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Twisted Morita–Mumford classes on braid groups

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Evaluating the twisted Morita–Mumford classes $\overline{h}_p$ (Kawazumi [12]) on the Artin braid group B_n, we give the stable algebraic independence of the $\overline{h}_p$'s on the automorphism group of the free group, $\mathrm{Aut}(F_n)$. This is sharper than the results obtained by restricting them to the mapping class group (Kawazumi [9]).

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Introduction

In the cohomological study of the mapping class group for a surface, the Morita–Mumford classes, $e_i = (-1)^{i+1}\kappa_i$, $i \geq 1$, [19; 17] play some important roles. As was proved by Miller [16] and Morita [17] independently, they are algebraically independent in the stable range $* < \frac{2}{3}g$. Madsen and Weiss [15] proved that the rational stable cohomology algebra of the mapping class groups, $H^*(\mathcal{M}_\infty; \mathbf{Q})$, is generated by the Morita–Mumford classes. The Morita–Mumford classes have twisted variants, $m_{i,j} \in H^{2i+j-2}(\mathcal{M}_{g,1}; \bigwedge^j H)$, $i, j \geq 0$, introduced by the author [11]. Here we denote by $\Sigma_{g,1}$ a 2–dimensional oriented compact connected C^∞ manifold of genus g with 1 boundary component, $\mathcal{M}_{g,1}$ its mapping class group, $\mathcal{M}_{g,1} := \pi_0 \mathrm{Diff}(\Sigma_{g,1}, \mathrm{id\ on\ } \partial\Sigma_{g,1})$, and H the integral first homology group of the surface $\Sigma_{g,1}$. The mapping class group $\mathcal{M}_{g,1}$ acts on H in an obvious way. The twisted variants also satisfy the algebraic independence. More precisely, the algebra $H^*(\mathcal{M}_{g,1}; \bigwedge^* H) \otimes \mathbf{Q}$ is the polynomial algebra in the set $\{m_{i,j};\ i \geq 0,\ j \geq 1, \text{ and } i + j \geq 2\}$ over the algebra $H^*(\mathcal{M}_{g,1}; \mathbf{Q})$ in the range where the total degree $\leq \frac{2}{3}g$ (Kawazumi [9, Theorem 1.C].) Hence, from the theorem of Madsen and Weiss [15] stated above, the algebra $H^*(\mathcal{M}_{g,1}; \bigwedge^* H) \otimes \mathbf{Q}$ is stably isomorphic to the polynomial algebra in the set $\{m_{i,j};\ i \geq 0,\ j \geq 0, \text{ and } i + j \geq 2\}$ over $\mathbf{Q}$. Similar results hold for any other symplectic coefficients (Kawazumi [9, Theorem 1.B].) Furthermore all the cohomology classes on the mapping class group obtained by contracting the coefficients of the twisted ones using the intersection pairing $H^{\otimes 2} \to \mathbf{Z}$ are exactly the algebra generated by the (original) Morita–Mumford classes e_i's (Morita [18], Kawazumi and Morita [13]).

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Some of the twisted ones have the advantage over the original ones of being defined on the automorphism group of a free group, which has the mapping class group and the braid group as proper subgroups. Let $n \geq 2$ be an integer, F_n a free group of rank n with free basis $x_1, x_2, \dots, x_n$

$$F_n = \langle x_1, x_2, \dots, x_n \rangle,$$

and $\mathrm{Aut}(F_n)$ the automorphism group of the group F_n. The Dehn–Nielsen theorem tells us the natural action of the group $\mathcal{M}_{g,1}$ on the free group $\pi_1(\Sigma_{g,1})$ of rank $2g$ induces an injective homomorphism $\mathcal{M}_{g,1} \to \mathrm{Aut}(F_{2g})$. In view of a theorem of Artin [2] the braid group B_n of n strings is embedded into the group $\mathrm{Aut}(F_n)$.

Now we denote by H and H^* the first integral homology and cohomology groups of the group F_n

$$H := H_1(F_n; \mathbf{Z}) = {F_n}^{\mathrm{abel}} = F_n/[F_n.F_n] \quad \text{and} \quad H^* := H^1(F_n; \mathbf{Z}) = \mathrm{Hom}(H, \mathbf{Z}),$$

respectively, on which the automorphism group $\mathrm{Aut}(F_n)$ acts in an obvious way. We write $[\gamma] := \gamma \bmod [F_n, F_n] \in H$ for $\gamma \in F_n$, and $X_i := [x_i] \in H$ for i, $1 \leq i \leq n$. In [12] we introduced cohomology classes

$$h_p \in H^p(\mathrm{Aut}(F_n); H^* \otimes H^{\otimes(p+1)}) \quad \text{and} \quad \overline{h}_p \in H^p(\mathrm{Aut}(F_n); H^{\otimes p})$$

for $p \geq 1$. Restricted to the mapping class group $\mathcal{M}_{g,1}$ they coincide with the twisted Morita–Mumford classes

$$\begin{aligned} &(p+2)!\, h_p|_{\mathcal{M}_{g,1}} = m_{0,p+2} \in H^p(\mathcal{M}_{g,1}; H^{\otimes(p+2)}), \quad \text{and} \\ &p!\, \overline{h}_p|_{\mathcal{M}_{g,1}} = -m_{1,p} \in H^p(\mathcal{M}_{g,1}; H^{\otimes p}). \end{aligned}$$

