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# On braid groups and homotopy groups

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This article is an exposition of certain connections between the braid groups, classical homotopy groups of the 2–sphere, as well as Lie algebras attached to the descending central series of pure braid groups arising as Vassiliev invariants of pure braids. Natural related questions are posed at the end of this article.

20F36, 55Q40; 20F40, 55U10, 18G30

## 1 Introduction

The purpose of this article is to give an exposition of certain connections between the braid groups (Artin [1], see Birman [4]) and classical homotopy groups which arises in joint work of Jon Berrick, Yan-Loi Wong and the authors [9; 10; 3; 38]. These connections emerge through several other natural contexts such as Lie algebras attached to the descending central series of pure braid groups arising as Vassiliev invariants of the pure braid groups as developed by T Kohno [23; 24].

The main feature of this article is to identify certain "non-standard" free subgroups of the braid groups via Vassiliev invariants of pure braids. A second feature is to indicate natural ways in which these subjects fit together with classical homotopy theory. This article is an attempt to draw together these connections.

Since this paper was submitted, other connections to principal congruence subgroups in natural matrix groups as well as other extensions have developed. The authors have taken the liberty of adding an additional Section 7 with some of these new connections.

Natural related questions are posed at the end of this article.

Although not yet useful for direct computations, there is a strong connection between braid groups and homotopy groups. The braids which naturally arise in this setting also give a large class of special knots and links arising from Brunnian braids as described below as well as by Stanford [37]. It is natural to wonder whether and how these fit together.

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## 2 Braid groups, Vassiliev invariants of pure braids and certain free subgroups of braid groups

The section addresses a naive construction with the braid groups arising as a "cabling" construction. This construction is interpreted in later sections in terms of the structure of braid groups, Vassiliev invariants of pure braids as developed by Toshitake Kohno [23; 24], associated Lie algebras and the homotopy groups of the 2–sphere (Berrick, Wong and the authors [9; 10; 3; 38]).

Let $B_k$ denote Artin's $k$–stranded braid group while $P_k$ denotes the pure $k$–stranded braid group, the subgroup of $B_k$ which corresponds to the trivial permutation of the endpoints of the strands. The group $P_k$ is the fundamental group of the configuration space of ordered $k$–tuples of distinct points in the plane

$$\mathrm{Conf}(\mathbb{R}^2, k)$$

for which $\mathrm{Conf}(M,k) = \{(m_1, m_2, \cdots, m_k) \in M^k \mid m_i \neq m_j \text{ for all } i \neq j\}$ for any space $M$.

The group $B_k$ is the fundamental group of the orbit space

$$\mathrm{Conf}(\mathbb{R}^2, k)/\Sigma_k$$

obtained from the natural, free (left)-action of the symmetric group on $k$ letters $\Sigma_k$. The $k$–stranded braid group of an arbitrary connected surface $S$, $B_k(S)$, is defined to be the fundamental group of the configuration space of unordered $k$–tuples of distinct points in $S$, $\pi_1\mathrm{Conf}(S,k)/\Sigma_k$. The pure braid group $P_k(S)$ is defined to be the fundamental group

$$\pi_1\mathrm{Conf}(S,k).$$

The pure braid groups $P_k$ and $P_k(S^2)$ are intimately related to the loop space of the 2–sphere as elucidated below in the Section 4. Similar properties are satisfied for any sphere as described in Section 7.

To start to address this last point, first consider the free group on $N$ letters $F_N = F[y_1, \cdots, y_N]$ together with elements $x_i$ for $1 \le i \le N$ in $P_{N+1}$ given by the naive "cabling" pictured in Figure 1 below. The braid $x_1$ with $N = 1 = i$ in Figure 1 is Artin's generator $A_{1,2}$ of $P_2$. The braids $x_i$ for $1 \leq i \leq N$ in Figure 1 yield homomorphisms from a free group on $N$ letters $F_N = F[y_1, \cdots, y_N]$ to $P_{N+1}$

$$\Theta_N\colon F[y_1, \cdots, y_N] \to P_{N+1}$$

defined on generators $y_i$ in $F_N$ by the formula

$$\Theta_N(y_i) = x_i.$$

The maps $\Theta_N$ are the subject of thew authors' papers [9; 10] where it is shown $\Theta_N\colon F_N \to P_{N+1}$ is faithful for every $N$. Three natural questions arise: (1) Why would one want to know whether $\Theta_n$ is faithful, (2) are there sensible applications and (3) why is $\Theta_n$ faithful? The answers to these three questions provide the main content of this expository article.

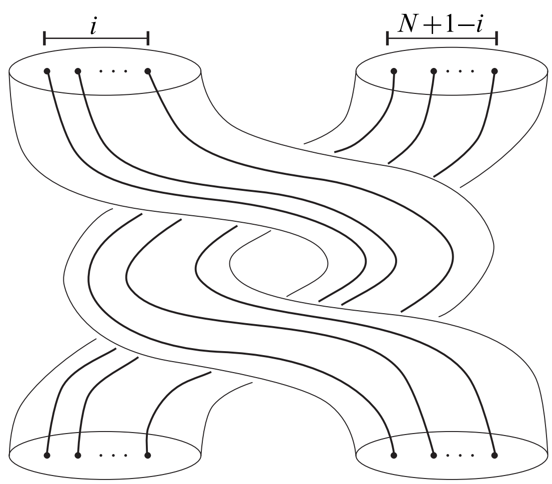


Figure 1: The braid $x_i$ in $P_{N+1}$

## 3 On $\Theta_n$

This section addresses one reason why the map $\Theta_n$ is faithful [9; 10]. The method of proof is to appeal to the structure of the Lie algebras obtained from the descending central series for both the source and the target of $\Theta_n$. The structure of these Lie algebras is reviewed below.

Recall the descending central series of a discrete group $\pi$ given by

$$\pi = \Gamma^1(\pi) \geq \Gamma^1(\pi) \geq \cdots$$

where $\Gamma^i(\pi)$ is the subgroup of $\pi$ generated by all commutators

$$[\cdots[x_1, x_2]x_3]\cdots]x_t]$$

for $t \geq i$ with $x_i \in \pi$. The group $\Gamma^i(\pi)$ is a normal subgroup of $\pi$ with the successive sub-quotients

$$gr_i(\pi) = \Gamma^i(\pi)/\Gamma^{i+1}(\pi)$$

which are abelian groups having additional structure as follows (cf Magnus, Karrass and Solitar [25]).

Consider the direct sum of all of the $gr_i(\pi) = \Gamma^i(\pi)/\Gamma^{i+1}(\pi)$ denoted

$$gr_*(\pi) = \oplus_{i\geq 1}\Gamma^i(\pi)/\Gamma^{i+1}(\pi).$$

The commutator function

$$[-,-]\colon \pi \times \pi \to \pi$$

given by

$$[x,y] = xyx^{-1}y^{-1},$$

passes to quotients to give a bilinear map

$$[-,-]\colon gr_s(\pi) \otimes_{\mathbb{Z}} gr_t(\pi) \to gr_{s+t}(\pi)$$

which satisfies both the antisymmetry law and Jacobi identity for a Lie algebra. (One remark about definitions: The abelian group $gr_*(\pi)$ is both a graded abelian group and a Lie algebra, but not a graded Lie algebra as the sign conventions do not work properly in this context. This situation can be remedied by doubling all degrees of elements in $gr_*(\pi)$.)

