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Basis-conjugating automorphisms of a free group and associated Lie algebras

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Let $F_n = \langle x_1, \ldots, x_n \rangle$ denote the free group with generators $\{x_1, \ldots, x_n\}$. Nielsen and Magnus described generators for the kernel of the canonical epimorphism from the automorphism group of F_n to the general linear group over the integers. In particular among them are the automorphisms $\chi_{k,i}$ which conjugate the generator x_k by the generator x_i leaving the x_j fixed for $j \neq k$. A computation of the cohomology ring as well as the Lie algebra obtained from the descending central series of the group generated by $\chi_{k,i}$ for $i < k$ is given here. Partial results are obtained for the group generated by all $\chi_{k,i}$.

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1 Introduction

Let π be a discrete group with $\mathrm{Aut}(\pi)$ the automorphism group of π. Consider the free group F_n generated by n letters $\{x_1, x_2, \cdots, x_n\}$. The kernel of the natural map

$$\mathrm{Aut}(F_n) \to GL(n, \mathbb{Z})$$

is denoted IA_n. Nielsen, and Magnus gave automorphisms which generate IA_n as a group [20; 16; 17]. These automorphisms are named as follows:

- $\chi_{k,i}$ for $i \neq k$ with $1 \leq i, k \leq n$, and
- $\theta(k; [s,t])$ for k, s, t distinct integers with $1 \leq k, s, t \leq n$ and $s < t$.

The definition of the map $\chi_{k,i}$ is given by the formula

$$\chi_{k,i}(x_j) = \begin{cases} x_j & \text{if } k \neq j, \\ (x_i^{-1})(x_k)(x_i) & \text{if } k = j. \end{cases}$$

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Thus the map $\chi_{k,i}^{-1}$ satisfies the formula

$$\chi_{k,i}^{-1}(x_j) = \begin{cases} x_j & \text{if } k \neq j, \\ (x_i)(x_k)(x_i{}^{-1}) & \text{if } k = j. \end{cases}$$

The map $\theta(k;[s,t])$ is defined by the formula

$$\theta(k;[s,t])(x_j) = \begin{cases} x_j & \text{if } k \neq j, \\ (x_k) \cdot ([x_s, x_t]) & \text{if } k = j. \end{cases}$$

for which the commutator is given by $[a,b] = a^{-1} \cdot b^{-1} \cdot a \cdot b$. Thus the map $\theta(k;[s,t])^{-1}$ satisfies the formula

$$\theta(k;[s,t])^{-1}(x_j) = \begin{cases} x_j & \text{if } k \neq j, \\ (x_k) \cdot ([x_s, x_t]^{-1}) & \text{if } k = j. \end{cases}$$

Consider the subgroup of IA_n generated by the $\chi_{k,i}$, the group of basis conjugating automorphisms of a free group. This subgroup has topological interpretations. First of all it is the pure group of motions of n unlinked circles in S^3 (Goldsmith [9] and Jensen, McCammond and Meier [12]) and because of this it is known as the "group of loops". On the other hand it is also the pure braid–permutation group. This is explained at the end of this section. This group is denoted $P\Sigma_n$ in [12]. McCool gave a presentation for it [18]. This presentation is listed in Theorem 1.1 below. The subgroup of $P\Sigma_n$ generated by the $\chi_{k,i}$ for $i < k$ is denoted $P\Sigma_n^+$ here and is called the "upper triangular McCool group" in [3].

The purpose of this article is to determine the natural Lie algebra structure obtained from the descending central series for $P\Sigma_n^+$ together with related information for $P\Sigma_n$ as well as the structure of the cohomology ring of $P\Sigma_n^+$. One motivation for the work here is that the groups $P\Sigma_n$ and $P\Sigma_n^+$ are natural as well as accessible cases arising as analogues of work in D Johnson [13], S Morita [19], D Hain [10], B Farb [6], N Kawazumi [14], T Kohno [15], C Jensen, J McCammond, and J Meier [12], T Sakasai [22], T Satoh [23], A Pettet [21] and Y Ihara [11]. In those works the Johnson filtration is used frequently rather than the descending central series. The techniques here for addressing these Lie algebras are due to T Kohno [15] and M Falk and R Randell [5].

The cohomology of $P\Sigma_n$ was computed by C Jensen, J McCammond, and J Meier [12]. N Kawazumi [14], T Sakasai [22], T Satoh [23] and A Pettet [21] have given related cohomological information for IA_n. The integral cohomology of the natural direct limit of the groups $\mathrm{Aut}(F_n)$ is given in work of S Galatius [8].

The main results here arise from McCool's presentation which is stated next.

Theorem 1.1 *A presentation of $P\Sigma_n$ is given by generators $\chi_{k,j}$ together with the following relations.*

(1) $\chi_{i,j} \cdot \chi_{k,j} \cdot \chi_{i,k} = \chi_{i,k} \cdot \chi_{i,j} \cdot \chi_{k,j}$ *for* i, j, k *distinct.*

(2) $[\chi_{k,j}, \chi_{s,t}] = 1$ *if* $\{j,k\} \cap \{s,t\} = \varnothing$.

(3) $[\chi_{i,j}, \chi_{k,j}] = 1$ *for* i, j, k *distinct.*

(4) $[\chi_{i,j} \cdot \chi_{k,j}, \chi_{i,k}] = 1$ *for* i, j, k *distinct (redundantly).*

In what follows below, $gr^*(\pi)$ denotes the associated graded Lie algebra obtained from the descending central series of a discrete group π. Work of T Kohno [15], as well as M Falk, and R Randell [5] provide an important description of these Lie algebras for many groups π, one of which is the pure braid group on n strands P_n. The Lie algebra $gr^*(P_n)$, basic in Kohno's work, gave an important ingredient in his analysis of Vassiliev invariants of pure braids in terms of iterated integrals [15]. A presentation for this Lie algebra is given by the quotient of the free Lie algebra $L[B_{i,j} | 1 \le i < j \le k]$ generated by elements $B_{i,j}$ with $1 \le i < j \le k$ modulo the "infinitesimal braid relations" or "horizontal $4T$ relations" given by the following three relations:

(1) If $\{i,j\} \cap \{s,t\} = \varnothing$, then $[B_{i,j}, B_{s,t}] = 0$.

(2) If $i < j < k$, then $[B_{i,j}, B_{i,k} + B_{j,k}] = 0$.

(3) If $i < j < k$, then $[B_{i,k}, B_{i,j} + B_{j,k}] = 0$.

The results below use the methods of Kohno, and Falk–Randell to obtain information about $P\Sigma_n$, and $P\Sigma_n^+$ as well. One feature is that the Lie algebras given by $gr^*(P\Sigma_n)$ and $gr^*(P\Sigma_n^+)$ satisfy two of the "horizontal $4T$ relations".

The next theorem is technical, but provides the foundation required to prove the main results here; the proof is given in Section 5.

