Filtering the fiber of the pinch map

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A new type of Hopf invariant is described for the fiber of the pinch map:

$$F \longrightarrow X \cup CA \xrightarrow{\pi} SA$$

and this is used to study the boundary map in the fibration sequence of Cohen, Moore and Neisendorfer:

$$\Omega^2 S^{2n+1} \xrightarrow{\partial_n} \Omega F_n \longrightarrow \Omega P^{2n+1} \longrightarrow \Omega S^{2n+1}.$$

The boundary map is shown to be compatible with the Hopf invariant and a filtration of the spliting is obtained.

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1 Introduction

In [3], Cohen, Moore and Neisendorfer analyzed the homotopy type of the mod p^r Moore space

$$P^{2n+1} = S^{2n} \cup_{p^r} e^{2n+1}$$

by constructing a fibration sequence

$$\Omega^2 S^{2n+1} \xrightarrow{\partial_n} \Omega F_n \longrightarrow \Omega P^{2n+1} \longrightarrow \Omega S^{2n+1}.$$

A key result is that both ΩF_n and ΩP^{2n+1} contain a factor which is the loop space on a one point union of Moore spaces that are at least $4n - 2$ connected. Removing this factor (which can be inductively considered later) leaves a fibration sequence:

$$\cdots \longrightarrow \Omega^2 S^{2n+1} \xrightarrow{\overline{\overline{\partial}}} \overline{F}_n \longrightarrow T^{2n+1} \longrightarrow \Omega S^{2n+1}$$

and they show that

$$\overline{F}_n \cong S^{2n-1} \times \prod_{k \geq 1} S^{2np^k-1}\{p^{r+1}\}.$$

An important remaining question is to understand the components of $\overline{\overline{\partial}}$:

$$\overline{\partial}_0 \colon \Omega^2 S^{2n+1} \longrightarrow S^{2n-1}$$

$$\overline{\partial}_k \colon \Omega^2 S^{2n+1} \longrightarrow S^{2np^k-1}\{p^{r+1}\}.$$

$\overline{\partial}_0$ is well understood and plays a key role in homotopy theory. This is the noted
"Cohen–Moore–Neisendorfer map". It fits into a homotopy commutative diagram:

$$
\begin{array}{ccc}
S^{2n-1} & \xrightarrow{\ p\ } & S^{2n-1} \\
\downarrow & \overset{\overline{\partial}_0}{\nearrow} & \downarrow \\
\Omega^2 S^{2n+1} & \xrightarrow{\ p\ } & \Omega^2 S^{2n+1}
\end{array}
$$

the consequence of which is that

$$
p \cdot \pi_k(S^{2n+1}) \subset E^2 \pi_{k-2}(S^{2n-1}) \qquad k > 0
$$

from which the exponent result follows:

$$
p^n \pi_k(S^{2n+1}) = 0 \qquad k > 0.
$$

There is much interest in understanding $\overline{\partial}_k$ for $k > 0$; in particular, it is not known
whether they are all null homotopic (see Anick and Gray [1], Cohen, Moore and
Neisendorfer [3], Gray [5; 6] and Gray and Theriault [7]). In case $r > 1$, Neisendorfer
[10; 11] has shown that $\overline{\partial}_k = 0$ for $k \geq 1$, so our emphasis will be on the case $r = 1$.

A remarkable observation of Cohen, Moore and Neisendorfer [2] is that there is an
isomorphism of Hopf algebras (see Appendix A):

$$
H_*(\Omega^2 S^{2n+1}; \mathbb{Z}/p) \cong H_*(\overline{F}_n; \mathbb{Z}/p).
$$

However, the induced homomorphism $(\overline{\partial})_* = 0$.

The intention of this work is to consider constructions involving $\overline{F}_n$ which are analogous
to constructions involving $\Omega^2 S^{2n+1}$. We will be able to do this compatibly with the
map $\overline{\partial}$. In particular, we will look at the filtration of Selick [12] and the classifying
space of the double suspension [5]. Our results will follow from the construction of a
new "Hopf invariant" type map:

$$
F_n \xrightarrow{\ h\ } F_{np}
$$

which induces an epimorphism in p–local homology. Recall that there is a p–local
fibration sequence:

$$
S^{2n}_{(p^s-1)} \longrightarrow S^{2n}_\infty \xrightarrow{\ H_{p^s}\ } S^{2np^s}_\infty.
$$

Here we are using James' notation X_∞ for the reduced product space and $X_{(r)}$ for
the subspace of words of length less than or equal to r. We also write $F_{(r)}$ for the

analogous filtration of the relative James construction for F (Gray [4]). By analogy, we compare $F_{(p^s-1)}$ to the fiber Δ_s of the Hopf invariant h^s:

$$\Delta_s \longrightarrow F_n \xrightarrow{h^s} F_{np^s}$$

1.1 Theorem $\Omega\Delta_s \simeq \Omega F_{(p^s-1)} \times \Omega X_s$ where X_s is a wedge of mod p^r Moore spaces.

This will appear as Theorem 8.1. These same ideas lead to:

1.2 Theorem If $k > 1$, there is a homotopy commutative diagram:

$$
\begin{array}{ccc}
\Omega^2 S^{2n+1} & \xrightarrow{\overline{\partial}_{n(k)}} & S^{2np^k-1}\{p^{r+1}\} \\
\gamma_k \downarrow \cong & & \uparrow \overline{\partial}_{np^k-1}(1) \\
\Omega^2 S^{2n+1} & \xrightarrow{\Omega H_{p^k-1}} & \Omega^2 S^{2np^{k-1}+1}
\end{array}
$$

where H_{p^k} is a James–Hopf invariant and γ_k is a homotopy equivalence.

1.3 Theorem The map $\overline{\partial}_{n(k)}\colon \Omega^2 S^{2n+1} \longrightarrow S^{2np^k-1}\{p^{r+1}\}$ is homotopic to a composition:

$$\Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n \xrightarrow{\epsilon} S^{2np^k-1}\{p^{r+1}\}.$$

These results follow from equation (**9.2**) and Corollary 10.3 respectively, in the sequel.

Throughout this paper all spaces will be assumed to be localized at a fixed prime $p > 2$ and all homology will be p–local unless otherwise indicated.

2 Filtration of $\Omega^2 S^{2n+1}$

$H_*(\Omega^2 S^{2n+1}; \mathbb{Z}/p)$ is a free commutative algebra on generators of dimensions $2np^i - 1$, $2np^{i+1} - 2$ for $i \geq 0$. Selick [12] has described a sequence of H–spaces whose homology filters $H_*(\Omega^2 S^{2n+1}; \mathbb{Z}/p)$ by successively adding one generator at each stage. Let us write $S^{2n}_{(k)} = J_k(S^{2n})$ for the kth filtration of the James construction $J(S^{2n}) = S^{2n}_\infty$. Then the space capturing the first $2s$ generators of

$H_*(\Omega^2 S^{2n+1}; \mathbb{Z}/p)$ is precisely $\Omega S^{2n}_{(p^s-1)}$. Selick [12] defines spaces G_s which contain the first $2s+1$ generators. We define G_s by a diagram of fibrations:

$$
\begin{array}{ccccc}
G_s & \longrightarrow & S^{2np^s-1} & \xrightarrow{\alpha} & S^{2n}_{(p^s-1)} \\
\downarrow & & \downarrow & & \| \\
\Omega^2 S^{2n+1} \longrightarrow \Omega^2 S^{2np^s+1} & \longrightarrow & S^{2n}_{(p^s-1)} \longrightarrow \Omega S^{2n+1} \xrightarrow{H_{np^s}} \Omega S^{2np^s+1} \\
\downarrow & & \downarrow{\scriptstyle v} & & \\
BW_{np^s} & == & BW_{np^s} & &
\end{array}
$$

where $\alpha\colon S^{2np^s-1} \to S^{2n}_{(p^s-1)}$ is the attaching map for the $2np^s$ cell of S^{2n}_∞, H_{np^s} is the appropriate James–Hopf invariant, and BW_{np^s} is the classifying space for the double suspension. Since G_s is the fiber of $v \circ \Omega H_{np^s}$, it is an H space if $p > 3$ (Gray [5, Proposition 6]). G_s could also be described as the fiber of a Toda–Hopf invariant $\Omega S^{2n}_{(p^{s+1}-1)} \xrightarrow{H'} \Omega S^{2np^{s+1}-1}$ using the techniques of Gray [5] and Moore and Neisendorfer [9]).