The associated graded Lie algebra obtained from the descending central series for the target yields Vassiliev invariants of pure braids by work of Kohno [23; 24]. This Lie algebra has been used by both Kohno and Drinfel'd [13] in their work on the KZ equations. The Lie algebra obtained from the descending central series of the free group $F_N$ is a free Lie algebra by a classical result due to P Hall [18]; see also Serre [35].

The proof described next yields more information than just the fact that $\Theta_N$ is faithful. The method of proof gives a natural connection of Vassiliev invariants of braids to a classical spectral sequence abutting to the homotopy groups of the 2–sphere. Sections 5, 6 and 7 below provide an elucidation of this interconnection.

A discrete group $\pi$ is said to be residually nilpotent provided

$$\bigcap_{i\geq 1}\Gamma^i(\pi)=\{\text{identity}\}$$

where $\Gamma^i(\pi)$ denotes the $i$-th stage of the descending central series for $\pi$. Examples of residually nilpotent groups are free groups, and $P_n$.

**Lemma 3.1**

(1) *Assume that $\pi$ is a residually nilpotent group. Let*

$$\alpha\colon \pi\to G$$

*be a homomorphism of discrete groups such that the morphism of associated graded Lie algebras*

$$\mathrm{gr}_*(\alpha)\colon \mathrm{gr}_*(\pi)\to \mathrm{gr}_*(G)$$

*is a monomorphism. Then $\alpha$ is a monomorphism.*

(2) *If $\pi$ is a free group, and $\mathrm{gr}_*(\alpha)$ is a monomorphism, then $\alpha$ is a monomorphism.*

Thus one step in the proof of Theorem 3.2 below is to describe the map

$$\Theta_n\colon F[y_1,y_2,\cdots,y_n]\to P_{n+1}$$

on the level of associated graded Lie algebras

$$\mathrm{gr}_*(\Theta_n)\colon \mathrm{gr}_*(F[y_1,y_2,\cdots,y_n])\to \mathrm{gr}_*(P_{n+1}).$$

Recall Artin's generators $A_{i,j}$ for $P_{n+1}$ together with the projections of the $A_{i,j}$ to $\mathrm{gr}_*(P_{n+1})$ labeled $B_{i,j}$ [9; 10].

**Theorem 3.2** *The induced morphism of Lie algebras*

$$\mathrm{gr}_*(\Theta_n)\colon \mathrm{gr}_*(F[y_1,y_2,\cdots,y_n])\to \mathrm{gr}_*(P_{n+1})$$

*satisfies the formula*

$$\mathrm{gr}_*(\Theta_n)(y_q)=\Sigma_{1\leq i\leq n-q+1<j\leq n+1}B_{i,j}.$$

Examples of this theorem are listed next.

**Example 3.3**

(1) If $q = 1$, then
$$\mathrm{gr}_*(\Theta_n)(y_1) = B_{1,n+1} + B_{2,n+1} + \cdots + B_{n,n+1}.$$
Thus if $q = 1$, and $n = 3$,
$$\mathrm{gr}_*(\Theta_3)(y_1) = B_{1,4} + B_{2,4} + B_{3,4}.$$

(2) If $q = 2$, then
$$\mathrm{gr}_*(\Theta_n)(y_2) = (B_{1,n+1} + B_{2,n+1} + \cdots + B_{n-1,n+1}) + (B_{1,n} + B_{2,n} + \cdots + B_{n-1,n}).$$
Thus if $q = 2$, and $n = 3$,
$$\mathrm{gr}_*(\Theta_3)(y_2) = (B_{1,4} + B_{2,4}) + (B_{1,3} + B_{2,3}).$$

(3) In general,
$$\mathrm{gr}_*(\Theta_n)(y_q) = V_{n-q+2} + V_{n-q+3} + \cdots + V_{n+1}$$
where
$$V_r = B_{1,r} + B_{2,r} + \cdots + B_{r-1,r}.$$
Thus if $q = 3$, and $n = 3$,
$$\mathrm{gr}_*(\Theta_3)(y_3) = B_{1,2} + B_{1,3} + B_{1,4}.$$

To determine the map of Lie algebras with a more global view, the structure of the Lie algebra $\mathrm{gr}_*(P_n)$ is useful, and is given as follows. Let $L[S]$ denote the free Lie algebra over $\mathbb{Z}$ generated by a set $S$. The next theorem was given in work of Kohno [23; 24], and Falk and Randell [17].

**Theorem 3.4** *The Lie algebra $\mathrm{gr}_*(P_n)$ is the quotient of the free Lie algebra generated by $B_{i,j}$ for $1 \le i < j \le n$ modulo the* infinitesimal braid relations *(also called the* horizontal 4T relations *or* Yang–Baxter–Lie relations*)*
$$\mathrm{gr}_*(P_n) = L[B_{i,j} \mid 1 \le i < j \le n]/I$$
*where $I$ denotes the 2–sided (Lie) ideal generated by the infinitesimal braid relations as listed next:*

(1) $[B_{i,j}, B_{s,t}] = 0$, *if* $\{i,j\} \cap \{s,t\} = \varnothing$

(2) $[B_{i,j}, B_{i,s} + B_{s,j}] = 0$

(3) $[B_{i,j}, B_{i,t} + B_{j,t}] = 0$

A computation with these maps gives the following result of [9; 10] for which further connections are elucidated in sections 5 and 7.

**Theorem 3.5** *The maps $\Theta_n\colon F[y_1, y_2, \cdots, y_n] \to P_{n+1}$ on the level of associated graded Lie algebras*

$$\mathrm{gr}_*(\Theta_n)\colon \mathrm{gr}_*(F[y_1, y_2, \cdots, y_n]) \to \mathrm{gr}_*(P_{n+1})$$

*are monomorphisms. Thus the maps $\Theta_n$ are monomorphisms.*

**Remark 3.6** Two remarks concerning $\Theta_n$ are given next.

(1) That $\Theta_n$ is a monomorphism identifies $F[y_1, y_2, \cdots, y_n]$ as a free subgroup of rank $n$ in $P_{n+1}$. However, there are other, natural free groups of rank $n$ in $P_{n+1}$. These arise from the fibrations of Fadell and Neuwirth given by projection maps $p_i\colon \mathrm{Conf}(\mathbb{R}^2, n+1) \to \mathrm{Conf}(\mathbb{R}^2, n)$ which delete the $i$-th coordinate and have fibre of the homotopy type of an $n$–fold wedge of circles $\vee_n S^1$ (Fadell and Neuwirth [16]).
Let $q_i\colon P_{n+1} \to P_n$ denote the map induced by $p_i$ on the level of fundamental groups. The kernel of $q_i$ is a free group of rank $n$.
The image of $\Theta_n$ has a contrasting feature: Any composite of the natural projection maps $d_I\colon P_{n+1} \to P_2$ precomposed with $\Theta_n$,

$$F[y_1, y_2, \cdots, y_n] \xrightarrow{\Theta_n} P_{n+1} \xrightarrow{d_I} P_2,$$

is a surjection.

