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The homology of spaces of polynomials with roots of bounded multiplicity

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Let $P^l_{k,n}$ be the space consisting of monic complex polynomials $f(z)$ of degree k and such that the number of n–fold roots of $f(z)$ is at most l. In this paper, we determine the integral homology groups of $P^l_{k,n}$.

55P35; 55R20, 58D15

1 Introduction

In [1], Arnol'd studied a space $P^l_{k,n}$ consisting of monic complex polynomials $f(z)$ of degree k and such that the number of n–fold roots of $f(z)$ is at most l. In particular, he calculated the first five nontrivial integral homology groups of $P^l_{k,n}$. The purpose of this paper is, using another approach, to determine $H_*(P^l_{k,n};\mathbb{Z})$ completely.

Let $C_k(\mathbb{C})$ denote the configuration space of unordered k–tuples of distinct points in $\mathbb{C}$. The study of the topology of $C_k(\mathbb{C})$ originated in [1]. For that purpose, Arnol'd performed an induction for $P^l_{k,n}$ with making k larger and l smaller while n being fixed. Then one obtains information on $P^l_{k,n}$ for all k, n and l. In particular, setting $n = 2$ and $l = 0$, we obtain information on $C_k(\mathbb{C})$. (Strictly speaking, Arnol'd considered the complement $S^{2k} - P^l_{k,n}$ instead of $P^l_{k,n}$.)

Using this induction, Arnol'd calculated the first five nontrivial integral homology groups of $P^l_{k,n}$. (See Theorem 3.1 for $n = 2$.) But because of problems involved in the induction, it seems difficult to calculate further homology groups. Then we naturally encounter the following problem: how to determine $H_*(P^l_{k,n};\mathbb{Z})$.

The purpose of this paper is to give an answer to the problem. Our main results will be stated in Section 3. (See Theorems 3.3 and 3.7.) Here we summarize how the groups $H_*(P^l_{k,n};\mathbb{Z})$ are determined.

Theorem 1.1 *Let $J^l(S^{2n-2})$ be the l-th stage of the James construction which builds ΩS^{2n-1}, and let $W^l(S^{2n-2})$ be the homotopy theoretic fiber of the inclusion $J^l(S^{2n-2}) \hookrightarrow \Omega S^{2n-1}$. Then:*

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(i) (a) *The homomorphism*

$$H_*(P^l_{k,n};\mathbb{Z}) \to H_*(P^l_{k+1,n};\mathbb{Z})$$

which is induced from the natural inclusion $P^l_{k,n} \hookrightarrow P^l_{k+1,n}$ *is a monomorphism onto a direct summand.*

(b) *There is a stable homotopy equivalence*

$$P^l_{\infty,n} \underset{s}{\simeq} W^l(S^{2n-2}).$$

(ii) *The homology groups* $H_*(W^l(S^{2n-2});\mathbb{Z})$ *are determined. In particular, all higher* p*–torsions are determined for all primes* p*.*

(iii) *For each* $x \in H_*(W^l(S^{2n-2});\mathbb{Z})$*, the least* k *such that* x *is contained in* $H_*(P^l_{k,n};\mathbb{Z})$ *is determined.*

Remark 1.2 For $l = 0$, Theorem 1.1 is already well-known. First, about Theorem 1.1 (i) (a), the inclusion $P^0_{k,n} \hookrightarrow P^0_{k+1,n}$, which is called a stabilization map, was constructed by Guest, Kozlowski and Yamaguchi in [7; 8]. Moreover, the induced homomorphism $H_*(P^0_{k,n};\mathbb{Z}) \to H_*(P^0_{k+1,n};\mathbb{Z})$ was studied in [8]. Second, about Theorem 1.1 (i) (b) and (iii) for $l = 0$, Guest, Kozlowski and Yamaguchi [7] and independently Kallel [9] established a more precise result. (See Theorem 2.2.)