Constructing an analogous filtration for $\overline{F}_n$ is not difficult. The key result is in the compatibility of the two filtrations.

3   Proof of Theorem CMN

In this section we give a brief summary of the proof of the main result of Cohen, Moore and Neisendorfer [3].

Theorem CMN   *There is a diagram of fibration sequences:*

$$
\begin{array}{ccccc}
\Omega F_n & \longrightarrow & \Omega P^{2n+1} & \longrightarrow & \Omega S^{2n+1} \\
\downarrow & & \downarrow & & \| \\
S^{2n-1} \times V_n & \longrightarrow & T^{2n+1} & \longrightarrow & \Omega S^{2n+1} \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i'} & & \\
P_n & \xrightarrow{\;=\;} & P_n & & \\
\downarrow{\scriptstyle \overline{\phi}} & & \downarrow & & \\
F_n & \longrightarrow & P^{2n+1} & \xrightarrow{\;p\;} & S^{2n+1}
\end{array}
$$

where the maps i and i' are null homotopic, p is the pinch map, P_n is a one point union of mod p^r Moore spaces of dimension $\geq 4n$ and

$$V_n = \prod_{k \geq 1} S^{2np^k - 1}\{p^{r+1}\}.$$

Here $S^m\{d\}$ is the fiber of the degree d map on S^m. The inessentiality of i and i' implies that

$$\Omega F_n \simeq S^{2n-1} \times V_n \times \Omega P_n$$
$$\Omega P^{2n+1} \simeq T^{2n+1} \times \Omega P_n.$$

Sketch of the proof Both the mod p^r homotopy and the mod p homology of ΩP^{2n+1} have differential Lie algebra structures, and the Hurewicz map is a Lie algebra homomorphism. Furthermore ΩF_n has an extended ideal structure. These structures are obtained from the Samelson product and the rth Bockstein $\beta^{(r)}$ ($\beta^{(i)} = 0$ for $i < r$).

$H_*(\Omega P^{2n+1}; \mathbb{Z}/p) = U(L)$ where L is a free Lie algebra on $v \in H_{2n}(\Omega P^{2n+1}; \mathbb{Z}/p)$ and $u = \beta^{(r)}(v)$ and $U(L)$ is the universal enveloping algebra. $L^{(0)} \subset L$ is the Lie ideal generated by $x_i = ad^{i-1}(v)(u)$ for $i \geq 1$ and $H_*(\Omega F_n; \mathbb{Z}/p) = U(L^{(0)})$.

Furthermore the suspensions of the x_i,

$$\sigma(x_i) \in H_{2ni}(F_n; \mathbb{Z}/p) \cong \mathbb{Z}/p$$

are non zero. u and v lie in the image of the mod p^r Hurewicz homomorphism, so all of L and hence $L^{(0)}$ does as well

$$L^{(0)} \subset \mathrm{im}\{\pi_*(\Omega F_n; \mathbb{Z}/p^r) \longrightarrow H_*(\Omega F_n; \mathbb{Z}/p)\}$$

using the extended ideal structure in homotopy.

Both in homotopy and homology, $\beta^{(r)}(x_{p^k}) = 0$, so one may construct an extension:

$$P^{2np^k - 1}(p^{r+1}) \xrightarrow{\delta_k} \Omega F_n$$

of the mod p^r homotopy class

$$P^{2np^k - 1} \longrightarrow \Omega F_n$$

representing x_{p^k} in homology. The only property of the maps δ_k that is needed is that their Hurewicz image is x_{p^k}. From these maps the authors construct

$$S^{2np^k - 1}\{p^{r+1}\} \xrightarrow{e} \Omega P^{2np^k}(p^{r+1}) \xrightarrow{\Omega \bar{\delta}_k} \Omega F_n$$

where $\bar{\delta}_k$ is the adjoint of δ_k and e is a particular map described in the next section. These maps are assembled via loop multiplication to obtain:

$$\Theta \colon S^{2n-1} \times V_n \longrightarrow \Omega F_n$$

The Bockstein $\beta^{(r)}$ is trivial in $H_*(S^{2n-1} \times V_n; \mathbb{Z}/p)$ but not in $H_*(\Omega F_n; \mathbb{Z}/p)$. However Θ induces an isomorphism in the Bockstein homology of these homology groups (see Appendix A for a calculation of the Bockstein homology).

Next the authors construct a sequence of sub-Lie algebras $L^{(k+1)} \subset L^{(k)} \subset L^{(0)}$ via short exact sequences of Lie algebras:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & L^{(1)} & \longrightarrow & L^{(0)} & \longrightarrow & \langle \tau_0 \rangle & \longrightarrow 0 \\
0 & \longrightarrow & L^{(k+1)} & \longrightarrow & L^{(k)} & \longrightarrow & \langle \tau_k, \sigma_k \rangle & \longrightarrow 0.
\end{array}
$$

Here $\langle \tau_0 \rangle$ and $\langle \tau_{k_1} \sigma_k \rangle$ are free commutative Lie algebras generated by $\tau_k = x_{p^k}$ of dimension $2np^k - 1$ and

$$\sigma_k = \frac{1}{2p} \sum_{i=1}^{p^k-1} \binom{p^k}{i} [x_i, x_{p^k-i}] \in L^{(0)}.$$

This is possible since τ_k is a generator and σ_k is decomposable in $L^{(k-1)}$ but not in $L^{(k)}$ since $\frac{1}{2p} \binom{p^k}{p^{k-1}}$ is a p–local unit and $x_{p^{k-1}} = \tau_{k-1} \notin L^{(k)}$.

It follows that there is a split short exact sequence of differential Hopf algebras:

$$0 \longrightarrow U(L^{(k+1)}) \longrightarrow U(L^{(k)}) \longrightarrow H_*(S^{2np^k-1}\{p^{r+1}\}; \mathbb{Z}/p) \to 0$$

and hence

$$U(L^{(\infty)}) \otimes H_p(S^{2n-1} \times V_n; \mathbb{Z}/p) \cong U(L^{(0)}) = H_*(\Omega F_n; \mathbb{Z}/p)$$

where $L^{(\infty)} = \bigcap_{k \geq 0} L^{(k)}$.