(2) The combinatorial behavior of the map $\mathrm{gr}_*(\Theta_n)$ is intricate even though the definition is elementary as well as natural. For example, various powers of 2 arise in the computation of the map

$$\mathrm{gr}_*(\Theta_n)\colon \mathrm{gr}_*(F[y_1, y_2, \cdots, y_n]) \to \mathrm{gr}_*(P_{n+1})$$

for $n > 2$. One example is listed next.

**Example 3.7** $\Theta_3([[[y_1, y_2]y_3]y_2]) = -[[[\gamma_1, \gamma_2]\gamma_3]\gamma_2] + 2[[[\gamma_1, \gamma_3]\gamma_2]\gamma_2] + \delta$ where $\delta$ is independent of the other terms with $\gamma_1 = B_{1,4} + B_{2,4} + B_{3,4}$, $\gamma_2 = B_{3,4}$ and $\gamma_3 = B_{2,4} + B_{3,4}$. At first glance, these elements may appear to be ”random“. However, this formula represents a systematic behavior which arises naturally from kernels of certain morphisms of Lie algebras.

The crucial feature which makes the computations effective is the “infinitesimal braid relations”. In addition, the behavior of the map $\mathrm{gr}_*(\Theta_n)$ is more regular after restricting to certain sub-Lie algebras arising in the third stage of the descending central series [9; 10]. Finally, the maps $\Theta_n$ also induce monomorphisms of restricted Lie algebras on passage to the Lie algebras obtained from the mod-$p$ descending central series [9; 10].

## 4 Simplicial objects, and $\Delta$–objects

Some basic constructions which are part of an algebraic topologist's toolkit are described in this section (Moore [31], Curtis [11], Bousfield and Kan [5], May [27], and Rourke and Sanderson [32]).

One of the great insights in classical homotopy theory, due to Moore and then 'extended' by Kan as well as Rourke and Sanderson is that not only are homology groups a combinatorial invariant, but so are homotopy groups. The basic combinatorial framework is that of a simplicial set developed in [31; 11; 5; 27] and a $\Delta$–set developed in [32] both of which model the combinatorics of a simplicial complex.

**Definition 4.1** A $\Delta$–*set* is a collection of sets

$$K_\bullet = \{K_0, K_1, \cdots\}$$

with functions, *face operations,*

$$d_i\colon K_t \to K_{t-1} \text{ for } 0 \le i \le t$$

which satisfy the identities

$$d_i d_j = d_{j-1} d_i$$

if $i < j$. A $\Delta$–object in a small category $\mathcal{C}$ is a $\Delta$–set $K_\bullet = \{K_0, K_1, \cdots\}$ with the $K_i$ given by objects in $\mathcal{C}$ and the maps $d_i\colon K_t \to K_{t-1}$ given by morphisms in $\mathcal{C}$. Thus, a $\Delta$–*group* is a $\Delta$–set for which all $d_i\colon K_t \to K_{t-1}$ are group homomorphisms.

Let $\Delta[n]$ denote the $n$–simplex with $\delta_i\colon \Delta[n] \to \Delta[n+1]$ the inclusion of the $i$-th face. Assume that each set $S_i$ is given the discrete topology unless otherwise stated. The geometric realization of a $\Delta$–set $K_\bullet$ is given by

$$|K_\bullet| = (\amalg K_n \times \Delta[n])/\sim$$

where $\sim$ is the equivalence relation generated by requiring that if $x \in K_{n+1}$ and $\alpha \in \Delta[n]$, then

$$(d_i(x), \alpha) \sim (x, \delta_i(\alpha)).$$

**Example 4.2** Examples of $\Delta$–sets are given next.

(1) A choice of $\Delta$–set with exactly one 0–simplex and one 1–simplex has geometric realization given by the circle.

(2) A natural example of a $\Delta$–group arises from the pure braid groups $P_n(S) = \pi_1\mathrm{Conf}(S,n)$ for a path-connected surface $S$ [3]. Define

$$\Delta_\bullet(S) \quad \text{by} \quad \Delta_n(S) = P_{n+1}(S)$$

the $(n+1)$-st pure braid group for the surface $S$.
There are $n+1$ homomorphisms

$$d_i\colon P_{n+1}(S) \to P_n(S)$$

for $0 \le i \le n$ obtained by deleting the $(i+1)$-st strand in $P_{n+1}(S)$. The homomorphisms $d_i$ are induced on the level of fundamental groups of configuration spaces by the projection maps

$$p_{i+1}\colon \mathrm{Conf}(S,n+1) \to \mathrm{Conf}(S,n)$$

given be deleting the $(i+1)$-st coordinate and satisfy the identities required for a $\Delta$–group.
In case $S = \mathbb{CP}^1 = S^2$, the associated $\Delta$–group gives basic information about the homotopy groups of the 2–sphere [3]. In case $S_g$ is a closed orientated surface, the $\Delta$–group $\Delta_\bullet(S_g)$ does not admit the structure of a simplicial group as given in the next Definition 4.3.

Simplicial sets are defined next.

**Definition 4.3** A *simplicial set* is

(1) a $\Delta$–set $K_\bullet = \{K_0, K_1, \cdots\}$ together with

(2) functions, *degeneracy operations,*

$$s_j\colon K_t \to K_{t+1} \text{ for } 0 \le j \le t$$

which satisfy the *simplicial identities*

$$d_i d_j = d_{j-1} d_i \text{ if } i < j,$$

$$s_i s_j = s_{j+1} s_i \text{ if } i \le j \text{ and}$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j, \\ \text{identity} & \text{if } i = j \text{ or } i = j+1, \\ s_j d_{i-1} & \text{if } i > j+1. \end{cases}$$

A simplicial-object in a small category $\mathcal{C}$ is a simplicial-set $K_\bullet = \{K_0, K_1, \cdots\}$ with the $K_i$ given by objects in $\mathcal{C}$ for which both face maps $d_i\colon K_t \to K_{t-1}$ and degeneracies $s_j\colon K_t \to K_{t+1}$ are given by morphisms in $\mathcal{C}$.

Thus, a *simplicial-group*

$$G_\bullet = \{G_0, G_1, \cdots\}$$

is a simplicial-set for which all of the $G_i$ are groups with face and degeneracies given by group homomorphisms.

**Example 4.4** Two examples of simplicial sets are given next.

(1) The simplicial 1–simplex $\Delta[1]$ has two 0–simplices $\langle 0\rangle$ and $\langle 1\rangle$. The $n$–simplices of $\Delta[1]$ are sequences $\langle 0^i, 1^{n+1-i}\rangle$ for $0 \le i \le n+1$. All of the non-degenerate simplices are $\langle 0\rangle$, $\langle 1\rangle$, and $\langle 0, 1\rangle$.

