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Homotopy algebra of open–closed strings

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This paper is a survey of our previous works on open–closed homotopy algebras, together with geometrical background, especially in terms of compactifications of configuration spaces (one of Fred's specialities) of Riemann surfaces, structures on loop spaces, etc. We newly present Merkulov's geometric A_∞–structure [48] as a special example of an OCHA. We also recall the relation of open–closed homotopy algebras to various aspects of deformation theory.

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Dedicated to Fred Cohen in honor of his 60th birthday

1 Introduction

Open–closed homotopy algebras (OCHAs) (Kajiura and Stasheff [37]) are inspired by Zwiebach's open–closed string field theory [62], which is presented in terms of decompositions of moduli spaces of the corresponding Riemann surfaces. The Riemann surfaces are (respectively) spheres with (closed string) punctures and disks with (open string) punctures on the boundaries. That is, from the viewpoint of conformal field theory, classical closed string field theory is related to the conformal plane $\mathbb{C}$ with punctures and classical open string field theory is related to the upper half plane H with punctures on the boundary. Thus classical closed string field theory has an L_∞–structure (Zwiebach [61], Stasheff [56], Kimura, Stasheff and Voronov [40]) and classical open string field theory has an A_∞–structure (Gaberdiel and Zwiebach [13], Zwiebach [62], Nakatsu [50] and Kajiura [35]). The algebraic structure, we call it an OCHA, that the classical open–closed string field theory has is similarly interesting since it is related to the upper half plane H with punctures both in the bulk and on the boundary.

In operad theory (see Markl, Shnider, and Stasheff [46]), the relevance of the little disk operad to closed string theory is known, where a (little) disk is related to a closed string puncture on a sphere in the Riemann surface picture above. The homology of the little disk operad defines a Gerstenhaber algebra (Cohen [7], Getzler and Jones

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[17]), in particular, a suitably compatible graded commutative algebra structure and graded Lie algebra structure. The framed little disk operad is in addition equipped with a BV–operator which rotates the disk boundary S^1. The algebraic structure on the homology is then a BV–algebra (Getzler [16]), where the graded commutative product and the graded Lie bracket are related by the BV–operator. Physically, closed string states associated to each disk boundary S^1 are constrained to be the S^1–invariant parts, the kernel of the BV–operator. This in turn leads to concentrating on the Lie algebra structure, where two disk boundaries are identified by *twist*-sewing as Zwiebach did [61]. On the other hand, he worked at the chain level ('off shell'), discovering an L_∞–structure. This was important since the multi-variable operations of the L_∞–structure provided correlators of closed string field theory. Similarly for open string theory, the little interval operad and associahedra are relevant, the homology corresponding to a graded associative algebra, but the chain level reveals an A_∞–structure giving the higher order correlators of open string field theory.

The corresponding operad for the open–closed string theory is the Swiss-cheese operad (Voronov [60]) that combines the little disk operad with the little interval operad; it was inspired also by Kontsevich's approach to deformation quantization. The algebraic structure at the homology level has been analyzed thoroughly by Harrelson [28]. In contrast, our work in the open–closed case is at the level of strong homotopy algebra, combining the known but separate L_∞– and A_∞–structures.

In our earlier work, we defined such a homotopy algebra and called it an *open–closed homotopy algebra* (OCHA) [37]. In particular, we showed that this description is a homotopy invariant algebraic structure, ie, that it transfers well under homotopy equivalences or quasi-isomorphisms. Also, we showed that an open–closed homotopy algebra gives us a general scheme for deformation of open string structures (A_∞–algebras) by closed strings (L_∞–algebras).

In this paper, we aim to explain a background for OCHAs, the aspect of moduli spaces of Riemann surfaces. Also, we present the relation of OCHAs with Merkulov's geometric A_∞–structures [48; 49].

An open–closed homotopy algebra consists of a direct sum of graded vector spaces $\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_o$. It has an L_∞–structure on $\mathcal{H}_c$ and reduces to an A_∞–algebra if we set $\mathcal{H}_c = 0$. Moreover, the operations that intertwine the two are a generalization of the strong homotopy analog of H Cartan's notion of a Lie algebra $\mathfrak{g}$ acting on a differential graded algebra E (Cartan [6], Flato, Gerstenhaber and Voronov [11]). In Section 2, we start from discussing the moduli space aspects and the associated operad (tree graph) structures for A_∞–algebras, L_∞–algebras, and then OCHAs, together with recalling other descriptions by multi-variable operations and coderivation differentials. In a more physically oriented paper [38], we gave an alternative interpretation in the language of homological vector fields on a supermanifold.

One of the key theorems in homotopy algebra is the minimal model theorem which was first proved for A_∞–algebras by Kadeishvili [34]. The minimal model theorem states the existence of minimal models for homotopy algebras analogous to Sullivan's minimal models [58] for differential graded commutative algebras introduced in the context of rational homotopy theory. In Section 3 we re-state the minimal model theorem for our open–closed homotopy algebras.

As suggested by Merkulov, his geometric A_∞–structure [48] is a special example of an OCHA. In Section 4, we present a new formulation of an OCHA in Merkulov's framework.

In Section 5, we recall the relation of open–closed homotopy algebras to various aspects of deformation theory and the relevant moduli spaces and in Section 6, return to the relation to the motivating string theory.

There is a distinction between the historical grading used in defining A_∞– and L_∞–structures and the more recent one common in the physics literature. They are related by (de)suspension of the underlying graded vector spaces. Since we emphasize the versions in terms of a single differential of degree one on the relevant 'standard construction', we will only occassionally refer to the older version, primarily for ungraded strictly associative or Lie algebras or strict differential graded algebras. The distinction does influence the exposition, but the only importance technically is the signs that occur. However, the detailed signs are conceptually unimportant (although crucial in calculations), so we indicate them here only with $\pm$, the precise details being in [37; 38].

We restrict our arguments to the case that the characteristic of the field k is zero. We further let $k=\mathbb{C}$ for simplicity.

2 Strong homotopy algebra

2.1 Topology of based loop spaces

An open–closed homotopy algebra [37] is a strong homotopy algebra (or ∞–algebra) which combines two typical strong homotopy algebras, an A_∞–algebra and an L_∞–algebra.

An A_∞–algebra was introduced [53] as a structure exemplified by the chains on the based loop space $Y := \Omega X$ of a topological space X with a base point $x_0 \in X$. A based loop $x \in Y := \Omega X$ is a map $x\colon [0,1] \to X$ such that $x(0) = x(1) = x_0$. The based loop space Y forms a group-like space, where the product

$$m_2\colon Y \times Y \to Y$$

is given naturally by connecting two based loops as usual. The product m_2 is not associative but there exists a homotopy between $m_2(m_2 \times 1)$ and $m_2(1 \times m_2)$ described by an interval K_3 (Figure 1 (a))

$$m_3\colon K_3 \times Y \times Y \times Y \longrightarrow Y\ .$$

In a similar way, we can consider possible operations of $(Y)^{\times 4} \to Y$ constructed from m_2 and m_3. These connect to form a map on the boundary of a pentagon, which can be extended to a pentagon K_4 (Figure 1 (b)), providing a higher homotopy operation:

$$m_4\colon K_4 \times Y^{\times 4} \longrightarrow Y\ .$$

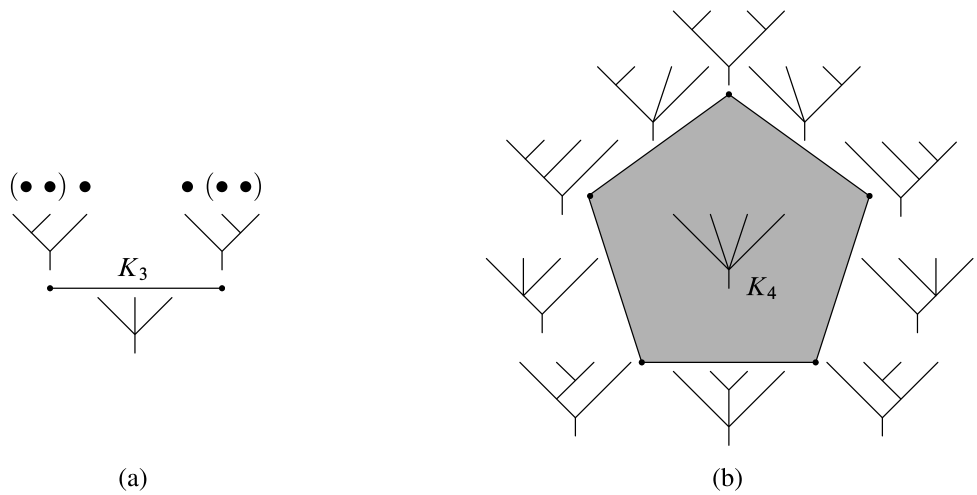


Figure 1: (a) An interval as associahedra K_3 (b) A pentagon as associahedra K_4

Repeating this procedure leads to higher dimensional polytopes K_n, of dimension $(n-2)$ [53], now called *associahedra* since the vertices correspond to all ways of associating a string of n letters. For $Y = \Omega X$, we have higher homotopies

$$m_n\colon K_n \times (Y)^{\times n} \longrightarrow Y$$

extending maps on the boundary of K_n defined by compositions of the m_k for $k < n$. Then, a topological space Y equipped with the structures $\{m_n, K_n\}_{n \geq 2}$ as above is called an A_∞–space.

2.2 Compactification of moduli spaces of disks with boundary punctures

Although it was not noticed for many years, the associahedra K_n can be obtained as the moduli space of the real compactification of the configuration space of $(n-2)$ distinct points in an interval or to a real compactification $\underline{\mathcal{M}}_{n+1}$ of the moduli space $\mathcal{M}_{n+1}$ of a disk with $(n+1)$ points on the boundary (Figure 2 (a)).