Consequently $H_t(U(L^{(\infty)}); \beta^{(r)}) = 0$ for $t > 0$ and hence $L^{(\infty)}$ has a basis consisting of classes $\{x_\alpha, \beta^{(r)} x_\alpha\}$. Since $x_\alpha \in L^{(\infty)} \subset L^{(0)}$ one can choose maps

$$\phi_\alpha \colon P^{n_\alpha} \longrightarrow \Omega F_n$$

whose homology image is x_α and $\beta^{(r)} x_\alpha$ using the extended ideal structure. Assembling these one produces a one point union of mod p^r Moore spaces, P_n and a map:

$$\bar{\phi} \colon P_n \longrightarrow F_n$$

such that the homology image of

$$(\Omega\overline{\phi})_* \colon H_*(\Omega P_n; \mathbb{Z}/p) \longrightarrow H_*(\Omega F_n; \mathbb{Z}/p)$$

is exactly $U(L^{(\infty)})$. From this the authors produce a homotopy equivalence

$$S^{2n-1} \times V_n \times \Omega P_n \longrightarrow \Omega F_n$$

by multiplying the maps Θ and $\overline{\phi}$ via the H space structure in ΩF_n. Let $\overline{F}_n$ be the fiber of $\overline{\phi}$. Then

$$S^{2n-1} \times V_n \xrightarrow{\Theta} \Omega F_n \xrightarrow{\partial} \overline{F}_n$$

is a homotopy equivalence and i is inessential. This completes the outline of the proof. □

4 Combinatorial description of F_n

In [4] a combinatorial description of the fiber of the pinch map:

$$F \longrightarrow X \cup CA \xrightarrow{\pi} SA$$

was given in the spirit of the James construction X_∞ for ΩSX. The model, designated $(X, A)_\infty$ consists of all words in X_∞ with the property that all letters after the first letter are required to lie in A, where $A \subset X$. Alternatively, this can be described by a push out diagram:

(4.1)

$$
\begin{array}{ccc}
X \times A_\infty & \longrightarrow & (X, A)_\infty \\
\uparrow & & \uparrow \\
A \times A_\infty & \longrightarrow & A_\infty
\end{array}
$$

4.2 Proposition [4] *There is a map $(X, A)_\infty \to F$ which is a homotopy equivalence when the inclusion $A \subset X$ is a cofibration.*

There is an action of the monoid A_∞ on $(X, A)_\infty$ and $(X, A)_\infty$ can be thought of as a universal space in the following sense. If Y is any space on which A_∞ acts and $g \colon X \to Y$ is any map such that $g(a) = a \cdot *$ for some point $* \in Y$, there is a unique A_∞ equivariant map

$$g_\infty \colon (X, A)_\infty \to Y.$$

(See [4, 4.2]). The map $(X, A)_\infty \to F$ is constructed from the action $A_\infty \times F \to \Omega SA \times F \to F$.

The orbit space of $(X, A)_\infty$ under the action of A_∞ is X/A and we may use the universal property to establish the following diagram

(4.3)

$$
\begin{array}{ccc}
(X, A)_\infty & \xrightarrow{\ e\ } & X_\infty \\
\downarrow{\scriptstyle \rho} & & \downarrow \\
X/A & \longrightarrow & (X/A)_\infty
\end{array}
$$

Note that the inclusion $e\colon (X/A)_\infty \to X_\infty$ is the unique A_∞ equivariant extensions of the inclusion of X in X_∞.

Example 4.4 Let $A = S^{2n-2} \subset P^{2n-1}(p^t) = X$. Then $(X, A)_\infty$ is the homotopy fiber of the map of degree p^t

$$
(X, A)_\infty = S^{2n-1}\{p^t\} \longrightarrow S^{2n-1} \xrightarrow{\ p^t\ } S^{2n-1}
$$

and the map $e\colon S^{2n-1}\{p^t\} \longrightarrow (P^{2n-1}(p^t))_\infty \cong \Omega P^{2n}(p^t)$ is uniquely determined as a ΩS^{2n-1} equivariant map extending the inclusion of $P^{2n-1}(p^t)$.

One of the main features of the construction $(X, A)_\infty$ is that we can define functorial Hopf invariants using the same formulas as in James [8]. In particular we have a pointwise commutative diagram:

$$
\begin{array}{ccc}
A_\infty & \xrightarrow{\ H_k\ } & (A^{(k)})_\infty \\
\downarrow & & \downarrow \\
(X, A)_\infty & \xrightarrow{\ H_k\ } & (X \wedge A^{(k-1)})_\infty \\
\downarrow & & \downarrow \\
X_\infty & \xrightarrow{\ H_k\ } & (X^{(k)})_\infty
\end{array}
$$

It would be desirable to construct functorial compressions of these maps:

$$
h\colon (X, A)_\infty \longrightarrow (X \wedge A^{(k-1)}, A^{(k)})_\infty \subset (X \wedge A^{(k-1)})_\infty
$$

but I have been unable to do this. In the next section we will construct a map h of this form, but we have no knowledge of how it relates to H_k.

5 Construction of $h\colon F_n \longrightarrow F_{np}$

In this section we will define a kind of Hopf invariant which is key for the results of this paper.

5.1 Theorem *Suppose (X, A) is a suspension pair and $r \geq 1$. Then there is a map:*

$$h \colon (X, A)_\infty \longrightarrow (X \wedge A^{(r-1)}, A^{(r)})_\infty$$

such that the composition:

$$X \times A_\infty \longrightarrow (X, A)_\infty \overset{h}{\longrightarrow} (X \wedge A^{(r-1)}, A^{(r)})_\infty \overset{\rho}{\longrightarrow} X \wedge A^{(r-1)}/A^{(r)}$$

is homotopic to the composition:

$$X \times A_\infty \overset{1 \times H_{r-1}}{\longrightarrow} X \times A_\infty^{(r-1)} \longrightarrow X \wedge A_\infty^{(r-1)} \overset{\alpha}{\longrightarrow} X \wedge A^{(r-1)} \longrightarrow X \wedge A^{(r-1)}/A^{(r)}$$

where $H_{r-1} \colon A_\infty \to A_\infty^{(r-1)}$ is any map and α uses the suspension structure of X to collapse $X \wedge A_\infty^{(r-1)}$ to $X \wedge A^{(r-1)}$.

5.2 Corollary *Let F_n be the fiber of the pinch map*

$$p \colon P^{2n+1}(p^r) \longrightarrow S^{2n+1}$$

for each n and H_{r-1} be any choice of James–Hopf invariants. Then there is a map:

$$h \colon F_n \longrightarrow F_{rn}$$

such that the composition:

$$S^{2n} \times \Omega S^{2n+1} \longrightarrow F_n \overset{h}{\longrightarrow} F_{rn} \longrightarrow P^{2rn+1}$$

is homotopic to:

$$S^{2n} \times \Omega S^{2n+1} \overset{1 \times H_{r-1}}{\longrightarrow} S^{2n} \times \Omega S^{2(r-1)n+1} \longrightarrow S^{2rn} \longrightarrow P^{2rn+1};$$

in particular, $h_* \colon H_{2rn}(F_n) \longrightarrow H_{2rn}(F_{rn})$ is an isomorphism.

Note It is an easy calculation to see that

$$H_i(F_n) = \begin{cases} \mathbb{Z}_{(p)} & n \mid i \\ 0 & n \nmid i \end{cases}$$

and the map $F_n \overset{\pi}{\longrightarrow} P^{2n+1}(p^r)$ is reduction mod p^r in homology. In particular we have defined a map:

$$F_n \longrightarrow P^{2nr+1}(p^r)$$

for each $r \geq 1$ which is onto in p–local homology.

Proof of Theorem 5.1 We begin by constructing a map

$$\mu\colon (X,A)_\infty \longrightarrow (X,A)_\infty/A_\infty \longrightarrow X \times A_\infty/A \rtimes A_\infty \equiv X \rtimes A_\infty/A \rtimes A_\infty$$

which follows from the push out diagram (4.0). This map is functorial and we have a commutative diagram:

$$
\begin{array}{ccc}
(X,A)_\infty & \xrightarrow{\;\mu\;} & X \rtimes A_\infty/A \rtimes A_\infty \\
\uparrow & & \uparrow \\
X \times A_\infty & \longrightarrow & X \rtimes A_\infty
\end{array}
$$

using the functorial property, we have the commutative diagram:

$$
\begin{array}{ccc}
(X,A)_\infty & \longrightarrow & X \rtimes A_\infty/A \rtimes A_\infty \\
\downarrow & & \downarrow \\
(CX,A)_\infty & \longrightarrow & CX \rtimes A_\infty/A \rtimes A_\infty \simeq S(A \rtimes A_\infty)
\end{array}
$$

however $(CX, A)_\infty$ is the fiber of $CX/A \longrightarrow SA$ and hence is contractible. Consequently there is a lifting γ of μ:

$$
\begin{array}{ccccc}
& (X \rtimes A_\infty, A \rtimes A_\infty)_\infty & \xrightarrow{\;\epsilon\;} & (X \wedge A_\infty, A \wedge A_\infty)_\infty \\
\gamma \nearrow & \;\downarrow{\rho_1} & & \;\downarrow{\rho_2} \\
(X,A)_\infty \xrightarrow{\;\mu\;} & X \rtimes A_\infty/A \rtimes A_\infty & \longrightarrow & X \wedge A_\infty/A \wedge A_\infty.
\end{array}
$$