(2) The simplicial circle $S^1$ is a quotient of the simplicial 1–simplex $\Delta[1]$ obtained by identifying $\langle 0\rangle$ and $\langle 1\rangle$. There are exactly two equivalence classes of non-degenerate simplices given by $\langle 0\rangle$, and $\langle 0, 1\rangle$. Furthermore, the simplicial circle $S^1$ is given in degree $k$ by
 (a) a single point $\langle 0\rangle$ in case $k = 0$, and
 (b) $n+1$ points $\langle 0^i, 1^{n+1-i}\rangle$ for $0 \le i < n+1$ in case $k = n$ for which $\langle 0^{n+1}\rangle$ and $\langle 1^{n+1}\rangle$ are identified.

 In what follows below, it is useful to label these simplices by

$$y_{n+1-i} = \langle 0^i, 1^{n+1-i}\rangle$$

 for $0 < i \le n+1$ with $y_0 = s_0^n(\langle 0\rangle)$.
 In addition, classical, elegant constructions for the standard simplicial $n$–simplex $\Delta[n]$ as well as the $n$–sphere are given in [5; 11; 27].

Homotopy groups are defined for any $\Delta$–set, but the definition admits a simple description for simplicial sets which satisfy an additional condition.

**Definition 4.5** A simplicial set $K_\bullet$ is said to satisfy the *extension condition* if for every set of $n$–simplices $x_0, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1} \in K_n$ which satisfy the compatibility condition

$$d_i x_j = d_{j-1} x_i, \quad i < j, \quad i \neq k, \quad j \neq k,$$

there exists an $(n+1)$–simplex $x \in K_{n+1}$ such that $d_i x = x_i$ for $i \neq k$. A simplicial set which satisfies the extension condition is also called a *Kan complex*.

For each Kan complex $K_\bullet$ with base-point $* \in K_0$ and $s_0^n(*) \in K_n$, the $n$-th homotopy set $\pi_n(K_\bullet, *)$ is defined for $n \ge 1$ as the equivalences classes of simplices $a \in K_n$, denoted $[a]$, such that

(1) $d_i(a) = s_0^{n-1}(*)$ for all $0 \leq i \leq n$, and

(2) two such simplices $a, a' \in K_n$ are equivalent provided there exists a simplex $y \in K_{n+1}$ such that $d_{n+1}y = a'$, $d_n y = a$, and $d_i y = s_0^n(*)$ for all $0 \leq i < n$.

If $K_\bullet$ satisfies the extension condition, and $n \geq 1$, then $\pi_n(K_\bullet, *)$ is a group. In case $n \geq 2$, $\pi_n(K_\bullet, *)$ is also abelian.

Basic examples of (i) Kan complexes as well as (ii) $\Delta$–groups which do not admit the structure of a Kan complex are given next.

**Example 4.6** A simplicial group $G_\bullet = \{G_0, G_1, \cdots\}$ always satisfies the extension condition as shown in [31].

(1) If $G_\bullet = \{G_0, G_1, \cdots\}$ is a simplicial group, then the $n$-th homotopy group of $G_\bullet$ is the quotient group

$$\pi_n(G_\bullet) = Z_n / B_n$$

for which
$$Z_n = \bigcap_{0 \leq i \leq n} \mathrm{Ker}(d_i\colon G_n \to G_{n-1}),$$

and
$$B_n = d_0\left(\bigcap_{1 \leq i \leq k+1} \mathrm{Ker}(d_i\colon G_{n+1} \to G_n)\right).$$

(2) Additional, related information is stated next for certain $\Delta$–groups $\Delta_\bullet = \{\Delta_0, \Delta_1, \cdots\}$ which are not necessarily simplicial groups.
   (a) In this case, the set of left cosets $\pi_n(\Delta_\bullet) = Z_n / B_n$ is still well-defined.
   (b) The left cosets $Z_n / B_n$ may not admit a natural structure as a group. One case occurs for $S_g$ a closed, oriented surface with $\Delta_\bullet = \Delta_\bullet(S_g)$ as given in Example 4.2 and in [3].
   (c) The special case of $S = S^2$ has further properties given in in [3] and below.

An example of a simplicial group obtained naturally from Artin's pure braid groups is described next.

**Example 4.7** Consider $\Delta$–groups with $\Delta_n(S) = P_{n+1}(S)$ as given in Example 4.2 for surfaces $S$. Specialize to the surface

$$S = \mathbb{R}^2.$$

In this case, there are also $n+1$ homomorphisms

$$s_i\colon P_{n+1} \to P_{n+2}$$

obtained by "doubling" the $(i+1)$-st strand. The homomorphisms $s_i$ are induced on the level of fundamental groups by the maps for configuration spaces

$$\mathbb{S}_i\colon \mathrm{Conf}(\mathbb{R}^2, n+1) \to \mathrm{Conf}(\mathbb{R}^2, n+2)$$

defined by the formula

$$\mathbb{S}_i(x_1, \cdots, x_{n+1}) = (x_1, \cdots, x_{i+1}, \lambda(x_{i+1}), x_{i+2}, \cdots, x_{n+1})$$

where $\lambda(x_{i+1}) = x_{i+1} + (\epsilon, 0)$ for $(\epsilon, 0)$ a point in $\mathbb{R}^2$ with

$$\epsilon = (1/2) \cdot \min_{t \neq i+1} ||x_{i+1} - x_t||.$$

The homomorphisms $d_i$ and $s_j$ satisfy the simplicial identities [9; 10; 3].

Thus the pure groups in case $S = \mathbb{R}^2$ provide an example of a simplicial group denoted

$$\mathrm{AP}_\bullet \quad \text{with} \quad \mathrm{AP}_n = P_{n+1}$$

for $n = 0, 1, 2, 3, \cdots$.

Consider a pointed topological space $(X, *)$. The pointed loop space of $X$, $\Omega(X)$, has a natural multiplication coming from "loop sum" which is not associative, but homotopy associative. Milnor proved that the loop space of a connected simplicial complex is homotopy equivalent to a topological group [28]. James [22] proved that the loop space of the suspension of a connected CW–complex is naturally homotopy equivalent to a free monoid as explained by Hatcher [19, page 282]. Milnor realized that the James construction could be translated directly into the the language of simplicial groups as described next [30].

**Definition 4.8** Let $K_\bullet$ denote a pointed simplicial set (with base-point $* \in K_0$ and $s_0^n(*) \in K_n$). Define *Milnor's free simplicial group* $F[K]_\bullet$ in degree $n$ by

$$F[K]_n = F[K_n]/s_0^n(*) = 1.$$

Then $F[K]_\bullet$ is a simplicial group with face and degeneracy operations given by the natural multiplicative extension of those for $K_\bullet$. In addition, the face and degeneracy operations applied to a generator is either another generator or the identity element.

