \frac{n}{j+1}} \{d\}_k & \text{if } i = n - 2j \\ \bigoplus_{d|n, d \leq \frac{n}{j+1}} \{d\}_{n-1} & \text{if } i = n - 2j - 1 \end{cases}$$

To perform our computation we will use a method quite similar to [15], namely the complex introduced in [32], and the spectral sequence induced by a natural filtration.

Recall from [32] that the complex that compute the cohomology of G_{B_n} over $R_{q,t}$ is given as follows:

$$C_n^* = \bigoplus_{\Gamma \subset I_n} R \cdot \Gamma$$

where I_n denote the set $\{1, \dots, n\}$ and the graduation is given by $|\Gamma|$.

The set I_n corresponds to the set of nodes of the Dynkin diagram of B_n and in particular the last element, n , corresponds to the last node.

It is useful to consider also the analog complex \overline{C}_n^* for the cohomology of G_{A_n} on the local system $R_{q,t}$. In this case the action associated to a standard generator is always the $(-q)$ -multiplication and so the complex \overline{C}_n^* and its cohomology are free as $\mathbb{Q}[t^{\pm 1}]$ -modules. The complex \overline{C}_n^* is isomorphic to C_n^* as an R -module. In both complexes the coboundary map is

$$(3) \quad \delta(q,t)(\Gamma) = \sum_{j \in I_n \setminus \Gamma} (-1)^{\sigma(j,\Gamma)} \frac{W_{\Gamma \cup \{j\}}(q,t)}{W_{\Gamma}(q,t)} (\Gamma \cup \{j\})$$

where $\sigma(j, \Gamma)$ is the number of elements of Γ that are less than j in the natural ordering. In the case A_n , $W_{\Gamma}(q, t)$ is the Poincaré polynomial of the parabolic subgroup

$W_\Gamma \subset A_n$ generated by the elements in the set Γ , with weight q for each standard generator, while in the case B_n $W_\Gamma(q, t)$ is the Poincaré polynomial of the parabolic subgroup $W_\Gamma \subset B_n$ generated by the elements in the set Γ , with weight q for the first $n - 1$ generators and t for the last generator.

Using Proposition 2.8 we can give an explicit computation of the coefficients appearing in (3). For any $\Gamma \subset I_n$, let $\bar{\Gamma}$ be the subgraph of the Dynkin diagram B_n which is spanned by Γ . Recall that if $\bar{\Gamma}$ is a connected component of the Dynkin diagram of B_n without the last element, then

$$W_\Gamma(q, t) = [m + 1]_q!,$$

where $m = |\Gamma|$. If $\bar{\Gamma}$ is connected and contains the last element of B_n , then by Proposition 2.8

$$W_\Gamma(q, t) = [2m]_{q,t}!!,$$

where $m = |\Gamma|$.

If $\bar{\Gamma}$ is the union of several connected components of the Dynkin diagram, $\bar{\Gamma} = \bar{\Gamma}_1 \cup \dots \cup \bar{\Gamma}_k$, then $W_\Gamma(q, t)$ is the product

$$\prod_{i=1}^k W_{\Gamma_i}(q, t)$$

of the factors corresponding to the different components.

If $j \notin \Gamma$ we can write $\bar{\Gamma}(j)$ for the connected component of $\overline{\Gamma \cup \{j\}}$ containing j . Suppose that $m = |\Gamma(j)|$ and i is the number of elements in $\Gamma(j)$ greater than j . Then, if $n \in \Gamma(j)$ we have

$$\frac{W_{\Gamma \cup \{j\}}(q, t)}{W_\Gamma(q, t)} = \left[\begin{matrix} m \\ i \end{matrix} \right]_{q,t}'$$

and

$$\frac{W_{\Gamma \cup \{j\}}(q, t)}{W_\Gamma(q, t)} = \left[\begin{matrix} m + 1 \\ i + 1 \end{matrix} \right]_q$$

otherwise.