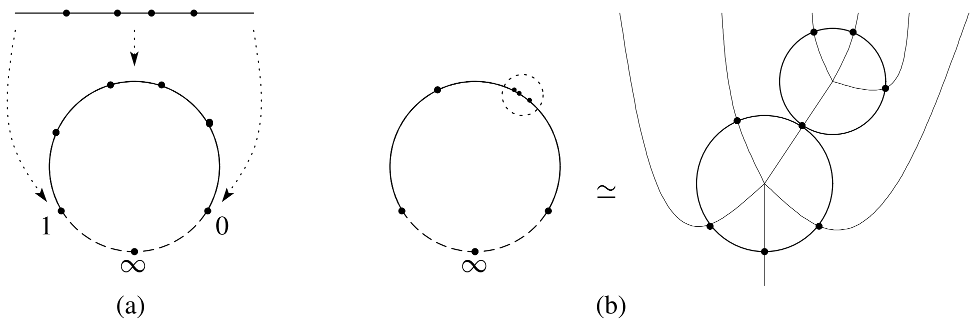


Figure 2: (a) The identification of the interval with $(n-2)$ points on it with the boundary of the disk with $(n+1)$ points on the boundary
(b) The correspondence of the compactification of the moduli spaces with tree graphs; the case of a boundary component of the compactified moduli space $\mathcal{M}_7$

This compactification can be related directly to the planar tree operad (Figure 2 (b)). $\mathcal{M}_{n+1}$ is described as the configuration space of $(n+1)$–punctures on $S^1 \sim \mathbb{R}\cup\{\infty\}$ divided by conformal transformations. In the case in which the Riemann surface is the disk, the conformal transformations form $SL(2,\mathbb{R})$, whose element $g\in SL(2,\mathbb{R})$ acts on $\mathbb{R}\cup\{\infty\}$ as

$$g(x)=\frac{ax+b}{cx+d}\ ,\quad x\in(\mathbb{R}\cup\{\infty\})\ ,\quad g:=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{R})\ . \tag{1}$$

This degree of freedom can be killed by fixing three points on the boundary. Usually we set the three points at 0, 1 and ∞. We take the point ∞ as the 'root edge'.

Then, the interval is identified with the arc between 0 and 1 as in Figure 2 (a). Thus, we obtain:

$$\mathcal{M}_{n+1}=\{(t_2,\dots,t_{n-1})\mid 0<t_2<t_3<\cdots<t_{n-1}<1\}\ . \tag{2}$$

The real compactifications $\underline{\mathcal{M}}_{n+1}$ of Axelrod and Singer [1] (the real analog of the Fulton–MacPherson compactification [12]), that is, the compactifications of $\mathcal{M}_{n+1}$

with real codimension one boundaries, are in fact combinatorially homeomorphic to the Stasheff associahedron K_n. For instance, we rather obviously have:

- for $n=2$, $\mathcal{M}_{2+1} \simeq \{pt\} \simeq \bar{\mathcal{M}}_{2+1} \simeq K_2$,
- for $n=3$, $\mathcal{M}_{3+1} \simeq \{t_2 \mid 0 < t_2 < 1\}$ and $\bar{\mathcal{M}}_{3+1} \simeq K_3 \simeq$ the closed interval.

For $n>3$, the particulars of the compactification process account for the compactification being combinatorially homeomorphic to K_4 rather than to the closed simplex.

2.3 Tree formulation

There are some advantages to indexing the maps m_k and their compositions by planar rooted trees (as originally suggested by Frank Adams around 1960, when trees would have had to be inserted in manuscripts by hand); e.g. m_k will correspond to the *corolla* M_k with k leaves all attached directly to the root. The composite $m_k \bullet_i m_l$ then corresponds to grafting the root of M_l to the i-th leaf of M_k, reading from left to right (see Figure 3). (Thus $\bullet_i$ is a precise analog of Gerstenhaber's $\circ_i$, although the correspondence was not observed for a couple of decades.) This is the essence of the planar rooted tree operad [46].

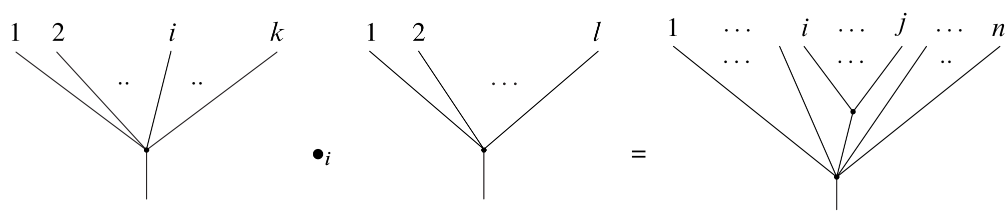


Figure 3: The grafting $M_k \bullet_i M_l$ of the l–corolla M_l to the i-th leaf of k–corolla M_k, where $j = i+l-1$ and $n = k+l-1$

Multilinear maps compose in just this way, so relations (5) can be phrased as saying we have a map from planar rooted trees to multilinear maps respecting the $\bullet_i$ 'products', the essence of a map of operads [46]. This was originally observed in terms of the vector spaces of chains on a based loop space, but abstracted as follows: Let $\mathcal{A}_\infty(n)$, $n \geq 1$, be a graded vector space spanned by planar rooted trees with n leaves with identity $e \in \mathcal{A}_\infty(1)$. For a planar rooted tree $T \in \mathcal{A}_\infty(n)$, its grading is introduced as the number of the vertices contained in T, which we denote by $v(T)$. A tree $T \in \mathcal{A}_\infty(n)$, $n \geq 2$, with $v(T)=1$ is the corolla M_n. Any tree T with $v(T)=2$ is obtained by the grafting of two corollas as in Figure 3. Grafting of any two trees is defined in a similar way, with an appropriate sign rule, and any tree T with $v(T) \geq 2$

can be obtained recursively by grafting a corolla to a tree T' with $v(T') = v(T) - 1$. One can define a differential d of degree one, which acts on each corolla as

$$d\,(M_n) = -\sum_{k,l\geq 2,\ k+l=n+1}\ \sum_{i=1}^{k} M_k \bullet_i M_l \tag{3}$$

and extends to one on $\mathcal{A}_\infty := \oplus_{n\geq 1}\mathcal{A}_\infty(n)$ by the following rule:

$$d(T \bullet_i T') = d(T) \bullet_i T' + (-1)^{v(T)} T \bullet_i d(T')\ .$$

If we introduce the contraction of internal edges, that is, indicate by $T' \to T$ that T is obtained from T' by contracting an internal edge, the differential is equivalently given by

$$d(T) = \sum_{T'\to T} \pm T'$$

with an appropriate sign $\pm$. Thus, one obtains a dg operad $\mathcal{A}_\infty$, which is known as the A_∞*–operad*.

An algebra A over $\mathcal{A}_\infty$ is obtained by a representation $\phi\colon \mathcal{A}_\infty(k) \to \mathrm{Hom}(A^{\otimes k}, A)$, ie, a map ϕ compatible with the $\bullet_i$'s and also the differentials. We denote by m_k the image $\phi(M_k)$ of M_k by ϕ. Then, for each corolla we have

$$\sum_{k+l=n+1}\ \sum_{i=1}^{k} \pm m_k \bullet_i m_l = 0, \tag{4}$$

where we now write m_1 for d.

This then becomes the definition.

Definition 2.1 (A_∞–algebra – strongly homotopy associative algebra – [53]) Let A be a $\mathbb{Z}$–graded vector space $A = \oplus_{r\in\mathbb{Z}} A^r$ and suppose that there exists a collection of degree one multilinear maps

$$\mathfrak{m} := \{m_k\colon A^{\otimes k} \to A\}_{k\geq 1}\ .$$

$(A, \mathfrak{m})$ is called an A_∞*–algebra* when the multilinear maps m_k satisfy the following relations

$$\sum_{k+l=n+1}\ \sum_{i=1}^{k} \pm m_k(o_1,\dots,o_{i-1}, m_l(o_i,\dots,o_{i+l-1}), o_{i+l},\dots,o_n) = 0 \tag{5}$$

for $n \geq 1$.

A *weak* or *curved* A_∞–algebra consists of a collection of degree one multilinear maps

$$\mathfrak{m} := \{m_k\colon A^{\otimes k} \to A\}_{k\geq 0}$$

satisfying the above relations, but for $n \geq 0$ and in particular with $k, l \geq 0$.

Remark 2.2 The 'weak' version is fairly new, apparently first in papers of Getzler and Jones (and Petrack) [18; 19], then was adopted by Zwiebach in the L_∞–context [61] and later used to study what physicists refer to as a 'background' for string field theory. The map $m_0\colon \mathbb{C} \to A$ is regarded as an element $m_0(1) \in A$. The augmented relation then implies that $m_0(1)$ is a cycle, but $m_1 m_1$ need no longer be 0, rather $m_1 m_1 = \pm m_2(m_0 \otimes 1) \pm m_2(1 \otimes m_0)$.

Remark 2.3 Recall, as mentioned earlier, that the component maps would have m_k of degree $(k-2)$ in the original formulation.

Definition 2.4 (A_∞–morphism) For two A_∞–algebras $(A, \mathfrak{m})$ and $(A', \mathfrak{m}')$, a collection of degree zero (degree preserving) multilinear maps

$$\{f_k\colon A^{\otimes k} \to A'\}_{k\geq 1}$$

is called an A_∞*–morphism* $\{f_k\}_{k\geq 1}\colon (A, \mathfrak{m}) \to (A', \mathfrak{m}')$ iff it satisfies the following relations:

$$\begin{aligned}\sum_{1\leq k_1<k_2\cdots<k_j=n} m'_j(f_{k_1}(o_1, .., o_{k_1}), f_{k_2-k_1}(o_{k_1+1}, .., o_{k_2}),\\ \dots, f_{n-k_{j-1}}(o_{k_{j-1}+1}, .., o_n))\\ = \sum_{k+l=n+1}\sum_{i=1}^{k} \pm f_k(o_1, \dots, o_{i-1}, m_l(o_i, .., o_{i+l-1}), o_{i+l}, \dots, o_n)\end{aligned} \tag{6}$$

for $n \geq 1$. In particular, if $f_1\colon A \to A'$ induces an isomorphism between the cohomologies $H(A)$ and $H(A')$, the A_∞–morphism is called an A_∞*–quasi-isomorphism*.