Finally, we note that the homology groups $H_*(C_k(\mathbb{C});\mathbb{Z}/p)$ were determined later, using other approaches, by Fuks for $p = 2$ [6] and by F Cohen for odd primes p [3]. F Cohen also determined Steenrod operations.

This paper is organized as follows. In Section 2 we summarize previous results on $P^l_{k,n}$ which imply Theorem 1.1 (i). In Section 3 we first recall Arnol'd's results in Theorem 3.1. Our main result for $n = 2$ is Theorem 3.3, which generalizes Theorem 3.1. Theorem 3.7 is a generalization of Theorem 3.3 for general n.

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2 Previous results

As in Section 1, we set

$$P^l_{k,n} = \{f(z) : f(z) \text{ is a monic complex polynomial of degree } k \\ \text{and such that the number of } n\text{–fold roots of } f(z) \text{ is at most } l\}.$$

Since $P^l_{k,n} = \mathbb{C}^k$ for $k < n(l+1)$, we can assume that $k \geq n(l+1)$.

On the other hand, let $\mathrm{Rat}_k(\mathbb{C}P^{n-1})$ denote the space of based holomorphic maps of degree k from the Riemannian sphere S^2 to the complex projective space $\mathbb{C}P^{n-1}$. The basepoint condition we assume is that $f(\infty) = [1, \dots, 1]$. Such holomorphic maps are given by rational functions:

$$\mathrm{Rat}_k(\mathbb{C}P^{n-1}) = \{(p_1(z), \dots, p_n(z)) : \text{each } p_i(z) \text{ is a monic degree–}k \text{ polynomial} \\ \text{and such that there are no roots common to all } p_i(z)\}.$$

The study of the topology of $\mathrm{Rat}_k(\mathbb{C}P^{n-1})$ originated in Segal's paper [13], where it is proved that the natural inclusion $\mathrm{Rat}_k(\mathbb{C}P^{n-1}) \hookrightarrow \Omega^2_k \mathbb{C}P^{n-1} \simeq \Omega^2 S^{2n-1}$ is a homotopy equivalence up to dimension $k(2n-3)$.

Later, F Cohen et al determined the stable homotopy type of $\mathrm{Rat}_k(\mathbb{C}P^{n-1})$ as follows:

Theorem 2.1 [4; 5] *Let*

$$\Omega^2 S^{2n-1} \underset{s}{\simeq} \bigvee_{1 \le j} D_j(S^{2n-3})$$

be Snaith's stable splitting. Then there is a stable homotopy equivalence

$$\mathrm{Rat}_k(\mathbb{C}P^{n-1}) \underset{s}{\simeq} \bigvee_{j=1}^{k} D_j(S^{2n-3}).$$

In particular, combining Theorem 2.1 for $n = 2$ with the stable splitting of $C_k(\mathbb{C})$ (Brown and Peterson [2]), we have

$$C_k(\mathbb{C}) \underset{s}{\simeq} \mathrm{Rat}_{[\frac{k}{2}]}(\mathbb{C}P^1). \tag{2–1}$$

Guest, Kozlowski and Yamaguchi and independently Kallel generalized (2–1) as follows:

Theorem 2.2 [7; 9] *For $n \ge 3$, there is a homotopy equivalence*

$$P^0_{k,n} \simeq \mathrm{Rat}_{[\frac{k}{n}]}(\mathbb{C}P^{n-1}).$$

Remarks 2.3 (i) It is proved by Guest, Kozlowski and Yamaguchi in [8] that the (modified) jet map $P^0_{k,n} \to \mathrm{Rat}_k(\mathbb{C}P^{n-1})$ defined by

$$f(z) \mapsto (f(z), f(z) + f'(z), \dots, f(z) + f^{(n-1)}(z))$$

is a homotopy equivalence up to dimension $(2n-3)\left[\frac{k}{n}\right]$ if $n \geq 3$, and a homology equivalence up to dimension $(2n-3)\left[\frac{k}{n}\right]$ if $n = 2$.