**Example 4.9** An example of $F[K]_\bullet$ is given by $K_\bullet = S^1_\bullet$ the simplicial circle. Notice that $F[S^1]_\bullet$ in degree $n$ is isomorphic to the free group $F[S^1]_n = F[y_1, \cdots, y_n]$ by Example 4.4.

Milnor defined the geometric realization of a simplicial set $K_\bullet = \{K_0, K_1, \cdots\}$ for which the underlying topology of $K_\bullet$ is discrete [29]. Recall the inclusion of the $i$-th face $\delta_i\colon \Delta[n-1] \to \Delta[n]$ togther with the projection maps to the $j$-th face $\sigma_j\colon \Delta[n+1] \to \Delta[n]$ [5; 11; 27].

**Definition 4.10** The *geometric realization of* $K_\bullet$ is

$$|K_\bullet| = (\amalg K_n \times \Delta[n])/\sim$$

where $\sim$ denotes the equivalence relation generated by requiring

(1) if $x \in K_{n+1}$ and $\alpha \in \Delta[n]$, then $(d_i(x), \alpha) \sim (x, \delta_i(\alpha))$, and

(2) if $y \in K_n$ and $\beta \in \Delta[n+1]$, then $(x, \sigma_j(\beta)) \sim (s_j(x), \beta)$.

**Theorem 4.11** *If $K_\bullet$ is a reduced simplicial set (that is $K_0$ is equal to a single point $\{*\}$), then the geometric realization $|F[K]_\bullet|$ is homotopy equivalent to $\Omega\Sigma|K_\bullet|$. Thus the homotopy groups of $F[K]_\bullet$ (as given in [31] and Example 4.6) are isomorphic to the homotopy groups of the space $\Omega\Sigma|K_\bullet|$.*

**Example 4.12** Consider the special case of $K_\bullet = S^1_\bullet$. Then the geometric realization $|F[S^1]_\bullet|$ is homotopy equivalent to $\Omega S^2$, and there are isomorphisms

$$\pi_n F[S^1]_\bullet \to \pi_n \Omega S^2 \cong \pi_{n+1} S^2.$$

A partial synthesis of this information is given sections 5, 6 and 7.

## 5 Pure braid groups of surfaces as simplicial groups and $\Delta$–groups

The homomorphism $\Theta_n\colon F[y_1, y_2, \cdots, y_n] \to P_{n+1}$ which arises from the cabling operation described in Figure 1 satisfies the following properties.

(1) The homomorphisms $\Theta_n\colon F[y_1, y_2, \cdots, y_n] \to P_{n+1}$ give a morphism of simplicial groups

$$\Theta\colon F[S^1]_\bullet \to \mathrm{AP}_\bullet$$

for which the homomorphism $\Theta_n$ is the restriction of $\Theta$ to $F[S^1]_n$.

(2) By Theorem 3.5, the homomorphisms $\Theta_n\colon F[y_1, y_2, \cdots, y_n] \to P_{n+1}$ are monomorphisms and so the morphism $\Theta\colon F[S^1] \to \mathrm{AP}_\bullet$ is a monomorphism of simplicial groups.

(3) There is exactly one morphism of simplicial groups $\Theta$ with the property that $\Theta_1(y_1) = A_{1,2}$.

Thus, the picture given in Figure 1 is a description for generators of $F[S^1]_n$ in the simplicial group $F[S^1]_\bullet$. These features are summarized next.

**Theorem 5.1** *The homomorphisms $\Theta_n\colon F[y_1, y_2, \cdots, y_n] \to P_{n+1}$ ("pictured" in Figure 1) give the unique morphism of simplicial groups*

$$\Theta\colon F[S^1]_\bullet \to \mathrm{AP}_\bullet$$

*with $\Theta_1(y_1) = A_{1,2}$. The map $\Theta$ is an embedding. Hence the $n$-th homotopy group of $F[S^1]$, isomorphic to $\pi_{n+1}(S^2)$, is a natural sub-quotient of $\mathrm{AP}_\bullet$. Furthermore, the smallest sub-simplicial group of $\mathrm{AP}_\bullet$ which contains the element $\Theta_1(y_1) = A_{1,2}$ is isomorphic to $F[S^1]_\bullet$.*

On the other-hand, the homotopy sets for the $\Delta$–group $\Delta_\bullet(S^2)$ are also giving the homotopy groups of the 2–sphere via a different occurrence of $F[S^1]_\bullet$. The homeomorphism of spaces

$$\mathrm{Conf}(S^2, k) \to PGL(2, \mathbb{C}) \times \mathrm{Conf}(S^2 - Q_3, k-3)$$

for $k \geq 3$ and where $Q_3$ denotes a set of there distinct points in $S^2$ is basic for the next Theorem [3].

**Theorem 5.2** *If $S = S^2$ and $n \geq 4$, then there are isomorphisms*

$$\pi_n(\Delta_\bullet(S^2)) \to \pi_n(S^2).$$

The descriptions of homotopy groups implied by these Theorems admit interpretations in terms of classical, well-studied features of the braid groups as given in the next section. An extension to all spheres is given in [9; 10] as pointed out in Section 7.

## 6 Brunnian braids, "almost Brunnian" braids, and homotopy groups

The homotopy groups of a simplicial group, or the homotopy sets of a $\Delta$–group admit a combinatorial description as pointed out in Example 4.6. These homotopy sets are the set of left cosets $Z_n/B_n$ where $Z_n$ is the group of $n$–cycles and $B_n$ is the group of $n$–boundaries for the $\Delta$–group.

Recall Example 4.2 in which the $\Delta$–group $\Delta_\bullet(S)$ is specified by $\Delta_n(S) = P_{n+1}(S)$ the $(n+1)$–stranded pure braid group for a connected surface $S$. The main point of this section is that the $n$–cycles $Z_n$ are given by the "Brunnian braids" while the $n$–boundaries $B_n$ are given by the "almost Brunnian braids", subgroups considered next which are also important in other applications (Magnum and Stanford [26]).

**Definition 6.1** Consider the $n$–stranded pure braid group for any (connected) surface $S$, the fundamental group of $\mathrm{Conf}(S,n)$. The group of Brunnian braids $\mathrm{Brun}_n(S)$ is the subgroup of $P_n(S)$ given by those braids which become trivial after deleting any single strand. That is,

$$\mathrm{Brun}_n(S) = \bigcap_{0 \le i \le n-1} \mathrm{Ker}(d_i\colon P_n(S) \to P_{n-1}(S)).$$

The "almost Brunnian" $(n+1)$–stranded braid group is

$$\mathrm{QBrun}_{n+1}(S) = \bigcap_{1 \le i \le n} \mathrm{Ker}(d_i\colon P_{n+1}(S) \to P_n(S)).$$

The subgroup $\mathrm{QBrun}_{n+1}(S)$ of $P_{n+1}(S)$ consists of those braids which are trivial after deleting any one of the strands $2, 3, \cdots, n+1$, but not necessarily the first.