A_∞–quasi-isomorphisms play important roles from the homotopy algebraic point of view (see Section 3).

2.4 Coalgebra formulation

The maps m_k can be assembled into a single map, also denoted $\mathfrak{m}$, from the tensor space $T^c A = \oplus_{k\geq 0} A^{\otimes k}$ to A with the convention that $A^{\otimes 0} = \mathbb{C}$. The grading implied by having the maps m_k all of degree one is the usual grading on each $A^{\otimes k}$. We can regard T^cA as the tensor *coalgebra* by defining

$$\triangle(o_1\otimes\cdots\otimes o_n)=\Sigma_{p=0}^{n}(o_1\otimes\cdots\otimes o_p)\otimes(o_{p+1}\otimes\cdots\otimes o_n)\ .$$

A map $f\in\mathrm{Hom}(T^cA,T^cA)$ is a *graded coderivation* means $\triangle f=(f\otimes\mathbf{1}+\mathbf{1}\otimes f)\triangle$, with the appropriate signs and dual to the definition of a graded derivation of an algebra. Here $\mathbf{1}$ denotes the identity $\mathbf{1}\colon A\to A$. We then identify $\mathrm{Hom}(T^cA,A)$ with $\mathrm{Coder}(T^cA)$ by lifting a multilinear map as a coderivation [55]. Analogously to the situation for derivations, the composition graded commutator of coderivations is again a coderivation; this graded commutator corresponds to the *Gerstenhaber bracket* on $\mathrm{Hom}(T^cA,A)$ [14; 55]. Notice that this involves a shift in grading since Gerstenhaber uses the traditional Hochschild complex grading. Thus $\mathrm{Coder}(T^cA)$ is a graded Lie algebra and in fact a dg Lie algebra with respect to the bar construction differential, which corresponds to the Hochschild differential on $\mathrm{Hom}(T^cA,A)$ in the case of an associative algebra (A,m) [14]. Using the bracket, the differential can be written as $[m,\ ]$.

The advantage of this point of view is that the component maps m_k assemble into a single map $\mathfrak{m}$ in $\mathrm{Coder}(T^cA)$ and relations (5) can be summarized by

$$[\mathfrak{m},\mathfrak{m}]=0\quad\text{ or, equivalently, }\quad D^2=0\ ,$$

where $D=[\mathfrak{m},\ ]$. In fact, $\mathfrak{m}\in\mathrm{Coder}(T^cA)$ is an A_∞–algebra structure on A iff $[\mathfrak{m},\mathfrak{m}]=0$ *and* $\mathfrak{m}$ has no constant term, $m_0=0$. If $m_0\neq 0$, the structure is a weak A_∞–algebra. The A_∞–morphism components similarly combine to give a single map of dg coalgebras $\mathfrak{f}\colon T^cA\to T^cA'$, $(\mathfrak{f}\otimes\mathfrak{f})\triangle=\triangle\mathfrak{f}$. In particular, equation (6) is equivalent to $\mathfrak{f}\circ\mathfrak{m}=\mathfrak{m}'\circ\mathfrak{f}$.

2.5 L_∞–algebras

Since an ordinary Lie algebra $\mathfrak{g}$ is regarded as ungraded, the defining bracket is regarded as skew-symmetric. If we regard $\mathfrak{g}$ as all of degree one, then the bracket would be graded symmetric. For dg Lie algebras and L_∞–algebras, we need graded symmetry, which refers to the usual symmetry with signs determined by the grading. The basic relation is

$$\tau\colon x\otimes y\mapsto(-1)^{|x||y|}y\otimes x\ . \tag{7}$$

The sign of a permutation of n graded elements, is defined by

$$\sigma(c_1,\ldots,c_n)=\pm(c_{\sigma(1)},\ldots,c_{\sigma(n)}), \tag{8}$$

where the sign $\pm$ is given by what is called the Koszul sign of the permutation.

Definition 2.5 (Graded symmetry) A *graded symmetric multilinear map* of a graded vector space V to itself is a linear map $f\colon V^{\otimes n} \to V$ such that for any $c_i \in V$, $1 \le i \le n$, and any $\sigma \in \mathfrak{S}_n$ (the permutation group of n elements), the relation

$$f(c_1, \dots, c_n) = \pm f(c_{\sigma(1)}, \dots, c_{\sigma(n)})$$

holds, where $\pm$ is the Koszul sign above.

The *graded symmetric coalgebra* $C(V)$ on a graded vector space V is defined as the *sub*coalgebra $C(V) \subset T^c V$ consisting of the graded symmetric elements in each $V^{\otimes n}$.

Definition 2.6 By a (k,l)*–unshuffle* of $c_1, \dots, c_n$ with $n = k + l$ is meant a permutation σ such that for $i < j \le k$, we have $\sigma(i) < \sigma(j)$ and similarly for $k < i < j \le k + l$. We denote the subset of (k,l)–unshuffles in $\mathfrak{S}_{k+l}$ by $\mathfrak{S}_{k,l}$ and by $\mathfrak{S}_{k+l=n}$, the union of the subgroups $\mathfrak{S}_{k,l}$ with $k + l = n$.

Definition 2.7 (L_∞–algebra (strong homotopy Lie algebra) [44]) Let L be a graded vector space and suppose that a collection of degree one graded symmetric linear maps $\mathfrak{l} := \{l_k\colon L^{\otimes k} \to L\}_{k \ge 1}$ is given. $(L, \mathfrak{l})$ is called an L_∞*–algebra* iff the maps satisfy the following relations

$$\sum_{\sigma \in \mathfrak{S}_{k+l=n}} \pm l_{1+l}(l_k(c_{\sigma(1)}, \dots, c_{\sigma(k)}), c_{\sigma(k+1)}, \dots, c_{\sigma(n)}) = 0 \tag{9}$$

for $n \ge 1$, where $\pm$ is the Koszul sign (8) of the permutation $\sigma \in \mathfrak{S}_{k+l=n}$.

A *weak* L_∞*–algebra* consists of a collection of degree one graded symmetric linear maps $\mathfrak{l} := \{l_k\colon L^{\otimes k} \to L\}_{l \ge 0}$ satisfying the above relations, but for $n \ge 0$ and with $k, l \ge 0$.

Remark 2.8 The alternate definition in which the summation is over all permutations, rather than just unshuffles, requires the inclusion of appropriate coefficients involving factorials.

Remark 2.9 A dg Lie algebra is expressed as the desuspension of an L_∞–algebra $(L, \mathfrak{l})$ where l_1 and l_2 correspond to the differential and the Lie bracket, respectively, and higher multilinear maps $l_3, l_4, \dots$ are absent.

Remark 2.10 For the 'weak/curved' version, remarks analogous to those for weak A_∞–algebras apply, and similarly for morphisms.

In a similar way as in the A_∞ case, an L_∞–algebra $(L, \mathfrak{l})$ is described as a coderivation $\mathfrak{l}\colon C(L) \to C(L)$ satisfying $(\mathfrak{l})^2 = 0$. Also, for two L_∞–algebras $(L, \mathfrak{l})$ and $(L', \mathfrak{l}')$, an L_∞–morphism is defined as a coalgebra map $\mathfrak{f}\colon C(L) \to C(L')$, where $\mathfrak{f}$ consists of graded symmetric multilinear maps $f_k\colon L^{\otimes k} \to L'$ of degree zero with $k \geq 1$, satisfying $\mathfrak{l}' \circ \mathfrak{f} = \mathfrak{f} \circ \mathfrak{l}$.

The tree operad description of L_∞–algebras uses non-planar rooted trees with leaves numbered $1, 2, \ldots$ arbitrarily [46]. Namely, a non-planar rooted tree can be expressed as a planar rooted tree but with arbitrary ordered labels for the leaves. In particular, corollas obtained by permuting the labels are identified (Figure 4). Let $\mathcal{L}_\infty(n)$, $n \geq 1$,

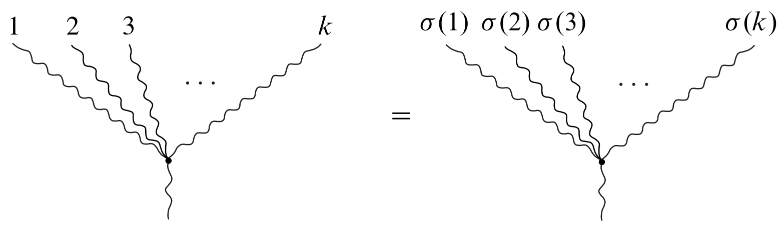


Figure 4: Nonplanar k–corolla L_k corresponding to l_k. Since edges are non-planar, it is symmetric with respect to the permutation of the edges.

be a graded vector space generated by those non-planar rooted trees of n leaves. For a tree $T \in \mathcal{L}_\infty(n)$, a permutation $\sigma \in \mathfrak{S}_n$ of the labels for leaves generates a different tree in general, but sometimes the same one because of the symmetry of the corollas above. The grafting, $\circ_i$, to the leaf labelled i is defined as in the planar case in subsection 2.3, and any non-planar rooted tree is obtained by grafting corollas $\{L_k\}_{k\geq 2}$ recursively, as in the planar case, together with the permutations of the labels for the leaves. A degree one differential $d\colon \mathcal{L}_\infty(n) \to \mathcal{L}_\infty(n)$ is given in a similar way; for $T' \to T$ indicating that T is obtained from T' by the contraction of an internal edge,

$$d(T) = \sum_{T' \to T} \pm T' \,,$$

and $d(T \circ_i T') = d(T) \circ_i T' + (-1)^{v(T)} T \circ_i d(T')$ again holds. Thus, $\mathcal{L}_\infty := \oplus_{n\geq 1} \mathcal{L}_\infty(n)$ forms a dg operad, called the *L_∞–operad*.