We now apply the homotopy commutative diagram:

$$
\begin{array}{ccccc}
(X \wedge A_\infty, A \wedge A_\infty)_\infty & \xrightarrow{\;\beta\;} & (X \wedge A_\infty^{(r-1)}, A \wedge A_\infty^{(r-1)})_\infty & \xrightarrow{\;\alpha_\infty\;} & (X \wedge A^{(r-1)}, A^{(r)})_\infty \\
\;\downarrow{\rho_2} & & \;\downarrow{\rho_3} & & \;\downarrow{\rho} \\
X \wedge A_\infty/A \wedge A_\infty & \xrightarrow{\;\beta'\;} & X \wedge A_\infty^{(r-1)}/A \wedge A_\infty^{(r-1)} & \xrightarrow{\;\alpha'\;} & X \wedge A^{(r-1)}/A^{(r)}
\end{array}
$$

where β and β' are defined using $H_{r-1}\colon A_\infty \to A_\infty^{(r-1)}$. We now define $h = \alpha_\infty \beta \epsilon \gamma$. Clearly the composition:

$$X \times A_\infty \longrightarrow (X,A)_\infty \xrightarrow{\;h\;} (X \wedge A^{(r-1)}, A^{(r)})_\infty \xrightarrow{\;\rho\;} X \wedge A^{(r-1)}/A^{(r)}$$

is homotopic to the composition on the left and top in the diagram:

$$
\begin{array}{ccccc}
X \wedge A_\infty/A \wedge A_\infty & \xrightarrow{\;\beta'\;} & X \wedge A_\infty^{(r-1)}/A \wedge A_\infty^{(r-1)} & \xrightarrow{\;\alpha'\;} & X \wedge A^{(r-1)}/A^{(r)} \\
\uparrow & & \uparrow & & \uparrow \\
X \times A_\infty & \xrightarrow{1 \times H_{r-1}} X \times A_\infty^{(r-1)} \xrightarrow{\;\epsilon'\;} X \wedge A_\infty^{(r-1)} & \xrightarrow{\;\alpha\;} & X \wedge A^{(r-1)}
\end{array}
$$

and consequently to the composite along the bottom and the right as well.   □

5.3 Proposition *The map* $h: F_n \longrightarrow F_{np^s}$ *induces an epimorphism in* p*–local homology.*

Proof We use the map $\Omega S^{2n+1} \xrightarrow{\partial} F_n$ which has degree p in H^{2nk} for each $k > 0$ to calculate the cup product structure in $H^*(F_n)$. We choose generators $e_i \in H^{2ni}(F_n)$ such that $\partial^*(e_i)$ is p times the generator of $H^{2ni}(\Omega S^{2n+1})$ dual to the ith power of a homology generator in $H_{2n}(\Omega S^{2n+1})$. Then it is easy to see that

$$e_i e_j = p \binom{i+j}{i} e_{i+j}.$$

Let us designate $d_i \in H^{2nip^s}(F_{np^s})$ for the corresponding generator; then

$$h^*(d_1) = u_1 e_{p^s}$$

for some p local unit u_1 by Corollary 5.2. We show that

$$h^*(d_i) = u_i e_{ip^s}$$

where u_i is a p–local unit for each $i \geq 1$ by induction. Using the product structure we have

$$pi\, d_i = d_1 d_{i-1},$$

so $pi h^*(d_i) = h^*(d_1 d_{i-1}) = h^*(d_1) h^*(d_{i-1})$

$$= (u_1 e_{p^s})(u_{i-1} e_{(i-1)p^s})$$

$$= pu_1 u_{i-1} \binom{ip^s}{p^s} e_{ip^s}.$$

It suffices to show that $\frac{1}{i}\binom{ip^s}{p^s}$ is a p–local unit.

Now let $v_p(m)$ be the exponent of p in m and $[x]$ be the greatest integer less than or equal to x. Then

$$v_p(n!) = \sum_{i \geq 1} \left[\frac{n}{p^i}\right].$$

Consequently $v_p((p^s i)!) = p^{s-1}i + p^{s-2}i + \cdots + i + v_p(i!)$ so

$$v_p\left(\binom{p^s i}{p^s}\right) = v_p((p^s i)!) - v_p((p^s(i-1))!) - v_p(p^s!)$$

$$= v_p(i!) - v_p((i-1)!)$$

$$= v_p(i).$$

and we are done.   □

By Corollary 5.2, the composition

$$\Omega S^{2n+1} \xrightarrow{\partial} F_n \xrightarrow{h} F_{np^s} \longrightarrow P^{2np^2+1}$$

is null homotopic, so there is a lifting $\widetilde{H}_{p^s}$

$$
\begin{array}{ccc}
\Omega S^{2n+1} & \xrightarrow{\widetilde{H}_{p^s}} & \Omega S^{2np^s+1} \\
\partial \downarrow & & \downarrow \partial \\
F_n & \xrightarrow{\quad h \quad} & F_{np^s}.
\end{array}
$$

5.4 Proposition $(\widetilde{H}_{p^s})_* \colon H_*(\Omega S^{2n+1}) \to H_*(\Omega S^{2np^s+1})$ *is an epimorphism.*

This follows since both maps labelled ∂ have degree p in homology and h_* is an epimorphism.

In particular, the fiber of $\widetilde{H}_{p^s}$ is $S^{2n}_{(p^s-1)}$ as if $\widetilde{H}_{p^s}$ were the James–Hopf invariant H_{p^s}. Both $\Omega \widetilde{H}_{p^s}$ and ΩH_{p^s} can be placed in the fibration sequence induced by the inclusion $S^{2n}_{(p^s-1)} \subset \Omega S^{2n+1}$, so there is an equivalence $\gamma_s \colon \Omega^2 S^{2n+1} \to \Omega^2 S^{2n+1}$ such that

$$\Omega \widetilde{H}_{p^s} = \Omega H_{p^s} \circ \gamma_s$$

hence we have:

5.5 Corollary   *There is a homotopy commutative diagram*

$$
\begin{array}{ccc}
\Omega^2 S^{2n+1} & \xrightarrow{\partial_n} & \Omega F_n \\
\Omega \widetilde{H}_{p^s} \downarrow & & \downarrow \Omega h^s \\
\Omega^2 S^{2np^s+1} & \xrightarrow{\partial_{np}} & \Omega F_{np^s}
\end{array}
$$

where $\Omega \widetilde{H}_{p^s} \sim \Omega H_{p^s} \circ \gamma_s$ *for some homotopy equivalence* $\gamma_s \colon \Omega^2 S^{2n+1} \longrightarrow \Omega^2 S^{2n+1}$.

6 A Filtered Decomposition

Let Δ_s be the homotopy fiber of $h^s\colon F_n \to F_{np^s}$. In this section we will compare the decompositions of ΩF_n and ΩF_{np^s} and prove

6.1 Theorem $\Omega \Delta_s \cong S^{2n-1} \times \displaystyle\prod_{1 \le k < s} S^{2np^k-1}\{p^{r+1}\} \times \Omega S^{2np^s-1} \times \Omega R_s$ where R_s is a wedge of mod p^r Moore spaces.

Let $\Psi\colon S^{2n-1} \times V_n \to S^{2np-1} \times V_{np}$ be defined by first projecting onto V_n and then applying

$$V_n = S^{2np-1}\{p^2\} \times V_{np} \xrightarrow{\rho \times 1} S^{2np-1} \times V_{np}$$

where $\rho\colon S^{2np-1}\{p^2\} \to S^{2np-1}$ is the projection.