Theorem 1.2 *There exist homomorphisms*

$$\pi\colon P\Sigma_n \to P\Sigma_{n-1}$$

defined by

$$\pi(\chi_{k,i}) = \begin{cases} \chi_{k,i} & \text{if } i < n, \text{ and } k < n, \\ 1 & \text{if } i = n \text{ or } k = n. \end{cases}$$

The homomorphism $\pi\colon P\Sigma_n \to P\Sigma_{n-1}$ is an epimorphism. The kernel of π denoted K_n is generated by the elements $\chi_{n,i}$ and $\chi_{j,n}$ for $1 \le i, j \le n-1$. Furthermore, this extension is split and the conjugation action of $P\Sigma_{n-1}$ on $H_1(K_n)$ is trivial.

In addition, the homomorphism $\pi\colon P\Sigma_n \to P\Sigma_{n-1}$ *restricts to a homomorphism*

$$\pi|_{P\Sigma_n^+}\colon P\Sigma_n^+ \to P\Sigma_{n-1}^+$$

which is an epimorphism. The kernel of $\pi|_{P\Sigma_n^+}$ *denoted* K_n^+ *is a free group generated by the elements* $\chi_{n,i}$ *for* $1 \leq i \leq n-1$. *Furthermore, this extension is split and the conjugation action of* $P\Sigma_{n-1}^+$ *on* $H_1(K_n^+)$ *is trivial.*

Remark 1 After this paper was submitted, the authors learned that the result in Theorem 1.2 stating that $\pi\colon P\Sigma_n \to P\Sigma_{n-1}$ is a split epimorphism was proved earlier by Bardakov in [1] where other natural properties are developed. In addition, the feature that K_n is a semi-direct product as stated in Theorem 1.3 below was also proved in [1].

Provided that it is clear from the context, the notation $\chi_{k,i}$ is used ambiguously to denote both the element $\chi_{k,i}$ in $P\Sigma_n$, or in $P\Sigma_n^+$ when defined, as well as the equivalence class of $\chi_{k,i}$ in $gr^1(P\Sigma_n) = H_1(P\Sigma_n)$ or in $H_1(P\Sigma_n^+)$ when defined. Partial information concerning the Lie algebra $gr^*(P\Sigma_n)$ is given next.

Theorem 1.3 *There is a split short exact sequence of Lie algebras*

$$0 \to gr^*(K_n) \to gr^*(P\Sigma_n) \to gr^*(P\Sigma_{n-1}) \to 0.$$

The relations (1)–(4) are satisfied on the level of Lie algebras:

(1) *If* $\{i,j\} \cap \{s,t\} = \varnothing$, *then* $[\chi_{j,i}, \chi_{s,t}] = 0$.

(2) *If* i, j, k *are distinct, then* $[\chi_{i,k}, \chi_{i,j} + \chi_{k,j}] = 0$.

(3) *If* i, j, k *are distinct, the element* $[\chi_{k,i}, \chi_{j,i} + \chi_{j,k}]$ *is non-zero.*

(4) *If* i, j, k *are distinct,* $[\chi_{i,j}, \chi_{k,j}] = 0$.

Furthermore, there is a split epimorphism

$$\gamma\colon K_n \to \oplus_{n-1}\mathbb{Z}$$

with kernel denoted Λ_n *together with a split short exact sequence of Lie algebras*

$$0 \longrightarrow \mathbb{L}_n \longrightarrow gr^*(K_n) \xrightarrow{gr^*(\gamma)} \oplus_{n-1}\mathbb{Z} \longrightarrow 0$$

where $\mathbb{L}_n$ *is the Lie algebra kernel of* $gr^*(\gamma)$.

The same methods give a complete description for the cohomology algebra of $P\Sigma_n^+$ as well as the Lie algebra $gr^*(P\Sigma_n^+)$. A further application to be given later is a substantial contribution to the cohomology of each of the Johnson filtrations of IA_n.

To express the answers, the notation $\chi^*_{k,i}$ is used to denote the dual basis element to $\chi_{k,i}$, namely

$$\chi^*_{k,i}(\chi_{s,t}) = \begin{cases} 1 & \text{if } k = s \text{ and } i = t \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1.4 *The cohomology algebra of $P\Sigma_n^+$ satisfies the following properties.*

(1) *Each graded piece $H^k(P^+\Sigma_n)$ is a finitely generated, torsion-free abelian group.*

(2) *If $1 \le k \le n$, a basis for $H^k P\Sigma_n^+$ is given by $\chi^*_{i_1,j_1} \cdot \chi^*_{i_2,j_2} \cdots \chi^*_{i_k,j_k}$ where $2 \le i_1 < i_2 < \cdots < i_k \le n$, and $1 \le j_t < i_t$ for all t.*

(3) *A complete set of relations (assuming graded commutativity, and associativity) is given by*
- ${\chi^*_{i,k}}^2 = 0$ *for all $i > k$, and*
- $\chi^*_{i,j}[\chi^*_{i,k} - \chi^*_{j,k}] = 0$ *for $k < j < i$.*

The Lie algebra obtained from the descending central series for $P\Sigma_n^+$, $gr^*(P\Sigma_n^+)$, is additively isomorphic to a direct sum of free sub-Lie algebras

$$\bigoplus_{2 \le k \le n} L[\chi_{k,1}, \chi_{k,2}, \cdots, \chi_{k,k-1}],$$

with

- $[\chi_{k,j}, \chi_{s,t}] = 0$ *if $\{j,k\} \cap \{s,t\} = \varnothing$,*
- $[\chi_{k,j}, \chi_{s,j}] = 0$ *if $\{s,k\} \cap \{j\} = \varnothing$ and*
- $[\chi_{i,k}, \chi_{i,j} + \chi_{k,j}] = 0$ *for $j < k < i$.*

Remark 2 The structure of the cohomology algebra described in Theorem 1.4 corresponds to the structure of the algebra $H^*(P\Sigma_n, \mathbb{Z})$, as given in Jensen, McCammond and Meier [12], under the map induced by the canonical inclusion $P\Sigma_n^+ \subset P\Sigma_n$. Namely, the elements $\chi^*_{k,i}$ with $k < i$ are mapped to zero and relations of $H^*(P\Sigma_n, \mathbb{Z})$ become the relation (3) of Theorem 1.4.

Remark 3 Theorem 1.4 does not rule out the possibility that $P\Sigma_n^+$ is isomorphic to the pure braid group P_n. Notice that $P\Sigma_3^+$ is isomorphic to $\mathbb{Z} \times F[\chi_{3,1}, \chi_{3,2}]$ and thus P_3 where a generator of $\mathbb{Z}$ is given by $\chi_{2,1} \cdot \chi_{3,1}$. In fact, Theorem 1.4 implies that after suspending the classifying spaces of both $P\Sigma_n^+$ and P_n exactly once, these suspended classifying spaces are homotopy equivalent as they are both finite bouquets of spheres with the same dimensions.

Let $U[\mathcal{L}]$ denote the universal enveloping algebra of a Lie algebra $\mathcal{L}$. Since the Euler–Poincaré series for $U[L[\chi_{k,1}, \chi_{k,2}, \cdots, \chi_{k,k-1}]]$ is $1/(1-(k-1)t)$, the next corollary follows at once.