An algebra L over $\mathcal{L}_\infty$ obtained by a map $\phi\colon \mathcal{L}_\infty(k) \to \mathrm{Hom}(L^{\otimes k}, L)$ then forms an L_∞–algebra $(L, \mathfrak{l})$.

2.6 Compactification of moduli spaces of spheres with punctures

As A_∞–algebras can be described in terms of compactifications of moduli spaces of configurations of points on an interval, so, with some additional subtlety, L_∞–algebras can be described in terms of compactifications of moduli spaces of configurations of points on a Riemann sphere. The compactification corresponding to an L_∞–structure is the *real* compactification $\underline{\mathcal{M}}_{0,n}$ of the moduli spaces $\mathcal{M}_{0,n}$ of spheres with n punctures ([40], see also [61]). Here we use underbar in order to distinguish it from the complex compactification by Deligne–Knudsen–Mumford which is more familiar and often denoted by $\bar{\mathcal{M}}_{0,n}$. Also, we attach the lower index 0 indicating genus zero, in order to distinguish the real compactification of the moduli spaces of punctured spheres from that of punctured disks in subsection 2.2.

The moduli space $\mathcal{M}_{0,n}$ is defined as the configuration space of n points on a sphere $\simeq \mathbb{C}\cup\{\infty\}$ modulo the $SL(2,\mathbb{C})$ action

$$w'(w)=\frac{aw+b}{cw+d}\ ,\quad w\in\mathbb{C}\cup\{\infty\}\ ,\quad \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{C}).$$

This $SL(2,\mathbb{C})$ action allows us to fix three points; usually $0,1$ and ∞.

For $n=3$, the moduli space $\mathcal{M}_{0,2+1}$ is a point, so is its real compactification $\underline{\mathcal{M}}_{0,2+1}\simeq\{pt\}$. For $n=4$, one has

$$\mathcal{M}_{0,4}\ \simeq(\mathbb{C}\cup\infty)-\{0,1,\infty\},$$

which is the configuration space of four points $0,1,w,\infty$ with the subtraction of the 'diagonal'. The real compactification of $\mathcal{M}_{0,4}$ looks as in Figure 5: $\underline{\mathcal{M}}_{0,4}$ has

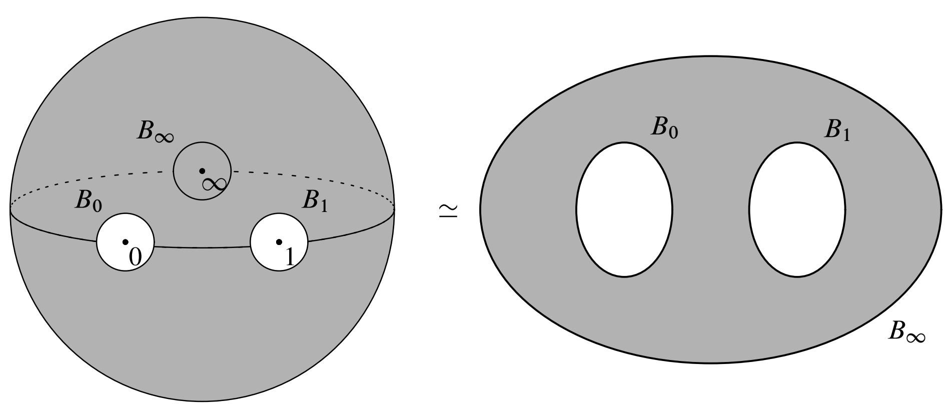


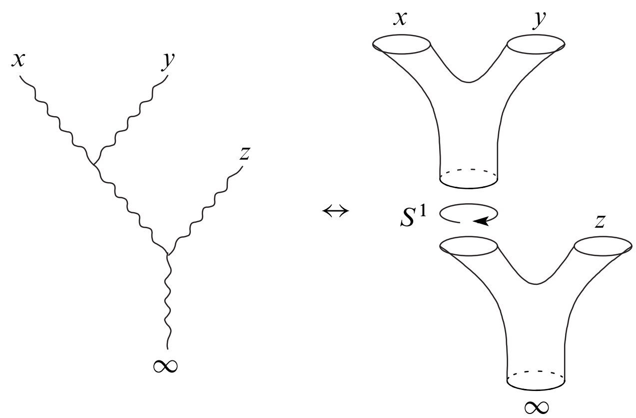
Figure 5: The real compactification $\underline{\mathcal{M}}_{0,3+1}$ of $\mathcal{M}_{0,3+1}$

codim$_{\mathbb{R}}=1$ boundaries B_0, B_1, B_∞. If we associate points $0,1,w$ to x,y,z and ∞

to the root edge, we get the correspondence:

$$\begin{aligned} B_0 &\leftrightarrow \pm[[x,z],y] \\ B_1 &\leftrightarrow \pm[[y,z],x] \\ B_\infty &\leftrightarrow \pm[[x,y],z]\,. \end{aligned}$$

Inspired by closed string field theory, this can be seen in terms of 'grafting' tubular neighborhoods of trees with freedom of a full S^1 of rotations of the boundaries which are to be identified:



Now consider the relative homology groups of the compactified moduli spaces modulo those on the lower dimensional strata. These give a version of the L_∞–operad. Corresponding to the relation $\partial(\underline{\mathcal{M}}_{0,4}) = B_0 \coprod B_1 \coprod B_\infty$, we obtain:

$$d(l_3)(x,y,z) = [[x,y],z] \pm [[y,z],x] \pm [[z,x],y]\,.$$

Notice that $\underline{\mathcal{M}}_{0,n}$ is not contractible for $n \geq 4$. In general, $\underline{\mathcal{M}}_{0,n}$ is a manifold with corners (as were the associahedra) of real dimension $2n-6$, but the strata are not generaly cells, as they were for the associahedra. Thus, to define the dg L_∞–operad, we use the homology of strata relative to boundary [40]. On the other hand, if we are concerned only with the corresponding homology operad, we need only use the little disks operad and Fred's configuration space calculations [7].

2.7 Open–closed homotopy algebra (OCHA)

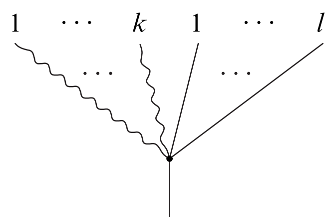
For our open–closed homotopy algebra, we consider a graded vector space $\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_o$ in which $\mathcal{H}_c$ will be an L_∞–algebra and $\mathcal{H}_o$, an A_∞–algebra.

An OCHA is inspired by the compactification of the moduli spaces of punctured Riemann surfaces (Riemann surfaces with marked points) or the decomposition of the moduli spaces as is done in constructing string field theory. More precisely, an OCHA should be an algebra over the DG operad of chains of the compactified moduli spaces of the corresponding Riemann surfaces. In this paper, we first present the DG–operad which we call the open–closed operad $\mathcal{OC}_\infty$. An OCHA obtained as a representation of $\mathcal{OC}_\infty$ has various interesting structures also from purely algebraic points of view. In particular, an OCHA can be viewed as a generalization of various known algebras. We shall discuss this after presenting the definition of an OCHA. Before giving the explicit definition in terms of 'algebraic' formulas, we look at the tree description.

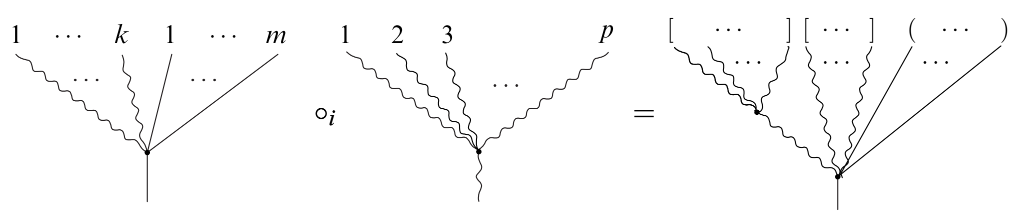
2.8 The tree description

We associated the k–corolla M_k of planar rooted trees to the multilinear map m_k of an A_∞–algebra, and the k–corolla L_k of non-planar rooted trees to the graded symmetric multilinear map l_k of an L_∞–algebra. For an OCHA $(\mathcal{H}, \mathfrak{l}, \mathfrak{n})$, we introduce the (k,l)–corolla $N_{k,l}$

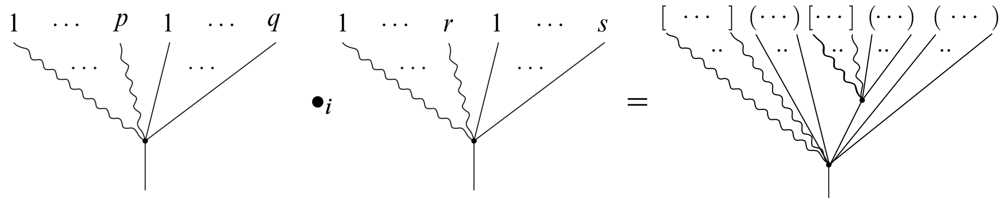
(10) $N_{k,l} =$



which is defined to be partially symmetric (non-planar); only symmetric with respect to the k leaves. We express symmetric leaves as wiggly edges and planar (= non-symmeteric) leaves as straight edges as before. Let us consider such corollas for $2k + l + 1 \geq 3$. As we shall explain further later, this constraint is motivated by the stable moduli space of a disk with k–punctures interior and $(l+1)$–punctures on the boundary of the disk. We also consider non-planar corollas $\{L_k\}_{k\geq 2}$. The planar k–corolla M_k is already included as $N_{0,k}$. Since we have two kinds of edges, we have two kinds of grafting. We denote by $\circ_i$ (resp. $\bullet_i$) the grafting of a wiggly (resp. straight) root edge to an i-th wiggly (resp. straight) edge, respectively. For these corollas, we have three types of composite; in addition to the composite $L_{1+k} \circ_i L_l$ in $\mathcal{L}_\infty$, there is a composite $N_{k,m} \circ_i L_p$ described by



where in the right hand side the labels are given by $[i,\dots,i+p-1][1,\dots,i-1,i+p,\dots,p+k-1](1,\dots,m)$, and the composite $N_{p,q}\bullet_i N_{r,s}$



with labels $[1,\dots,p](1,\dots,i-1)[p+1,\dots,p+r](i,\dots,i+s-1)(i+s,\dots,q+s-1)$. To these resulting trees, grafting of a corolla L_k or $N_{k,l}$ can be defined in a natural way, and we can repeat this procedure. Let us consider tree graphs obtained in this way, that is, by grafting the corollas l_k and $n_{k,l}$ recursively, together with the action of permutations of the labels for closed string leaves. Each of them has a wiggly or straight root edge. The tree graphs with wiggly root edges, with the addition of the identity $e_c\in\mathcal{L}_\infty(1)$, generate $\mathcal{L}_\infty$ as stated in subsection 2.5. On the other hand, the tree graphs with both wiggly and straight edges are new.