(ii) Kallel [10] generalized $P^0_{k,n}$ as follows: let $F^d(\mathbb{R}^m, k)$ be the space of ordered k–tuples of vectors in $\mathbb{R}^m$ so that no vector occurs more than d times in the k–tuple. We set $C^d(\mathbb{R}^m, k) = F^d(\mathbb{R}^m, k)/\Sigma_k$. Then $C^1(\mathbb{R}^m, k)$ is the usual configuration space and $C^{n-1}(\mathbb{R}^2, k) \cong P^0_{k,n}$. Recall that using $F^1(\mathbb{R}^m, k)$, May, Milgram and Segal constructed a combinatorial model for $\Omega^m \Sigma^m X$, where X is a connected CW–complex. Using $F^d(\mathbb{R}^m, k)$, Kallel [10] generalized the model for general d. He also considered the case when X is disconnected. In particular, setting $m = 2$, $d = n-1$ and $X = S^0$ in his result, he recovered the homotopy and homology equivalences $P^0_{\infty,n} \simeq \Omega^2 S^{2n-1}$ for $n \geq 3$ and $n = 2$, respectively. (See Theorem 2.2 and (2–1) for these equivalences.)

(iii) For $n \geq 2$, a stable homotopy equivalence

$$P^0_{k,n} \underset{s}{\simeq} \mathrm{Rat}_{\left[\frac{k}{n}\right]}(\mathbb{C}P^{n-1}) \tag{2–2}$$

was proved by Vassiliev in [14]. Theorem 2.2 is a stronger version of (2–2) for $n \geq 3$.

We consider generalizations of Theorems 2.1 and 2.2. We set

$$\begin{aligned} X^l_{k,n} = \{(p_1(z), \ldots, p_n(z)) :\ & \text{each } p_i(z) \text{ is a monic degree–}k \text{ polynomial} \\ & \text{and such that there are at most } l \text{ roots common to all } p_i(z)\}. \end{aligned}$$

Theorem 2.4 (Kamiyama [11]) *Let $J^l(S^{2n-2})$ denote the l-th stage of the James construction which builds ΩS^{2n-1}, and let $W^l(S^{2n-2})$ be the homotopy theoretic fiber of the inclusion $J^l(S^{2n-2}) \hookrightarrow \Omega S^{2n-1}$. Let*

$$W^l(S^{2n-2}) \underset{s}{\simeq} \bigvee_{1 \leq j} D_j \xi^l(S^{2n-2})$$

be a generalization of Snaith's stable splitting. (See Wong [15] and Kamiyama [11].) Then, there is a stable homotopy equivalence

$$X^l_{k,n} \underset{s}{\simeq} \bigvee_{j=1}^{k} D_j \xi^l(S^{2n-2}).$$

Theorem 2.5 (Kamiyama [12]) *For $l \geq 1$ and $n \geq 2$, there is a homotopy equivalence*

$$P^l_{k,n} \simeq X^l_{\left[\frac{k}{n}\right],n}.$$

Note that Theorem 1.1 (i) are consequences of Theorems 2.4 and 2.5.

3 The main results

In order to simplify notation, we first consider the case $n = 2$, which is of particular interest to us. Since $P^l_{k,2} = \mathbb{C}^k$ for $k < 2l + 2$, we assume that $k \geq 2l + 2$.

Arnol'd proved the following:

Theorem 3.1 [1]

(i) *For* $1 \leq j \leq 2l$, *we have* $H_j(P^l_{k,2}; \mathbb{Z}) = 0$.