Corollary 1.5 *The Euler–Poincaré series for $U[P\Sigma_n^+]$ is equal to*

$$\prod_{1\le k\le n-1} 1/(1-kt).$$

Definition 1.6 Let $\mathbb{M}_n$ denote the smallest subalgebra of $H^*(IA_n;\mathbb{Z})$ such that

(1) $\mathbb{M}_n$ surjects to $H^*(P\Sigma_n^+)$ (that such a surjection exists follows from Theorem 1.7) and

(2) $\mathbb{M}_n$ is closed respect to the conjugation action of $GL(n,\mathbb{Z})$ on $H^*(IA_n;\mathbb{Z})$.

Theorem 1.7 *The natural inclusion $j\colon P\Sigma_n^+ \to IA_n$ composed with the abelianization map $A\colon IA_n \to IA_n/[IA_n, IA_n] = \oplus_{n\binom{n}{2}}\mathbb{Z}$ given by*

$$P\Sigma_n^+ \xrightarrow{\ j\ } IA_n \xrightarrow{\ A\ } IA_n/[IA_n, IA_n] = H_1(IA_n)$$

induces a split epimorphism in integral cohomology. Thus the integral cohomology of $P\Sigma_n^+$ is a direct summand of the integer cohomology of IA_n (a summand which is not invariant under the action of $GL(n,\mathbb{Z})$). Furthermore, the image of

$$A^*\colon H^*(IA_n/[IA_n, IA_n]) \to H^*(IA_n)$$

contains $\mathbb{M}_n$.

In addition, the suspension of the classifying space $BP\Sigma_n^+$, $\Sigma(BP\Sigma_n^+)$, is a retract of $\Sigma(BIA_n)$ and there is an induced map

$$\theta\colon BIA_n \to \Omega\Sigma(BP\Sigma_n^+)$$

which factors the Freudenthal suspension $E\colon BP\Sigma_n^+ \to \Omega\Sigma(BP\Sigma_n^+)$ given by the composite

$$BP\Sigma_n^+ \to BIA_n \to \Omega\Sigma(BP\Sigma_n^+).$$

Remark 4 Properties of the image of A^* are addressed in work of N Kawazumi [14], T Sakasai [22], T Satoh [23] and A Pettet [21]. An analogous map $BIA_n \to \Omega\Sigma(BP\Sigma_n)$ is constructed in work of C Jensen, J McCammond and J Meier [12].

The remainder of this introduction is devoted to the structure of the *braid–permutation group* BP_n introduced by R Fenn, R Rimányi and C Rourke [7]. The group BP_n is defined as the subgroup of $\mathrm{Aut}(F_n)$ generated by ξ_i and σ_i, where the action of an element ϕ in $\mathrm{Aut}(F_n)$ is from the right with

$$(x_j)\xi_i = \begin{cases} x_{i+1} & j=i, \\ x_i & j=i+1, \\ x_j & \text{otherwise;} \end{cases}$$

$$(x_j)\sigma_i = \begin{cases} x_{i+1} & j = i, \\ x_{i+1}^{-1} x_i x_{i+1} & j = i+1, \\ x_j & \text{otherwise.} \end{cases}$$

The group BP_n is presented by the set of generators ξ_i and σ_i for $1 \le i \le n-1$, and by the relations:

(1)
$$\begin{cases} \xi_i^2 = 1, & \\ \xi_i \xi_j = \xi_j \xi_i & |i-j| > 1, \\ \xi_i \xi_{i+1} \xi_i = \xi_{i+1} \xi_i \xi_{i+1}; & \end{cases}$$

(2)
$$\begin{cases} \sigma_i \sigma_j = \sigma_j \sigma_i & |i-j| > 1, \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}; & \end{cases}$$

(3)
$$\begin{cases} \xi_i \sigma_j = \sigma_j \xi_i & |i-j| > 1, \\ \xi_i \xi_{i+1} \sigma_i = \sigma_{i+1} \xi_i \xi_{i+1}, & \\ \sigma_i \sigma_{i+1} \xi_i = \xi_{i+1} \sigma_i \sigma_{i+1}, & \end{cases}$$

The group BP_n is also characterized [7] as the the subgroup of $\mathrm{Aut}(F_n)$ consisting of automorphism $\phi \in \mathrm{Aut}(F_n)$ of *permutation–conjugacy type* which satisfy

(4)
$$(x_i)\phi = w_i^{-1} x_{\lambda(i)} w_i$$

for some word $w_i \in F_n$ and permutation $\lambda \in \Sigma_n$ the symmetric group on n letters.

Theorem 1.8 *The group BP_n is the semi-direct product of the symmetric group on n–letters Σ_n and the group $P\Sigma_n$ with a split extension*

$$1 \longrightarrow P\Sigma_n \longrightarrow \mathrm{BP}_n \longrightarrow \Sigma_n \longrightarrow 1.$$

Theorems 1.3, 1.4 and 1.8 provide some information about the cohomology as well as Lie algebras associated to BP_n.

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2 Projection maps $P\Sigma_n \to P\Sigma_{n-1}$

Consider the map

$$p\colon F_n \to F_{n-1}$$

defined by the formula

$$p(x_j) = \begin{cases} x_j & \text{if } j \leq n-1, \\ 1 & \text{if } j = n. \end{cases}$$

In addition, let $\nu\colon P\Sigma_n^+ \to P\Sigma_n$ denote the natural inclusion.

Theorem 2.1 *The projection maps $p\colon F_n \to F_{n-1}$ induce homomorphisms*

$$\pi\colon P\Sigma_n \to P\Sigma_{n-1}$$

given by

$$\pi(\chi_{k,i}) = \begin{cases} \chi_{k,i} & \text{if } i < n, \text{ and } k < n, \\ 1 & \text{if } i = n \text{ or } k = n. \end{cases}$$

Furthermore, these homomorphisms restrict to

$$\pi\colon P\Sigma_n^+ \to P\Sigma_{n-1}^+$$

together with a commutative diagram:

$$\begin{array}{ccc} P\Sigma_n^+ & \xrightarrow{\pi} & P\Sigma_{n-1}^+ \\ \nu\downarrow & & \downarrow\nu \\ P\Sigma_n & \xrightarrow{\pi} & P\Sigma_{n-1} \end{array}$$

Proof Consider the following commutative diagrams:

$$\begin{array}{ccc} F_n & \xrightarrow{\chi_{n,j}} & F_n \\ p\downarrow & & \downarrow p \\ F_{n-1} & \xrightarrow{1} & F_{n-1} \end{array}$$

$$\begin{array}{ccc} F_n & \xrightarrow{\chi_{j,n}} & F_n \\ p\downarrow & & \downarrow p \\ F_{n-1} & \xrightarrow{1} & F_{n-1} \end{array}$$

If $k, j < n$, then the following diagram commutes:

$$\begin{array}{ccc} F_n & \xrightarrow{\chi_{j,k}} & F_n \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ F_{n-1} & \xrightarrow{\chi_{j,k}} & F_{n-1} \end{array}$$

Thus, if any of

$$i,k < n \quad \text{or} \quad i = n \quad \text{or} \quad k = n$$

hold, then the functions $\chi_{k,i}$ restrict to isomorphisms of F_{n-1}. The restriction is evidently compatible with composition of isomorphisms. Hence there is an induced homomorphism

$$\pi\colon P\Sigma_n \to P\Sigma_{n-1}$$

given by

$$\pi(\chi_{k,i}) = \begin{cases} \chi_{k,i} & \text{if } i < n \text{ and } k < n, \\ 1 & \text{if } i = n \text{ or } k = n. \end{cases}$$

These homomorphisms are compatible with the inclusion maps $\nu\colon P\Sigma_n^+ \to P\Sigma_n$, and the theorem follows. □

3 On automorphisms of $P\Sigma_n$, and $P\Sigma_n^+$

The conjugation action of $\mathrm{Aut}(F_n)$ on itself restricted to certain natural subgroups of $\mathrm{Aut}(F_n)$ has $P\Sigma_n$, and $P\Sigma_n^+$ as characteristic subgroups. The purpose of this section is to give two such natural subgroups. One of these subgroups is used below to determine the relations in the cohomology algebra for $P\Sigma_n^+$.