**Example 6.2** Consider the simplicial group $\mathrm{AP}_\bullet$ with

$$\mathrm{AP}_n = P_{n+1}$$

for $n = 0, 1, 2, 3, \cdots$ as given in Example 4.7.

In this case, notice that that the map $d_0\colon \mathrm{QBrun}_{k+2} \to \mathrm{Brun}_{k+1}$ is a split epimorphism. Thus the homotopy groups of the simplicial group $\mathrm{AP}_\bullet$ are all trivial.

An inspection of definitions gives that

(1) the group of $n$–cycles of $\Delta_\bullet(S)$, $Z_n(S)$, is precisely $\mathrm{Brun}_{n+1}(S)$ while

(2) the group of $n$–boundaries, $B_n(S)$, is exactly $d_0(\mathrm{QBrun}_{n+2}(S))$.

This feature is recorded next as a lemma.

**Lemma 6.3** *Let $S$ denote a connected surface with associated $\Delta$–group $\Delta_\bullet(S)$ (as given in Example 4.2). Then the following hold.*

(1) *The group of $n$–cycles $Z_n(S)$ is $\mathrm{Brun}_{n+1}(S)$.*

(2) *The group of boundaries $B_n(S)$ is $d_0(\mathrm{QBrun}_{n+2}(S))$.*

(3) *There is an isomorphism*

$$\pi_k(\mathrm{AP}_\bullet) \to \mathrm{Brun}_{k+1}/d_0(\mathrm{QBrun}_{k+2}).$$

*Furthermore,* $\pi_k(\mathrm{AP}_\bullet)$ *is the trivial group.*

(4) *There is an isomorphism of left cosets which is natural for pointed embeddings of connected surfaces* $S$

$$\pi_k(\Delta_*(S)) \to \mathrm{Brun}_{k+1}(S)/d_0(\mathrm{QBrun}_{k+2}(S)).$$

Properties of the $\Delta$–group for the 2–sphere $S = \mathbb{CP}^1 = S^2$ is the main subject of [3] where the next result is proven.

**Theorem 6.4** *If* $S = S^2$ *and* $k \geq 4$*, then*

$$\pi_k(\Delta_\bullet(S^2)) = \mathrm{Brun}_{k+1}(S^2)/d_0(\mathrm{QBrun}_{k+2}(S^2))$$

*is a group which is isomorphic to the classical homotopy group* $\pi_k(S^2)$*.*

*Furthermore, there is an exact sequence of groups*

$$1 \to \mathrm{Brun}_{k+2}(S^2) \to \mathrm{Brun}_{k+1}(\mathbb{R}^2) \to \mathrm{Brun}_{k+1}(S^2) \to \pi_k(S^2) \to 1.$$

The next lemma follows by a direct check of the long exact homotopy sequence obtained from the Fadell–Neuwirth fibrations for configuration spaces [16; 15].

**Lemma 6.5** *If* $S$ *is a surface not homeomorphic to either* $S^2$ *or* $\mathbb{RP}^2$*, and* $k \geq 3$*, then* $\mathrm{Brun}_k(S)$ *and* $\mathrm{QBrun}_k(S)$ *are free groups. If* $S$ *is any surface, and* $k \geq 4$*, then* $\mathrm{Brun}_k(S)$ *and* $\mathrm{QBrun}_k(S)$ *are free groups.*

**Example 6.6** One classical example of a Brunnian braid group is $\mathrm{Brun}_4(S^2)$ which is isomorphic to the principle congruence subgroup of level 4 in $PSL(2,\mathbb{Z})$ as given in Section 7 below.

One question below in Section 8 is to consider properties of the free groups obtained from the intersections $\Theta_k(F_k) \cap \mathrm{Brun}_{k+1}$ as well as $\Theta_k(F_k) \cap d_0(\mathrm{QBrun}_{k+2})$ where $\Theta_k\colon F[y_1, y_2, \cdots, y_k] \to P_{k+1}$ is the homomorphism in Section 3. These groups are precisely the cycles and boundaries for $F[S^1]_\bullet$.

**Lemma 6.7** *If* $k \geq 3$*, then* $\Theta_k(F_k) \cap \mathrm{Brun}_{k+1}$ *as well as* $\Theta_k(F_k) \cap d_0(\mathrm{QBrun}_{k+2})$ *are countably infinitely generated free groups.*

The standard Hall collection process or natural variations can be used to give inductive recipes rather than closed forms for generators. T Stanford has given a related elegant exposition of the Hall collection process [37]. The analogous process was applied by Cohen and Levi [7] to give group theoretic models for iterated loop spaces.

The connection of the homotopy groups of $S^2$ as well as the Lie algebra attached to the descending central series of the pure braid groups is discussed next.

**Theorem 6.8** *The group $\Theta_k(F_k) \cap d_0(\mathrm{QBrun}_{k+2})$ is a normal subgroup of $\Theta_k(F_k) \cap \mathrm{Brun}_{k+1}$. There are isomorphisms*

$$\Theta_k(F_k) \cap \mathrm{Brun}_{k+1} / \Theta_k(F_k) \cap d_0(\mathrm{QBrun}_{k+2}) \to \pi_{k+1} S^2.$$

The method of proving that the maps $\Theta_n \colon F[y_1, y_2, \cdots, y_n] \to P_{n+1}$ are monomorphisms via Lie algebras admits an interpretation in terms of classical homotopy theory. The method is to filter both simplicial groups $F[S^1]_\bullet$ and $\mathrm{AP}_\bullet$ via the descending central series, and then to analyze the natural map on the level of associated graded Lie algebras.

On the other-hand, the Lie algebra arising from filtering any simplicial group by its' descending central series gives the $E^0$–term of the Bousfield–Kan spectral sequence for the simplicial group in question [5]. Similarly, filtering via the mod- $p$ descending central series gives the classical unstable Adams spectral sequence [5; 11; 38].

Thus the method of proof of Theorem 3.5 is precisely an analysis of the behavior of the natural map $\Theta \colon F[S^1]_\bullet \to \mathrm{AP}_\bullet$ on the level of the $E^0$–term of the Bousfield–Kan spectral sequence. This method exhibits a close connection between Vassiliev invariants of pure braids and these natural spectral sequences. The next result is restatement of Theorem 3.5 proven in [9; 10].

**Corollary 6.9** *The maps $\Theta_n \colon F[y_1, y_2, \cdots, y_n] \to P_{n+1}$ on the level of associated graded Lie algebras*

$$\mathrm{gr}_*(\Theta_n) \colon \mathrm{gr}_*(F[y_1, y_2, \cdots, y_n]) \to \mathrm{gr}_*(P_{n+1})$$

*are monomorphisms. Thus the maps $\Theta_n$ induce embeddings on the level of the $E^0$–term of the Bousfield–Kan spectral sequences for $E^0(\Theta) \colon E^0(F[S^1]_\bullet) \to E^0(\mathrm{AP}_\bullet)$.*

## 7 Other connections

Several further connections, outlined next, emerged after this article was submitted.