6.2 Lemma *With appropriate choices of the maps Θ_n and Θ_{np} there is a homotopy commutative diagram:*

$$
\begin{array}{ccc}
S^{2n-1} \times V_n & \xrightarrow{\ \Theta_n\ } & \Omega F_n \\
\downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \Omega h} \\
S^{2np-1} \times V_{np} & \xrightarrow{\ \Theta_{np}\ } & \Omega F_{np}.
\end{array}
$$

Proof Given choices $\bar\delta_k\colon P^{2np^k}(p^{r+1}) \to F_n$ inducing epimorphisms in p–local cohomology, define $\bar\delta_k(np)$ as the composition:

$$P^{2np^{k+1}}(p^{r+1}) \xrightarrow{\ \bar\delta_{k+1}\ } F_n \xrightarrow{\ h\ } F_{np}$$

From this we construct

$$
\begin{array}{ccccc}
S^{2np^{k+1}-1}\{p^{r+1}\} & \xrightarrow{\ e\ } & \Omega P^{2np^{k+1}}(p^{r+1}) & \xrightarrow{\ \Omega\bar\delta_{k+1}\ } & \Omega F_n \\
\downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \Omega h} \\
S^{2np^{k+1}-1}\{p^{r+1}\} & \xrightarrow{\ e\ } & \Omega P^{2np^{k+1}}(p^{r+1}) & \xrightarrow{\ \Omega\bar\delta_k(np)\ } & \Omega F_{np}
\end{array}
$$

where $k > 0$, and

$$
\begin{array}{ccccc}
S^{2np-1}\{p^{r+1}\} & \longrightarrow & \Omega P^{2np}(p^{r+1}) & \xrightarrow{\ \Omega\bar\delta_1\ } & \Omega F_n \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \Omega h} \\
S^{2np-1} & \longrightarrow & \Omega S^{2np} & \longrightarrow & \Omega F_{np}
\end{array}
$$

using diagram (**4.3**). Multiplying these together in order gives the result.   □

6.3 Lemma   *The map* $\bar{\phi}_n\colon P_n \to F_n$ *can be chosen so that* $P_n = P_{np} \vee Q_n$ *and there is a homotopy commutative diagram:*

$$
\begin{array}{ccc}
P_{np} \vee Q_n & \xrightarrow{\;\bar{\phi}_n\;} & F_n \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle h} \\
P_{np} & \xrightarrow{\;\bar{\phi}_{np}\;} & F_{np}
\end{array}
$$

where p *is the projection.*

Proof   Since $h\colon H_*(F_n; \mathbb{Z}/p) \to H_*(F_{np}; \mathbb{Z}/p)$ is onto the same holds for

$$
U(L^{(0)}(n)) \cong H_*(\Omega F_n; \mathbb{Z}/p) \to H_*(\Omega F_{np}; \mathbb{Z}/p) \cong U(L^{(0)}(np))
$$

in fact the generators $x_{ip} \in L^{(0)}(n)$ satisfy

$$
h_*(x_{ip}(n)) = u_i x_i(np)
$$

where u_i is a p–local unit (see Proposition 5.3).

Now given a basis $\{x_\alpha, \beta^{(r)} x_\alpha\}$ for $L^{(\infty)}(np)$, each x_α is a Lie bracket in the $x_i(np)$ and this element consequently lifts to the same bracket in $x_{ip}(n)$. Thus these liftings are linearly independent and can be chosen as part of a basis. They are all in the image of the mod p^r Hurewicz homomorphism. Thus we have chosen generators for $L^{(\infty)}(n)$ which sort into those which are lifting of the generators for $L^{(\infty)}(np)$ and the others. Realizing these via the Hurewicz homomorphism gives the maps $\bar{\phi}_n$. By a change in basis for Q_n, we can assume that the map $p\colon P_n \to P_{np}$ is trivial on Q_n.

We now use Lemma 6.3 to construct the following ladder of fibrations:

(**6.4**)
$$
\begin{array}{ccccccc}
\Omega F_n & \xrightarrow{\;\bar{\partial}_n\;} & \overline{F}_n & \longrightarrow & P_n & \xrightarrow{\;\bar{\phi}_n\;} & F_n \\
\downarrow & & \downarrow{\scriptstyle \gamma^s} & & \downarrow & & \downarrow{\scriptstyle h^s} \\
\Omega F_{np^s} & \xrightarrow{\;\bar{\partial}_{np^s}\;} & \overline{F}_{np^s} & \longrightarrow & P_{np^s} & \xrightarrow{\;\bar{\phi}_{np^s}\;} & F_{np^s}
\end{array}
$$

and we use Lemma 6.2 to construct compatible equivalences:

(**6.5**)
$$
\begin{array}{ccccc}
S^{2n-1} \times V_n & \longrightarrow & \Omega F_n & \xrightarrow{\;\partial_n\;} & \overline{F}_n \\
\downarrow{\scriptstyle \phi^s} & & \downarrow{\scriptstyle \Omega h^s} & & \downarrow{\scriptstyle \gamma^s} \\
S^{2np^s-1} \times V_{np^s} & \longrightarrow & \Omega F_{np^s} & \longrightarrow & \overline{F}_{np^s}.
\end{array}
$$

Taking fibers vertically in (**6.4**), we have a fibration sequence:

$$\Omega\Delta_s \to K_s \to R_s \to \Delta_s$$

where $R_s = (Q_n \vee \cdots \vee Q_{pn^s-1}) \rtimes \Omega P_{np^s}$ is a wedge of mod p^r Moore spaces and

$$K_s = S^{2n-1} \times \prod_{1 \leq k < s} S^{2np^k-1}\{p^{r+1}\} \times \Omega S^{2np^s-1}.$$

We use the splitting in (**6.5**) to construct a splitting

$$K_s \to \Omega\Delta_s \to K_s.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

7 Decomposition of $\Omega F_{(p^s-1)}$

Let $F_{(p^s-1)}$ be the $2np^s - 1$ skeleton of F; since the mod p^r homotopy classes

$$P^{2ni-1} \longrightarrow \Omega F_n$$

representing x_i in homology factor through $F_{(p^s-1)}$ when $i < p^s$, the suspension map

$$\sigma\colon \widetilde{H}_*(\Omega F_{(p^s-1)}; Z/p) \longrightarrow H_*(F_{(p^s-1)}; k)$$

is onto. By Proposition A.3 of the Appendix, $H_*(\Omega F_{(p^s-1)}; \mathbb{Z}/p)$ is a tensor algebra on classes x_i of dimension $2ni - 1$ for $i < p^s$.

7.1 Theorem $\Omega F_{(p^s-1)} \simeq S^{2n-1} \times \prod_{1 \leq k < s} S^{2np^k-1}\{p^{r+1}\} \times \Omega S^{2np^s-1} \times \Omega Q_s$
where $Q_s \subset Q_{s+1} \subset \cdots \subset P_n$ is a wedge of mod p^r Moore spaces for some choice of P_n in [3].

Proof This proceeds along the same lines as [3]. The maps

$$\bar{\delta}_k\colon P^{2np^k} \to F$$

factor through $F_{(p^s-1)}$ when $k < s$ as does the restriction

$$\bar{\delta}_s\Big|_{S^{2np^s-1}}\colon S^{2np^s-1} \to F.$$

We thus construct a map

$$\Theta_s \colon K_s = S^{2n-1} \times \prod_{1 \leq k < s} S^{2np^k - 1}\{p^{r+1}\} \times \Omega S^{2np^s - 1} \to \Omega F_{(p^s - 1)}$$

as before. The Bocksteins $\beta^{(i)}$ for $i < r$ are trivial and Θ_s induces an isomorphism in the homology under the r th Bockstein. To see this observe that the Bockstein homology spectral sequence of $H_*(\Omega F_{(p^s - 1)}; \mathbb{Z}/p)$ is a restriction of the spectral sequence for $H_*(\Omega F; \mathbb{Z}/p)$ (see Appendix A). $H_*(\Omega F_{(p^s - 1)}; \mathbb{Z}/p) = U(L_s^{(0)})$ where $L_s^{(0)}$ is a free Lie algebra generated by x_i of dimension $2ni - 1$ for $1 \leq i < p^s$.