A choice of generators for $\mathrm{Aut}(F_n)$ is listed next. Let σ denote an element in the symmetric group on n letters Σ_n which acts naturally on F_n by permutation of coordinates with ξ_i given by the transposition $(i, i+1)$ (see the end of the previous section). Thus

$$\xi_i(x_j) = \begin{cases} x_j & \text{if } \{j\} \cap \{i, i+1\} = \varnothing, \\ x_{i+1} & \text{if } j = i \text{ and} \\ x_i & \text{if } j = i+1. \end{cases}$$

Next consider

$$\tau_i\colon F_n \to F_n$$

which sends x_i to x_i^{-1}, and fixes x_j for $j \neq i$. Thus

$$\tau_i(x_j) = \begin{cases} x_j & \text{if } j \neq i \text{ and} \\ x_i^{-1} & \text{if } j = i. \end{cases}$$

The elements τ_i, and ξ_j for $1 \leq i, j \leq n$ generate the "signed permutation group" in $\mathrm{Aut}(F_n)$; this group, also known as the wreath product $\Sigma_n \wr \mathbb{Z}/2\mathbb{Z}$, embeds in $GL(n, \mathbb{Z})$ via the natural map $\mathrm{Aut}(F_n) \to GL(n, \mathbb{Z})$. In this wreath product, a Coxeter group of type B_n, the elements τ_i can be expressed in terms of τ_1 and ξ_i.

Let δ denote the automorphism of F_n which sends x_1 to x_1x_2 while fixing x_i for $i > 1$. It follows from [17] that $\mathrm{Aut}(F_n)$ is generated by the elements

- ξ_i for $1 \leq i \leq n$,
- τ_i for $1 \leq i \leq n$ and
- δ.

It is natural to consider the action of some of these elements on $P\Sigma_n$, and $P\Sigma_n^+$ by conjugation.

Proposition 3.1 *Subgroups of the automorphism groups of $P\Sigma_n^+$, and $P\Sigma_n$ are listed as follows.*

(1) *Conjugation by the elements τ_i for $1 \leq i \leq n$ leaves $P\Sigma_n^+$ invariant. Thus $\oplus_n \mathbb{Z}/2\mathbb{Z}$ is isomorphic to a subgroup of $\mathrm{Aut}(P\Sigma_n^+)$ with induced monomorphisms*

$$\theta\colon \oplus_n \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(P\Sigma_n^+)$$

obtained by conjugating an element in $P\Sigma_n^+$ by the τ_i. The conjugation action of τ_i on the elements $\chi_{s,t}$ is specified by the formulas

$$\tau_i(\chi_{s,t})\tau_i^{-1} = \begin{cases} \chi_{s,t} & \text{if } \{i\} \cap \{s,t\} = \varnothing, \\ \chi_{s,i}^{-1} & \text{if } t = i, \\ \chi_{i,t} & \text{if } s = i \end{cases}$$

for $s > t$.

(2) *Conjugation by the elements τ_i and ξ_i for $1 \leq i \leq n$ leaves $P\Sigma_n$ invariant. Thus $\Sigma_n \wr \mathbb{Z}/2\mathbb{Z}$ is isomorphic to a subgroup of $\mathrm{Aut}(P\Sigma_n)$ with induced monomorphisms*

$$\theta\colon \Sigma_n \wr \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(P\Sigma_n)$$

obtained by conjugating an element in $P\Sigma_n$ by the τ_i and ξ_i. The conjugation action of τ_i and ξ_i on the $\chi_{s,t}$ is specified by the formulas

$$\tau_i(\chi_{s,t})\tau_i^{-1} = \begin{cases} \chi_{s,t} & \text{if } \{i\} \cap \{s,t\} = \varnothing, \\ \chi_{s,i}^{-1} & \text{if } t = i, \\ \chi_{i,t} & \text{if } s = i, \end{cases}$$

and

$$\xi_i(\chi_{s,t})\xi_i^{-1} = \begin{cases} \chi_{s,t} & \text{if } \{i, i+1\} \cap \{s,t\} = \varnothing, \\ \chi_{i+1,t} & \text{if } s = i \text{ and } t \neq i+1, \\ \chi_{i+1,i} & \text{if } s = i \text{ and } t = i+1, \\ \chi_{i,t} & \text{if } s = i+1 \text{ and } t \neq i, \\ \chi_{i,i+1} & \text{if } s = i+1 \text{ and } t = i, \\ \chi_{s,i+1} & \text{if } s \neq i+1 \text{ and } t = i, \\ \chi_{s,i} & \text{if } s \neq i \text{ and } t = i+1, \end{cases}$$

for all $s \neq t$.

Proof The proof of this proposition is a direct computation with details omitted. □

4 On certain subgroups of IA_n

The purpose of this section is to consider the subgroups

(1) $\mathcal{G}_n$ of $P\Sigma_n$ generated by the elements $\chi_{n,i}$ and $\chi_{j,n}$ for $1 \le i, j \le n-1$ and

(2) $\mathcal{G}_n^+$ of $P\Sigma_n^+$ generated by the elements $\chi_{n,i}$ for $1 \le i \le n-1$.

Proposition 4.1 (1) *The following relations are satisfied in $P\Sigma_n$.*

(i) $\chi_{i,j}^{-1} \cdot \chi_{n,j} \cdot \chi_{i,j} = \chi_{n,j}$ *with $i, j < n$.*

(ii) $\chi_{i,k}^{-1} \cdot \chi_{n,j} \cdot \chi_{i,k} = \chi_{n,j}$ *with $\{i,k\} \cap \{n,j\} = \varnothing$.*

(iii) $\chi_{j,k}^{-1} \cdot \chi_{n,j} \cdot \chi_{j,k} = \chi_{n,k} \cdot \chi_{n,j} \cdot \chi_{n,k}^{-1}$ *with $k, j < n$.*

(iv) $\chi_{i,k}^{-1} \cdot \chi_{j,n} \cdot \chi_{i,k} = \chi_{j,n}$ *with $\{i,k\} \cap \{n,j\} = \varnothing$.*

(v) $\chi_{j,i}^{-1} \cdot \chi_{j,n} \cdot \chi_{j,i} = \chi_{n,i} \cdot \chi_{j,n} \cdot \chi_{n,i}^{-1}$ *with $i, j < n$.*

(vi) $\chi_{i,j}^{-1} \cdot \chi_{j,n} \cdot \chi_{i,j} = (\chi_{n,j} \cdot \chi_{i,n}^{-1} \cdot \chi_{n,j}^{-1}) \cdot (\chi_{i,n} \cdot \chi_{j,n})$ *with $i, j < n$.*

(2) *The group $\mathcal{G}_n$ is a normal subgroup of $P\Sigma_n$ and is the kernel of the projection*

$$\pi\colon P\Sigma_n \to P\Sigma_{n-1}.$$

Thus $\mathcal{G}_n = K_n$ as given in Theorem 2.1.

