

The homology of spaces of polynomials with roots of bounded multiplicity

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Let $P_{k,n}^l$ be the space consisting of monic complex polynomials $f(z)$ of degree k and such that the number of n -fold roots of $f(z)$ is at most l . In this paper, we determine the integral homology groups of $P_{k,n}^l$.

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1 Introduction

In [1], Arnol'd studied a space $P_{k,n}^l$ consisting of monic complex polynomials $f(z)$ of degree k and such that the number of n -fold roots of $f(z)$ is at most l . In particular, he calculated the first five nontrivial integral homology groups of $P_{k,n}^l$. The purpose of this paper is, using another approach, to determine $H_*(P_{k,n}^l; \mathbb{Z})$ completely.

Let $C_k(\mathbb{C})$ denote the configuration space of unordered k -tuples of distinct points in \mathbb{C} . The study of the topology of $C_k(\mathbb{C})$ originated in [1]. For that purpose, Arnol'd performed an induction for $P_{k,n}^l$ with making k larger and l smaller while n being fixed. Then one obtains information on $P_{k,n}^l$ for all k, n and l . In particular, setting $n = 2$ and $l = 0$, we obtain information on $C_k(\mathbb{C})$. (Strictly speaking, Arnol'd considered the complement $S^{2k} - P_{k,n}^l$ instead of $P_{k,n}^l$.)

Using this induction, Arnol'd calculated the first five nontrivial integral homology groups of $P_{k,n}^l$. (See [Theorem 3.1](#) for $n = 2$.) But because of problems involved in the induction, it seems difficult to calculate further homology groups. Then we naturally encounter the following problem: how to determine $H_*(P_{k,n}^l; \mathbb{Z})$.

The purpose of this paper is to give an answer to the problem. Our main results will be stated in [Section 3](#). (See [Theorems 3.3](#) and [3.7](#).) Here we summarize how the groups $H_*(P_{k,n}^l; \mathbb{Z})$ are determined.

Theorem 1.1 *Let $J^l(S^{2n-2})$ be the l -th stage of the James construction which builds ΩS^{2n-1} , and let $W^l(S^{2n-2})$ be the homotopy theoretic fiber of the inclusion $J^l(S^{2n-2}) \hookrightarrow \Omega S^{2n-1}$. Then:*

(i) (a) *The homomorphism*

$$H_*(P_{k,n}^l; \mathbb{Z}) \rightarrow H_*(P_{k+1,n}^l; \mathbb{Z})$$

which is induced from the natural inclusion $P_{k,n}^l \hookrightarrow P_{k+1,n}^l$ is a monomorphism onto a direct summand.

(b) *There is a stable homotopy equivalence*

$$P_{\infty,n}^l \underset{s}{\simeq} W^l(S^{2n-2}).$$

(ii) *The homology groups $H_*(W^l(S^{2n-2}); \mathbb{Z})$ are determined. In particular, all higher p -torsions are determined for all primes p .*

(iii) *For each $x \in H_*(W^l(S^{2n-2}); \mathbb{Z})$, the least k such that x is contained in $H_*(P_{k,n}^l; \mathbb{Z})$ is determined.*

Remark 1.2 For $l = 0$, [Theorem 1.1](#) is already well-known. First, about [Theorem 1.1](#) (i) (a), the inclusion $P_{k,n}^0 \hookrightarrow P_{k+1,n}^0$, which is called a stabilization map, was constructed by Guest, Kozłowski and Yamaguchi in [7; 8]. Moreover, the induced homomorphism $H_*(P_{k,n}^0; \mathbb{Z}) \rightarrow H_*(P_{k+1,n}^0; \mathbb{Z})$ was studied in [8]. Second, about [Theorem 1.1](#) (i) (b) and (iii) for $l = 0$, Guest, Kozłowski and Yamaguchi [7] and independently Kallel [9] established a more precise result. (See [Theorem 2.2](#).)