To complete the proof we construct Lie algebras $L_s^{(k)} \subset L_s^{(0)}$ which are compatible with the subalgebras $L^{(k)}$ of $L^{(0)}$. First we examine the construction of $L^{(k)}$. (See Section 2). Recall that we have short exact sequences:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & L^{(1)} & \longrightarrow & L^{(0)} & \longrightarrow & \langle \tau_0 \rangle & \longrightarrow & 0 \\
0 & \longrightarrow & L^{(k+1)} & \longrightarrow & L^{(k)} & \xrightarrow{\tau_{k+1}} & \langle \tau_k, \sigma_k \rangle & \longrightarrow & 0
\end{array}
$$

where π_{k+1} is any map of Lie algebras such that

$$\pi_{k+1}(\sigma_k) = \sigma_k$$
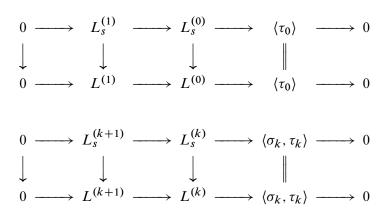$$\pi_{k+1}(\tau_k) = \tau_k$$

where $\sigma_k, \tau_k \in L^{(k)} \subset L^{(0)}$ are given by the formulas:

$$\tau_k = x_{p^k}$$

$$\sigma_k = \frac{1}{2p} \sum_{i=1}^{p^k - 1} \binom{p^k}{i} [x_i, x_{p^k - i}].$$

Nothing is said about the value of π_{k+1} on the other generators. We need to be more specific at this point. Let us define the weight of an element in a free Lie algebra to be the minimal number of brackets in any term; in particular, $L^{(0)}$ is free on generators x_i, we define $\omega(x_i) = 1, \omega[x, y] = \omega(x) + \omega(y)$ and $\omega(\Sigma a_i) = \min \omega(a_i)$. For an element $z \in L^{(k)}$ we define the weight of z to be the weight considered as an element of $L^{(0)}$. Thus $\omega(\tau_k) = 1$ and $\omega(\sigma_k) = 2$. We further specify the Lie algebra homomorphism π_{k+1} by demanding that $\pi_{k+1}(z) = 0$ if $\omega(z) > 2$. Now define $L_s^{(k)} = L_s \cap L^{(k)}$

for $k \leq s$. Since $\sigma_k, \tau_k \in L_s^{(k)}$, we have short exact sequences:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & L_s^{(1)} & \longrightarrow & L_s^{(0)} & \longrightarrow & \langle \tau_0 \rangle & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & L^{(1)} & \longrightarrow & L^{(0)} & \longrightarrow & \langle \tau_0 \rangle & \longrightarrow & 0
\end{array}
$$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & L_s^{(k+1)} & \longrightarrow & L_s^{(k)} & \longrightarrow & \langle \sigma_k, \tau_k \rangle & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & L^{(k+1)} & \longrightarrow & L^{(k)} & \longrightarrow & \langle \sigma_k, \tau_k \rangle & \longrightarrow & 0
\end{array}
$$

for $k < s$.

Define $L_s^{(s+1)}$ to be the kernel of $L_s^{(s)} \to \langle \sigma_s \rangle \to 0$. In fact we have $L_s^{(s+1)} \subset L^{(\infty)}$. To see this we need to show that $\pi_{r+1}(L_s^{(s+1)}) = 0$ for $r \geq s$. The generators of $L_s^{(s+1)}$ of filtration 1 are of the form x_i for $i < p^s$ and those of filtration 2 are of the form $[x_i, x_j]$ for $i, j < p^s$. None of these have dimension $2np^s - 1$, so $L_s^{(s+1)}$ lies in the kernel of the composition $L^{(s+1)} \to \langle \tau_s, \sigma_s \rangle \to \langle \tau_s \rangle$. Consequently $L_s^{(s+1)} \subset L^{(s+1)}$. Similarly for $r > s$ the generators of $L_s^{(s+1)}$ of weight 1 and 2 have dimensions $\leq 4n(p^s - 1)$ and consequently their images are 0 in $\langle \sigma_r, \tau_r \rangle$ for $r > s$.

It follows that we may first choose a basis for $L_s^{(s+1)}$ and then choose a basis for $L^{(\infty)}$ containing these elements. This completes the proof. $\qquad \square$

7.2 Corollary *There is a commutative diagram of fibration sequences:*

$$
\begin{array}{ccccccc}
\Omega F_n & \longrightarrow & V_n & \overset{*}{\longrightarrow} & P_n & \longrightarrow & F_n \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Omega F_{(p^{s+1}-1)} & \longrightarrow & K_{s+1} & \overset{*}{\longrightarrow} & Q_{s+1} & \longrightarrow & F_{(p^{s+1}-1)} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Omega F_{(p^s-1)} & \longrightarrow & K_s & \overset{*}{\longrightarrow} & Q_s & \longrightarrow & F_{(p^s-1)}
\end{array}
$$

where all the vertical maps are mod p homology monomorphisms and each Q_s is a wedge of mod p^r Moore spaces.

8 On the sequence $F_{(p^s-1)} \to F_n \to F_{np^s}$

Recall that there is a p–local fibration sequence

$$S^{2n}_{(p^s-1)} \to S^{2n}_\infty \xrightarrow{H_{p^s}} S^{2np^s}_\infty.$$

By analogy, we compare $F_{(p^s-1)}$ to the fiber Δ_s of the Hopf invariant h^s:

$$\Delta_s \to F_n \xrightarrow{h^s} F_{np^s}$$

8.1 Theorem $\Omega\Delta_s \simeq \Omega F_{(p^s-1)} \times \Omega X_s$ where X_s is a wedge of mod p^r Moore spaces.

Proof For dimensional reasons, the inclusion $F_{(p^s-1)} \subset F_n$ lifts to Δ_s. Consider the pull back diagram:

$$
\begin{array}{ccccccccc}
\Omega Y_s & \longrightarrow & \Omega F_{(p^s-1)} & \longrightarrow & K_s & \longrightarrow & Y_s & \longrightarrow & F_{(p^s-1)} \\
\downarrow & & \downarrow & & \downarrow= & & \downarrow & & \downarrow \\
\Omega R_s & \longrightarrow & \Omega\Delta_s & \longrightarrow & K_s & \xrightarrow{\;*\;} & R_s & \longrightarrow & \Delta_s.
\end{array}
$$

The map $K_s \to \Omega\Delta_s \to \Omega F_n$ constructed in Section 6 is obtained from maps:

$$
\begin{array}{lll}
\bar{\delta}_k\colon P^{2np^k} \longrightarrow & F_n & \qquad k < s \\
S^{2np^s-1} \longrightarrow P^{2np^s} & \xrightarrow{\bar{\delta}_s} F_n &
\end{array}
$$

and all of these maps factor through $F_{(p^s-1)}$ for dimensional reasons. Consequently the retraction $K_s \to \Omega\Delta_s$ factors through $\Omega F_{(p^s-1)}$ as well and hence the map $K_s \to Y_s$ is null homotopic. Consequently $\Omega Q_s \simeq \Omega Y_s$ and the map $Q_s \to F_{(p^s-1)}$ lifts to an equivalence $Q_s \simeq Y_s$. Now the inclusion $Q_s \to P_n$ factors through R_s, so $R_s = Q_s \vee E_s$ and we have a diagram of fibrations:

$$
\begin{array}{ccccc}
K_s & \longrightarrow & Q_s & \longrightarrow & F_{(p^s-1)} \\
=\downarrow & & \downarrow & & \downarrow \\
K_s & \longrightarrow & Q_s \vee E_s & \longrightarrow & \Delta_s.
\end{array}
$$

