On the Rothenberg–Steenrod spectral sequence for the mod 2 cohomology of classifying spaces of spinor groups

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We compute the cotorsion product of the mod 2 cohomology of spinor group $\mathrm{spin}(n)$, which is the E_2–term of the Rothenberg–Steenrod spectral sequence for the mod 2 cohomology of the classifying space of the spinor group $\mathrm{spin}(n)$. As a consequence of this computation, we show the non-collapsing of the Rothenberg–Steenrod spectral sequence for $n \geq 17$.

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1 Introduction

Let n be a fixed integer greater than or equal to 9. In [9], Quillen computed the mod 2 cohomology of the classifying space $B\mathrm{Spin}(n)$ using the Leray–Serre spectral sequence associated with the fiber bundle $B\pi\colon B\mathrm{Spin}(n) \to BSO(n)$. In terms of the Hurwitz–Radon number h given by

$$\begin{array}{rl} 4\ell & \text{if } n = 8\ell+1, \\ 4\ell+1 & \text{if } n = 8\ell+2, \\ 4\ell+2 & \text{if } n = 8\ell+3 \text{ or } 8\ell+4, \\ 4\ell+3 & \text{if } n = 8\ell+5,\ 8\ell+6,\ 8\ell+7 \text{ or } 8\ell+8, \end{array}$$

Quillen's result is stated as follows:

Theorem 1.1 (Quillen) *As a graded $\mathbb{F}_2$–algebra, we have*

$$H^*(B\mathrm{Spin}(n);\mathbb{F}_2) = \mathbb{F}_2[w_2,\dots,w_n]/J \otimes \mathbb{F}_2[z],$$

where $J = (v_0,\dots,v_{h-1})$, $v_0 = w_2$, $v_k = \mathrm{Sq}^{2^{k-1}}\cdots\mathrm{Sq}^1 w_2$ for $1 \leq k \leq h-1$ and $\deg z = 2^h$. Moreover, $v_0,\dots,v_{h-1}$ is a regular sequence and the Poincaré series is given by

$$\prod_{k=0}^{h-1}(1-t^{2^k+1}) \Big/ \left\{(1-t^{2^h})\prod_{k=2}^{n}(1-t^k)\right\}.$$

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On the other hand, the Rothenberg–Steenrod spectral sequence can often be the most powerful tool for computing the mod p cohomology of the classifying space BG from the mod p cohomology of the underlying connected compact Lie group G. Its E_2–term is given by the cotorsion product

$$\mathrm{Cotor}_{H^*(G;\mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p)$$

and it converges to the mod p cohomology of the classifying space BG. Recently, we proved in [3] the non-degeneracy of the Rothenberg–Steenrod spectral sequence for the mod 3 cohomology of the classifying space BE_8 of the exceptional Lie group E_8. Until this paper all computational results in literature indicated that the Rothenberg–Steenrod spectral sequence collapses at the E_2–level. Although it is not in literature, it has been a folklore to experts for a long time that the Rothenberg–Steenrod spectral sequence for the mod 2 cohomology of the classifying space $B\mathrm{Spin}(n)$ does not collapse at the E_2–level for some n. In the case $n = 2^{s-1} + 1$, for example, it is easy to compute the cotorsion product. Since the mod 2 cohomology of $\mathrm{Spin}(2^{s-1} + 1)$ is a primitively generated Hopf algebra, its cotorsion product is a polynomial algebra $\mathbb{F}_2[w_k] \otimes \mathbb{F}_2[z']$ where $4 \le k \le 2^{s-1}$, $k \ne 2^\ell + 1$ $(\ell = 1, \dots, s-2)$ and $\deg z' = 2^s$. However, the mod 2 cohomology of $B\mathrm{Spin}(2^{s-1} + 1)$ is not a polynomial algebra for $s \ge 5$. So, comparing their Poincaré series, it is easy to deduce that the Rothenberg–Steenrod spectral sequence does not collapse at the E_2–level. In this paper, through the computation of the cotorsion product

$$\mathrm{Cotor}_{H^*(\mathrm{Spin}(n);\mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2)$$

for all $n \ge 9$, we give a proof for the non-degeneracy of the Rothenberg–Steenrod spectral sequence for all $n \ge 17$.

Let s be an integer such that

$$2^{s-1} < n \le 2^s.$$

In Section 2, we define an integer h' for $n \ge 9$. Using the integers s and h', our main result is stated as follows:

Theorem 1.2 *Let $A = H^*(\mathrm{Spin}(n); \mathbb{F}_2)$. Suppose that $n \ge 9$. Then, we have an isomorphism of graded $\mathbb{F}_2$–algebras*

$$\mathrm{Cotor}_A(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2[w_2, \dots, w_n]/J' \otimes \mathbb{F}_2[z'],$$

where $J' = (v_0, \dots, v_{h'-1})$, $v_0 = w_2$,

$$v_k = \underbrace{\mathrm{Sq}^0 \cdots \mathrm{Sq}^0}_{k\text{–times}} v_0 \quad (k = 1, \dots, s-1),$$

$$v_s = \sum_{i+j=2^{s-1}} w_{2i+1} w_{2j},$$

and

$$v_{s+k} = \mathrm{Sq}^{2^{k-1}} \cdots \mathrm{Sq}^1 v_s \quad (k \geq 1).$$

Moreover, the sequence $v_0, \dots, v_{h'-1}$ *is a regular sequence and the Poincaré series of the cotorsion product is given by*

$$\prod_{k=0}^{h'-1} (1 - t^{2^k+1}) \Big/ \left\{ (1 - t^{2^{h'}}) \prod_{k=2}^{n} (1 - t^k) \right\}.$$

A caution is called for; the action of Steenrod squares in Theorem 1.2 is the one defined for the cotorsion product. It is not the one induced by the action of Steenrod squares on $A = H^*(\mathrm{Spin}(n); \mathbb{F}_2)$. In particular, Sq^0 is not the identity homomorphism. We recall the action of Steenrod square on the cotorsion product in Section 4. After defining the integer h', we prove the following proposition in Section 2.