Finally, we note that the homology groups $H_*(C_k(\mathbb{C}); \mathbb{Z}/p)$ were determined later, using other approaches, by Fuks for $p = 2$ [6] and by F Cohen for odd primes p [3]. F Cohen also determined Steenrod operations.

This paper is organized as follows. In [Section 2](#) we summarize previous results on $P_{k,n}^l$ which imply [Theorem 1.1](#) (i). In [Section 3](#) we first recall Arnol'd's results in [Theorem 3.1](#). Our main result for $n = 2$ is [Theorem 3.3](#), which generalizes [Theorem 3.1](#). [Theorem 3.7](#) is a generalization of [Theorem 3.3](#) for general n .

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2 Previous results

As in [Section 1](#), we set

$$P_{k,n}^l = \{f(z) : f(z) \text{ is a monic complex polynomial of degree } k \\ \text{and such that the number of } n\text{-fold roots of } f(z) \text{ is at most } l\}.$$

Since $P_{k,n}^l = \mathbb{C}^k$ for $k < n(l+1)$, we can assume that $k \geq n(l+1)$.

On the other hand, let $\text{Rat}_k(\mathbb{C}P^{n-1})$ denote the space of based holomorphic maps of degree k from the Riemannian sphere S^2 to the complex projective space $\mathbb{C}P^{n-1}$. The basepoint condition we assume is that $f(\infty) = [1, \dots, 1]$. Such holomorphic maps are given by rational functions:

$$\text{Rat}_k(\mathbb{C}P^{n-1}) = \{(p_1(z), \dots, p_n(z)) : \text{each } p_i(z) \text{ is a monic degree-}k \text{ polynomial and such that there are no roots common to all } p_i(z)\}.$$

The study of the topology of $\text{Rat}_k(\mathbb{C}P^{n-1})$ originated in Segal's paper [13], where it is proved that the natural inclusion $\text{Rat}_k(\mathbb{C}P^{n-1}) \hookrightarrow \Omega_k^2 \mathbb{C}P^{n-1} \simeq \Omega^2 S^{2n-1}$ is a homotopy equivalence up to dimension $k(2n-3)$.

Later, F Cohen et al determined the stable homotopy type of $\text{Rat}_k(\mathbb{C}P^{n-1})$ as follows:

Theorem 2.1 [4; 5] *Let*

$$\Omega^2 S^{2n-1} \underset{s}{\simeq} \bigvee_{1 \leq j} D_j(S^{2n-3})$$

be Snaith's stable splitting. Then there is a stable homotopy equivalence

$$\text{Rat}_k(\mathbb{C}P^{n-1}) \underset{s}{\simeq} \bigvee_{j=1}^k D_j(S^{2n-3}).$$

In particular, combining **Theorem 2.1** for $n = 2$ with the stable splitting of $C_k(\mathbb{C})$ (Brown and Peterson [2]), we have

$$(2-1) \quad C_k(\mathbb{C}) \underset{s}{\simeq} \text{Rat}_{[\frac{k}{2}]}(\mathbb{C}P^1).$$

Guest, Kozłowski and Yamaguchi and independently Kallel generalized (2-1) as follows:

Theorem 2.2 [7; 9] *For $n \geq 3$, there is a homotopy equivalence*

$$P_{k,n}^0 \simeq \text{Rat}_{[\frac{k}{n}]}(\mathbb{C}P^{n-1}).$$

Remarks 2.3 (i) It is proved by Guest, Kozłowski and Yamaguchi in [8] that the (modified) jet map $P_{k,n}^0 \rightarrow \text{Rat}_k(\mathbb{C}P^{n-1})$ defined by

$$f(z) \mapsto (f(z), f(z) + f'(z), \dots, f(z) + f^{(n-1)}(z))$$

is a homotopy equivalence up to dimension $(2n - 3) \left\lfloor \frac{k}{n} \right\rfloor$ if $n \geq 3$, and a homology equivalence up to dimension $(2n - 3) \left\lfloor \frac{k}{n} \right\rfloor$ if $n = 2$.