Definition 2.11 We denote by $\mathcal{N}_\infty(k;l)$, the graded vector space generated by rooted tree graphs with k wiggly leaves and l straight leaves. In particular, we formally add the identity e_o generating $\mathcal{N}_\infty(0;1)$ and a corolla $N_{1,0}$ generating $\mathcal{N}_\infty(1;0)$. The tree operad relevant for OCHAs is then $\mathcal{OC}_\infty:=\mathcal{L}_\infty\oplus\mathcal{N}_\infty$.

In fact, $\mathcal{OC}_\infty$ is an example of a *colored* operad [5; 46; 59]. For each tree $T\in\mathcal{OC}_\infty$, its grading is given by the number of vertices $v(T)$.

For trees in $\mathcal{OC}_\infty$, let $T'\to T$ indicate that T is obtained from T' by contracting a wiggly or a straight internal edge. A degree one differential $d\colon\mathcal{OC}_\infty\to\mathcal{OC}_\infty$ is given by

$$d(T)=\sum_{T'\to T}\pm T'\,, \tag{11}$$

so that the following compatibility holds:

$$\begin{aligned}d(T\circ_i T')&=d(T)\circ_i T'+(-1)^{v(T)}T\circ_i d(T'),\\ d(T\bullet_i T'')&=d(T)\bullet_i T''+(-1)^{v(T)}T\bullet_i d(T'')\,.\end{aligned}$$

Thus, $\mathcal{OC}_\infty$ forms a dg colored operad.

2.9 Formal definition of OCHA

For two $\mathbb{Z}$–graded vector spaces $\mathcal{H}_c$ and $\mathcal{H}_o$, an open–closed homotopy algebra $(\mathcal{H} := \mathcal{H}_c \oplus \mathcal{H}_o, \mathfrak{l}, \mathfrak{n})$ is an algebra over the operad $\mathcal{OC}_\infty$. An algebra $\mathcal{H} := \mathcal{H}_c \oplus \mathcal{H}_o$ over $\mathcal{OC}_\infty$ is obtained by a representation

$$\phi\colon \mathcal{L}_\infty(k) \to \mathrm{Hom}(\mathcal{H}_c^{\otimes k}, \mathcal{H}_c)\ , \qquad \phi\colon \mathcal{N}_\infty(k;l) \to \mathrm{Hom}((\mathcal{H}_c)^{\otimes k} \otimes (\mathcal{H}_o)^{\otimes l}, \mathcal{H}_o)$$

which is compatible with respect to the grafting $\circ_i$, $\bullet_i$ and the differential d. Here, regarding elements in both $\mathrm{Hom}(\mathcal{H}_c^{\otimes k}, \mathcal{H}_c)$ and $\mathrm{Hom}((\mathcal{H}_c)^{\otimes k} \otimes (\mathcal{H}_o)^{\otimes l}, \mathcal{H}_o)$ as those in $\mathrm{Coder}(C(\mathcal{H}_c) \otimes T^c(\mathcal{H}_o))$, the differential on the algebra side is given by

$$d := [l_1 + n_{0,1},\ \ ], \tag{12}$$

where both l_1 and $n_{0,1}$ are the canonical lift of differentials $l_1\colon \mathcal{H}_c \to \mathcal{H}_c$ and $n_{0,1}\colon \mathcal{H}_o \to \mathcal{H}_o$ on the corresponding graded vector spaces. On the other hand, in addition to the differential $d\colon \mathcal{L}_\infty \to \mathcal{L}_\infty$ defining the L_∞–structure, we have the differential $d\colon \mathcal{N}_\infty \to \mathcal{N}_\infty$ (11) which acts on the corolla $N_{k,l}$ as

$$d(N_{n,m}) = \sum_{k+p=n+1} \sum_i N_{k,m} \circ_i L_p + \sum_{p+r=n, q+s=m+1} \sum_i N_{p,q} \bullet_i N_{r,s}.$$

By combining this with equation (12), one can write down the conditions for an OCHA:

$$\begin{aligned} 0 = &\sum_{\sigma \in \mathfrak{S}_{p+r=n}} \pm n_{1+r,m}\big((l_p \otimes \mathbf{1}_c^{\otimes r} \otimes \mathbf{1}_o^{\otimes m})(c_{\sigma(I)}; o_1, \dots, o_m)\big) \\ &+ \sum_{\substack{\sigma \in \mathfrak{S}_{p+r=n} \\ i+s+j=m}} \pm n_{p,i+1+j}\left((\mathbf{1}_c^{\otimes p} \otimes \mathbf{1}_o^{\otimes i} \otimes n_{r,s} \otimes \mathbf{1}_o^{\otimes j})(c_{\sigma(I)}; o_1, \dots, o_m)\right), \end{aligned}$$

where $c_1, \dots, c_n$ and $o_1, \dots, o_m$ are homogeneous elements in $\mathcal{H}_c$ and $\mathcal{H}_o$, respectively, and the signs $\pm$ are the Koszul sign (8) of σ.

More explicitly:

Definition 2.12 (Open–Closed Homotopy Algebra (OCHA) [37]) An *open–closed homotopy algebra (OCHA)* $(\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_o, \mathfrak{l}, \mathfrak{n})$ consists of an L_∞–algebra $(\mathcal{H}_c, \mathfrak{l})$ and a family of maps $\mathfrak{n} = \{n_{p,q}\colon \mathcal{H}_c^{\otimes p} \otimes \mathcal{H}_o^{\otimes q} \to \mathcal{H}_o\}$ of degree one for $p, q \geq 0$ with the exception of $(p,q)=(0,0)$ satisfying the compatibility conditions:

$$\begin{aligned} 0 = &\sum_{\sigma\in\mathfrak{S}_{p+r=n}} \pm n_{1+r,m}(l_p(c_{\sigma(1)},\dots,c_{\sigma(p)}),c_{\sigma(p+1)},\dots,c_{\sigma(n)};o_1,\dots,o_m) \\ &+\sum_{\sigma\in\mathfrak{S}_{p+r=n},\ i+s+j=m} \pm n_{p,i+1+j}(c_{\sigma(1)},..,c_{\sigma(p)};o_1,..,o_i, \\ &\qquad\qquad n_{r,s}(c_{\sigma(p+1)},..,c_{\sigma(n)};o_{i+1},..,o_{i+s}),o_{i+s+1},..,o_m) \end{aligned} \tag{13}$$

for homogeneous elements $c_1,\dots,c_n\in\mathcal{H}_c$ and $o_1,\dots,o_m\in\mathcal{H}_o$ with the full range $n,m\geq 0$, $(n,m)\neq(0,0)$. The signs $\pm$ are given in [37].

A *weak/curved OCHA* consists of a weak L_∞–algebra $(\mathcal{H}_c,\mathfrak{l})$ with a family of maps $\mathfrak{n}=\{n_{p,q}\colon \mathcal{H}_c^{\otimes p}\otimes\mathcal{H}_o^{\otimes q}\to\mathcal{H}_o\}$, of degree one, now for $p,q\geq 0$ satisfying the analog of the above relation.

An open–closed homotopy algebra includes various sub-structures or reduces to various simpler structures as particular cases. The substructure $(\mathcal{H}_c,\mathfrak{l})$ is by definition an L_∞–algebra and $(\mathcal{H}_o,\{m_k:=n_{0,k}\})$ forms an A_∞–algebra. A nontrivial structure obtained as a special case of OCHAs is the action of $\mathcal{H}_c$ as an L_∞–algebra on $\mathcal{H}_o$ as a dg vector space. Lada and Markl ([43] Definition 5.1) provide the definition of an L_∞–module where one can see the structure as satisfying the relations for a Lie module 'up to homotopy'. This is the appropriate strong homotopy version of the action of an ordinary Lie algebra L on a vector space M, also described as M being a module over L or a representation of L. If we set $n_{p,0}=0$ for all $p\geq 1$, the substructure $(\mathcal{H},\{n_{p,1}\})$ makes $\mathcal{H}_o$ an L_∞–module over $(\mathcal{H}_c,\mathfrak{l})$. Thus we can also speak of $\mathcal{H}_o$ as a *strong homotopy module* over $\mathcal{H}_c$ or as a *strong homotopy representation* of $\mathcal{H}_c$ (cf [54]).

On the other hand, $n_{1,q}$ with $q\geq 1$ forms a strong homotopy derivation [45] with respect to the A_∞–algebra $(\mathcal{H}_o,\{m_k\}_{k\geq 1})$. Moreover, we have the strong homotopy version of an algebra A over a Lie algebra L, that is, an action of L by derivations of A, so that the L_∞–map $L\to\mathrm{End}(A)$ takes values in the Lie sub-algebra $\mathrm{Der}A$.

In his ground breaking "Notions d'algèbre différentielle; $\cdots$ " [6], Henri Cartan formalized several dg algebra notions related to his study of the deRham cohomology of principal fibre bundles, in particular, that of a Lie group G acting in ('dans') a differential graded algebra E. The action uses only the Lie algebra $\mathfrak{g}$ of G. Cartan's action includes both the graded derivation d, the Lie derivative $\theta(X)$ and the inner derivative $i(X)$ for $X\in\mathfrak{g}$. The concept was later reintroduced by Flato, Gerstenhaber and Voronov [11] under the name *Leibniz pair*.