Proposition 1.3 *For* $9 \leq n \leq 16$*, we have* $h' = h$*. For* $n \geq 17$*, we have* $h' < h$*.*

Thus, we have the following theorem.

Theorem 1.4 *For* $n \leq 16$*, the Rothenberg–Steenrod spectral sequence for the mod* 2 *cohomology* $H^*(B\mathrm{Spin}(n); \mathbb{F}_2)$ *collapses at the* E_2*–level. For* $n \geq 17$*, the Rothenberg–Steenrod spectral sequence for the mod* 2 *cohomology* $H^*(B\mathrm{Spin}(n); \mathbb{F}_2)$ *does not collapse at the* E_2*–level.*

The cotorsion products appear in other settings. There exist spectral sequences converging to the mod p cohomology of classifying spaces of loop groups as well as to the one of classifying spaces of finite Chevalley groups. Both spectral sequences have the same E_2–term:

$$\mathrm{Cotor}_{H^*(G;\mathbb{F}_p)}(\mathbb{F}_p, H^*(G; \mathbb{F}_p)).$$

In the case $G = \mathrm{Spin}(10)$, $p = 2$, the computation of the above cotorsion product is done in Kuribayashi, Mimura and Nishimoto [4] using the twisted tensor product. However, it seems to be not so easy to carry out their computation for $n > 10$. In this paper, we use the change-of-rings spectral sequence and Steenrod squares as our tools. We hope that the computation done in this paper can shed some light on the computation of the cotorsion products

$$\mathrm{Cotor}_{H^*(G;\mathbb{F}_p)}(\mathbb{F}_p, H^*(G; \mathbb{F}_p)).$$

In Section 2, we define integers s, t, m, m', ε, h' and sets C, D, E and prove some elementary properties of these integers and sets as well as Proposition 1.3. We use these integers and sets in order to describe generators and relations of cotorsion products in Section 5. In Section 3, we give a naive criterion for a sequence in a polynomial ring over a field to be a regular sequence in terms of Gröbner bases. In Section 4, we recall some results on the Steenrod squares acting on cotorsion products and the change-of-rings spectral sequence. In Section 5, we prove Theorem 1.2 using the results in Sections 3 and 4.

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2 Integers s, t, h'

In this section, for a given integer $n \geq 9$, we define integers s, t, m, m', ε, h' and sets C, D, E and prove some elementary properties of these integers and sets. We use these integers, sets and their properties in Section 5 in order to describe generators and relations, in particular v_{s+k} in Theorem 1.2, of cotorsion products. We do not use the results in this section until Section 5. Throughout this section, we assume that n is a fixed integer greater than or equal to 9.

To begin with, we define integers s, t, m, m' and ε. For a positive integer k, let $\alpha(k)$ be the number of 1's in the binary expansion of k. Let s be an integer such that

$$2^{s-1} < n \leq 2^s.$$

For $n < 2^s - 2$, let t be an integer such that

$$2^s - 2^t - 1 \leq n < 2^s - 2^{t-1} - 1,$$

and for $n = 2^s, 2^s - 1, 2^s - 2$, let $t = 1$.

Let us consider a set of integers

$$E = \{k \in \mathbb{Z} \mid 2 \leq k \leq n, \alpha(k-1) \geq 2\},$$

and its subset

$$D = \{k \in \mathbb{Z} \mid k \leq n, 2^s - k + 1 \leq n, \alpha(k-1) \geq 2, \alpha(2^s - k) \geq 2\}.$$

It is easy to verify the following proposition.

Proposition 2.1 *The set D is empty if and only if $n = 2^{s-1} + 1$.*

Proof Since $n \geq 9$, we may assume that $s \geq 4$. Let $k = 2^{s-1} + 2$. Then, we have $\alpha(k-1) = 2$ and $\alpha(2^s - k) = s - 2 \geq 2$. Thus, if $n \geq 2^{s-1} + 2$, we have $k \in D$. If $n = 2^{s-1} + 1$ and $k' \in D$, then $2^s - (2^{s-1} + 1) + 1 \leq k' \leq 2^{s-1} + 1$. So, we have $k' = 2^{s-1}$ or $2^{s-1} + 1$. Since $\alpha(2^s - 2^{s-1}) = 1$ and $\alpha((2^{s-1} + 1) - 1) = 1$, $2^{s-1}, 2^{s-1} + 1 \notin D$. Therefore, D is empty. □

When D is not empty, let m be the greatest integer in D, put

$$m' = 2^{s-t}(2^s - m) + 1,$$

and let us define ε as follows:

$$\begin{aligned} \varepsilon &= 0 \ \text{ if } m' > n, \\ \varepsilon &= 1 \ \text{ if } m' \leq n. \end{aligned}$$

We also define h' as follows:

$$\begin{aligned} h' &= s && \text{if } D = \varnothing, \\ h' &= 2s - t + \varepsilon && \text{if } D \neq \varnothing. \end{aligned}$$

Next, we prove Proposition 1.3 by computing h' for $9 \leq n \leq 32$ and by showing that the inequality $h' < h$ holds for $n \geq 33$.

