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Odd-primary homotopy exponents of compact simple Lie groups

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We note that a recent result of the second author yields upper bounds for odd-primary homotopy exponents of compact simple Lie groups which are often quite close to the lower bounds obtained from v_1–periodic homotopy theory.

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1 Statement of results

The homotopy p–exponent of a topological space X, denoted $\exp_p(X)$, is the largest e such that some homotopy group $\pi_i(X)$ contains a $\mathbb{Z}/p^e$–summand.[1] In work dating back to 1989, the first author and collaborators have obtained lower bounds for $\exp_p(X)$ for all compact simple Lie groups X and all primes p by using v_1–periodic homotopy theory. Recently, the second author [12] proved a general result, stated here as Lemma 2.1, which can yield upper bounds for homotopy exponents of spaces which map to a sphere. In this paper, we show that these two bounds often lead to a quite narrow range of values for $\exp_p(X)$ when p is odd and X is a compact simple Lie group.

Our first new result, which will be proved in Section 2, combines Lemma 2.1 with a classical result of Borel and Hirzebruch.

1.1 Theorem *Let p be odd.*

(a) *If $n < p^2 + p$, then $\exp_p(\mathrm{SU}(n)) \le n - 1 + \nu_p((n-1)!)$.*

(b) *If $n \ge p^2 + 1$, then $\exp_p(\mathrm{SU}(n)) \le n + p - 3 + \binom{\lfloor \frac{n-2}{p-1} \rfloor - p + 2}{2}$.*

Here and throughout, $\nu_p(-)$ denotes the exponent of p in an integer, p is an odd prime, and $\lfloor x \rfloor$ denotes the integer part of x. All spaces are localized at p. It is useful to note the elementary fact that

$$\nu_p(m!) = \lfloor \tfrac{m}{p} \rfloor + \lfloor \tfrac{m}{p^2} \rfloor + \cdots ,$$

[1] Some authors (eg [12]) say that p^e is the homotopy p–exponent.

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and the well-known fact that $\nu_p(m!) \le \lfloor \frac{m-1}{p-1} \rfloor$.

Theorem 1.1(a) compares nicely with the following known result.

1.2 Theorem

(a) (Davis and Sun [7, 1.1]) *For any prime* p, $\exp_p(\mathrm{SU}(n)) \ge n-1+\nu_p(\lfloor \frac{n}{p} \rfloor !)$.

(b) (Davis and Yang [8, 1.8]) *If* p *is odd,* $1 \le t < p$, *and* $tp-t+2 \le n \le tp+1$, *then* $\exp_p(\mathrm{SU}(n)) \ge n$.

Thus we have the following corollary, which gives the only values of $n > p$ in which the precise value of $\exp_p(\mathrm{SU}(n))$ is known.

1.3 Corollary *If* p *is an odd prime, and* $n = p+1$ *or* $n = 2p$, *then* $\exp_p(\mathrm{SU}(n)) = n$.

When $n = p+1$, this was known (although perhaps never published) since, localized at p, we have $\mathrm{SU}(p+1) \simeq B(3, 2p+1) \times S^5 \times \cdots \times S^{2p-1}$, the exponent of which follows from Proposition 1.4 together with the result of Cohen, Moore, and Neisendorfer [5] that if p is odd, then $\exp_p(S^{2n+1}) = n$. Here and throughout, $B(2n+1, 2n+1+q)$ denotes an S^{2n+1}–bundle over S^{2n+1+q} with attaching map α_1 a generator of $\pi_{2n+q}(S^{2n+1})$, and $q = 2p-2$. Note also that the result of [5] implies that if $n \le p$, then $\exp_p(\mathrm{SU}(n)) = \exp_p(S^3 \times \cdots \times S^{2n-1}) = n-1$.

1.4 Proposition *If* p *is odd, then* $\exp_p(B(3, 2p+1)) = p+1$, *while if* $n > 1$, *then* $n+p-1 \le \exp_p(B(2n+1, 2n+1+q)) \le n+p$.