We need only the $\theta(X)$ (which we denote $\rho(X)$ since by θ we denote the image by ρ of an element $X \in \mathfrak{g}$), then the algebraic structure is an example of a dg algebra over a dg Lie algebra $\mathfrak{g}$. Its higher homotopy extension a mathematician would construct by the usual procedures of strong homotopy algebra leads to the following definition (see the Appendix by M Markl in [37]):

Definition 2.13 (A_∞–algebra over an L_∞–algebra) Let L be an L_∞–algebra and A an A_∞–algebra which as a dg vector space is an sh-L module. That A is an *A_∞–algebra over L* means that the module structure map $\rho\colon L \to \mathrm{End}(A)$, regarded as in $\mathrm{Coder}(T^c A)$, extends to an L_∞–map $L \to \mathrm{Coder}(T^c A)$.

An A_∞–algebra over an L_∞–algebra defined as above is an OCHA $(\mathcal{H}, \mathfrak{l}, \mathfrak{n})$ with $n_{p,0} = 0$ for $p \geq 1$.

Given two OCHAs $(\mathcal{H}, \mathfrak{l}, \mathfrak{n})$ and $(\mathcal{H}', \mathfrak{l}', \mathfrak{n}')$, an OCHA morphism from $(\mathcal{H}, \mathfrak{l}, \mathfrak{n})$ to $(\mathcal{H}', \mathfrak{l}', \mathfrak{n}')$ is defined by a collection of degree zero multilinear maps $f_k\colon (\mathcal{H}_c)^{\otimes k} \to \mathcal{H}'_c$, $k \geq 1$, and $f_{k,l}\colon (\mathcal{H}_c)^{\otimes k} \otimes (\mathcal{H}_o)^{\otimes l} \to \mathcal{H}'_o$, $k, l > 0$, $(k,l) \neq (0,0)$, satisfying certain conditions [37]. In particular, $\{f_k\}_{k \geq 1}$ forms an L_∞–morphism from $(\mathcal{H}_c, \mathfrak{l})$ to $(\mathcal{H}'_c, \mathfrak{l}')$. The notion of *OCHA–quasi-isomorphisms* is defined as OCHA–morphisms such that both $f_1\colon \mathcal{H}_c \to \mathcal{H}'_c$ and $f_{0,1}\colon \mathcal{H}_o \to \mathcal{H}'_o$ induce isomorphisms on the cohomologies.

An OCHA $(\mathcal{H}, \mathfrak{l}, \mathfrak{n})$ has a coalgebra description in terms of a degree one codifferential constructed from $\mathfrak{l}$ and $\mathfrak{n}$ on the tensor coalgebra of $\mathcal{H}$ (see [37]). Hoefel [30] has recently shown the following:

Theorem 2.14 (Hoefel [30]) *OCHAs are characterized as being given by all coderivations of degree 1 and square zero on* $\mathrm{Coder}(C(\mathcal{H}_c) \otimes T^c(\mathcal{H}_o))$.

Then, two OCHAs $(\mathcal{H}, \mathfrak{l}, \mathfrak{n})$ and $(\mathcal{H}', \mathfrak{l}', \mathfrak{n}')$, an OCHA–morphism $\mathfrak{f}\colon (\mathcal{H}, \mathfrak{l}, \mathfrak{n}) \to (\mathcal{H}', \mathfrak{l}', \mathfrak{n}')$ is described as a coalgebra map $\mathfrak{f}\colon C(\mathcal{H}_c) \otimes T^c(\mathcal{H}_o) \to C(\mathcal{H}'_c) \otimes T^c(\mathcal{H}'_o)$ satisfying $(\mathfrak{l}' + \mathfrak{n}') \circ \mathfrak{f} = \mathfrak{f} \circ (\mathfrak{l} + \mathfrak{n})$, where $\mathfrak{l} + \mathfrak{n}$ is the codifferential constructed from the OCHA structures $\mathfrak{l}$ and $\mathfrak{n}$.

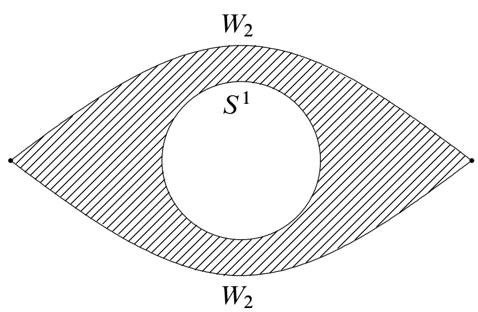
Also, one can describe an OCHA dually in terms of a supermanifold, see [38].

2.10 Examples of the moduli space description

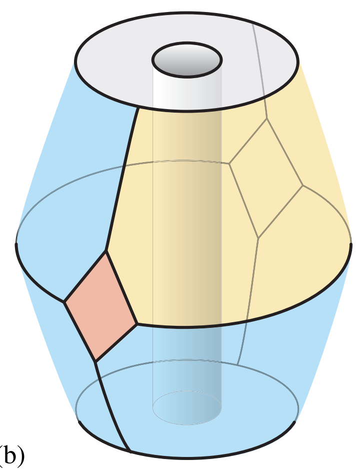
An OCHA should be an algebra over the DG operad of relative chains of the compactified moduli spaces of the corresponding punctured Riemann surfaces. The strata of the compactified moduli space can be labelled by the trees of the $\mathcal{OC}_\infty$–operad. In this direction, Hoefel discusses more carefully the details of these structures [29].

Let us consider a moduli space corresponding to the open–closed case. For $n_{p,q}$, the corresponding moduli space is that of a disk with p–punctures in the bulk (interior) and $(q+1)$–punctures on the boundary. For $p=0$, as we saw, $\{m_q = n_{0,q}\}$ forms an A_∞–structure, and the corresponding moduli spaces are the associahedra. The moduli space corresponding to the operation $\{n_{1,q}\}$ with one closed string input is the same as the *cyclohedra* $\{W_{q+1}\}$, which is the moduli space of configuration space of points on S^1 modulo rotation discussed by Bott and Taubes (see [46, page 241] and [57]).

The cyclohedra $\{W_n\}$ are contractible polytopes. However, the moduli spaces corresponding to $n_{p,q}$ with $p \geq 2$ are not contractible in general. Let us consider the moduli space corresponding to $n_{2,q}$: the disk with two two interior punctures (= closed strings). For $q=0$, the moduli space is described as in Figure 6 (a), which is what is called 'The Eye' in the paper on deformation quantization by Kontsevich [42]. For $n_{2,1}$, the compactified moduli space is topologically a solid torus as in Figure 6 (b) (this figure was made by S Devadoss [9]).



(a)



(b)

Figure 6: (a) The Eye (b) The solid torus

Recall that the facets (codim one faces) of the associahedra are products of associahedra. For cyclohedra, the facets are products of cyclohedra and associahedra. The analog holds for the open–closed compactified moduli spaces (which currently are nameless), although they are not polytopes.

2.11 Cyclic structure

We can also consider an additional *cyclic structure* on open–closed homotopy algebras. See [37; 38] for the definition. The cyclic structure can be defined in terms of *symplectic* inner products. These inner products are essential to the description of the Lagrangians appearing in string field theory (see [36]). The string theory motivation for this additional structure is that punctures on the boundary of the disk inherit a cyclic order from the orientation of the disk and the operations are to respect this cyclic structure, just as the L_∞–structure reflects the symmetry of the punctures in the interior of the disk or on the sphere.

Let us explain briefly the cyclicity in the case of A_∞–algebras. In terms of trees, the distinction between the root and the leaves can be absorbed by regarding these edges as cyclically ordered.

From the viewpoint of the moduli spaces $\mathcal{M}_{n+1}$ of punctured disks, representing $\mathcal{M}_{n+1}$ as in equation (2), the cyclic action is an automorphism $g\colon \mathcal{M}_{n+1}\to\mathcal{M}_{n+1}$, where g is an $SL(2,\mathbb{R})$ transformation (1) such that

$$\begin{aligned}(\infty, 0=t_1,t_2,\dots,t_{n-1},t_n=1)\mapsto &(g(\infty),g(0),g(t_2),\dots,g(t_{n-1}),g(1))\\ &=(0,t_2,\dots,t_{n-1},1,\infty).\end{aligned}$$

One can compactify $\mathcal{M}_{n+1}$ so that the cyclic action extends to the one on $\bar{\mathcal{M}}_{n+1}$. In this way, one can consider a cyclic action on the associahedra. This cyclic action for the associahedra is discussed in [20]. Thus, from the viewpoint of Riemann surfaces, ie, string theory, taking the cyclic action into account for the associahedra is very natural.

This can also be seen visually from the planar trees associated to disks with points on the boundary, cf Figure 2 (b). Correspondingly, for an A_∞–algebra $(A,\mathfrak{m})$, a cyclic structure is defined by a (nondegenerate) inner product $\omega\colon A\otimes A\to\mathbb{C}$ of fixed integer degree satisfying

$$\omega(m_n(o_1,\dots,o_n),o_{n+1})=\pm\omega(m_n(o_2,\dots,o_n,o_{n+1}),o_1)$$

for any homogeneous elements $o_1,\dots,o_{n+1}\in A$ (see [46]).

In a similar way using a nondegenerate inner product, a cyclic structure is defined for an L_∞–algebra and then for an OCHA [37; 38].

3 The minimal model theorem

Homotopy algebras are designed to have homotopy invariant properties. A key and useful theorem in homotopy algebras is then the minimal model theorem, proved by

Kadeishvili for A_∞–algebras [34]. For an A_∞–algebra $(A,\mathfrak{m})$, the minimal model theorem states that there exists another A_∞–algebra $(H(A),\mathfrak{m}')$ on the cohomologies of (A,m_1) and an A_∞–quasi-isomorphism from $(H(A),\mathfrak{m}')$ to $(A,\mathfrak{m})$. Since $\mathfrak{m}'$ is an A_∞–structure on the cohomology $H(A)$, the differential m'_1 is trivial; such an A_∞–algebra is called *minimal*.