(ii) *For* $2l + 1 \leq j \leq 2l + 5$, *the groups* $H_j(P^l_{k,2}; \mathbb{Z})$ *are cyclic and the orders are given by the following table.*

Table 1: The orders of the groups $H_j(P^l_{k,2}; \mathbb{Z})$ $(2l + 1 \leq j \leq 2l + 5)$

$k \setminus j$	$2l+1$	$2l+2$	$2l+3$	$2l+4$	$2l+5$
$2l+2, 2l+3$	∞	0	0	0	0
$2l+4, 2l+5$	∞	$l+2$	0	0	0
$2l+6, 2l+7$	∞	$l+2$	$2/(l+1)$	$(l+3)/2$	0
$2l+8, 2l+9$	∞	$l+2$	$2/(l+1)$	$((l+3)/2)(2/(l+1))$	$3/(l+1)$
$2l+10, 2l+11$	∞	$l+2$	$2/(l+1)$	$((l+3)/2)(2/(l+1))$	$6/(l+1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	∞	$l+2$	$2/(l+1)$	$((l+3)/2)(2/(l+1))$	$6/(l+1)$

Here we introduce the notation

$$a/b = \frac{a}{\gcd(a, b)},$$

where $\gcd(a, b)$ *is the greatest common divisor of the integers* a *and* b.

In order to state our main results, we prepare some notation.

Definition 3.2 Let p be a prime.

(i) We write l as $l = p^m q$ such that

$$q = \sum_{\nu=0}^{N} a_\nu p^\nu,$$

where $0 \leq a_\nu \leq p - 1$ and $a_N \neq 0$, $a_0 \neq 0$.

(ii) For q in (i), we consider terms of the form

$$(p-1)\sum_{\nu=j}^{i} p^{\nu}.$$

We take such terms as large as possible, whence we have $a_\nu = p-1$ $(j \le \nu \le i)$ and $a_{i+1} \neq p-1$, $a_{j-1} \neq p-1$. Assume that all possible pairs (i, j) for q are given by

$$(i_\alpha, j_\alpha), \quad 1 \le \alpha \le r,$$

where we arrange them as $j_\alpha \ge i_{\alpha+1} + 2$.

(iii) For $1 \le \alpha \le r$, we set

$$u_\alpha = \sum_{\nu=i_\alpha+1}^{N} a_\nu p^\nu.$$

(iv) We set

$$d_\alpha = 2\left(p^m u_\alpha + p^{m+i_\alpha+1} - 1\right).$$

(v) We set

$$\mu_\alpha = i_\alpha - j_\alpha + 2.$$

Our main result for $n = 2$ is then:

Theorem 3.3 *Let p be a prime. Then all higher p–torsions in $H_*(W^l(S^2);\mathbb{Z})$ are given as follows.*

(i) *If $m \ge 1$, then*
  (a) *For $1 \le \alpha \le r$, $H_{d_\alpha}(W^l(S^2);\mathbb{Z})$ contains $\mathbb{Z}/p^{\mu_\alpha}$ as a direct summand.*
  (b) *For each α, the least k such that the higher p–torsion in (a) appears as a direct summand in $H_{d_\alpha}(P^l_{k,2};\mathbb{Z})$ is*

$$k = d_\alpha + 2.$$

(ii) *If $m = 0$, then we omit the case $\alpha = r$ from (i).*

Remark 3.4 We can determine all p–torsions of order exactly p in $H_*(P^l_{k,2};\mathbb{Z})$ from the following facts: all p–torsions in $H_*(W^l(S^2);\mathbb{Z})$ of order exactly p are determined from the Bockstein operation on $H_*(W^l(S^2);\mathbb{Z}/p)$, and $H_*(P^l_{k,2};\mathbb{Z}/p)$ is a subspace of $H_*(W^l(S^2);\mathbb{Z}/p)$ (see Proposition 3.6). Hence using Theorem 3.3, we know the groups $H_*(P^l_{k,2};\mathbb{Z})$ completely.

Example 3.5 We consider the case

$$l = p^m(p-1)\left(\sum_{\nu=j_1}^{i_1} p^\nu + \sum_{\nu=j_2}^{i_2} p^\nu\right).$$

(i) If $m \geq 1$, then there are 2 higher p–torsions:

(a) For $k \geq 2p^{m+i_1+1}$,

$$H_{2(p^{m+i_1+1}-1)}(P^l_{k,2};\mathbb{Z})$$

contains $\mathbb{Z}/p^{i_1-j_1+2}$ as a direct summand.