Proof This just combines Bendersky, Davis and Mimura [3, 1.3] for the lower bound and Theriault [12, 2.1] for the upper bound. □

Upper and lower bounds for the p–exponents of $\mathrm{Sp}(n)$ and $\mathrm{Spin}(n)$ can be extracted from Theorems 1.1 and 1.2 using long-known relationships of their p–localizations to that of appropriate $\mathrm{SU}(m)$. Indeed, Harris [9] showed that there are p–local equivalences

$$\mathrm{SU}(2n) \simeq \mathrm{Sp}(n) \times (\mathrm{SU}(2n)/\mathrm{Sp}(n)) \tag{1.5}$$

$$\mathrm{Spin}(2n+1) \simeq \mathrm{Sp}(n) \tag{1.6}$$

$$\mathrm{Spin}(2n+2) \simeq \mathrm{Spin}(2n+1) \times S^{2n+1}. \tag{1.7}$$

Combining this with Theorems 1.1 and 1.2 leads to the following corollary.

1.8 Corollary *Let* p *be odd.*

(1) $\exp_p(\mathrm{Spin}(2n+2)) = \exp_p(\mathrm{Spin}(2n+1)) = \exp_p(\mathrm{Sp}(n)) \le \exp_p(\mathrm{SU}(2n))$, *which is bounded according to Theorem 1.1.*

(2) $\exp_p(\mathrm{Sp}(n)) \ge 2n - 1 + \nu_p(\lfloor \frac{2n}{p} \rfloor !)$.

(3) *If* $1 \le t < p$, *and* $tp - t + 2 \le 2n \le tp + 1$, *then* $\exp_p(\mathrm{Sp}(n)) \ge 2n$.

Proof The second and third parts of (1) are immediate from **(1.6)** and **(1.5)**, while the first equality of (1) follows from **(1.7)** and the fact that $\exp_p(\mathrm{Spin}(2n+1)) \ge \exp_p(S^{2n+1})$, which is a consequence of part (2) and **(1.6)**. For parts (2) and (3), we need to know that the homotopy classes yielding the lower bounds for $\exp_p(\mathrm{SU}(2n))$ given in Theorem 1.2 come from its $\mathrm{Sp}(n)$ factor in **(1.5)**. To see this, we first note that in Bendersky and Davis [1, 1.2] it was proved that, if p is odd and k is odd, then

(1.9) $$v_1^{-1}\pi_{2k}(\mathrm{Sp}(n); p) \approx v_1^{-1}\pi_{2k}(\mathrm{SU}(2n); p).$$

These denote the p–primary v_1–periodic homotopy groups, which appear as summands of actual homotopy groups. The proofs of [7, 1.1] and [8, 1.8], which yielded Theorem 1.2, were obtained by computing $v_1^{-1}\pi_{2k}(\mathrm{SU}(n); p)$ for certain $k \equiv n-1 \bmod 2$. When applied to $\mathrm{SU}(2n)$, these groups are in $v_1^{-1}\pi_{2k}(\mathrm{SU}(2n); p)$ with k odd, and so by **(1.9)** they appear in the $\mathrm{Sp}(n)$ factor. □

For all (X, p) with X an exceptional Lie group and p an odd prime, except $(E_7, 3)$ and $(E_8, 3)$, we can make an excellent comparison of bounds for $\exp_p(X)$ using results in the literature. We use splittings of the torsion-free cases tabulated in [3, 1.1], but known much earlier (Mimura, Nishida and Toda [10]). In Table 1, we list the range of possible values of $\exp_p(X)$ when the precise value is not known. We also list the factor in the product decomposition which accounts for the exponent. Finally, in cases in which the exponent bounds do not follow from results already discussed, we provide references. Here $B(n_1, \dots, n_r)$ denotes a space built from fibrations involving p–local spheres of the indicated dimensions and equivalent to a factor in a p–localizaton of a special unitary group or quotient of same. Also, $B_2(3, 11)$ denotes a sphere-bundle with attaching map α_2, and W denotes a space constructed by Wilkerson and shown in [11, 1.1] to fit into a fibration $\Omega K_5 \to B(27, 35) \to W$. Finally, K_3 and K_5 denotes Harper's space as described in Bendersky and Davis [2] and Theriault [12].