(3) *The group $\mathcal{G}_n^+$ is a normal subgroup of $P\Sigma_n^+$ and is the kernel of the projection*

$$\pi|_{P\Sigma_n^+}\colon P\Sigma_n^+ \to P\Sigma_{n-1}^+.$$

Thus $\mathcal{G}_n^+ = K_n^+$ as given in Theorem 2.1.

Proof

Consider the elements $\chi_{i,j}^{-1} \cdot \chi_{t,n} \cdot \chi_{i,j}$ and $\chi_{i,j}^{-1} \cdot \chi_{n,t} \cdot \chi_{i,j}$ for various values of t together with McCool's relations as listed in Theorem 1.1:

(1) $\chi_{i,j} \cdot \chi_{k,j} \cdot \chi_{i,k} = \chi_{i,k} \cdot \chi_{i,j} \cdot \chi_{k,j}$ for i, j, k distinct.

(2) $[\chi_{k,j}, \chi_{s,t}] = 1$ if $\{j,k\} \cap \{s,t\} = \varnothing$.

(3) $[\chi_{i,j}, \chi_{k,j}] = 1$ for i, j, k distinct.

The verification of part (1) of the proposition breaks apart into six natural cases where the first three are given by the conjugation action on $\chi_{n,j}$ while the second three are given by the conjugation action on $\chi_{j,n}$.

(i) $\chi_{i,j}^{-1} \cdot \chi_{n,j} \cdot \chi_{i,j} = \chi_{n,j}$ by formula (3) with $i, j < n$.

(ii) $\chi_{i,k}^{-1} \cdot \chi_{n,j} \cdot \chi_{i,k} = \chi_{n,j}$ by formula (2) with $\{i,k\} \cap \{n, j\} = \varnothing$.

(iii) $\chi_{j,k}^{-1} \cdot \chi_{n,j} \cdot \chi_{j,k} = \chi_{n,k} \cdot \chi_{n,j} \cdot \chi_{n,k}^{-1}$ by formulas (1) and (3) with $k, j < n$.

(iv) $\chi_{i,k}^{-1} \cdot \chi_{j,n} \cdot \chi_{i,k} = \chi_{j,n}$ by formula (2) with $\{i,k\} \cap \{n, j\} = \varnothing$.

(v) $\chi_{j,i}^{-1} \cdot \chi_{j,n} \cdot \chi_{j,i} = \chi_{n,i} \cdot \chi_{j,n} \cdot \chi_{n,i}^{-1}$ by formulas (1) and (3) with $i, j < n$.

(vi) $\chi_{i,j}^{-1} \cdot \chi_{j,n} \cdot \chi_{i,j} = (\chi_{n,j} \cdot \chi_{i,n}^{-1} \cdot \chi_{n,j}^{-1}) \cdot (\chi_{i,n} \cdot \chi_{j,n})$ by formulas (1) and (3) with $i, j < n$.

A sketch of formula (iv) is listed next for convenience of the reader. Assume that i, j, n are distinct.

(a) $\chi_{i,n} \cdot \chi_{j,n} \cdot \chi_{i,j} = \chi_{i,j} \cdot \chi_{i,n} \cdot \chi_{j,n}$.

(b) $\chi_{j,n} \cdot \chi_{i,j} = \chi_{i,n}^{-1} \cdot \chi_{i,j} \cdot \chi_{i,n} \cdot \chi_{j,n}$.

(c) $\chi_{i,j}^{-1} \cdot \chi_{j,n} \cdot \chi_{i,j} = (\chi_{i,j}^{-1} \cdot \chi_{i,n}^{-1} \cdot \chi_{i,j}) \cdot (\chi_{i,n} \cdot \chi_{j,n})$.

(d) By formula (v) above, $\chi_{i,j}^{-1} \cdot \chi_{i,n} \cdot \chi_{i,j} = \chi_{n,j} \cdot \chi_{i,n} \cdot \chi_{n,j}^{-1}$.

(e) Thus $\chi_{i,j}^{-1} \cdot \chi_{j,n} \cdot \chi_{i,j} = (\chi_{n,j} \cdot \chi_{i,n} \cdot \chi_{n,j}^{-1}) \cdot (\chi_{i,n} \cdot \chi_{j,n})$ and formula (vi) follows.

Thus

(1) $\mathcal{G}_n$ is a normal subgroup of $P\Sigma_n$ and

(2) $\mathcal{G}_n^+$ is a normal subgroup of $P\Sigma_n^+$

by inspection of the previous relations (i–vi).

Next notice that $\mathcal{G}_n$ is in the kernel of the projection $\pi\colon P\Sigma_n \to P\Sigma_{n-1}$. Denote by σ the canonical section $\sigma\colon P\Sigma_{n-1} \to P\Sigma_n$ with $\sigma(\chi_{j,i}) = \chi_{j,i}$. Every element W in $P\Sigma_n$ is equal to a product $W_{n-1}\cdot X_n$ where W_{n-1} is in the image of the section σ applied to $P\Sigma_{n-1}$ and X_n is in $\mathcal{G}_n$ by inspection of the relations relations (i–vi). Thus, the kernel of π is generated by all conjugates of the elements $\chi_{n,i}$ and $\chi_{j,n}$ for $1 \le i, j \le n-1$ which coincides with $\mathcal{G}_n$. A similar assertion and proof applies to $\mathcal{G}_n^+$ using relations (iv, v, vi). □

The following result [15; 5] will be used below to derive the structure of certain Lie algebras in this article.

Theorem 4.2 *Let*

$$1 \to A \to B \to C \to 1$$

be a split short exact sequence of groups for which conjugation by C induces the trivial action on $H_1(A)$. Then there is a split short exact sequence of Lie algebras

$$0 \to gr^*(A) \to gr^*(B) \to gr^*(C) \to 0.$$

To apply Theorem 4.2, features of the local coefficient system in homology for the projection maps $\pi\colon P\Sigma_n \to P\Sigma_{n-1}$ and $\pi|_{P\Sigma_n^+}\colon P\Sigma_n^+ \to P\Sigma_{n-1}^+$ are obtained next.