Here H and H^* are isomorphic to each other as $\mathcal{M}_{g,1}$ modules because of the intersection pairing of the surface $\Sigma_{g,1}$. The class $p!\overline{h}_p$ can be regarded as an element in $H^p(\mathrm{Aut}(F_n); \bigwedge^p H)$.

In this note we confine ourselves to studying the behavior of $\overline{h}_p$'s restricted to the braid group B_n, and consider the rational coefficients

$$H_{\mathbf{Q}} := H \otimes_{\mathbf{Z}} \mathbf{Q} \quad \text{and} \quad H^*_{\mathbf{Q}} := H^* \otimes_{\mathbf{Z}} \mathbf{Q}.$$

In this paper we prove the following result:

Theorem 1 *The cohomology classes $\overline{h}_p$'s are algebraically independent in the algebra $H^*(B_n; \bigwedge^* H_{\mathbf{Q}})$ in the range where the total degree $\leq n$.*

Here the total degree of $\overline{h}_p$ is defined to be $2p$. Theorem 1 implies the algebraic independence on the automorphism group $\mathrm{Aut}(F_n)$. This is sharper than that obtained by restricting them to the mapping class group $\mathcal{M}_{g,1}$ [9, Theorem 1.C], where the range is given by the inequality the total degree $\leq \frac{2}{3}g = \frac{1}{3}n$.

Theorem 1 was announced in [10]. Its proof given in Section 3 is based on some kind of primitiveness of the $\overline{h}_p$'s (Proposition 1.2) and the evaluation of $\overline{h}_{n-1}$ on the pure braid group of n strings, P_n (Lemma 2.4). In Section 4 we will give some remarks on the cohomology of the automorphism group $\mathrm{Aut}(F_n)$.

1 Twisted Morita–Mumford classes on the automorphism group $\mathrm{Aut}(F_n)$

Throughtout this paper we denote by $C^*(G;M)$ the normalized standard complex of a group G with values in a G–module M, and use the Alexander–Whitney cup product $\cup$: $C^*(G;M_1) \otimes C^*(G;M_2) \to C^*(G;M_1 \otimes M_2)$. Moreover we denote by $Z^p(G;M)$, $p \geq 0$, the p–cocycles in the cochain complex $C^*(G;M)$.

Now we recall the definition of the twisted cohomology classes h_p and $\overline{h}_p$ on the automorphism group $\mathrm{Aut}(F_n)$ for $p \geq 1$. The semi-direct product

$$\overline{A_n} := F_n \rtimes \mathrm{Aut}(F_n)$$

admits an extension of groups

$$F_n \xrightarrow{\iota} \overline{A_n} \xrightarrow{\pi} \mathrm{Aut}(F_n) \tag{1–1}$$

given by $\iota(\gamma) = (\gamma, 1)$ and $\pi(\gamma,\varphi) = \varphi$ for $\gamma \in F_n$ and $\varphi \in \mathrm{Aut}(F_n)$. The map $k_0 : \overline{A_n} \to H$, $(\gamma,\varphi) \mapsto [\gamma]$, satisfies the cocycle condition. We write also k_0 for the cohomology class $[k_0] \in H^1(\overline{A_n};H)$. For each $p \geq 1$ we define h_p by the image of the $(p+1)$-st power of the cohomology class k_0 under the Gysin map of the extension (1–1)

$$h_p := \pi_\sharp({k_0}^{\otimes(p+1)}) \in H^p(\mathrm{Aut}(F_n); H^* \otimes H^{\otimes(p+1)}) \tag{1–2}$$

[12]. Contracting the coefficients by the $\mathrm{GL}(H)$–homomorphism

$$r_p\colon H^* \otimes H^{\otimes(p+1)} \to H^{\otimes p}, \quad f \otimes v_0 \otimes v_1 \otimes \cdots \otimes v_p \mapsto f(v_0) v_1 \otimes \cdots \otimes v_p, \tag{1–3}$$

we define

$$\overline{h}_p := {r_p}_*(h_p) \in H^p(\mathrm{Aut}(F_n); H^{\otimes p}). \tag{1–4}$$

The p-th exterior power $k_0{}^p = p!k_0{}^{\otimes p}$ can be regarded as a cohomology class with coefficients in $\bigwedge^p H$. Hence, if we consider the rational coefficients $H_{\mathbf{Q}}$, we may regard $\overline{h}_p$ as a cohomology class in $H^p(\mathrm{Aut}(F_n); \bigwedge^p H_{\mathbf{Q}})$.