Now the fiber of $Q_s \to Q_s \vee E_s$ is $\Omega(E_s \rtimes \Omega Q_s)$ which is a retract of $\Omega(Q_s \vee E_s) = \Omega R_s$ and hence a retract of $\Omega\Delta_s$. This completes the proof with $X_s = E_s \rtimes \Omega Q_s$. $\square$

9 Factorization of $\overline{\partial}$

In this section we will consider the components of

$$\overline{\partial}_n\colon \Omega^2 S^{2n+1} \longrightarrow S^{2n-1} \times V_n = S^{2n-1} \times \prod_{k\geq 1} S^{2np^k-1}\{p^{r+1}\}$$

we will write $\partial_n(k)\colon \Omega^2 S^{2n+1} \longrightarrow S^{2np^k-1}\{p^{r+1}\}$ for the kth component, $k > 0$ and $\overline{\partial}_n(0)$ for the projection $\Omega^2 S^{2n+1} \longrightarrow S^{2n-1}$.

We begin by combining the diagram from Corollary 5.5:

$$\begin{array}{ccccc}
\Omega^2 S^{2n+1} & \xrightarrow{\partial_n} & \Omega F_n & \longrightarrow & \overline{F}_n \\
{\scriptstyle \Omega\widetilde{H}_{p^s}}\downarrow & & \downarrow{\scriptstyle \Omega h^s} & & \downarrow{\scriptstyle h'} \\
\Omega^2 S^{2np^s+1} & \xrightarrow{\partial_{np}} & \Omega F_{np^s} & \longrightarrow & \overline{F}_{np^s}
\end{array}$$

with the equivalences from Lemma 6.2

$$\begin{array}{ccccc}
S^{2n-1} \times V_n & \xrightarrow{\Theta_n} & \Omega F_n & \longrightarrow & \overline{F}_n \\
\downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \Omega h} & & \downarrow{\scriptstyle h'} \\
S^{2np-1} \times V_{np} & \xrightarrow{\Theta_{np}} & \Omega F_{np} & \longrightarrow & \overline{F}_{np}
\end{array}$$

where h' is defined in Lemma 6.3, to get a homotopy commutative diagram:

$$\begin{array}{ccc}
\Omega^2 S^{2n+1} & \xrightarrow{\overline{\partial}_n} & S^{2n-1} \times V_n \\
{\scriptstyle \Omega\widetilde{H}_{p^s}}\downarrow & & \downarrow{\scriptstyle \psi} \\
\Omega^2 S^{2np^s+1} & \xrightarrow{\overline{\partial}_{np^s}} & S^{2np^s-1} \times V_{np^s}.
\end{array}$$

From this we see that

$$\overline{\partial}_n(k+1) = \overline{\partial}_{np^k}(1) \circ \Omega H_{p^k} \circ \gamma_k, \qquad k \geq 0$$

while

(9.1)
$$\begin{array}{ccc}
\Omega^2 S^{2n+1} & \xrightarrow{\overline{\partial}_n(1)} & S^{2np-1}\{p^{r+1}\} \\
{\scriptstyle \Omega\widetilde{H}_p}\downarrow & & \downarrow{\scriptstyle p} \\
\Omega^2 S^{2np+1} & \xrightarrow{\overline{\partial}_{np}(0)} & S^{2np-1}.
\end{array}$$

Putting these together gives:

(**9.2**)
$$\bar{\partial}_n = \bar{\partial}_n(0) + \sum_{k=0}^{\infty} \bar{\partial}_{np^{k-1}}(1) \circ \Omega H_{p^{k-1}} \circ \gamma_k$$

where $\gamma_k \colon \Omega^2 S^{2n+1} \to \Omega^2 S^{2n+1}$ is an equivalence. Equation (**9.2**) proves Theorem 1.2.

Remark If we had better control over the relationship between the Hopf invariants h^s and the James–Hopf invariant, we could remove the factor γ_k.

9.3 Proposition *Suppose $\bar{\partial}_n(1) \colon \Omega^2 S^{2n+1} \to S^{2np-1}\{p^{r+1}\}$ is null homotopic for each n. Then*

(a) $\bar{\partial}_n$ *factors through S^{2n-1}*

(b) *The loops on the pth James–Hopf invariant:*

$$\Omega H_p \colon \Omega^2 S^{2n+1} \longrightarrow \Omega^2 S^{2np+1}$$

is homotopic to a composition:

$$\Omega^2 S^{2n+1} \longrightarrow \Omega^2 P^{2np+1} \xrightarrow{\Omega^2 p} \Omega^2 S^{2np+1}$$

(c) *The fiber B of $\bar{\partial}_{np}(0) \colon \Omega^2 S^{2np+1} \to S^{2np-1}$ is a retract of $\Omega^2 P^{2np+1}$*

(d) $B \simeq BW_n$.

Proof (a) follows directly from (**9.2**). To verify (b), note that by hypothesis

$$\Omega H_p \colon \Omega^2 S^{2n+1} \longrightarrow \Omega^2 S^{2np+1}$$

lifts so the fiber B of $\bar{\partial}_{np}(0)$. The following diagram provides a map from B to $\Omega^2 P^{2np+1}$:

$$
\begin{array}{ccc}
B & \longrightarrow & \Omega^2 P^{2np+1} \\
\downarrow & & \downarrow \\
\Omega^2 S^{2np+1} & \xrightarrow{\ 1\ } & \Omega^2 S^{2np+1} \\
\downarrow{\scriptstyle \bar{\partial}_{np}(0)} & & \downarrow \\
S^{2np-1} & \longrightarrow & \Omega F_{np}
\end{array}
$$

(c) follows since $\Omega F_n \simeq S^{2np-1} \times V_{np} \times \Omega P_{np}$, so

$$\Omega^2 P^{2np+1} \simeq B \times \Omega V_{np} \times \Omega^2 P_{np}.$$

To verify (d), note that $S^2 BW_n$ is a retract of $S^2 \Omega^2 S^{2n+1}$ by [5], so there is a map $S^2 BW_n \to P^{2np+1}$ which is non zero in homology, It's adjoint provides an equivalence:

$$BW_n \to \Omega^2 P^{2np+1} \to B. \qquad \square$$

10 Relationship with BW_n

In this section we describe a factorization of $\overline{\partial}$ using the classifying space BW_n for the double suspension. Recall from [5]

10.1 Proposition *Suppose* $\alpha\colon S^{2n} \to Y$ *and* $\beta\colon S^{2n-1}Y \to S^{4n-1}$ *are maps such that* $\beta \circ S^{2n-1}\alpha\colon S^{4n-1} \to S^{2n-1}Y \to S^{4n-1}$ *is homotopic to the identity. Then there is a fibration sequence:*

$$S^{2n-1} \xrightarrow{\overline{\alpha}} \Omega Y \longrightarrow B.$$

This is natural in the following sense: Suppose we are also given

$$\alpha'\colon S^{2n} \longrightarrow Z$$
$$\beta'\colon S^{2n-1}Z \longrightarrow S^{4n-1}$$
$$f\colon Y \longrightarrow Z$$

such that $\beta \sim \beta' \circ S^{2n-1}f$ *and* $\alpha' \sim f\alpha$.