(ii) Kallel [10] generalized $P_{k,n}^0$ as follows: let $F^d(\mathbb{R}^m, k)$ be the space of ordered k -tuples of vectors in \mathbb{R}^m so that no vector occurs more than d times in the k -tuple. We set $C^d(\mathbb{R}^m, k) = F^d(\mathbb{R}^m, k)/\Sigma_k$. Then $C^1(\mathbb{R}^m, k)$ is the usual configuration space and $C^{n-1}(\mathbb{R}^2, k) \cong P_{k,n}^0$. Recall that using $F^1(\mathbb{R}^m, k)$, May, Milgram and Segal constructed a combinatorial model for $\Omega^m \Sigma^m X$, where X is a connected CW-complex. Using $F^d(\mathbb{R}^m, k)$, Kallel [10] generalized the model for general d . He also considered the case when X is disconnected. In particular, setting $m = 2$, $d = n - 1$ and $X = S^0$ in his result, he recovered the homotopy and homology equivalences $P_{\infty,n}^0 \simeq \Omega^2 S^{2n-1}$ for $n \geq 3$ and $n = 2$, respectively. (See Theorem 2.2 and (2-1) for these equivalences.)

(iii) For $n \geq 2$, a stable homotopy equivalence

$$(2-2) \quad P_{k,n}^0 \underset{s}{\simeq} \text{Rat}_{\lfloor \frac{k}{n} \rfloor}(\mathbb{C}P^{n-1})$$

was proved by Vassiliev in [14]. Theorem 2.2 is a stronger version of (2-2) for $n \geq 3$.

We consider generalizations of Theorems 2.1 and 2.2. We set

$$X_{k,n}^l = \{(p_1(z), \dots, p_n(z)) : \text{each } p_i(z) \text{ is a monic degree-}k \text{ polynomial and such that there are at most } l \text{ roots common to all } p_i(z)\}.$$

Theorem 2.4 (Kamiyama [11]) *Let $J^l(S^{2n-2})$ denote the l -th stage of the James construction which builds ΩS^{2n-1} , and let $W^l(S^{2n-2})$ be the homotopy theoretic fiber of the inclusion $J^l(S^{2n-2}) \hookrightarrow \Omega S^{2n-1}$. Let*

$$W^l(S^{2n-2}) \underset{s}{\simeq} \bigvee_{1 \leq j} D_j \xi^l(S^{2n-2})$$

be a generalization of Snaith's stable splitting. (See Wong [15] and Kamiyama [11].) Then, there is a stable homotopy equivalence

$$X_{k,n}^l \underset{s}{\simeq} \bigvee_{j=1}^k D_j \xi^l(S^{2n-2}).$$

Theorem 2.5 (Kamiyama [12]) *For $l \geq 1$ and $n \geq 2$, there is a homotopy equivalence*

$$P_{k,n}^l \simeq X_{\lfloor \frac{k}{n} \rfloor, n}^l.$$

Note that [Theorem 1.1](#) (i) are consequences of [Theorems 2.4](#) and [2.5](#).

3 The main results

In order to simplify notation, we first consider the case $n = 2$, which is of particular interest to us. Since $P_{k,2}^l = \mathbb{C}^k$ for $k < 2l + 2$, we assume that $k \geq 2l + 2$.

Arnol'd proved the following:

Theorem 3.1 [1]

- (i) For $1 \leq j \leq 2l$, we have $H_j(P_{k,2}^l; \mathbb{Z}) = 0$.
- (ii) For $2l + 1 \leq j \leq 2l + 5$, the groups $H_j(P_{k,2}^l; \mathbb{Z})$ are cyclic and the orders are given by the following table.