Proposition 4.3 (1) *The natural conjugation action of $P\Sigma_{n-1}$ on $H_1(K_n)$ is trivial. Thus there is a split short exact sequence of Lie algebras*

$$0 \to gr^*(K_n) \to gr^*(P\Sigma_n) \to gr^*(P\Sigma_{n-1}) \to 0.$$

(2) *The natural conjugation action of $P\Sigma_{n-1}^+$ on $H_1(K_n^+)$ is trivial. Thus there is a split short exact sequence of Lie algebras*

$$0 \to gr^*(K_n^+) \to gr^*(P\Sigma_n^+) \to gr^*(P\Sigma_{n-1}^+) \to 0.$$

Proof As before, consider the elements $\chi_{i,j}^{-1}\cdot\chi_{n,t}\cdot\chi_{i,j}$ and $\chi_{i,j}^{-1}\cdot\chi_{t,n}\cdot\chi_{i,j}$ together with McCool's relations as given in Proposition 4.1 to obtain the following formulas.

(i) $\chi_{i,j}^{-1}\cdot\chi_{n,j}\cdot\chi_{i,j} = \chi_{n,j}$ with $i, j < n$.

(ii) $\chi_{i,k}^{-1}\cdot\chi_{n,j}\cdot\chi_{i,k} = \chi_{n,j}$ with $\{i,k\}\cap\{n,j\} = \varnothing$.

(iii) $\chi_{j,k}^{-1} \cdot \chi_{n,j} \cdot \chi_{j,k} = \chi_{n,j} \cdot (\chi_{n,j}^{-1} \cdot \chi_{n,k} \cdot \chi_{n,j} \cdot \chi_{n,k}^{-1})$ with $k, j < n$. Thus $\chi_{j,k}^{-1} \cdot \chi_{n,j} \cdot \chi_{j,k} = \chi_{n,j} \cdot ([\chi_{n,j}^{-1}, \chi_{n,k}])$.

(iv) $\chi_{i,k}^{-1} \cdot \chi_{j,n} \cdot \chi_{i,k} = \chi_{j,n}$ with $\{i,k\} \cap \{n,j\} = \varnothing$.

(v) $\chi_{j,i}^{-1} \cdot \chi_{j,n} \cdot \chi_{j,i} = \chi_{n,i} \cdot \chi_{j,n} \cdot \chi_{n,i}^{-1} = \chi_{j,n} \cdot \chi_{j,n}^{-1} \cdot \chi_{n,i} \cdot \chi_{j,n} \cdot \chi_{n,i}^{-1}$ with $i, j < n$. Thus $\chi_{j,i}^{-1} \cdot \chi_{j,n} \cdot \chi_{j,i} = \chi_{j,n} \cdot [\chi_{j,n}^{-1}, \chi_{n,i}]$.

(vi) $\chi_{i,j}^{-1} \cdot \chi_{j,n} \cdot \chi_{i,j} = (\chi_{n,j} \cdot \chi_{i,n}^{-1} \cdot \chi_{n,j}^{-1}) \cdot (\chi_{i,n} \cdot \chi_{j,n}) = [\chi_{n,j}, \chi_{i,n}^{-1}] \cdot \chi_{j,n}$ with $i, j < n$.

It then follows that the conjugation action of $\chi_{s,t}$ on the class of either $\chi_{n,j}$ or $\chi_{j,n}$ in $H_1(K_n)$ fixes that class (in $H_1(K_n)$). Thus there is a short exact sequence of Lie algebras $0 \to gr^*(K_n) \to gr^*(P\Sigma_n) \to gr^*(P\Sigma_{n-1}) \to 0$ by Theorem 4.2

A similar assertion and proof follows for $H_1(K_n^+)$ by inspection of formulas (iv,v,vi). The proposition follows. □

5 Proof of Theorem 1.2

The first part of Theorem 1.2 that the projection map $\pi\colon P\Sigma_n \to P\Sigma_{n-1}$ is an epimorphism with kernel generated by $\chi_{n,i}$ and $\chi_{j,n}$ for $1 \le i, j \le n-1$ follows from Proposition 4.1. Furthermore, this extension is split by the section $\sigma\colon P\Sigma_{n-1} \to P\Sigma_n$. That the local coefficient system is trivial for $H_1(K_n)$ follows from Proposition 4.3. (Note that it is possible that the local coefficient system is non-trivial for higher dimensional homology groups of K_n.) Similar properties are satisfied for $\pi|_{P\Sigma_n^+}\colon P\Sigma_n^+ \to P\Sigma_{n-1}^+$ by Propositions 4.1 and 4.3.

Consider the free group F_{n-1} with generators $x_1, \cdots, x_{n-1}$. There is a homomorphism

$$\Phi_{n-1}\colon F_{n-1} \to K_n^+$$

obtained by defining

$$\Phi_{n-1}(x_i) = \chi_{n,i}.$$

This homomorphism is evidently a surjection. To check that the subgroup K_n^+ is a free group generated by the elements $\chi_{n,i}$ for $1 \le i \le n-1$, it suffices to check that Φ_{n-1} is a monomorphism.

Observe that if

$$W = x_{i_1}^{\epsilon_1} \cdot x_{i_2}^{\epsilon_2} \cdots x_{i_r}^{\epsilon_r}$$

for $\epsilon_i = \pm 1$ is a word in F_{n-1}, then $\Phi_{n-1}(W)$ is an automorphism of F_n with

$$\Phi_{n-1}(W)(x_n) = W \cdot x_n \cdot W^{-1}.$$

Thus if W is in the kernel of Φ_{n-1}, then $W \cdot x_n \cdot W^{-1} = x_n$. Furthermore, if W is in F_{n-1}, then $W = 1$. Theorem 1.2 follows.

6 Proof of Theorem 1.3

Theorem 1.3 states that there a split short exact sequence of Lie algebras

$$0 \to gr^*(K_n) \to gr^*(P\Sigma_n) \to gr^*(P\Sigma_{n-1}) \to 0.$$

This follows from Proposition 4.3.

The next assertion is that certain relations are satisfied in the Lie algebra $gr^*(P\Sigma_n)$: If $\{i, j\} \cap \{s, t\} = \varnothing$, then $[\chi_{j,i}, \chi_{s,t}] = 0$ by one of McCool's relations in Theorem 1.1.

Next notice that one of the horizontal $4T$ relations follows directly from McCool's identity in Theorem 1.1. Since $[\chi_{i,j} \cdot \chi_{k,j}, \chi_{i,k}] = 1$ for i, j, k distinct, it follows that

$$[\chi_{i,k}, \chi_{i,j} + \chi_{k,j}] = 0$$

on the level of Lie algebras.

In addition, $[\chi_{k,j}, \chi_{s,j}] = 0$ on the level of Lie algebras if $\{s, k\} \cap \{j\} = \varnothing$ by inspection of Theorem 1.1.

That $[\chi_{k,i}, \chi_{j,i} + \chi_{j,k}]$ is non-zero follows from [4]. The details are omitted: they are a direct computation using the Johnson homomorphism together with structure for the Lie algebra of derivations of a free Lie algebra.

Next notice that $H_1(P\Sigma_n) = \oplus_{\binom{n}{2}}\mathbb{Z}$. Project it to the summand with basis $\chi_{i,n}$ for $1 \leq i \leq n-1$. Denote by γ the following composition

$$K_n \to P\Sigma_n \to H_1(P\Sigma_n) \to \oplus_{n-1}\mathbb{Z}.$$

It is evidently a split epimorphism as $[\chi_{i,n}, \chi_{j,n}] = 1$. The remaining properties follow by inspection. Theorem 1.3 follows.