Proof of Proposition 1.3 For $n \leq 32$, by direct computation, we have the following tables.

n	s	t	m	m'	ε	h'	ℓ	h
9	4	3	–	–	–	4	1	4
10	4	3	10	13	0	5	1	5
11	4	2	11	21	0	6	1	6
12	4	2	11	21	0	6	1	6
13	4	1	13	25	0	7	1	7
14	4	1	13	25	0	7	1	7
15	4	1	13	25	0	7	1	7
16	4	1	13	25	0	7	1	7

n	s	t	m	m'	ε	h'	ℓ	h
17	5	4	–	–	–	5	2	8
18	5	4	18	29	0	6	2	9
19	5	4	19	27	0	6	2	10
20	5	4	20	25	0	6	2	10
21	5	4	21	23	0	6	2	11
22	5	4	22	21	1	7	2	11
23	5	3	23	37	0	7	2	11
24	5	3	23	37	0	7	2	11
25	5	3	25	29	0	7	3	12
26	5	3	26	25	1	8	3	13
27	5	2	27	41	0	8	3	14
28	5	2	27	41	0	8	3	14
29	5	1	29	49	0	9	3	15
30	5	1	29	49	0	9	3	15
31	5	1	29	49	0	9	3	15
32	5	1	29	49	0	9	3	15

Next, we deal with the case $n \geq 33$. In this case, we may assume that $s \geq 6$. By the definition of t, we have $t \geq 1$. So, we have $\max\{2s - t + \varepsilon, s\} \leq 2s$. Therefore, it suffices to show the inequality $2s < h$. Assume that $n = 8\ell + r$ where $1 \leq r \leq 8$. Then, by the definition of s, we have

$$2^{s-1} < 8\ell + r \leq 8\ell + 8.$$

Hence, we have

$$2^{s-2} < 4\ell + 4.$$

Therefore, we obtain

$$h \geq 4\ell > 2^{s-2} - 4 \geq 2s$$

for $s \geq 6$ as required. □

We prove some elementary properties of D, say Propositions 2.2 and 2.3, which we need in the proof of Proposition 5.1.

Proposition 2.2 *Suppose that D is not empty. If $k \in D$, then $2^s - k + 1 \in D$.*

Proof It is easy to see that

(1) $2^s - k + 1 \leq n$,

(2) $2^s - (2^s - k + 1) + 1 = k \leq n$,

(3) $\alpha((2^s - k + 1) - 1) = \alpha(2^s - k) \geq 2$,

(4) $\alpha(2^s - (2^s - k + 1)) = \alpha(k - 1) \geq 2$. □

Proposition 2.3 *Suppose that D is not empty and $k \in D$. Then*:

(1) $2^{s-t+1}(k-1) + 1 > n$.

(2) *If $\varepsilon = 0$, then $2^{s-t}(k-1) + 1 > n$.*

Proof First, we prove (1). Since $2^s - k + 1$ is also in D, we have

$$2^s - n \leq k - 1.$$

Hence, we have

$$2^{s-t+1}(k-1) + 1 > 2^s + 2^{s-t+1} + 1 > n.$$

Next, we prove (2). Since $2^s - k + 1$ is also in D, by the definition of m, we have

$$2^s - k + 1 \leq m.$$

Thus, we have

$$2^s - m \leq k - 1.$$

Since $\varepsilon = 0$, we have

$$2^{s-t}(k-1)+1 \geq 2^{s-t}(2^s - m)+1 = m' > n. \qquad \square$$

It is clear that the number of integers in E is $n-s-1$. For $k = 0, \dots, s-t-1$, we define $\sigma(k)$ by

$$\sigma(k) = 2^s - 2^{s-1-k} - 1.$$

Let
$$C_0 = \{\sigma(k) \mid k = 0, \dots, s-t-1\}.$$

Then, it is easy to see that C_0 is a subset of E. For $k = s-t$, we define $\sigma(k)$ to be m if $\varepsilon = 1$. For $k = s-t+\varepsilon, \dots, n-s-2$, we define $\sigma(k)$ as follows:

$$\sigma(k) \in \{a \in E \mid a \notin C_0, a \neq m \text{ if } \varepsilon = 1\},$$

and then we have

$$\sigma(s-t+\varepsilon) < \cdots < \sigma(n-s-2).$$

Let $\tau(k) = 2^{s-1} + 2^k + 1$ for $k = 0, \dots, s-t-1$. Let $C = C_0 \cup C_1$, where

$$C_1 = \{\, \tau(k) \mid k = 0, \dots, s-t-1 \,\}.$$

What we need in the proof of Proposition 5.2 in Section 5 is the following Propositions 2.4 and 2.5. For the rest of this section, we assume that $n \geq 18$, $n \neq 2^{s-1}+1$ and $s \geq 5$.

Proposition 2.4 *Suppose that $n \geq 18$ and $n \neq 2^{s-1}+1$. Then, the integers $\sigma(k)$, $\tau(k)$ $(k = 0, \dots, s-t-1)$ are distinct from each other.*

Proof If $n \geq 18$, then $s \geq 5$, so that $s-1 > 3$. Since $(s-t)$ integers $\sigma(k)$ $(k = 0, \dots, s-t-1)$ in C_0 are distinct from each other, since $(s-t)$ integers $\tau(k)$ $(k = 0, \dots, s-t-1)$ in C_1 are also distinct from each other, and since $\alpha(\sigma(k)) = s-1$, $\alpha(\tau(k)) \leq 3$, we have that $C_0 \cap C_1 = \varnothing$ and that $(2s-2t)$ integers $\sigma(k)$, $\tau(k')$ are distinct from each other where $k, k' \in \{0, \dots, s-t-1\}$. $\square$

Proposition 2.5 *Suppose that $n \geq 18$ and $n \neq 2^{s-1}+1$. If $\varepsilon = 1$, then m, $m' \notin C$.*

The rest of this section is devoted to proving Proposition 2.5 above. Firstly, we prove that if $n \geq 18$ and if $n \in C$, then we have $\varepsilon = 0$.