Table 1: The orders of the groups $H_j(P_{k,2}^l; \mathbb{Z})$ ($2l + 1 \leq j \leq 2l + 5$)

$k \setminus j$	$2l + 1$	$2l + 2$	$2l + 3$	$2l + 4$	$2l + 5$
$2l + 2, 2l + 3$	∞	0	0	0	0
$2l + 4, 2l + 5$	∞	$l + 2$	0	0	0
$2l + 6, 2l + 7$	∞	$l + 2$	$2/(l + 1)$	$(l + 3)/2$	0
$2l + 8, 2l + 9$	∞	$l + 2$	$2/(l + 1)$	$((l + 3)/2)(2/(l + 1))$	$3/(l + 1)$
$2l + 10, 2l + 11$	∞	$l + 2$	$2/(l + 1)$	$((l + 3)/2)(2/(l + 1))$	$6/(l + 1)$
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
∞	∞	$l + 2$	$2/(l + 1)$	$((l + 3)/2)(2/(l + 1))$	$6/(l + 1)$

Here we introduce the notation

$$a/b = \frac{a}{\gcd(a, b)},$$

where $\gcd(a, b)$ is the greatest common divisor of the integers a and b .

In order to state our main results, we prepare some notation.

Definition 3.2 Let p be a prime.

- (i) We write l as $l = p^m q$ such that

$$q = \sum_{v=0}^N a_v p^v,$$

where $0 \leq a_v \leq p - 1$ and $a_N \neq 0, a_0 \neq 0$.

(ii) For q in (i), we consider terms of the form

$$(p-1) \sum_{v=j}^i p^v.$$

We take such terms as large as possible, whence we have $a_v = p-1$ ($j \leq v \leq i$) and $a_{i+1} \neq p-1$, $a_{j-1} \neq p-1$. Assume that all possible pairs (i, j) for q are given by

$$(i_\alpha, j_\alpha), \quad 1 \leq \alpha \leq r,$$

where we arrange them as $j_\alpha \geq i_{\alpha+1} + 2$.

(iii) For $1 \leq \alpha \leq r$, we set

$$u_\alpha = \sum_{v=i_\alpha+1}^N a_v p^v.$$

(iv) We set

$$d_\alpha = 2 \left(p^m u_\alpha + p^{m+i_\alpha+1} - 1 \right).$$

(v) We set

$$\mu_\alpha = i_\alpha - j_\alpha + 2.$$

Our main result for $n = 2$ is then:

Theorem 3.3 *Let p be a prime. Then all higher p -torsions in $H_*(W^l(S^2); \mathbb{Z})$ are given as follows.*

(i) *If $m \geq 1$, then*

(a) *For $1 \leq \alpha \leq r$, $H_{d_\alpha}(W^l(S^2); \mathbb{Z})$ contains $\mathbb{Z}/p^{\mu_\alpha}$ as a direct summand.*

(b) *For each α , the least k such that the higher p -torsion in (a) appears as a direct summand in $H_{d_\alpha}(P_{k,2}^l; \mathbb{Z})$ is*

$$k = d_\alpha + 2.$$

(ii) *If $m = 0$, then we omit the case $\alpha = r$ from (i).*

Remark 3.4 We can determine all p -torsions of order exactly p in $H_*(P_{k,2}^l; \mathbb{Z})$ from the following facts: all p -torsions in $H_*(W^l(S^2); \mathbb{Z})$ of order exactly p are determined from the Bockstein operation on $H_*(W^l(S^2); \mathbb{Z}/p)$, and $H_*(P_{k,2}^l; \mathbb{Z}/p)$ is a subspace of $H_*(W^l(S^2); \mathbb{Z}/p)$ (see [Proposition 3.6](#)). Hence using [Theorem 3.3](#), we know the groups $H_*(P_{k,2}^l; \mathbb{Z})$ completely.

Example 3.5 We consider the case

$$l = p^m(p-1) \left(\sum_{v=j_1}^{i_1} p^v + \sum_{v=j_2}^{i_2} p^v \right).$$

- (i) If $m \geq 1$, then there are 2 higher p -torsions:
 - (a) For $k \geq 2p^{m+i_1+1}$,

$$H_{2(p^{m+i_1+1}-1)}(P_{k,2}^l; \mathbb{Z})$$

contains $\mathbb{Z}/p^{i_1-j_1+2}$ as a direct summand.