7 Proof of Theorem 1.4

Proposition 4.3 gives that the action of $P\Sigma^+_{n-1}$ on $H_1(K^+_n)$ is trivial. There are two consequences of this fact.

The first consequence is that there is a split short exact sequence of Lie algebras

$$0 \to gr^*(K^+_n) \to gr^*(P\Sigma^+_n) \to gr^*(P\Sigma^+_{n-1}) \to 0$$

by Proposition 4.3 or [15; 5].

Notice that one of the horizontal $4T$ relations follows directly from McCool's identity in Theorem 1.1. As checked in the proof of Theorem 1.3, the relation

$$[\chi_{i,k}, \chi_{i,j} + \chi_{k,j}] = 0$$

is satisfied on the level of Lie algebras.

The remaining two relations $[\chi_{j,k}, \chi_{s,t}] = 0$ if $\{j,k\} \cap \{s,t\} = \varnothing$, and $[\chi_{j,k}, \chi_{s,j}] = 0$ if $\{s,k\} \cap \{j\} = \varnothing$ follow by inspection of McCool's relations. Thus the asserted structure of Lie algebra follows.

The second consequence is that the local coefficient system for the Lyndon–Hochschild–Serre spectral sequence of the extension

$$1 \to K_n^+ \to P\Sigma_n^+ \to P\Sigma_{n-1}^+ \to 1$$

has trivial local coefficients in cohomology. Since K_n^+ is a free group, it has torsion free cohomology which is concentrated in degrees at most 1. Thus the E_2–term of the spectral sequence splits as a tensor product

$$H^*(P\Sigma_{n-1}^+) \otimes_{\mathbb{Z}} H^*(K_n^+).$$

Since the extension is split, all differentials are zero, and the spectral sequence collapses at the E_2–term.

Note that $P\Sigma_2^+$ is isomorphic to the integers. Thus the cohomology of $P\Sigma_n^+$ is torsion free and a basis for the cohomology is given as stated in the theorem by inspection of the E_2–term of the spectral sequence by induction on n.

It thus remains to work out the product structure in cohomology which is asserted to be

- ${\chi_{k,i}^*}^2 = 0$ for all $i > k$, and
- $\chi_{i,j}^*[\chi_{i,k}^* - \chi_{j,k}^*] = 0$ for $k < j < i$,

for which $\chi_{i,k}^*$ is the basis element in cohomology dual to $\chi_{i,k}$.

Notice that ${\chi_{i,k}^*}^2 = 0$. It suffices to work out the relation $\chi_{3,2}^* \cdot [\chi_{3,1}^* - \chi_{2,1}^*] = 0$ in case $n = 3$ by the natural projection maps.

Consider the natural split epimorphism obtained by restriction to

$$\pi|_{P\Sigma_3^+} \colon P\Sigma_3^+ \to P\Sigma_2^+.$$

The kernel is a free group on two letters $\chi_{3,1}, \chi_{3,2}$ and the spectral sequence of the extension collapses. On the level of cohomology, it suffices to work out the value of $\chi^*_{3,1} \cdot \chi^*_{3,2}$. Thus the product $\chi^*_{3,1} \cdot \chi^*_{3,2}$ is equal to the linear combination $A\chi^*_{2,1} \cdot \chi^*_{3,1} + B\chi^*_{2,1} \cdot \chi^*_{3,2}$ for scalars A and B.

Next, consider McCool's relations (as stated in 1.1) which gives

$$\chi_{3,1} \cdot \chi_{2,1} \cdot \chi_{3,2} = \chi_{3,2} \cdot \chi_{3,1} \cdot \chi_{2,1}.$$

Thus the commutator $[\chi_{3,1} \cdot \chi_{2,1}, \chi_{3,2}]$ is 1 in $P\Sigma_3$. Since this commutator is trivial, there is an induced homomorphism

$$\rho\colon \mathbb{Z} \times \mathbb{Z} \to P\Sigma_3^+$$

defined by the equation

$$\rho((a,b)) = \begin{cases} \chi_{3,2} & \text{if } (a,b) = (1,0) \text{ and} \\ \chi_{3,1} \cdot \chi_{2,1} & \text{if } (a,b) = (0,1). \end{cases}$$

Let $(1,0)^*$ and $(0,1)^*$ denote the two associated natural classes in $H^1(\mathbb{Z} \times \mathbb{Z}; \mathbb{Z})$ which are dual to the natural homology basis. Notice that

- $\rho^*(\chi^*_{2,1}) = (0,1)^*$,
- $\rho^*(\chi^*_{3,2}) = (1,0)^*$ and
- $\rho^*(\chi^*_{3,1}) = (0,1)^*$.

Let $(1,1)^*$ denote the fundamental cycle of $H^2(\mathbb{Z} \times \mathbb{Z})$ given by the cup product $(1,0)^* \cdot (0,1)^*$. Consider

$$-(1,1)^* = \rho^*(\chi^*_{3,1} \cdot \chi^*_{3,2}) = \rho^*(A\chi^*_{2,1} \cdot \chi^*_{3,1} + B\chi^*_{2,1} \cdot \chi^*_{3,2}) = -B(1,1)^*.$$

Thus $B = 1$.

Next, consider the automorphisms of $P\Sigma_3^+$. Recall that there are automorphisms of $P\Sigma_n^+$ specified by the conjugation action of τ_i on the elements $\chi_{s,t}$ given by the formulas:

$$\tau_i(\chi_{s,t})\tau_i^{-1} = \begin{cases} \chi_{s,t} & \text{if } \{i\} \cap \{s,t\} = \varnothing, \\ \chi_{s,i}^{-1} & \text{if } t = i, \\ \chi_{i,t} & \text{if } s = i. \end{cases}$$

Notice that τ_1 leaves $P\Sigma_3^+$ invariant and is thus an automorphism of $P\Sigma_3^+$. Thus on the level of $H_1(P\Sigma_3^+)$, conjugation by τ_1 denoted ϕ_1, is given by the formula

(1) $\phi_{1*}(\chi_{2,1}) = -\chi_{2,1}$,

(2) $\phi_{1*}(\chi_{3,1}) = -\chi_{3,1}$ and

(3) $\phi_{1*}(\chi_{3,2}) = \chi_{3,2}$.

Thus on the level of cohomology,

(1) $\phi_1^*(\chi_{2,1}^*) = -\chi_{2,1}^*$,

(2) $\phi_1^*(\chi_{3,1}^*) = -\chi_{3,1}^*$ and

(3) $\phi_1^*(\chi_{3,2}^*) = \chi_{3,2}^*$.

Apply the automorphism ϕ_1^* to the equation $\chi_{3,1}^* \cdot \chi_{3,2}^* = A\chi_{2,1}^* \cdot \chi_{3,1}^* + B\chi_{2,1}^* \cdot \chi_{3,2}^*$ where $B = 1$ to obtain the following.