### Connection to principal congruence subgroups

One basic construction above is the Brunnian braid groups $\mathrm{Brun}_k(S)$. Recently, the authors have proven (unpublished) that the Brunnian braid group $\mathrm{Brun}_4(S^2)$ is isomorphic to the principal congruence subgroup of level 4 in $PSL(2,\mathbb{Z})$ [3].

This identification may admit an extension by considering the Brunnian braid groups $\mathrm{Brun}_{2g}(S^2)$ as natural subgroups of mapping class groups for genus $g$ surfaces. The subgroups $\mathrm{Brun}_{2g}(S^2)$ may embed naturally in $Sp(2g,\mathbb{Z})$ via classical surface topology using branched covers of the 2–sphere (work in progress).

### Connections to other spheres

The work above has been extended to all spheres as well as other connected CW–complexes [10]. One way in which other spheres arise is via the induced embedding of free products of simplicial groups

$$\Theta \amalg \Theta\colon F[S^1]_\bullet \amalg F[S^1]_\bullet \to \mathrm{AP}_\bullet \amalg \mathrm{AP}_\bullet.$$

The geometric realization of $F[S^1]_\bullet \amalg F[S^1]_\bullet$ is homotopy equivalent to $\Omega(S^2 \vee S^2)$ by Milnor's theorem stated above as 4.11.

Furthermore, $\Omega(S^2 \vee S^2)$ is homotopy equivalent to a weak infinite product of spaces $\Omega(S^n)$ for all $n > 1$.

### Connection to certain Galois groups

Consider automorphism groups $\mathrm{Aut}(H)$ where $H$ is $F_n$ or a completion of $F_n$ given by either the pro-finite completion $\widehat{F_n}$ or the pro-$\ell$ completion $(\widehat{F_n})_\ell$. Certain Galois groups $G$ are identified as natural subgroups of these automorphism groups by Belyĭ [2], Deligne [12], Drinfel'd [13; 14], Ihara [20; 21] and Schneps [33].

One example is Drinfel'd's Grothendieck–Teichmüller Galois group $G = \widehat{GT}$, a subgroup of $\mathrm{Aut}(\widehat{F_2})$.

Let $\mathrm{Der}(L^R[V_n])$ denote the Lie algebra of derivations of the free Lie algebra $L^R[V_n]$ where $V_n$ denotes a free module of rank $n$ over $R$ a commutative ring with identity. Two natural morphisms of Lie algebras which take values in $\mathrm{Der}(L^R[V_n])$ occur in this context as follows.

Properties of the infinitesimal braid relations as stated in Theorem 3.4 above give a natural second map

$$Ad\colon gr_*(P_{n+1}) \to \mathrm{Der}(L^{\mathbb{Z}}[V_n])$$

for which the kernel of $Ad$ is precisely the center of $gr_*(P_{n+1})$ (Cohen and Prassidis [8]). Combining this last fact with Theorem 3.2 gives properties of the composite morphism of Lie algebras

$$gr_*(F_n) \xrightarrow{\mathrm{gr}_*(\Theta_n)} gr_*(P_{n+1}) \xrightarrow{Ad} \mathrm{Der}(L^{\mathbb{Z}}[V_n]).$$

**Proposition 7.1** *If $n \geq 2$, the induced morphism of Lie algebras*

$$Ad \circ \mathrm{gr}_*(\Theta_n)\colon \mathrm{gr}_*(F[y_1, y_2, \cdots, y_n]) \to \mathrm{Der}(L^{\mathbb{Z}}[V_n])$$

*is a monomorphism.*

In addition, certain Galois groups $G$ above are filtered with induced morphisms of Lie algebras

$$gr_*(G) \to \mathrm{Der}(L^{\widehat{\mathbb{Z}}}[V_n])$$

where $\widehat{\mathbb{Z}}$ denotes the pro-finite completion of the integers. One example is $G = \widehat{GT}$ with

$$gr_*(\widehat{GT}) \to \mathrm{Der}(L^{\widehat{\mathbb{Z}}}[V_2])$$

as given in in [12; 20; 21; 34].

This raises the question of (i) whether the images of $Ad \circ \mathrm{gr}_*(\Theta_2)$ and $gr_*(\widehat{GT})$ in $\mathrm{Der}(L^{\widehat{\mathbb{Z}}}[V_2])$ have a non-trivial intersection and (ii) whether these intersections are meaningful.

## 8 Questions

The point of this section is to consider whether the connections between the braid groups and homotopy groups above are useful. Some natural as well as speculative problems are listed next.

(1) The combinatorial problem of distinguishing elements in the pure braid groups has been well-studied. For example, the Lie algebra associated to the descending central series of the pure braid group $P_n$ has been connected with Vassiliev theory and has been shown to be a complete set of invariants which distinguish all elements in $P_n$ [24]. Furthermore, these Lie algebras have been applied to other questions arising from the classical KZ–equations [23; 13] as well as the structure of certain Galois groups [20; 12; 13; 14].
Find invariants of braids up to the coarser equivalence relation given in Theorem 6.8. In particular, can one identify the subset of Vassiliev invariants of braids which are ”homotopy invariant”?

(2) Give combinatorial properties of the natural map $\mathrm{Brun}_{k+1}(\mathbb{R}^2) \to \mathrm{Brun}_{k+1}(S^2)$ which provide information about the cokernel. Two concrete problems are stated next.

(a) Give group theoretic reasons why the order of the 2–torsion in $\pi_*(S^2)$ is bounded above by 4 and why the $p$–torsion for an odd prime $p$ is bounded above by $p$.

(b) If $k+1 \geq 5$, the image of $\mathrm{Brun}_{k+1}(\mathbb{R}^2) \to \mathrm{Brun}_{k+1}(S^2)$ is a normal subgroup of finite index.
This fact follows from Serre's classical theorem that $\pi_k(S^2)$ is finite for $k > 3$ and Theorems 5.2 and 6.4 proven in [3].
Do natural features of the braid groups imply this result?

(3) Let $F_n$ denote the image of $\Theta_n(F_n)$. Observe that the groups $\mathrm{QBrun}_{n+2} \cap F_{n+1}$, and $\mathrm{Brun}_{n+1} \cap F_n$ are free. Furthermore, there is a short exact sequence of groups

$$1 \to F_n \cap d_0(\mathrm{QBrun}_{n+2}) \to F_n \cap \mathrm{Brun}_{n+1} \to \pi_{n+1} S^2 \to 1$$

as well as isomorphisms

$$F_n \cap \mathrm{Brun}_{n+1}/(F_n \cap d_0(\mathrm{QBrun}_{n+2})) \to \pi_{n+1} S^2$$

by Theorem 6.8.
Consider the Serre 5–term exact sequence for the group extension directly above to obtain information about the induced surjection

$$H_1(F_n \cap \mathrm{Brun}_{n+1}) \to \pi_{n+1}(S^2).$$

This 5–term exact sequence specializes to

$$\begin{array}{ccccc} H_2(\pi_{n+1}(S^2)) & \to & H_1(F_n \cap d_0(\mathrm{QBrun}_{n+2}))_{\pi_{n+1}(S^2)} & & \\ & \to & H_1(F_n \cap \mathrm{Brun}_{n+1}) & \to & \pi_{n+1}(S^2) \end{array}$$

where $A_\pi$ denotes the group of co-invariants of a $\pi$–module $A$. Thus $\pi_{n+1}(S^2)$ is a quotient of the free abelian group $H_1(F_n \cap \mathrm{Brun}_{n+1})$ with relations given by the image of the coinvariants $H_1(F_n) \cap d_0(\mathrm{QBrun}_{n+2}))_{\pi_{n+1}(S^2)}$.
Give combinatorial descriptions of the induced map on the level of the first homology groups

$$H_1(F_n \cap d_0(\mathrm{QBrun}_{n+2})) \to H_1(F_n \cap \mathrm{Brun}_{n+1}).$$