- (b) For $k \geq 2p^m(p^{i_1+1} - p^{j_1} + p^{i_2+1})$,

$$H_{2p^m(p^{i_1+1}-p^{j_1}+p^{i_2+1})-2}(P_{k,2}^l; \mathbb{Z})$$

contains $\mathbb{Z}/p^{i_2-j_2+2}$ as a direct summand.

- (ii) If $m = 0$, then we omit the case (b) from (i).

Proof of Theorem 3.3 (i) In order to prove (a), we determine $H_*(W^l(S^2); \mathbb{Z})$ by the following 2 steps.

- (1) Using the structure of $H_*(W^l(S^2); \mathbb{Z}/p)$, we determine the homological dimensions which have higher p -torsions.
- (2) Using the cohomology Serre spectral sequence for a fibration with coefficients in \mathbb{Z}/p , we determine the higher p -torsions.

(1) The structure of $H_*(W^l(S^2); \mathbb{Z}/p)$ was determined in [11] from the mod p Serre spectral sequence for the fibration

$$\Omega^2 S^3 \rightarrow W^l(S^2) \rightarrow J^l(S^2).$$

Let $x \in H_2(J^l(S^2); \mathbb{Z}/p)$ and $\iota \in H_1(\Omega^2 S^3; \mathbb{Z}/p)$ be the generators and we write $Q_1^t = Q_1 \cdots Q_1$ ($= t$ -times Q_1). In $H_*(W^l(S^2); \mathbb{Z}/p)$, the cases that the Bockstein operation is not clear are given as follows:

$$(3-1) \quad x^{p^m u_\alpha} \otimes Q_1^{m+i_\alpha+1}(\iota) \rightarrow x^{p^m v_\alpha} \otimes \beta Q_1^{m+j_\alpha}(\iota), \quad 1 \leq \alpha \leq r,$$

where we set

$$v_\alpha = \sum_{v=j_\alpha}^N a_v p^v.$$

(Note that by Definition 3.2, we have $v_\alpha = u_\alpha + \sum_{v=j_\alpha}^{i_\alpha} (p-1)p^v$. Note also that $v_\alpha = u_{\alpha+1}$ for $p = 2$.) Since

$$\deg(x^{p^m}v_\alpha \otimes \beta Q_1^{m+j_\alpha}(t)) = d_\alpha,$$

there is a higher p -torsion in $H_{d_\alpha}(W^l(S^2); \mathbb{Z})$. This completes (1).

(2) Consider the following homotopy commutative diagram:

$$\begin{array}{ccccc} W^l(S^2) & \longrightarrow & \tilde{J}^l(S^2) & \longrightarrow & \Omega S^3\langle 3 \rangle \\ \parallel & & \downarrow & & \downarrow \\ W^l(S^2) & \longrightarrow & J^l(S^2) & \longrightarrow & \Omega S^3 \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & K(\mathbb{Z}, 2) & \longrightarrow & K(\mathbb{Z}, 2) \end{array}$$

where $\tilde{J}^l(S^2)$ and $\Omega S^3\langle 3 \rangle$ are the homotopy theoretic fibers of the second and third columns respectively. Then the first row is a fibration and we consider the cohomology Serre spectral sequence for the fibration with coefficients in $\mathbb{Z}_{(p)}$. Note that $H^{d_\alpha+1}(W^l(S^2); \mathbb{Z}_{(p)})$ is determined if we calculate the cokernels of the differentials

$$(3-2) \quad d : E^{2ps, d_\alpha-2ps+1} \rightarrow E^{d_\alpha+2, 0}$$

for all possible $s \geq 1$. Since $H_q(W^l(S^2); \mathbb{Z}_{(p)}) = 0$ for $q \leq 2l$, we have the following restriction on s : $d_\alpha - 2ps + 1 \geq 2l + 1$, that is,

$$(3-3) \quad p^{m+j_\alpha} - 1 - \sum_{v=0}^{m+j_\alpha-2} b_v p^v \geq ps,$$

where $0 \leq b_v \leq p - 1$.