- $-\chi_{3,1}^* \cdot \chi_{3,2}^* = A\chi_{2,1}^* \cdot \chi_{3,1}^* - B\chi_{2,1}^* \cdot \chi_{3,2}^*$.
- $A\chi_{2,1}^* \cdot \chi_{3,1}^* - B\chi_{2,1}^* \cdot \chi_{3,2}^* = -[A\chi_{2,1}^* \cdot \chi_{3,1}^* + B\chi_{2,1}^* \cdot \chi_{3,2}^*]$ in a free abelian group of rank two with basis $\{\chi_{2,1}^* \cdot \chi_{3,1}^*, \chi_{2,1}^* \cdot \chi_{3,2}^*\}$.
- Hence $A = 0$ and $\chi_{3,2}^* \cdot [\chi_{3,1}^* - \chi_{2,1}^*] = 0$.
- It follows that $\chi_{i,j}^* \cdot [\chi_{i,k}^* - \chi_{j,k}^*] = 0$ for $k < j < i$ by a similar argument.

The Theorem follows.

8 Proof of Theorem 1.7

Recall that $H_1(IA_n)$ is a free abelian group of rank $n\binom{n}{2}$ with basis given by the equivalence classes of $\chi_{i,k}$ for $i \neq k$ with $1 \leq i, k \leq n$, and $\theta(k; [s,t])$ for k, s, t distinct positive natural numbers with $s < t$ [14; 6; 4; 23]. Thus the natural composite

$$P\Sigma_n^+ \longrightarrow IA_n \longrightarrow IA_n/[IA_n, IA_n]$$

is a split monomorphism on the level of the first homology group with image spanned by $\chi_{k,j}$ for $k > j$.

The cohomology algebra of $P\Sigma_n^+$ is generated as an algebra by elements of degree 1 given by $\chi_{k,j}^*$, for $k > j$ in the dual basis for $H_1(P\Sigma_n^+)$ by Theorem 1.4. Thus the composite $P\Sigma_n^+ \to IA_n \to IA_n/[IA_n, IA_n]$ is a split surjection in integral cohomology. That the image is not $GL(n,\mathbb{Z})$–invariant follows by an inspection. Since the composite

$$H^*(IA_n/[IA_n, IA_n]) \xrightarrow{A^*} H^*(IA_n) \longrightarrow H^*(P\Sigma_n^+)$$

is an epimorphism, and A^* is $GL(n,\mathbb{Z})$–equivariant, the image of A^* contains $\mathbb{M}_n$.

Notice that the classifying space $BIA_n/[IA_n, IA_n]$ is homotopy equivalent to a product of circles $(S^1)^{n\binom{n}{2}}$. Furthermore, the composite $P\Sigma_n^+ \to IA_n \to IA_n/[IA_n, IA_n]$ induces a split epimorphism on integral cohomology. That the map

$$BP\Sigma_n^+ \to (S^1)^{n\binom{n}{2}}$$

is split after suspending once follows directly from the next standard property of maps into products of spheres.

Lemma 8.1 *Let $f\colon X \to Y$ be a continuous map which satisfies the following properties.*

(1) *The space X is homotopy equivalent to a CW–complex,*

(2) *Y is a finite product of spheres of dimension at least 1 and*

(3) *the map f induces a split epimorphism on integral cohomology.*

Then

(1) *the suspension of X, $\Sigma(X)$, is a retract of $\Sigma(Y)$*

(2) *$\Sigma(X)$ is homotopy equivalent to a bouquet of spheres and*

(3) *the Freudenthal suspension $E\colon X \to \Omega\Sigma(X)$ factors through a map $Y \to \Omega\Sigma(X)$.*

Hence there is a map

$$\Theta\colon BIA_n \to \Omega\Sigma(P\Sigma_n^+)$$

which gives a factorization of the Freudenthal suspension. The Theorem follows.

9 Proof of Theorem 1.8

The purpose of this section is to give the proof of Theorem 1.8 along with other information. There is a homomorphism

$$\rho_n\colon \mathrm{BP}_n \to \Sigma_n$$

defined by $\rho(\sigma_i) = \xi_i = \rho(\xi_i)$. One way to see that such a homomorphism exists is to consider the pullback diagram

$$\begin{array}{ccc} \Xi_n & \xrightarrow{\pi} & \Sigma_n \\ {\scriptstyle i}\downarrow & & \downarrow \\ \mathrm{Aut}_n & \longrightarrow & GL(n,\mathbb{Z}) \end{array}$$

and to observe that BP_n is a subgroup of Ξ_n.

Recall that the group BP_n is characterized as the subgroup of $\mathrm{Aut}(F_n)$ consisting of automorphisms $\phi \in \mathrm{Aut}(F_n)$ of *permutation–conjugacy type* which satisfy

$$(x_i)\phi = w_i^{-1} x_{\lambda(i)} w_i \tag{5}$$

for some word $w_i \in F_n$ and permutation $\lambda \in \Sigma_n$ the symmetric group on n letters (where the action of an element ϕ in $\mathrm{Aut}(F_n)$ is from the right) [7]. Thus

$$((x_i)\phi\lambda^{-1}) = ((w_i^{-1})\lambda^{-1})(x_i)((w_i)\lambda^{-1}). \tag{6}$$

Furthermore $\sigma_i \circ \xi_i = \chi_{i+1,i}$ and so $\sigma_i = \chi_{i+1,i} \circ \xi_i^{-1}$. The group BP_n is thus generated by (1) Σ_n and (2) $P\Sigma_n$. Thus the kernel of the natural map $\rho_n\colon \mathrm{BP}_n \to \Sigma_n$ is $P\Sigma_n$. Theorem 1.8 follows.

10 Problems

(1) Is $P\Sigma_n^+$ isomorphic to P_n ? If $n = 2, 3$, the answer is clearly yes. Note added after this paper was submitted: Bardakov [2] has shown that if $n > 3$, the groups $P\Sigma_n^+$ and P_n are not isomorphic.

(2) Let $\mathcal{K}_n$ denote the subgroup of IA_n generated by all of the elements $\theta(k;[s,t])$. Identify the structure of $gr^*(\mathcal{K}_n)$.

(3) Are the natural maps

$$gr^*(P\Sigma_n^+) \to gr^*(IA_n),$$
$$gr^*(P\Sigma_n) \to gr^*(IA_n)$$

and/or

$$gr^*(\mathcal{K}_n) \to gr^*(IA_n)$$

monomorphisms ?

(4) Give the Euler–Poincaré series for $U[gr^*(IA_n) \otimes \mathbb{Q}]$ and $U[gr^*(P\Sigma_n) \otimes \mathbb{Q}]$ where $U[\mathcal{L}]$ denotes the universal enveloping algebra of a Lie algebra $\mathcal{L}$.

(5) Give the Euler–Poincaré series for $\mathbb{M}_n$.

(6) Observe that the kernel of the natural map out of the free product $P\Sigma_n * \mathcal{K}_n \to IA_n$ is a surjection with kernel a free group. The natural morphism of Lie algebras

$$\iota\colon gr^*(P\Sigma_n) \amalg gr^*(\mathcal{K}_n) \to gr^*(IA_n)$$

is an epimorphism (where $\amalg$ denotes the free product of Lie algebras). Is the kernel a free Lie algebra ?

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