1.10 Theorem *The homotopy p–exponents of exceptional Lie groups are as given in Table 1.*

Table 1: Homotopy exponents of exceptional Lie groups

X	p	$\exp_p(X)$	Factor	Reference
G_2	3	6	$B_2(3,11)$	[3, 1.3], [12, 2.2]
G_2	5	6	$B(3,11)$	
G_2	>5	5	S^{11}	
F_4, E_6	3	12	K_3	[2, 1.6], [12, 1.2]
F_4, E_6	$5,7$	$11,12$	$B(23-q,23)$	
F_4, E_6	11	12	$B(3,23)$	
F_4, E_6	>11	11	S^{23}	
E_7	5	$18,19,20$	$B(3,11,19,27,35)$	factor of SU(18)
E_7	7	$17,18,19$	$B(11,23,35)$	factor of SU(18)
E_7	$11,13$	$17,18$	$B(35-q,35)$	
E_7	17	18	$B(3,35)$	
E_7	>17	17	S^{35}	
E_8	5	$30,31$	W	[6, 1.1], [11, 1.2]
E_8	7	$29,30,31,32$	$B(23,35,47,59)$	[3, 1.4], Proposition 2.3
E_8	$11-23$	$29,30$	$B(59-q,59)$	
E_8	29	30	$B(3,59)$	
E_8	>29	29	S^{59}	

2 Proof of Theorem 1.1

In [12, Lemma 2.2], the second author proved the following result.

2.1 Lemma [12, 2.2,2.3] *Suppose there is a homotopy fibration*

$$F \to E \xrightarrow{q} S^{2n+1}$$

where E is simply-connected or an H–space and

$$|\operatorname{coker}(\pi_{2n+1}(E) \xrightarrow{q_*} \pi_{2n+1}(S^{2n+1}))| \le p^r.$$

Then $\exp_p(E) \le r + \max(\exp_p(F), n)$.

In [12, 2.2], it was required that E be an H–space, but [12, 2.3] noted that if E is not an H–space, the desired conclusion can be obtained by applying the loop-space functor to the fibration. We require E to be simply-connected so that we do not loop away a large fundamental group. We now use this lemma to prove Theorem 1.1.

Proof of Theorem 1.1 The proof is by induction on n. Let the odd prime p be implicit, and let $\mathrm{SU}'(n)$ denote the factor in the p–local product decomposition [10] of $\mathrm{SU}(n)$ which is built from spheres of dimension congruent to $2n-1 \bmod 2p-2$. By the induction hypothesis, the exponents of the other factors are $\leq$ the asserted amount. We will apply Lemma 2.1 to the fibration

$$\mathrm{SU}'(n-p+1) \to \mathrm{SU}'(n) \xrightarrow{q} S^{2n-1}.$$

In order to determine $|\operatorname{coker}(\pi_{2n-1}(\mathrm{SU}'(n)) \xrightarrow{q_*} \pi_{2n-1}(S^{2n-1}))|$, we use the classical result of Borel and Hirzebruch ([4, 26.7]) that

$$\pi_{2n-2}(\mathrm{SU}(n-1)) \approx \mathbb{Z}/(n-1)!.$$

When localized at p, it is clear that its p–component $\mathbb{Z}/p^{\nu_p((n-1)!)}$ must come from the $\mathrm{SU}'(n-p+1)$–factor in the product decomposition of $\mathrm{SU}(n-1)$, since $\pi_{2n-2}(\mathrm{SU}(n-1))$ is built from the classes $\alpha_i \in \pi_{2n-2}(S^{2n-1-iq})_{(p)}$. Thus