A similar problem arises with the the epimorphism $\mathrm{Brun}_{n+1}(S^2) \to \pi_n S^2$ with kernel in the image of $\mathrm{Brun}_{k+1}(\mathbb{R}^2)$ for $n+1 \geq 5$.

(4) This problem addresses a similarity between Vassiliev invariants and modular forms in the sense that both can be regarded as functions defined on certain braid groups. The initial ingredient here is given by the natural epimorphism

$$B_3 \to SL(2,\mathbb{Z})$$

with kernel given by the integers $\mathbb{Z}$. A generator for this kernel is equal to twice a generator of the center of $B_3$.

Recall that Shimura [36] gives isomorphisms

$$H^1(SL(2,\mathbb{Z});\mathbb{R}[x_1,x_2]) \to \mathcal{M},$$

and $$H^1(SL(2,\mathbb{Z});\mathrm{Sym}^k(x_1,x_2)) \to \mathcal{M}_{2k+2}$$

with the following properties.

(a) The action of $SL(2,\mathbb{Z})$ on $\mathbb{R}[x_1,x_2]$ is specified by the tautological representation on the two dimensional vector space with basis $x_1, x_2$ extended multiplicatively to the polynomial ring $\mathbb{R}[x_1,x_2]$.

(b) The module $\mathrm{Sym}^k(x_1,x_2)$ denotes the $SL(2,\mathbb{Z})$–submodule of $\mathbb{R}[x_1,x_2]$ given by polynomials of classical degree $k$.

(c) The vector space $\mathcal{M}_{2k+2}$ denotes the summand of the ring of modular forms of weight $2k+2$ (with Shimura's weight convention).

Also recall that the first cohomology group $H^1(SL(2,\mathbb{Z});\mathbb{R}[x_1,x_2])$ is given by the quotient of the module of crossed homomorphisms $SL(2,\mathbb{Z}) \to \mathbb{R}[x_1,x_2]$ modulo principal crossed homomorphisms. Thus elements in the classical ring of modular forms can be regarded as equivalence classes of certain functions defined on $SL(2,\mathbb{Z})$.

Furthermore, there are isomorphisms

$$H^1(B_3;\mathrm{Sym}^k(x_1,x_2)) \to E[u] \otimes \mathcal{M}_{2k+2}$$

where $E[u]$ denotes an exterior algebra with $u$ of degree 1. It is natural to ask whether and how crossed homomorphisms representing $H^1(B_3;\mathrm{Sym}^k(x_1,x_2))$ distinguish braids in $B_3$.

There is also a natural analogue for subgroups of $B_{2g+2}$ as follows. Determine $H^1(B_{2g+2};\mathbb{R}[x_1,\cdots,x_{2g}])$ where $B_{2g+2}$ acts via a natural symplectic representation on a vector space with basis $\{x_1,\cdots,x_{2g}\}$ the generating module for the polynomial ring and $\mathbb{R}[x_1,\cdots,x_{2g}]$. How do the natural crossed homomorphisms distinguish braids?

(5) Consider Brunnian braids $\mathrm{Brun}_k$. Fix a braid $\gamma$ with image in the $k$-th symmetric group $\Sigma_k$ given by a $k$–cycle. For any braid $\alpha$ in $\mathrm{Brun}_k$, the braid closure

of $\alpha \circ \gamma$ is a knot. Describe features of these knots or those obtained from the analogous constructions for $\Theta_k(F_{k-1}) \cap \mathrm{Brun}_k$. Where do these fit in Budney's description of the space of long knots [6]?

(6) Let $L[V]$ denote the free Lie algebra over the integers generated by the free abelian group $V$. Let $\mathrm{Der}(L[V])$ denote the Lie algebra of derivations of $L[V]$ and consider the classical adjoint representation

$$\mathrm{Ad}\colon L[V] \to \mathrm{Der}(L[V]).$$

Recall that the map $\Theta_k\colon F_k \to P_{k+1}$ induces a monomorphism of Lie algebras $\mathrm{gr}_*(\Theta_k)\colon \mathrm{gr}_*(F_k) \to \mathrm{gr}_*(P_{k+1})$ where $\mathrm{gr}_*(F_k)$ is isomorphic to the free Lie algebra $L[V_k]$ with $V_k$ a free abelian group of rank $k$. In addition, properties of the "infinitesimal braid relations" give a representation

$$\rho_k\colon \mathrm{gr}_*(P_{k+1}) \to \mathrm{Der}(L[V_k])$$

appearing in work on certain Galois groups [20] (with the integers $\mathbb{Z}$ replaced by the profinite completion of $\mathbb{Z}$) and, when restricted to the integers $\mathbb{Z}$, also addressed in [8].
Identify $F_k$ with $\Theta_k(F_k)$ in what follows below.
Give methods to describe combinatorial properties of the composite

$$\mathrm{gr}_*(F_k \cap \mathrm{Brun}_{k+1}) \xrightarrow{\mathrm{gr}_*(i_k)} \mathrm{gr}_*(F_k) \xrightarrow{\mathrm{gr}_*(\Theta_k)} \mathrm{gr}_*(P_{k+1}) \xrightarrow{\rho_k} \mathrm{Der}(L[V_k])$$

where $i_k\colon F_k \cap \mathrm{Brun}_{k+1} \to F_k$ is the natural inclusion. Let $\Phi_{k+1}$ denote this composite. When restricted to $\mathrm{gr}_*(F_k) = L[V_k]$, this map is a monomorphism. Give methods to describe the sub-quotient

$$\Phi_{k+1}(\mathrm{gr}_*(F_k \cap \mathrm{Brun}_{k+1}))/\Phi_{k+1}\mathrm{gr}_*(F_k \cap d_0(\mathrm{QBrun}_{k+2})).$$

(7) Assume that the pure braid groups $P_n(S)$ are replaced by either their pro-finite completion $\widehat{P_n(S)}$ or their pro-$\ell$ completions. Describe the associated changes for the homotopy groups arising in Theorems 5.2, 6.4, or 5.1. For example, is the torsion in these homotopy groups left unchanged by replacing $P_n(S)$ by $\widehat{P_n(S)}$?

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