The minimal model theorem holds also for an OCHA, which implies that an OCHA is also appropriately called a homotopy algebra:

Definition 3.1 (Minimal open–closed homotopy algebra) An OCHA $(\mathcal{H}=\mathcal{H}_c\oplus\mathcal{H}_o,\mathfrak{l},\mathfrak{n})$ is called *minimal* if $l_1=0$ on $\mathcal{H}_c$ and $n_{0,1}=0$ on $\mathcal{H}_o$.

Theorem 3.2 (Minimal model theorem for open–closed homotopy algebras) *For a given OCHA $(\mathcal{H},\mathfrak{l},\mathfrak{n})$, there exists a minimal OCHA $(H(\mathcal{H}),\mathfrak{l}',\mathfrak{n}')$ and an OCHA–quasi-isomorphism $\mathfrak{f}\colon (H(\mathcal{H}),\mathfrak{l}',\mathfrak{n}')\to(\mathcal{H},\mathfrak{l},\mathfrak{n})$.*

(See subsection 2.9 for the definition of an OCHA–quasi-isomorphism.)

Various stronger versions of this minimal model theorem hold for OCHAs [37], as for A_∞–algebras, L_∞–algebras, etc. One of them is the *homological perturbation theory* developed in particular on the homology of a differential graded algebra [23; 32; 27; 24; 25; 26] (see [47] for an application) and of a dg Lie algebra [33]. Another one is the *decomposition theorem* (see [39; 36]). As for classical A_∞, L_∞, etc cases, these theorems imply that OCHA–quasi-isomorphisms and in particular the one in Theorem 3.2 in fact give homotopy equivalence between OCHAs. This further implies the uniqueness of a minimal model for an OCHA $(\mathcal{H},\mathfrak{l},\mathfrak{n})$; a minimal OCHA $H(\mathcal{H})$ is unique up to an isomorphism on $H(\mathcal{H})$.

4 Geometric construction of OCHAs and Merkulov's structures

As mentioned, OCHAs admit a geometric expression in terms of supermanifolds as given in [38]. Here, instead of that, we give a partially supermanifold description in which an OCHA can be viewed as a 'geometric' weak A_∞–structure. This description is inspired by Merkulov's geometric A_∞– (and C_∞)–structures discussed as a generalization of Frobenius structures [48].

Definition 4.1 (Merkulov [49; 48] – paraphrased) A (Merkulov) *geometric A_∞–structure* on a graded manifold M with its tangent bundle TM is a collection of maps for $n\geq 1$:

(i) $\mu_n\colon \otimes^n_{\mathcal{O}_M} \mathcal{T}_M \to \mathcal{T}_M$,

(ii) where $\mu_1 := [\nu, \ ]$ is defined in terms of an element $\nu \in \mathcal{T}_M$ such that $[\nu,\nu]=0$, making the sheaf of sections $\mathcal{T}_M$ of TM into a sheaf of A_∞–algebras.

Condition (ii) means that ν is a homological vector field on $\mathcal{T}_M$, cf the dual supermanifold description of an A_∞–structure.

The Merkulov geometric A_∞–structure can be obtained as a special case of an OCHA $(\mathcal{H} := \mathcal{H}_c \oplus \mathcal{H}_o, \mathfrak{l}, \mathfrak{n})$ in which the $\mathbb{Z}$–graded supermanifold is an L_∞–algebra $\mathcal{H}_c$. With an eye toward the relevant deformation theory, we use the formal graded commutative power series ring denoted by $\mathbb{C}[[\psi]]$. More precisely, denote by $\{\mathbf{e}_i\}$ a basis of $\mathcal{H}_c$ and the dual base as ψ^i, where the degree of the dual basis is set by $\deg(\psi^i) = -\deg(\mathbf{e}_i)$. Then $\mathbb{C}[[\psi]]$ is the formal graded power series ring in the variables $\{\psi^i\}$.

Let us express the L_∞–structure $l_k\colon (\mathcal{H}_c)^{\otimes k} \to \mathcal{H}_c$, in terms of the bases:

$$l_k(\mathbf{e}_{i_1},\dots,\mathbf{e}_{i_k}) = \mathbf{e}_j c^j_{i_1\cdots i_k}\ .$$

Correspondingly, let us define an odd formal vector field on $\mathcal{H}_c$, that is, a derivation of $\mathbb{C}[[\psi]]$:

$$\delta_S = \frac{\overleftarrow{\partial}}{\partial \psi^j} c^j(\psi) = \sum_{k\geq 0} \frac{1}{k!} \frac{\overleftarrow{\partial}}{\partial \psi^j} c^j_{i_1\cdots i_k} \psi^{i_k}\cdots\psi^{i_1}\ , \qquad c^j(\psi) \in \mathbb{C}[[\psi]]\ . \tag{14}$$

Also, define a collection of multilinear structures on $\mathcal{H}_o$ parameterized by $\mathcal{H}_c$ as follows:

$$n_{i_1,\dots,i_p;q}(o_1,\dots,o_q) := n_{p,q}(\mathbf{e}_{i_1},\dots,\mathbf{e}_{i_p}; o_1,\dots,o_q) \tag{15}$$

for $p,q\geq 0$ with $p+q>0$. Then, let us define a new collection of multilinear maps on $\widetilde{\mathcal{H}}_o := \mathcal{H}_o \otimes \mathbb{C}[[\psi]]$ as follows:

$$\widetilde{m}_n := \sum_{k\geq 0} n_{i_1\cdots i_p,q} \psi^{i_p}\cdots\psi^{i_1}\colon\ (\widetilde{\mathcal{H}}_o)^{\otimes n} \to \widetilde{\mathcal{H}}_o\ , \qquad (n\neq 1)\ , \tag{16}$$

$$\widetilde{m}_1(\widetilde{o}) := \sum_{k\geq 0} n_{i_1\cdots i_p,q} \psi^{i_p}\cdots\psi^{i_1}(\widetilde{o}) - \delta_S(\widetilde{o}), \tag{17}$$

where the tensor product $\widetilde{o}^{\otimes n}$ is defined over $\mathbb{C}[[\psi]]$. One can see that $(\widetilde{\mathcal{H}}_o, \{\widetilde{m}_k\}_{k\geq 0})$ forms a weak A_∞–algebra over $\mathbb{C}[[\psi]]$. In fact, generalized to a more general base manifold $\mathcal{M}$, it may be plausible to call this a *geometric* weak A_∞–structure more general than Merkulov's, which can be regarded as the case in which $\mathcal{H}_o$ is the fiber of $T_0\mathcal{H}_c$, the tangent space of $\mathcal{H}_c$ at the origin of $\mathcal{H}_c$ and then we drop the second condition of his geometric A_∞–structure. Clearly an isomorphism of $T_0\mathcal{H}_c$ to $\mathcal{H}_c$

as graded vector spaces extends to an isomorphism from $\mathcal{T}_{\mathcal{H}_c}$ to $\widetilde{\mathcal{H}}_c := \mathcal{H}_c \otimes \mathbb{C}[[\psi]]$, cf [49], subsection 3.8.1. Under this identification, let us consider the particular case $n_{p,1} := \frac{1}{(p+1)!} l_{p+1}$. Thus, the differential $\widetilde{m}_1$ in equation (17) turns out to be

$$\widetilde{m}_1(\widetilde{c}) = -[\delta_S, \widetilde{c}], \qquad \widetilde{c} \in \widetilde{\mathcal{H}}_c.$$

Since he does not treat the 'weak' case, $n_{p,0}$ is of course zero for any p. The higher multilinear maps $\widetilde{m}_n\colon (\widetilde{\mathcal{H}}_c)^{\otimes n} \to \widetilde{\mathcal{H}}_c$, $n \geq 2$, are defined in the same way as in equation (16), only with the replacement of elements in $\widetilde{\mathcal{H}}_o$ by those in $\widetilde{\mathcal{H}}_c$. One can see that $(\widetilde{\mathcal{H}}_c, \widetilde{m})$ obtained as above coincides with the geometric A_∞–structure in subsection 3.8.1 of [49]. Then, the theorem in subsection 3.8.2 in [49] states that a geometric A_∞–structure is equivalent to a $\mathcal{G}erst_\infty$–algebra structure which is defined by certain relations described by tree graphs having both straight edges and wiggly edges as in subsection 3.6.1 of [49]. Unfortunately, there a straight (resp. wiggly) edge corresponds to an element in $\widetilde{\mathcal{H}}_o$ (resp. $\widetilde{\mathcal{H}}_c$), so his convention is the opposite of ours for OCHAs. Even taking this into account, his defining equation for a $\mathcal{G}erst_\infty$–algebra structure is superficially different from ours. This is because we have $n_{p,1} = \frac{1}{(p+1)!} l_{p+1}$ now; in the $\mathcal{G}erst_\infty$–algebra case, the L_∞–structure l_k (which is denoted by ν_k in [49]) is our l_k or our $k! n_{k-1,1}$. Remembering this fact, one can see that the $\mathcal{G}erst_\infty$–algebra condition in [49] is a special case of our OCHA condition.

The commutative version of the geometric A_∞–structure is called a geometric C_∞–structure [49; 48], which is a special G_∞–algebra and plays an important role in deformation theory. For a given geometric C_∞–structure, if we concentrate on the degree zero part of the graded vector space $\mathcal{H}_c$, the higher products are also concentrated on the one with degree zero. The resulting special geometric C_∞–structure is what is called an F–manifold, a generalization of a Frobenius manifold.

5 Applications of OCHAs to deformation theory

Consider an OCHA $(\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_o, \mathfrak{l}, \mathfrak{n})$. We will show how the combined structure implies the L_∞–algebra $(\mathcal{H}_c, \mathfrak{l})$ controls some deformations of the A_∞–algebra $(\mathcal{H}_o, \{m_k\}_{k\geq 1})$. We will further investigate the deformations of this control as $\mathcal{H}$ is deformed.