Sketch of proof of Theorem 3.1 It is convenient to represent generators $\Gamma \subset I_n$ by their characteristic functions $I_n \rightarrow \{0, 1\}$ so, simply by strings of 0's and 1's of length n .

We define a decreasing filtration F on the complex (C_n^*, δ) : $F^s C_n$ is the subcomplex generated by the strings of type $A1^s$ (ending with a string of $s-1$ s) and we have the inclusions

$$C_n = F^0 C_n \supset F^1 C_n \supset \dots \supset F^n C_n = R.1^n \supset F^{n+1} C_n = 0.$$

We have the following isomorphism of complexes:

$$(4) \quad (F^s C_n / F^{s+1} C_n) \simeq \overline{C}_{n-s-1}[s]$$

where \overline{C}_{n-s-1} is the complex for $G_{A_{n-s-1}}$ and the notation $[s]$ means that the degree is shifted by s .

The proof uses the spectral sequence E_* associated to the filtration F . The equality (4) tells us how the E_1 term of the spectral sequence looks like. In fact for $0 \leq s \leq n-2$ we have

$$(5) \quad E_1^{s,r} = H^r(G_{A_{n-s-1}}, R_{q,t}) = H^r(G_{A_{n-s-1}}, \mathbb{Q}[q^{\pm 1}]_q)[t^{\pm 1}]$$

since the t -action is trivial. For $s = n-1$ and $s = n$ the only non trivial elements in the spectral sequence are

$$(6) \quad E_1^{n-1,0} = E_1^{n,0} = R.$$

If we write $\{m\}[t^{\pm 1}]$ for the module $R/(\varphi_m(q))$, then the E_1 -term of the spectral sequence has a module $\{m\}[t^{\pm 1}]$ in position (s, r) if and only if one of the following condition is satisfied:

- a) $m \mid n-s-1$ and $r = n-s-2\frac{n-s-1}{m}$;
- b) $m \mid n-s$ and $r = n-s+1-2(\frac{n-s}{m})$.

We know the generators of these modules from [15]. Moreover (see formula (6)) we have modules R in position $(n-1, 0)$ and $(n, 0)$. The differentials are expressed in terms of modified binomials defined in formula (1). Then the proof is obtained by a subtle analysis of such differentials. \square

4 Some consequences

Theorem 3.1 gives the cohomology of G_{B_n} as well as (using Proposition 2.4) that of $G_{\tilde{A}_{n-1}}$ if we consider it only as $\mathbb{Q}[q^{\pm 1}]$ -module. As regards the rational cohomology, Proposition 2.4 translates into the following:

Proposition 4.1 *We have*

$$H_*(G_{\tilde{A}_{n-1}}, \mathbb{Q}) \cong H_*(G_{B_n}, \mathbb{Q}[t^{\pm 1}])$$

$$H^*(G_{\tilde{A}_{n-1}}, \mathbb{Q}) \cong H^*(G_{B_n}, \mathbb{Q}[[t^{\pm 1}]])$$

where the action of G_{B_n} on $\mathbb{Q}[t^{\pm 1}]$ (and on $\mathbb{Q}[[t^{\pm 1}]])$ is trivial for the generators $\epsilon_1, \dots, \epsilon_{n-1}$ and $(-t)$ -multiplication for the last generator $\bar{\epsilon}_n$.