Proposition 2.6 *Suppose that $n \geq 18$ and $n \neq 2^{s-1}+1$. If $\varepsilon = 1$, then we have $m = n$ and $2^{t-1}+1 < 2^s - n \leq 2^t + 1$.*

Proof We prove this proposition by showing that if $m \neq n$, then we have $\varepsilon = 0$. First, we deal with the case $n = 2^s$, $2^s - 1$ or $2^s - 2$. In this case, $t = 1$, $m = 2^s - 3$, $m' = 2^{s-1} \cdot 3 + 1 > 2^s + 1 > n$. Thus, we have $\varepsilon = 0$. So, without loss of generality, we may assume that $2^{s-1} + 2 \leq n \leq 2^s - 3$ and so we have

$$2^{t-1} + 1 < 2^s - n \leq 2^t + 1.$$

Suppose that $m \neq n$. Then, $\alpha(n-1) = 1$ or $\alpha(2^s - n) = 1$. The equality $\alpha(n-1) = 1$ holds if and only if $n = 2^{s-1} + 1$. Hence, $\alpha(2^s - n) = 1$. So we have $2^s - n = 2^t$, $m = 2^s - 2^t - 1$ and

$$m' = 2^{s-t}(2^t + 1) + 1 = 2^s + 2^{s-t} + 1 > n.$$

Hence, by definition, we have $\varepsilon = 0$. □

Proof of Proposition 2.5 By Proposition 2.6, we have $m = n$,

$$m' = 2^{s-t}(2^s - n) + 1$$

and

$$2^{t-1} + 1 < 2^s - n \leq 2^t + 1.$$

If $m \in C$ or if $m' \in C$, then one of the following conditions holds:

(1) $n = 2^s - 2^{s-1-k} - 1$,

(2) $n = 2^{s-1} + 2^k + 1$,

(3) $2^{s-t}(2^s - n) + 1 = 2^s - 2^{s-1-k} - 1$,

(4) $2^{s-t}(2^s - n) + 1 = 2^{s-1} + 2^k + 1$,

where $0 \leq k \leq s - t - 1$. We prove that it is not the case.

Case (1) We have $2^s - n = 2^{s-1-k} + 1$. So, we have $t = s - 1 - k$ and

$$m' - n = 2^{s-t}(2^t + 1) + 1 - (2^s - 2^{s-1-k} - 1) > 0.$$

This contradicts the assumption $\varepsilon = 1$.

Case (2) We have $2^s - n = 2^{s-1} - 2^k - 1$. So, one of the following statements holds:

(a) $t = s - 1$, $k < s - 2$ or

(b) $t = s - 2$, $k = s - 2$.

If $s - t = 1$ and $k < s - 2$, then $m' = 2^s - 2^{k+1} - 1$ and

$$m' - n = 2^{s-1} - 2^{k+1} - 2^k - 2.$$

If $s-t=2$ and $k=s-2$, then we have

$$m'-n=2^{s-1}-2^k-2.$$

In both cases, we have $m'-n>0$. This contradicts the assumption $\varepsilon=1$.

Case (3) We have

$$2^s-n=2^t-2^{(s-1-k)-(s-t)}-2^{1-(s-t)}.$$

By the definition of t, we have that $s-t>0$. Moreover, because of the assumption $k\le s-t-1$, we have $s-1-k>0$. Since 2^s-n is an integer, we have $s-t=1$ and $k=0$. So, we have $2^s-n=2^{s-2}-1$. This contradicts the inequality

$$2^{t-1}+1<2^s-n.$$

Case (4) We have

$$2^s-n=2^{t-1}+2^{k-(s-t)}.$$

Since 2^s-n is an integer, we have $k-(s-t)\ge 0$. This contradicts the assumption $0\le k\le s-t-1$.

Thus, any of the above four conditions (1), …, (4) does not hold. Hence, we have the desired result. □

3 Gröbner bases and regular sequences

In this section, we recall the notion of Gröbner bases and regular sequences. Let K be a field and let $R=K[x_1,\dots,x_n]$ be a polynomial ring over K in n variables $x_1,\dots,x_n$.

Firstly, we recall the definition of Gröbner basis and its elementary properties. We refer the reader to text books on Gröbner bases such as Adams and Loustaunau [1]. We assume that R has a fixed term order on the set of monomials of R. A term order is often called a monomial order in literature, see Eisenbud [2] for example. It is a total order on the set of monomials such that for monomials x, y, z:

$$z<xz<yz$$

if $x<y$ and $z\ne 1$. Let f be an element in R. We denote by $\mathrm{lp}(f)$ the leading power, or the leading monomial, of f and by $\mathrm{lt}(f)$ the leading term of f. In the case the coefficient field K is $\mathbb{F}_2$, the leading term and the leading monomial are the same. Let $G=\{g_1,\dots,g_r\}$ be a finite subset of R, where we assume that g_i's are nonzero and $g_i\ne g_j$ for $i\ne j$.

The subset G is called a Gröbner basis if each polynomial in the ideal $I = (g_1, \dots, g_r)$ has the leading term divisible by the leading term of g_k for some $g_k \in G$. A polynomial f is said to reduce to zero modulo G if and only if there exist $f_1, \dots, f_s \in R$ and $i_1, \dots, i_s \in \{1, \dots, r\}$ such that

$$f = \sum_{k=1}^{s} f_k g_{i_k},$$

where a scalar multiple of $\mathrm{lp}(f_1)\,\mathrm{lp}(g_{i_1})$ is a nonzero term in f, and for $k = 2, \dots, s$, a scalar multiple of $\mathrm{lp}(f_k)\,\mathrm{lp}(g_{i_k})$ is a nonzero term of

$$\mathrm{lp}(f - \sum_{\ell=1}^{k-1} f_\ell g_{i_\ell}).$$

It is clear from the definition of Gröbner basis that when $G = \{g_1, \dots, g_r\}$ is a Gröbner basis, a polynomial in R is in the ideal $(g_1, \dots, g_r)$ if and only if f reduces to zero modulo G.

The following theorem is known as the Buchberger criterion.