Then we have a homotopy commutative diagram:

$$
\begin{array}{ccccc}
S^{2n-1} & \xrightarrow{\overline{\alpha}} & \Omega Y & \longrightarrow & B \\
{\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle \Omega f} & & \downarrow{\scriptstyle \psi} \\
S^{2n-1} & \xrightarrow{\overline{\alpha'}} & \Omega Z & \longrightarrow & B'
\end{array}
$$

10.2 Corollary *There is a homotopy commutative diagram:*

$$
\begin{array}{ccccc}
S^{2n-1} & \longrightarrow & \Omega^2 S^{2n+1} & \longrightarrow & BW_n \\
\downarrow & & \downarrow & & \downarrow \\
S^{2n-1} & \longrightarrow & \Omega F_n & \longrightarrow & B
\end{array}
$$

Since S^{2n-1} is a retract of ΩF_n, $B \simeq V_n \times \Omega P_n$ and we have:

10.3 Corollary *There is a homotopy commutative diagram:*

$$
\begin{array}{ccc}
\Omega^2 S^{2n+1} & \longrightarrow & BW_n \\
\downarrow{\scriptstyle\partial} & & \downarrow \\
\Omega F_n & \longrightarrow & V_n.
\end{array}
$$

In particular, $\bar{\partial}_n(k)\colon \Omega^2 S^{2n+1} \to S^{2np^k-1}\{p^{r+1}\}$ factors through BW_n, proving Theorem 1.3.

Appendix A

This appendix has two purposes. We begin with a discussion of an observation that appears in the work of Cohen, Moore and Neisendorfer. We repeat it here because it sheds light on their results and is easily extended to the case of the filtrations in Section 7. In addition, we prove a general result about the homology of certain loop spaces which does not seem to be in the literature.

We begin by looking at the homology Serre spectral sequence for the fibering

$$
\Omega F_n \longrightarrow \Omega P^{2n+1} \xrightarrow{\ \Omega\pi\ } \Omega S^{2n+1}.
$$

This is a multiplicative spectral sequence with

$$
E^2_{p,q} \cong H_p(\Omega S^{2n+1}; \mathbb{Z}/p) \otimes H_q(\Omega F_n; \mathbb{Z}/p).
$$

Since all elements of $E^2_{p,0}$ are permanent cycles, $E^2 = E^\infty$. Restricting this fibration to $S^{2n}_{(k)}$ leads to a fibration

$$
\Omega F_n \longrightarrow E_k \longrightarrow S^{2n}_{(k)}
$$

and $H_*(E_k; \mathbb{Z}/p) \subset H_*(E_{k+1}; \mathbb{Z}/p) \subset \cdots \subset H_*(\Omega P^{2n+1}; \mathbb{Z}/p)$.

Thus $H_*(\Omega P^{2n+1}; \mathbb{Z}/p)$ is a filtered differential group using $\beta^{(r)}$. It has a spectral sequence:

$$
E^2_{p,q} = H_p(\Omega S^{2n+1}; \beta^{(r)}) \otimes H_q(H_*(\Omega F_n; \mathbb{Z}/p), \beta^{(r)})
$$

converging to $H_{p+q}(\Omega P^{2n+1}; \mathbb{Z}/p) = 0$ if $p + q > 0$. This gives a multiplicative spectral sequence:

$$
E^2_{p,q} = \mathbb{Z}_p[u_{2n}] \otimes H_q(H_*(\Omega F_n; \mathbb{Z}/p); \beta^{(r)}) \Rightarrow 0.
$$

This spectral sequence has the same form as the Serre Spectral sequence for the homology of the fibration:

$$\Omega^2 S^{2n+1} \longrightarrow P\Omega S^{2n+1} \longrightarrow \Omega S^{2n+1}.$$

The differentials are forced by the multiplicative structure and consequently these spectral sequences are isomorphic. From this we conclude

A.1 Proposition [3] $H_*(H_*(\Omega F_n; \mathbb{Z}/p); \beta^{(r)}) \cong H^*(\Omega^2 S^{2n+1}; \mathbb{Z}/p)$ *as algebras and co-algebras.*

A.2 Corollary $H_*(H_*(\Omega F_{(p^s-1)}; \mathbb{Z}/p); \beta^{(r)}) \cong H_*(\Omega S^{2n}_{(p^s-1)}; \mathbb{Z}/p).$

The later follows by restricting the spectral sequence to the first $p^s - 1$ columns.

A.3 Proposition *Suppose that X is simply connected and k is a field. Suppose that the suspension map:*

$$\sigma \colon \widetilde{H}_{r-1}(\Omega X; k) \longrightarrow \widetilde{H}_r(X; k)$$

is onto. Choose a right inverse τ to σ and let $T_* \subset \widetilde{H}_*(\Omega X; k)$ be the image of τ. Then $H_*(\Omega X; k)$ is the tensor algebra on T_*.

Proof We will use the Serre spectral sequence for the loop space fibration. Note that it is a spectral sequence of $H_*(\Omega X; k)$ modules. Observe that the suspension map can be factored

$$\widetilde{H}_{r-1}(\Omega X; k) = E^2_{0,r-1} \longrightarrow E^r_{0,r-1} \xleftarrow[\cong]{d_r} E^r_{r,0} \longrightarrow E^2_{r,0} = \widetilde{H}_r(X; k)$$

where the first homomorphism is onto and the last is $1-1$. By hypothesis, the total composite is onto so the last homomorphism is an isomorphism. In particular if $x \in E^2_{r,0}$, $d_i(x) = 0$ for $i < r$ and $d_r(x) = [\tau(x)] \in E^\tau_{0,r-1}$ which is non zero. Now define

$$F(r)_p = \begin{cases} H_p(X; k) & \text{if } p = 0 \text{ or } p \geq r \\ 0 & \text{if } 0 < p < r. \end{cases}$$

We claim that there is a quotient module $Q(r)$ of $H_*(\Omega X; k)$ such that

$$E^r_{p,q} \cong F(r)_p \otimes Q(r)_q.$$

This is certainly true when $r = 2$ with $Q(2) = H_*(\Omega X; k)$. We proceed by induction.

Because of this tensor product decomposition,

$$d^r(f \otimes q) = d^r(f \otimes 1)(1 \otimes q) = 1 \otimes \tau(f)q$$

and hence $d_{p,q}^r = 0$ unless $p = r$. That is, all non zero differentials terminate on the fiber. Furthermore

$$d_{r,q}^r\colon E_{r,q}^r \longrightarrow E_{0,q+r-1}^r$$

is a monomorphism since otherwise the kernel would need to be in the image of some differential. Consequently

$$\ker d_r = F(r+1) \otimes Q(r)$$
$$\lim d_r = F(r+1) \otimes T_* Q(r).$$

Now define $Q(r+1) = Q(r)/T_* Q(r)$ which completes the induction.

We observe that the spectral sequence is completely determined by τ. We complete the proof by applying the comparison theorem. Let Y be a wedge of spheres together with an isomorphism:

$$\phi\colon H_*(Y;k) \longrightarrow H_*(X;k).$$

We will define a map of spectral sequences:

$$E_{p,q}^r(Y) \longrightarrow E_{p,q}^r(X)$$

ϕ defines a isomorphism $F(\phi)\colon F(r)(Y) \longrightarrow F(r)(X)$. Since Y is a co-H space, there is a map $v\colon Y \longrightarrow S\Omega Y$; we define $\tau(Y)$ as the composite

$$\widetilde{H}_r(Y;k) \xrightarrow{\ v_*\ } \widetilde{H}_r(S\Omega Y;k) \cong \widetilde{H}_{r-1}(\Omega Y;k).$$

Now define $\Theta\colon H_*(\Omega Y;k) \longrightarrow H_*(\Omega X;k)$ is the ring homomorphism determined by

$$\Theta(\tau(Y)(x)) = \tau(x)(\phi(x)).$$

This defines a $H_*(\Omega Y;k)$ module homomorphism

$$E_{p,q}^2(Y) \longrightarrow E_{p,q}^2(X).$$

However at each stage

$$d^r(f \otimes q) = \begin{cases} 1 \otimes \tau(f)q & r = p \\ 0 & r \neq p. \end{cases}$$

So this defines a map of spectral sequences. Since they both converge to 0 and ϕ is an isomorphism, Θ is as well completing the proof. $\square$

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