Let $y_{2ps} \in H^{2ps}(\Omega S^3\langle 3 \rangle; \mathbb{Z}_{(p)})$ be a generator. Then a generator of $E^{2ps, d_\alpha-2ps+1}$ is mapped by d in (3-2) to $y_{2ps}y_{d_\alpha-2ps+2}$. It is easy to see that

$$(3-4) \quad y_{2ps}y_{d_\alpha-2ps+2} = \binom{p^m u_\alpha + p^{m+i_\alpha+1}}{ps} y_{d_\alpha+2}.$$

Consider the p -power component of the prime decomposition of the binomial coefficient in (3-4). Using (3-3), we see that the component is smallest when $ps = tp^{m+j_\alpha-1}$ ($1 \leq t \leq p - 1$) such that the p -power is $p^{i_\alpha-j_\alpha+2}$. Hence

$$H^{d_\alpha+1}(W^l(S^2); \mathbb{Z}_{(p)}) = \mathbb{Z}/p^{\mu_\alpha}$$

and Theorem 3.3 (i) (a) follows.

For [Theorem 3.3](#) (i) (b), we have the following:

Proposition 3.6 In $H_*(P_{k,2}^l; \mathbb{Z}/p)$, we define the weights of the homology classes x and ι (see (3–1)) to be 2. Then $H_*(P_{k,2}^l; \mathbb{Z}/p)$ is isomorphic to the subspace of $H^*(W^l(S^2); \mathbb{Z}/p)$ spanned by monomials of weight $\leq k$.

Proof The proposition is an easy consequence of Theorems 2.4 and 2.5. Note that it is reasonable to define the weights of x and ι to be 2 by the following reason: we have $H_*(W^l(S^2); \mathbb{Q}) = \wedge(x^l \otimes \iota)$. Since $P_{2l+2,2}^l \simeq S^{2l+1}$, the weight of $x^l \otimes \iota$ must be $2l + 2$. \square

Since the weight of $x^{p^m v_\alpha} \otimes \beta Q_1^{m+j_\alpha}(\iota)$ in (3–1) is $d_\alpha + 2$, [Theorem 3.3](#) (i) (b) follows.

(ii) For $m = 0$ and $\alpha = r$, the left-hand side of (3–1) is the mod p reduction of the generator of $H_{2l+1}(W^l(S^2); \mathbb{Z}) = \mathbb{Z}$ and the right-hand side is 0. Hence we must omit this case from (i). This completes the proof of [Theorem 3.3](#). \square

Finally we generalize [Theorem 3.3](#) for general n .

Theorem 3.7 We keep the notation of [Definition 3.2](#) except that we generalize d_α in (iv) as

$$d_{n,\alpha} = 2(n-1)p^m(u_\alpha + p^{i_\alpha+1}) - 2.$$

Then:

- (1) [Theorem 3.3](#) (i) (a) is generalized to the assertion that $H_{d_{n,\alpha}}(W^l(S^{2n-2}); \mathbb{Z})$ contains $\mathbb{Z}/p^{\mu_\alpha}$ as a direct summand.
- (2) About [Theorem 3.3](#) (i) (b), the least k such that the higher p -torsion in the above (1) appears as a direct summand in $H_{d_{n,\alpha}}(P_{k,n}^l; \mathbb{Z})$ is

$$k = \frac{n(d_{n,\alpha} + 2)}{2(n-1)}.$$

- (3) [Theorem 3.3](#) (ii) holds under these modifications.

Proof About x and ι in (3–1), we generalize that $x \in H_{2n-2}(J^l(S^{2n-2}); \mathbb{Z}/p)$ and $\iota \in H_{2n-3}(\Omega^2 S^{2n-1}; \mathbb{Z}/p)$ such that the weights of these elements are n . [Theorem 3.7](#) is clear from this. \square

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