Theorem 3.1 (Buchberger) *Let $G = \{g_1, \dots, g_r\}$ be a finite subset of R. Let*

$$S(g_i, g_j) = \frac{\mathrm{lcm}(\mathrm{lp}(g_i), \mathrm{lp}(g_j))}{\mathrm{lt}(g_i)} g_i - \frac{\mathrm{lcm}(\mathrm{lp}(g_i), \mathrm{lp}(g_j))}{\mathrm{lt}(g_j)} g_j,$$

where lcm *stands for the least common multiple. The set G is a Gröbner basis if and only if all $S(g_i, g_j)$ $(i \neq j)$ reduce to zero modulo G.*

Proof See the proof of Theorem 1.7.4 in [1]. □

We also recall the lemma below.

Lemma 3.2 *Let $g_1, g_2 \in R$ and suppose that both are nonzero. Let $d = \gcd(g_1, g_2)$. The following statements are equivalent*:

(1) $\mathrm{lp}(\frac{g_1}{d})$ *and* $\mathrm{lp}(\frac{g_2}{d})$ *are relatively prime*;

(2) $S(g_1, g_2)$ *reduces to zero modulo* $\{g_1, g_2\}$.

Proof See the proof of Lemma 3.3.1 in [1]. □

As an application of this lemma, by the Buchberger criterion, we have the following proposition.

Proposition 3.3 *Let $G = \{g_1, \dots, g_r\}$ be a finite set of polynomials in R. Suppose that the leading terms of g_i and g_j are relatively prime for $i \neq j$. Then, the set G is a Gröbner basis.*

Secondly, we recall the definition of a regular sequence. A sequence $g_1, \dots, g_r$ of polynomials in R is called a regular sequence if the multiplication by g_k induces a monomorphism

$$R \xrightarrow{\times g_1} R$$

for $k = 1$ and a monomorphism

$$R/(g_1, \dots, g_{k-1}) \xrightarrow{\times g_k} R/(g_1, \dots, g_{k-1})$$

for $k = 2, \dots, r$. If $g_1, \dots, g_r$ are homogeneous polynomials, then the Poincaré series of $R/(g_1, \dots, g_r)$ is given by

$$\prod_{k=1}^{r} \left(1 - t^{\deg g_k}\right) \Big/ \prod_{k=1}^{n} \left(1 - t^{\deg x_k}\right).$$

We need the following lemma in the proof of Proposition 5.2 in Section 5.

Lemma 3.4 *Suppose that $g_1, \dots, g_r$ are polynomials in R such that the leading monomials of g_i and g_j are relatively prime for $i \neq j$. Then, the sequence $g_1, \dots, g_r$ is a regular sequence.*

Proof Since R is an integral domain, it is clear that the multiplication by g_1 induces a monomorphism

$$R \to R.$$

For $k = 2, \dots, r$, by Proposition 3.3, $\{g_1, \dots, g_{k-1}\}$ is a Gröbner basis for $k = 2, \dots, r$. Suppose that $f \notin (g_1, \dots, g_{k-1})$ and that $g_k f \in (g_1, \dots, g_{k-1})$. Without loss of generality, we may assume that the leading term of f is not divisible by $\mathrm{lp}(g_i)$ where $i = 1, \dots, k-1$ and that the leading term $\mathrm{lp}(g_k)\,\mathrm{lp}(f)$ of $g_k f$ is divisible by some $\mathrm{lp}(g_i)$ where $i \in \{1, \dots, k-1\}$. Since $\mathrm{lp}(g_i)$ and $\mathrm{lp}(g_k)$ are relatively prime in R, we see that $\mathrm{lp}(f)$ is divisible by $\mathrm{lp}(g_i)$. It is a contradiction. Thus, we have that if $g_k f \in (g_1, \dots, g_{k-1})$, then $f \in (g_1, \dots, g_{k-1})$. □

4 Steenrod squares and the change-of-rings spectral sequence

In this section, we recall some facts on the action of Steenrod squares on cotorsion products and spectral sequences. We refer the reader to Singer's book [11].

Firstly, we recall the action of the Steenrod squares on the cotorsion product $\mathrm{Cotor}_A(\mathbb{F}_2,\mathbb{F}_2)$ for a connected Hopf algebra A over $\mathbb{F}_2$. Let

$$\phi\colon A \to A \otimes A$$

be the coproduct of A. Let $\bar{A}$ be the submodule generated by the positive degree elements. We denote by

$$\bar{\phi}\colon \bar{A} \to \bar{A} \otimes \bar{A}$$

the reduced coproduct. The cotorsion product $\mathrm{Cotor}_A(\mathbb{F}_2,\mathbb{F}_2)$ is a graded $\mathbb{F}_2$–algebra generated by elements $[x_1|\cdots|x_r]$ where we denote by $[x_1|\cdots|x_r]$ the element represented by $x_1 \otimes \cdots \otimes x_r \in \bar{A} \otimes \cdots \otimes \bar{A}$.

Theorem 4.1 below is a variant of Proposition 1.111 in Singer's book [11]. The unstable condition below immediately follows from the definition and the construction of Steenrod squares in [11]. It is also called Steenrod Operation Theorem A1.5.2 in Ravenel [10], which is a re-indexed form of 11.8 of May [5].