The cohomology of G_{B_n} over the module $\mathbb{Q}[t^{\pm 1}]$, with action as in Proposition 4.1, is computed by the complex C_n^* of Section 3 where we specialize q to -1 . So we use similar filtration and associated spectral sequence. Recall that the \mathbb{Q} cohomology of the braid group is of rank 1 in dimension 0, 1, and vanishes elsewhere. Then by using a formula analog to (5) we get

$$\begin{aligned} E_1^{s,r} &= \mathbb{Q}[t^{\pm 1}] \text{ if } 0 \leq s \leq n, r = 0 \text{ or } 0 \leq s \leq n-2, r = 1 \\ &= 0 \text{ otherwise} \end{aligned}$$

Next from formula (3) it follows

$$d_1^{s,r} = \{[s+1]_q (1+q^s t)\}_{\{q=-1\}}, r = 0, 1$$

so $d_1^{s,r} = 0$ for odd s while $d_1^{s,r} = 1+t$ for even s . It follows that in E_2 the odd columns are obtained from the same columns of E_1 dividing by $1+t$. The even columns vanish, except for n even it remains

$$E_2^{n-2,1} = E_2^{n,0} = \mathbb{Q}[t^{\pm 1}].$$

The only possible non vanishing boundaries are

$$d_2^{s,1} : E_2^{s,1} \rightarrow E_2^{s+2,0}$$

and these are of the form

$$\left[\begin{matrix} s+2 \\ s \end{matrix} \right]'_{[q=-1],t}.$$

Up to an invertible, the latter holds $(1+t)(1-t)$. Then d_2 vanishes except for $d_2^{n-2,1}$ in case n even. It follows that:

Theorem 4.2 *One has*

$$\begin{aligned} H^k(G_{B_n}, \mathbb{Q}[t^{\pm 1}]) &= \mathbb{Q}[t^{\pm 1}]/(1+t) & 1 \leq k \leq n-1 \\ H^n(G_{B_n}, \mathbb{Q}[t^{\pm 1}]) &= \mathbb{Q}[t^{\pm 1}]/(1+t) & \text{for odd } n \\ H^n(G_{B_n}, \mathbb{Q}[t^{\pm 1}]) &= \mathbb{Q}[t^{\pm 1}]/(1-t^2) & \text{for even } n. \end{aligned}$$

To obtain the rational cohomology of $G_{\tilde{A}_{n-1}}$ we need to apply the degree shift in Proposition 2.7.

Notice how Proposition 2.6 changes in the present situation.

Proposition 4.3 *We have*

$$H_*(G_{B_n}, \mathbb{Q}[t^{\pm 1}]) \cong H_*(G_{A_n}, V)$$

$$H^*(G_{B_n}, \mathbb{Q}[t^{\pm 1}]) \cong H^*(G_{A_n}, V)$$

where V is the representation of G_{A_n} defined in 2.5.

As a consequence we have:

Corollary 4.4 *Let V be the $(n + 1)$ -dimensional representation of the braid group Br_{n+1} defined in 2.5. Then the cohomology*

$$H^*(Br_{n+1}; V)$$

is given as in Theorem 4.2.

5 Related topological constructions

In [32] the orbit space of any Artin group of finite type, which is known to be a $K(\pi, 1)$ space [19], was shown to contract over an explicit polyhedron with explicit identifications on its faces (a construction based on [31] applied to Coxeter arrangements). As already suggested, few modifications are needed to obtain a similar description of the orbit space for Artin groups of infinite type (see also [10] for a different construction).

We briefly resume this construction.

Let (W, S) be a (finitely generated) Coxeter group, which we realize through the Tits representation as a group of (in general, non orthogonal) reflections in \mathbb{R}^n , where the base-chamber C_0 is the positive octant and S is the set of reflections with respect to the coordinate hyperplanes. (It is possible to consider more general representations; see Vinberg [41]). Let $U := W \cdot \overline{C_0}$ be the orbit of the closure of the base chamber (the *Tits cone*). Recall from [41] that:

- U is a convex cone in \mathbb{R}^n with vertex 0.
- $U = \mathbb{R}^n$ iff W is finite.
- $U^0 := \text{int}(U)$ is open in \mathbb{R}^n and a (relative open) facet $F \subset \overline{C_0}$ is contained in U^0 iff the stabilizer W_F is finite.