(b) For $k \geq 2p^m(p^{i_1+1}-p^{j_1}+p^{i_2+1})$,

$$H_{2p^m(p^{i_1+1}-p^{j_1}+p^{i_2+1})-2}(P^l_{k,2};\mathbb{Z})$$

contains $\mathbb{Z}/p^{i_2-j_2+2}$ as a direct summand.

(ii) If $m = 0$, then we omit the case (b) from (i).

Proof of Theorem 3.3 (i) In order to prove (a), we determine $H_*(W^l(S^2);\mathbb{Z})$ by the following 2 steps.

(1) Using the structure of $H_*(W^l(S^2);\mathbb{Z}/p)$, we determine the homological dimensions which have higher p–torsions.

(2) Using the cohomology Serre spectral sequence for a fibration with coefficients in $\mathbb{Z}_{(p)}$, we determine the higher p–torsions.

(1) The structure of $H_*(W^l(S^2);\mathbb{Z}/p)$ was determined in [11] from the mod p Serre spectral sequence for the fibration

$$\Omega^2 S^3 \to W^l(S^2) \to J^l(S^2).$$

Let $x \in H_2(J^l(S^2);\mathbb{Z}/p)$ and $\iota \in H_1(\Omega^2 S^3;\mathbb{Z}/p)$ be the generators and we write $Q_1^t = Q_1 \cdots Q_1$ ($=t$-times Q_1). In $H_*(W^l(S^2);\mathbb{Z}/p)$, the cases that the Bockstein operation is not clear are given as follows:

$$x^{p^m u_\alpha} \otimes Q_1^{m+i_\alpha+1}(\iota) \to x^{p^m v_\alpha} \otimes \beta Q_1^{m+j_\alpha}(\iota), \quad 1 \leq \alpha \leq r, \tag{3–1}$$

where we set

$$v_\alpha = \sum_{\nu=j_\alpha}^{N} a_\nu p^\nu.$$

(Note that by Definition 3.2, we have $v_\alpha = u_\alpha + \sum_{\nu=j_\alpha}^{i_\alpha}(p-1)p^\nu$. Note also that $v_\alpha = u_{\alpha+1}$ for $p=2$.) Since

$$\deg(x^{p^m v_\alpha} \otimes \beta Q_1^{m+j_\alpha}(\iota)) = d_\alpha,$$

there is a higher p–torsion in $H_{d_\alpha}(W^l(S^2);\mathbb{Z})$. This completes (1).

(2) Consider the following homotopy commutative diagram:

$$\begin{array}{ccccc}
W^l(S^2) & \longrightarrow & \tilde{J}^l(S^2) & \longrightarrow & \Omega S^3\langle 3\rangle \\
\| & & \downarrow & & \downarrow \\
W^l(S^2) & \longrightarrow & J^l(S^2) & \longrightarrow & \Omega S^3 \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & K(\mathbb{Z},2) & \longrightarrow & K(\mathbb{Z},2)
\end{array}$$

where $\tilde{J}^l(S^2)$ and $\Omega S^3\langle 3\rangle$ are the homotopy theoretic fibers of the second and third columns respectively. Then the first row is a fibration and we consider the cohomology Serre spectral sequence for the fibration with coefficients in $\mathbb{Z}_{(p)}$. Note that $H^{d_\alpha+1}(W^l(S^2);\mathbb{Z}_{(p)})$ is determined if we calculate the cokernels of the differentials

$$d : E^{2ps, d_\alpha - 2ps+1} \to E^{d_\alpha+2,0} \tag{3–2}$$

for all possible $s \geq 1$. Since $H_q(W^l(S^2);\mathbb{Z}_{(p)}) = 0$ for $q \leq 2l$, we have the following restriction on s: $d_\alpha - 2ps + 1 \geq 2l+1$, that is,

$$p^{m+j_\alpha} - 1 - \sum_{\nu=0}^{m+j_\alpha-2} b_\nu p^\nu \geq ps, \tag{3–3}$$

where $0 \leq b_\nu \leq p-1$.