Theorem 4.1 *With the notation above, for $p \geq 0$, $k \geq 0$, there exist homomorphisms*

$$\mathrm{Sq}^k\colon \mathrm{Cotor}_A^p(\mathbb{F}_2,\mathbb{F}_2) \to \mathrm{Cotor}_A^{p+k}(\mathbb{F}_2,\mathbb{F}_2)$$

satisfying

(1) *the unstable condition*:

$$\begin{aligned} &\mathrm{Sq}^0[x] = [x^2], \\ &\mathrm{Sq}^1[x] = [x|x] = [x]^2, \\ &\mathrm{Sq}^k[x] = 0 \qquad\qquad \text{for } k \geq 2; \end{aligned}$$

(2) *the Cartan formula*:

$$\mathrm{Sq}^k(xy) = \sum_{i+j=k,\, i,j\geq 0} (\mathrm{Sq}^i x)(\mathrm{Sq}^j y).$$

Note that $\mathrm{Sq}^0\colon \mathrm{Cotor}_A^p(\mathbb{F}_2,\mathbb{F}_2) \to \mathrm{Cotor}_A^p(\mathbb{F}_2,\mathbb{F}_2)$ is not the identity homomorphism.

Secondly, we recall the action of the Steenrod squares on the change-of-rings spectral sequence. Let us consider an extension of connected Hopf algebras:

$$\Gamma \to A \to \Lambda.$$

Then, there exists the change-of-rings spectral sequence

$$\{E_r^{p,q}, d_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1}\}$$

with the E_2–term

$$E_2^{p,q} = \mathrm{Cotor}_\Gamma^p(\mathbb{F}_2, \mathrm{Cotor}_A^q(\Gamma, \mathbb{F}_2)).$$

It converges to the cotorsion product $\mathrm{Cotor}_A(\mathbb{F}_2, \mathbb{F}_2)$ and is a first quadrant cohomology spectral sequence of graded $\mathbb{F}_2$–algebras.

The following is a combined form of Theorems 2.15 and 2.17 in Singer's book [11].

Theorem 4.2 *With the notation above, for all $p, q \geq 0$, $r \geq 2$, there exist homomorphisms*

$$\begin{aligned} &\mathrm{Sq}^k\colon E_r^{p,q} \to E_r^{p,q+k} && \text{if } 0 \leq k \leq q, \\ &\mathrm{Sq}^k\colon E_r^{p,q} \to E_{r+k-q}^{p+k-q,2q} && \text{if } q \leq k \leq q+r-2, \\ &\mathrm{Sq}^k\colon E_r^{p,q} \to E_{2r-2}^{p+k-q,2q} && \text{if } q+r-2 \leq k, \end{aligned}$$

such that

(1) *if $\alpha \in E_r^{p,q}$, then both $\mathrm{Sq}^k\alpha$ and $\mathrm{Sq}^k d_r\alpha$ survive to E_t, where*

$$\begin{aligned} &t = r && \text{if } 0 \leq k \leq q-r+1, \\ &t = 2r+k-q-1 && \text{if } q-r+1 \leq k \leq q, \\ &t = 2r-1 && \text{if } q \leq k; \end{aligned}$$

(2) *in E_t, we have*

$$d_t(\mathrm{Sq}^k\alpha) = \mathrm{Sq}^k d_r\alpha;$$

(3) *at the E_∞–level, Sq^k is compatible with the action of Sq^k on $\mathrm{Cotor}_A(\mathbb{F}_2, \mathbb{F}_2)$, that is, if we denote by*

$$\pi_{p,q}\colon F^p\mathrm{Cotor}_A^{p+q}(\mathbb{F}_2, \mathbb{F}_2) \to E_\infty^{p,q}$$

the edge homomorphism, then:

$$\begin{aligned} &\mathrm{Sq}^k\pi_{p,q} = \pi_{p,q+k}\mathrm{Sq}^k && \text{for } k \leq q \text{ and} \\ &\mathrm{Sq}^k\pi_{p,q} = \pi_{p+k-q,2q}\mathrm{Sq}^k && \text{for } k \geq q, \end{aligned}$$

where the Sq^k in the right hand-side of the above equalities are the one given in Theorem 4.1.

5 Cotorsion products

We refer the reader to the book of Mimura and Toda [7], Mimura [6] and their references for the cohomology of compact Lie groups. Recall that the mod 2 cohomology of $\mathrm{Spin}(n)$ is given as follows: Let E be the set E defined in Section 2. Let Δ be an

algebra generated by x_k with the relation $x_k^2 = x_{2k}$ where $x_k = 0$ if $k+1 \notin E$. As an algebra over $\mathbb{F}_2$, we have

$$H^*(\mathrm{Spin}(n); \mathbb{F}_2) = \Delta \otimes \Lambda(y_{2^s-1}).$$

The reduced coproduct $\bar{\phi}$ is given by

$$\bar{\phi}(x_k) = 0$$

for $k+1 \in E$ and

$$\bar{\phi}(y_{2^s-1}) = \sum_{i+j=2^{s-1}} x_{2i} \otimes x_{2j-1}.$$

In this section, by computing the change-of-rings spectral sequence associated with the extension of Hopf algebras:

$$\Delta \to H^*(\mathrm{Spin}(n); \mathbb{F}_2) \to \Lambda(y_{2^s-1}),$$

we prove Theorem 1.2. The subalgebra Δ is the image of the induced homomorphism

$$\pi^*\colon H^*(SO(n); \mathbb{F}_2) \to H^*(\mathrm{Spin}(n); \mathbb{F}_2).$$

The E_2-term of the spectral sequence is given by

$$\mathrm{Cotor}_\Delta(\mathbb{F}_2, \mathrm{Cotor}_{H^*(\mathrm{Spin}(n);\mathbb{F}_2)}(\Delta, \mathbb{F}_2)).$$

We call this spectral sequence the change-of-rings spectral sequence. As a matter of fact, it is nothing but the change-of-coalgebras spectral sequence in Section 2 of Moore and Smith [8]. It is also noted in [8] that the E_2–term is isomorphic to

$$\mathrm{Cotor}_\Delta(\mathbb{F}_2, \mathbb{F}_2) \otimes \mathrm{Cotor}_{\Lambda(y_{2^s-1})}(\mathbb{F}_2, \mathbb{F}_2).$$