$$\pi_{2n-2}(\mathrm{SU}'(n-p+1)) \approx \mathbb{Z}/p^{\nu_p((n-1)!)},$$

and the exact sequence

$$\pi_{2n-1}(\mathrm{SU}'(n)) \xrightarrow{q_*} \pi_{2n-1}(S^{2n-1}) \to \pi_{2n-2}(\mathrm{SU}'(n-p+1))$$

implies

$$\nu_p(|\operatorname{coker}(q_*)|) \leq \nu_p((n-1)!). \tag{2.2}$$

(a) By the induction hypothesis, $\exp_p(\mathrm{SU}'(n-p+1)) \leq n-p+\nu_p((n-p)!)$. By hypothesis, $n-p < p^2$ and hence $\nu_p((n-p)!) \leq p-1$. Thus $\exp_p(\mathrm{SU}'(n-p+1)) \leq n-1$, and so by Lemma 2.1 and (2.2)

$$\exp_p(\mathrm{SU}'(n)) \leq \nu_p(|\operatorname{coker}(q_*)|) + n - 1 \leq \nu_p((n-1)!) + n - 1,$$

as claimed.

(b) By (a), part (b) is true if $p^2+1 \leq n \leq p^2+p-1$. Let $n \geq p^2+p$, and assume the theorem is true for $\mathrm{SU}'(n-p+1)$. Then by Lemma 2.1 and the induction hypothesis

$$\exp_p(\mathrm{SU}'(n)) \leq \nu((n-1)!) + n - p + 1 + p - 3 + \binom{\lfloor \frac{n-p-1}{p-1} \rfloor - p + 2}{2}.$$

Note that even if $\exp_p(\mathrm{SU}'(n-p+1))$ happened to be less than $n-1$, our upper bound for it is $\geq n-1$, and so this bound for $\exp_p(\mathrm{SU}'(n))$ is still a correct deduction from Lemma 2.1.

Since $\nu_p((n-1)!) \leq \lfloor \frac{n-2}{p-1} \rfloor$, we obtain

$$\begin{aligned}
\exp_p(\mathrm{SU}'(n)) &\leq \left\lfloor \frac{n-2}{p-1} \right\rfloor + n - 2 + \binom{\lfloor \frac{n-2}{p-1} \rfloor - p + 1}{2} \\
&= \left\lfloor \frac{n-2}{p-1} \right\rfloor + n - 2 + \binom{\lfloor \frac{n-2}{p-1} \rfloor - p + 2}{2} - \binom{\lfloor \frac{n-2}{p-1} \rfloor - p + 1}{1} \\
&= n + p - 3 + \binom{\lfloor \frac{n-2}{p-1} \rfloor - p + 2}{2},
\end{aligned}$$

as desired. □

The result in part (b) could be improved somewhat by a more delicate numerical argument.

Part (b) of the following result was used in Table 1.

2.3 Proposition *Let $p = 7$.*

(a) $\exp_7(B(23, 35, 47)) \leq 25$.

(b) $\exp_7(B(23, 35, 47, 59)) \leq 32$.

Proof The thing that makes this require special attention is that these spaces are not a factor of an $\mathrm{SU}(n)$, because they do not contain an S^{11}. There are fibrations

$$B(23, 35) \to B(23, 35, 47) \to S^{47}$$

and

$$B(23, 35, 47) \to B(23, 35, 47, 59) \to S^{59}.$$

Since, localized at 7, $\pi_{46}(S^{23}) \approx \pi_{46}(S^{35}) \approx \mathbb{Z}/7$, we have $|\pi_{46}(B(23, 35))| \leq 7^2$, and similarly $|\pi_{58}(B(23, 35, 47))| \leq 7^3$. (In fact, it is easily seen that these are cyclic groups of the indicated order.) Using Lemma 2.1 and that $\exp_7(B(23, 35)) \leq 18$ by Proposition 1.4, we obtain

$$\exp_7(B(23, 35, 47)) \leq 2 + \max(18, 23) = 25,$$

and then

$$\exp_7(B(23, 35, 47, 59)) \leq 3 + \max(25, 29) = 32. \qquad \square$$

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