Let $y_{2ps} \in H^{2ps}(\Omega S^3\langle 3\rangle;\mathbb{Z}_{(p)})$ be a generator. Then a generator of $E^{2ps, d_\alpha-2ps+1}$ is mapped by d in (3–2) to $y_{2ps} y_{d_\alpha-2ps+2}$. It is easy to see that

$$y_{2ps} y_{d_\alpha-2ps+2} = \binom{p^m u_\alpha + p^{m+i_\alpha+1}}{ps} y_{d_\alpha+2}. \tag{3–4}$$

Consider the p–power component of the prime decomposition of the binomial coefficient in (3–4). Using (3–3), we see that the component is smallest when $ps = tp^{m+j_\alpha-1}$ $(1 \leq t \leq p-1)$ such that the p–power is $p^{i_\alpha - j_\alpha+2}$. Hence

$$H^{d_\alpha+1}(W^l(S^2);\mathbb{Z}_{(p)}) = \mathbb{Z}/p^{\mu_\alpha}$$

and Theorem 3.3 (i) (a) follows.

For Theorem 3.3 (i) (b), we have the following:

Proposition 3.6 *In $H_*(P^l_{k,2};\mathbb{Z}/p)$, we define the weights of the homology classes x and ι (see (3–1)) to be 2. Then $H_*(P^l_{k,2};\mathbb{Z}/p)$ is isomorphic to the subspace of $H^*(W^l(S^2);\mathbb{Z}/p)$ spanned by monomials of weight $\leq k$.*

Proof The proposition is an easy consequence of Theorems 2.4 and 2.5. Note that it is reasonable to define the weights of x and ι to be 2 by the following reason: we have $H_*(W^l(S^2);\mathbb{Q}) = \bigwedge(x^l \otimes \iota)$. Since $P^l_{2l+2,2} \simeq S^{2l+1}$, the weight of $x^l \otimes \iota$ must be $2l+2$. □

Since the weight of $x^{p^m v_\alpha} \otimes \beta Q_1^{m+j_\alpha}(\iota)$ in (3–1) is $d_\alpha + 2$, Theorem 3.3 (i) (b) follows.

(ii) For $m = 0$ and $\alpha = r$, the left-hand side of (3–1) is the mod p reduction of the generator of $H_{2l+1}(W^l(S^2);\mathbb{Z}) = \mathbb{Z}$ and the right-hand side is 0. Hence we must omit this case from (i). This completes the proof of Theorem 3.3. □

Finally we generalize Theorem 3.3 for general n.

Theorem 3.7 *We keep the notation of Definition 3.2 except that we generalize d_α in (iv) as*

$$d_{n,\alpha} = 2(n-1)p^m(u_\alpha + p^{i_\alpha+1}) - 2.$$

Then:

(1) *Theorem 3.3 (i) (a) is generalized to the assertion that $H_{d_{n,\alpha}}(W^l(S^{2n-2});\mathbb{Z})$ contains $\mathbb{Z}/p^{\mu_\alpha}$ as a direct summand.*

(2) *About Theorem 3.3 (i) (b), the least k such that the higher p–torsion in the above (1) appears as a direct summand in $H_{d_{n,\alpha}}(P^l_{k,n};\mathbb{Z})$ is*

$$k = \frac{n(d_{n,\alpha}+2)}{2(n-1)}.$$

(3) *Theorem 3.3 (ii) holds under these modifications.*

Proof About x and ι in (3–1), we generalize that $x \in H_{2n-2}(J^l(S^{2n-2});\mathbb{Z}/p)$ and $\iota \in H_{2n-3}(\Omega^2 S^{2n-1};\mathbb{Z}/p)$ such that the weights of these elements are n. Theorem 3.7 is clear from this. □

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