Let \mathcal{A} be the arrangement of reflection hyperplanes of W . Set

$$M(\mathcal{A}) := [U^0 + i\mathbb{R}^n] \setminus \bigcup_{H \in \mathcal{A}} H_{\mathbb{C}}$$

as the complement of the complexified arrangement. Notice that the group W acts freely on $M(\mathcal{A})$ so we can consider the orbit space

$$M(\mathcal{A})_W := M(\mathcal{A})/W.$$

The associated Artin group G_W is the fundamental group of the orbit space (see Brieskorn [4], Dũng [20] and van der Lek [40]).

Now take one point $x_0 \in C_0$; for any subset $J \subset S$ such that the parabolic subgroup W_J is finite, construct a $|J|$ -cell in U^0 as the ‘‘convex hull’’ of the W_J -orbit of x_0 in \mathbb{R}^n .

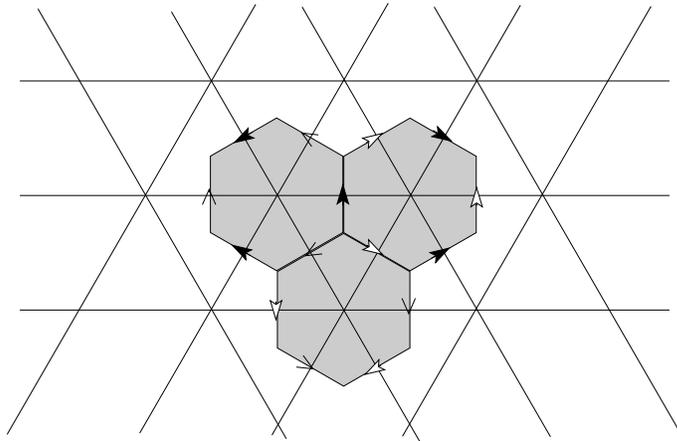


Figure 4: the space $K(G_{\tilde{A}_2}, 1)$ is given as union of 3 hexagons with edges glued according to the arrows (there are: 1 0-cell, 3 1-cells, 3 2-cells in the quotient).

So, we obtain a finite cell complex (see Figure 4) which is the union of (in general, different dimensional) polyhedra, corresponding to the maximal subsets J such that W_J is finite. Now take identifications on the faces of these polyhedra, the same as described in [32] for the finite case (they are shown in Figure 4 for the case \tilde{A}_2). We obtain a finite CW-complex X_W : it has a $|J|$ -cell for each $J \subset S$ such that W_J is finite.

We obtain as in [32]

Theorem 5.1 X_W is a deformation retract of the orbit space.

Remark When W is an affine group, the orbit space is known to be a $K(\pi, 1)$ for types \tilde{A}_n , \tilde{C}_n (see [29; 11; 10] for further classes).

Remark The standard presentation for G_W is quite easy to derive from the topological description of X_W ; we may thus recover Van del Lek's result [40].

Proposition 5.2 Let $K_W^{fin} := \{J \subset S : |W_J| < \infty\}$ with the natural structure of simplicial complex. Then the Euler characteristic of the orbit space (so, of the group G_W when such space is of type $k(\pi, 1)$) equals

$$\chi(K_W^{fin}).$$

In particular, if W is affine of rank $n + 1$ we have

$$\chi(M(\mathcal{A})_W) = \chi(K_W^{fin}) = 1 - \chi(S^{n-1}) = (-1)^n$$

Proof Last statement follows from the fact that K_W^{fin} contains all proper subsets of S ; thus:

$$H_*(K_W^{fin}) = \tilde{H}_{*-1}(S^{n-1}). \quad \square$$

Remark The cohomology of the orbit space in case \tilde{A}_n with trivial coefficients is deduced from Proposition 4.1 and from Theorem 4.2; that with local coefficients in the $G_{\tilde{A}_n}$ -module $\mathbb{Q}[q^{\pm 1}]$ is deduced from Theorem 3.1.

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