For the sake of notational simplicity, let

$$A = H^*(\mathrm{Spin}(n); \mathbb{F}_2)$$

and

$$B = H^*(SO(n); \mathbb{F}_2).$$

Firstly, we collect some results on $\mathrm{Cotor}_B(\mathbb{F}_2, \mathbb{F}_2)$ and the Rothenberg–Steenrod spectral sequence for the mod 2 cohomology of $BSO(n)$. As an algebra, B is generated by x_i with the relations $x_i^2 = x_{2i}$ where $x_i = 0$ for $i \geq n$. As a coalgebra, x_i $(i = 1, \dots, n-1)$ are primitive and B is primitively generated. So, the cotorsion product $\mathrm{Cotor}_B(\mathbb{F}_2, \mathbb{F}_2)$ is a polynomial algebra $\mathbb{F}_2[w_2, \dots, w_n]$ where w_{k+1} is represented by $[x_k] \in \mathrm{Cotor}_B^{1,k}(\mathbb{F}_2, \mathbb{F}_2)$. It is also clear that the Rothenberg-Steenrod spectral sequence collapses at the E_2–level and hence we have $H^*(BSO(n); \mathbb{F}_2) = \mathbb{F}_2[w_2, \dots, w_n]$,

where, by abuse of notation, we denote by w_{k+1} the element in $H^*(BSO(n);\mathbb{F}_2)$ represented by

$$w_{k+1} \in E_\infty^{1,k} = E_2^{1,k} = \mathrm{Cotor}_B^{1,k}(\mathbb{F}_2,\mathbb{F}_2).$$

Let $v_0 = w_2 \in \mathrm{Cotor}_B(\mathbb{F}_2,\mathbb{F}_2)$. For $1 \le k \le s-1$, let

$$v_k = \underbrace{\mathrm{Sq}^0 \cdots \mathrm{Sq}^0}_{k\text{-times}} v_0 \in \mathrm{Cotor}_B(\mathbb{F}_2,\mathbb{F}_2).$$

By the unstable condition in Theorem 4.1, we have $v_k = w_{2^k+1}$.

Let
$$v_s = \sum_{i+j=2^{s-1}} w_{2i+1} w_{2j},$$

where we assume that $i, j \ge 0$ and $w_0 = w_1 = 0$ and $w_i = 0$ for $i > n$. We define an element v_{s+k} in $\mathrm{Cotor}_B(\mathbb{F}_2,\mathbb{F}_2)$ for $k \ge 1$ by

$$v_{s+k} = \mathrm{Sq}^{2^{k-1}} \cdots \mathrm{Sq}^1 v_s.$$

Let
$$R = \mathbb{F}_2[w_2,\dots,w_n]/(v_0,\dots,v_{s-1})$$

be the polynomial ring generated by variables w_k where k ranges over the set E. This is isomorphic to the cotorsion product $\mathrm{Cotor}_\Delta(\mathbb{F}_2,\mathbb{F}_2)$.

We have the following proposition.

Proposition 5.1

(1) *The polynomial v_{2s-t+1} is zero in R.*

(2) *If $\varepsilon = 0$, then the polynomial v_{2s-t} is also zero in R.*

Proof Suppose that $w_i w_j$ is a nonzero term in v_s. By definition, it is easy to see that both i and j are in D. By the unstable condition and by the Cartan formula in Theorem 4.1 for $k \ge 1$, we have

$$\mathrm{Sq}^{2^{k-1}} \cdots \mathrm{Sq}^1 w_i w_j = w_i^{2^k} w_{2^k(j-1)+1} + w_{2^k(i-1)+1} w_j^{2^k}.$$

By Proposition 2.3, we have

$$\mathrm{Sq}^{2^{k-1}} \cdots \mathrm{Sq}^1 w_i w_j = 0$$

in the case $k \ge s-t$ or in the case $\varepsilon = 0$ and $k = s-t-1$. □

To prove Theorem 1.2, we need the following result.

Proposition 5.2 *If $n \geq 9$ and if $n \neq 2^{s-1}+1$, then the sequence $v_s, \dots, v_{h'-1}$ is a regular sequence in R.*

Proof Firstly, we deal with the case $10 \leq n \leq 16$. In this case, $s = 4$ and we have

$$v_4 = w_7 w_{10} + w_6 w_{11} + w_4 w_{13},\ \ v_5 = w_{13} w_{10}^2 + w_{11}^3 + w_7 w_{13}^2,\ \ v_6 = w_{13}^5,$$

where $w_i = 0$ for $n < i \leq 16$. We consider the degree reverse lexicographic order such that

$$w_4 > w_6 > w_7 > w_8 > w_{10} > w_{11} > w_{12} > w_{13} > w_{14} > w_{15} > w_{16}.$$

For $n = 13, 14, 15, 16$, we have $t = 1$ and $h' = 7$ and the leading terms of v_4, v_5, v_6 are $w_7 w_{10}$, w_{11}^3, w_{13}^5, respectively. So, by Lemma 3.4, we have the desired result. For $n = 11, 12$, we have $t = 2$, $h' = 6$ and the leading terms of $v_4 = w_7 w_{10} + w_6 w_{11}$, $v_5 = w_{11}^3$ are $w_7 w_{10}$, w_{11}^3, respectively. So, by Lemma 3.4, we have the desired result. For $n = 10$, we have $t = 3$, $h' = 5$ and it is clear that the sequence $v_4 = w_7 w_{10}$ is a regular sequence.

Next, we deal with the case $s \geq 5$, $n \neq 2^{s-1}+1$. In order to use Lemma 3.4, we need to define the term order on the set of monomials in R as follows: Suppose that

$$x = w_{\sigma(0)}^{i_0} \cdots w_{\sigma(n-s-2)}^{i_{n-s-2}}, \quad y = w_{\sigma(0)}^{j_0} \cdots w_{\sigma(n-s-2)}^{j_{n-s-2}}.$$

We define the weight of x by

$$w(x) = \sum_{\ell=0}^{s-t+\varepsilon-1} i_\ell.$$

We say $x > y$ if

(1) $w(x) > w(y)$ or

(2) $w(x) = w(y)$ and there is an integer k such that $i_\ell = j_\ell$ for $\ell < k$ and $i_k > j_k$.