We first review some of the basics of deformation theory from a homotopy algebra point of view. The philosophy of deformation theory which we follow (due originally, we believe, to Grothendieck [1] cf [52; 21; 8]) regards any deformation theory as 'controlled' by a dg Lie algebra $\mathfrak{g}$ (unique up to homotopy type as an L_∞–algebra).

[1] See [10] for an extensive annotated bibliography of deformation theory.

For the deformation theory of an (ungraded) associative algebra (A, m) [15] the standard controlling dg Lie algebra is $\mathrm{Coder}(T^c A)$ with the graded commutator as the graded Lie bracket [55]. Under the identification (including a shift in grading) of $\mathrm{Coder}(T^c A)$ with $\mathrm{Hom}(T^c A, A)$ (which is the Hochschild cochain complex), this bracket is identified with the Gerstenhaber bracket and the differential with the Hochschild differential, which can be written as $[m,\ ]$ [14].

The generalization to a differential graded associative algebra is straightforward; the differential is now: $[d_A + m_2, \quad]$. For an A_∞–algebra, the differential similarly generalizes to $[\mathfrak{m},\ ]$.

Deformations of A correspond to certain elements of $\mathrm{Coder}(T^c A)$, namely those that are solutions of an *integrability* equation, now known more commonly as a *Maurer–Cartan* equation.

Definition 5.1 (The classical Maurer–Cartan equation) In a dg Lie algebra $(\mathfrak{g}, d, [\ ,\ ])$, the *classical Maurer–Cartan equation* is

$$d\theta + \frac{1}{2}[\theta, \theta] = 0 \quad \text{for } \theta \in \mathfrak{g}^1. \tag{18}$$

For an A_∞–algebra $(A, \mathfrak{m})$ and $\theta \in \mathrm{Coder}^1(T^c A)$, a deformed A_∞–structure is given by $\mathfrak{m} + \theta$ iff

$$(\mathfrak{m} + \theta)^2 = 0\ .$$

Teasing this apart, since we start with $\mathfrak{m}^2 = 0$, we have equivalently

$$D\theta + \frac{1}{2}[\theta, \theta] = 0, \tag{19}$$

the Maurer–Cartan equation of the dg Lie algebra $(\mathrm{Coder}(T^c A), D, [\ ,\ ])$ (Here D is the natural differential on $\mathrm{Coder}(T^c A) \subset \mathrm{End}(T^c A)$, ie $D\theta = [\mathfrak{m}, \theta]$.)

For L_∞–algebras, the analogous remarks hold, substituting the Chevalley–Eilenberg complex for that of Hochschild, ie, using $\mathrm{Coder}\ C(L) \simeq \mathrm{Hom}(C(L), L)$.

Definition 5.2 (The strong homotopy Maurer–Cartan equation) In an L_∞–algebra $(L, \mathfrak{l})$, the *(generalized) Maurer–Cartan equation* is

$$\sum_{k \geq 1} \frac{1}{k!} l_k(\bar{c}, \dots, \bar{c}) = 0$$

for $\bar{c} \in L^0$. [2]

[2] Note that the degree of $\bar{c}$ is zero since a dg Lie algebra is precisely a special L_∞–algebra after a suitable degree shifing called the *suspension* and then $\mathfrak{g}^1 = L^0$.

We denote the set of solutions of the Maurer–Cartan equation as $\mathcal{MC}(L,\mathfrak{l})$ or more simply $\mathcal{MC}(L)$.

Now, since an OCHA can be thought of as a generalization of an A_∞–algebra over an L_∞–algebra (Definition 2.13), one has:

Theorem 5.3 [37; 38] *An OCHA $(\mathcal{H} := \mathcal{H}_c \oplus \mathcal{H}_o, \mathfrak{l}, \mathfrak{n})$ is equivalent to an L_∞–morphism from $(\mathcal{H}_c, \mathfrak{l})$ to $(\mathrm{Coder}(T^c(\mathcal{H}_o)), D = [\mathfrak{m},\ ], [\ ,\ ])$, where $\mathfrak{m}$ is the codifferential on $\mathrm{Coder}(T^c(\mathcal{H}_o))$ corresponding to $\{m_k = n_{0,k}\}_{k\geq 1}$.*

Since it is known that an L_∞–morphism preserves the solutions of the Maurer–Cartan equations, we obtain the following:

Theorem 5.4 *For an OCHA $(\mathcal{H} := \mathcal{H}_c \oplus \mathcal{H}_o, \mathfrak{l}, \mathfrak{n})$, a Maurer–Cartan element $\overline{c} \in \mathcal{M}(\mathcal{H}_c, \mathfrak{l})$ gives a deformation of $(\mathcal{H}_o, \mathfrak{m} := \{n_{0,k}\}_{k\geq 1})$ as a* weak *A_∞–algebra.*

For a dg Lie algebra, there is the notion of gauge transformation. A gauge transformation defines an equivalence relation $\sim$ between elements in L; two elements in L are equivalent iff they are related by a gauge transformation. In particular, gauge transformations preserves the solution space $\mathcal{MC}(L)$. Thus, the quotient space of $\mathcal{MC}(L)$ by the equivalence relation $\sim$ is well-defined:

$$\mathcal{M}(L) := \mathcal{MC}(L)/\sim .$$

The moduli space of deformations is defined as this $\mathcal{M}(L)$. In particular, one has isomorphic spaces $\mathcal{M}(L)$ for any L of the same L_∞–homotopy type. Thus, deformation theory is in general controlled by an L_∞ homotopy class of a dg Lie algebra.

In general, to construct or even show the existence of an L_∞–morphism is a very difficult problem. However, there exists a pair of a dg Lie algebra and an L_∞–algebra where the existence of an L_∞–morphism between them is guaranteed in some sense. We shall explain this below.

6 Homotopy algebra and string theory

Let us start from a 'definition' of string theory which provides the main motivation for obtaining a pair of a dg Lie algebra and an L_∞–algebra together with an L_∞–morphism between them.

A string is a one dimensional object whose 'trajectory' (worldsheet) is described as a Riemann surface (two-dimensional object). In string theory, the most fundamental quantities are the scattering amplitudes of strings. The most general Riemann surfaces to be concerned with are those with genera, boundaries, and punctures on the boundaries and/or in the interior, where a puncture on a boundary (resp. in the interior) corresponds to an open (resp. closed) string insertion. For a fixed appropriate field theory on Riemann surfaces, the scattering amplitudes are obtained by choosing a suitable compactification of the moduli spaces of punctured Riemann surfaces; the scattering amplitudes are integrals over the compactified moduli spaces. The collection of the scattering amplitudes obtained as above are endowed with special algebraic structures, associated to the stratifications of the compactifications of the moduli spaces. In this sense, a definition of a string theory is the pair of an operad (associated to the compactified moduli spaces) and a representation of the operad (an algebra over the operad).

In particular, if we consider a *sigma-model* on a Riemann surface, ie, a field theory whose fields are maps from the Riemann surfaces to a target space M, the representation obtained by the field theory has some information about the geometry of M.

As we explained, an A_∞–algebra $(A,\mathfrak{m})$ is obtained as a representation of the A_∞–operad $\mathcal{A}_\infty$ on A. So a deformation of $(A,\mathfrak{m})$ is a deformation of a representation of the A_∞–operad $\mathcal{A}_\infty$ on the fixed graded vector space A.

Here, recall that the A_∞–operad $\mathcal{A}_\infty$ is a structure which is associated to the real compactification of the moduli spaces of disks with punctures on the boundaries.

On the other hand, a representation of the L_∞–operad $\mathcal{L}_\infty$ on a fixed graded vector space L is an L_∞–algebra $(L,\mathfrak{l})$.

Now, if we fix a tree open string theory and a tree closed string theory on Riemann surfaces, we obtain an A_∞–algebra $(\mathcal{H}_o,\mathfrak{m})$ and the dg Lie algebra $(\mathrm{Coder}(T^c(\mathcal{H}_o)), D, [\ ,\ ])$ controlling its deformation for the tree open string theory, and also an L_∞–algebra $(\mathcal{H}_c,\mathfrak{l})$ for the tree closed string. Moreover, the tree open–closed string system provides a representation of the $\mathcal{OC}_\infty$–operad on $\mathcal{H}:=\mathcal{H}_c\oplus\mathcal{H}_o$, that is, an OCHA $(\mathcal{H},\mathfrak{l},\mathfrak{n})$. Together with Theorem 5.3, by considering an appropriate tree open–closed string (field) theory, one can get a pair of a dg Lie algebra and an L_∞–algebra together with a non-trivial L_∞–morphism betweeen them.

One example of this is Kontsevich's set-up [42] for deformation quantization [3; 4] (see [38]). Furthermore, as L_∞–algebras $\mathcal{H}_c$ of closed string (field) theories, one can consider for instance the dg Lie algebra controlling the extended deformation of complex structures [2] (which is what is called the B–model in string theory (cf [38])), or the dg Lie algebra controlling deformation of generalized complex structure [22], etc.

Then, OCHAs should guarantee the existence of corresponding deformations of the A_∞–structures. Currently, it should be a very interesting problem to descibe explicitly such a deformation of an A_∞–structure as was done in the case of $*$–product for deformation quantization [42]. Some attempts in this direction can be found in [31] for B–twisted topological strings and [51] for the case of generalized complex structures.

In these situations, which are related to mirror symmetry, the L_∞–algebra corresponding to the tree closed string theory is (homotopy equivalent to) trivial, which implies that the deformation of the corresponding A_∞–structure is unobstructed (see [38]). In such situations, deformation of A_∞–structures can be thought of as a (homological) algebraic description of an L_∞–algebra $(\mathcal{H}_c, \mathfrak{l})$ describing deformation of a geometry (see homological mirror symmetry by Kontsevich [41]).

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