Since $2^k(2^s - \sigma(\ell)) + 1 > n$ for $\ell < k$, we have $w_{2^k(2^s-\sigma(\ell))+1} = 0$ for $\ell < k$. So, we obtain

$$v_{s+k} \equiv \sum_{\ell=k}^{s-t+\varepsilon-1} w_{\sigma(\ell)}^{2^k} w_{2^k(2^s-\sigma(\ell))+1}$$

modulo terms with weight less than 2^k. The leading terms of $v_s, \dots, v_{2s-t-1}$ are $w_{\sigma(0)} w_{\tau(0)}, \dots, w_{\sigma(s-t-1)}^{2^{s-t-1}} w_{\tau(s-t-1)}$ and the leading term of v_{2s-t} is $w_m^{2^{s-t}} w_{m'}$ if $\varepsilon = 1$. By Proposition 2.4, we have

$$\gcd(w_{\sigma(k)}^{2^k} w_{\tau(k)}, w_{\sigma(k')}^{2k'} w_{\tau(k')}) = 1$$

for $k \neq k' \in C_0$ and, by Proposition 2.5, we have

$$\gcd(w_{\sigma(k)}^{2^k} w_{\tau(k)}, w_m^{2^{s-t}} w_{m'}) = 1$$

for $k \in C_0$ when $\varepsilon = 1$. Therefore, by Lemma 3.4, we have that the sequence $v_s, \dots, v_{2s-t+\varepsilon-1}$ is a regular sequence. □

By abuse of notation, we identify the above

$$R = H^*(BSO(n); \mathbb{F}_2)/(v_0, \dots, v_{s-1}) = \mathrm{Cotor}_\Delta(\mathbb{F}_2, \mathbb{F}_2)$$

with the image of

$$B\pi^*\colon H^*(BSO(n); \mathbb{F}_2) \to H^*(B\mathrm{Spin}(n); \mathbb{F}_2)$$

and with $E_2^{*,0}$ in the change-of-rings spectral sequence. Thus, we have

$$E_2^{*,*} = R \otimes \mathbb{F}_2[\zeta],$$

where $\zeta \in E_2^{0,1}$ is the element represented by $[y_{2^s-1}]$. Now, we complete the proof of Theorem 1.2.

Proof of Theroem 1.2 Let us consider the cobar resolution

$$\bar{A} \xrightarrow{d} \bar{A} \otimes \bar{A} \xrightarrow{d} \bar{A} \otimes \bar{A} \otimes \bar{A} \to \cdots.$$

It is clear that

$$d(y_{2^s-1}) = \sum_{i+j=2^{s-1}} x_{2i} \otimes x_{2j-1}$$

and so the element

$$v_s = \sum_{i+j=2^{s-1}} w_{2i+1} w_{2j}$$

is zero in $\mathrm{Cotor}_A(\mathbb{F}_2, \mathbb{F}_2)$. Therefore, $v_s \in E_2^{2,0}$ is equal to $d_2(\zeta)$. Hence, by Theorem 4.2, we have that both $\mathrm{Sq}^{2^{k-1}} \cdots \mathrm{Sq}^1 \zeta \in E_2^{0,2^k}$ and $\mathrm{Sq}^{2^{k-1}} \cdots \mathrm{Sq}^1 d_2\zeta \in E_2^{2^k+1,0}$ survive to the E_{2^k+1}–term and

$$d_{2^k+1} \mathrm{Sq}^{2^{k-1}} \cdots \mathrm{Sq}^1 \zeta = \mathrm{Sq}^{2^{k-1}} \cdots \mathrm{Sq}^1 d_2\zeta \in E_{2^k+1}^{0,2^k}.$$

For $k = 1, \dots, h' - s - 1$, we have, by the unstable condition,

$$\mathrm{Sq}^{2^{k-1}} \cdots \mathrm{Sq}^1 \zeta = \zeta^{2^k}$$

and, by definition,

$$\mathrm{Sq}^{2^{k-1}} \cdots \mathrm{Sq}^1 d_2\zeta = v_{s+k}.$$

Since $v_s, \dots, v_{h'-1}$ is a regular sequence in R and since $E_2 = R \otimes \mathbb{F}_2[\zeta]$, we have, for $k = 1, \dots, h'-s-1$,

$$E_{2^k+1} = \cdots = E_{2^{k-1}+2} = R/(v_s, \dots, v_{s+k-1}) \otimes \mathbb{F}_2[\zeta^{2^k}].$$

Moreover, we have

$$E_\infty = E_{2^{h'-s-1}+2} = R/(v_s, \dots, v_{h'-1}) \otimes \mathbb{F}_2[\zeta^{2^{h'-s}}].$$

It is clear that an algebra homomorphism

$$\varphi\colon H^*(BSO(n); \mathbb{F}_2) \otimes \mathbb{F}_2[z'] \to H^*(B\mathrm{Spin}(n); \mathbb{F}_2)$$

defined by $\varphi(w_k \otimes 1) = B\pi^*(w_k)$ and $\varphi(1 \otimes z') = z''$, where z'' represents $\zeta^{2^{h'-s}} \in E_\infty^{0,2^{h'-s}}$, induces an isomorphism

$$R \otimes \mathbb{F}_2[z']/(v_s \otimes 1, \dots, v_{h'-1} \otimes 1) \to \mathrm{Cotor}_A(\mathbb{F}_2, \mathbb{F}_2).$$

So there is no extension problem and it completes the proof of Theorem 1.2. □

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