A Magnus expansion θ of the free group F_n gives an explicit cocycle representing the class h_p. The completed tensor algebra generated by H, $\widehat{T} = \widehat{T}(H) := \prod_{m=0}^{\infty} H^{\otimes m}$, has a decreasing filtration of two-sided ideals $\widehat{T}_p := \prod_{m \geq p} H^{\otimes m}$, $p \geq 1$. It should be remarked that the subset $1 + \widehat{T}_1$ is a subgroup of the multiplicative group of the algebra $\widehat{T}$. We call a map $\theta\colon F_n \to 1 + \widehat{T}_1$ a *Magnus expansion* of the free group F_n, if $\theta\colon F_n \to 1 + \widehat{T}_1$ is a group homomorphism, and if $\theta(\gamma) \equiv 1 + [\gamma] \pmod{\widehat{T}_2}$ for any $\gamma \in F_n$. We write $\theta(\gamma) = \sum_{m=0}^{\infty} \theta_m(\gamma)$, $\theta_m(\gamma) \in H^{\otimes m}$. The m-th component $\theta_m\colon F_n \to H^{\otimes m}$ is a map, but *not* a group homomorphism. A Magnus expansion $\mathrm{std}\colon F_n \to 1 + \widehat{T}_1$ is defined by $\mathrm{std}(x_i) := 1 + X_i$, $1 \leq i \leq n$. Here we denote $X_i := [x_i] \in H$, the homology class of the generator x_i. We call it *the standard Magnus expansion*. As is described in classical references, the value $\mathrm{std}(\gamma)$ for any word $\gamma \in F_n$ is explicitly computed by means of Fox' free differentials. All the results of this paper can be derived from the expansion std.

We define a map $\tau_1^\theta\colon \mathrm{Aut}(F_n) \to H^* \otimes H^{\otimes 2}$ by

$$\tau_1^\theta(\varphi)[\gamma] = \theta_2(\gamma) - |\varphi|^{\otimes 2}\theta_2(\varphi^{-1}(\gamma)) \in H^{\otimes 2} \tag{1–5}$$

for $\gamma \in F_n$ and $\varphi \in \mathrm{Aut}(F_n)$. Here $|\varphi| \in \mathrm{GL}(H)$ is the automorphism of $H = F_n{}^{\mathrm{abel}}$ induced by φ. This map τ_1^θ satisfies the cocycle condition [12, Lemma 2.1]. Now we introduce a $\mathrm{GL}(H)$–homomorphism

$$\varsigma_p\colon (H^* \otimes H^{\otimes 2})^{\otimes p} = \mathrm{Hom}(H, H^{\otimes 2})^{\otimes p} \to \mathrm{Hom}(H, H^{\otimes (p+1)}) = H^* \otimes H^{\otimes (p+1)}$$

for each $p \geq 1$. If $p \geq 2$, we define

$$\begin{aligned} &\varsigma_p(u_{(1)} \otimes u_{(2)} \otimes \cdots \otimes u_{(p-1)} \otimes u_{(p)}) \\ &:= \left(u_{(1)} \otimes 1_{H^{\otimes(p-1)}}\right) \circ \left(u_{(2)} \otimes 1_{H^{\otimes(p-2)}}\right) \circ \cdots \circ \left(u_{(p-1)} \otimes 1_H\right) \circ u_{(p)}, \end{aligned} \tag{1–6}$$

where $u_{(i)} \in \mathrm{Hom}(H, H^{\otimes 2}) = H^* \otimes H^{\otimes 2}$, $1 \leq i \leq p$. In the case $p = 1$, we define $\varsigma_1 := 1_{H^* \otimes H^{\otimes 2}}$. Then we have:

Theorem 1.1 [12, Theorem 4.1]

$$h_p = \varsigma_{p*}([\tau_1^\theta]^{\otimes p}) \in H^p(\mathrm{Aut}(F_n); H^* \otimes H^{\otimes(p+1)})$$

for any Magnus expansion θ and each $p \geq 1$. In the case $p = 1$ we have $[\tau_1^\theta] = h_1 \in H^1(\mathrm{Aut}(F_n); H^* \otimes H^{\otimes 2})$.

Some kind of primitiveness of the cohomology classes h_p and $\overline{h}_p$ follows from the theorem. We write simply $A_n := \mathrm{Aut}(F_n)$ for the remainder of the section. Suppose $n_1 + n_2 \le n$. Let A_{n_2} act on the words in the letters $x_{n_1+1}, x_{n_1+2}, \dots, x_{n_1+n_2}$ in an obvious way. Then we have a natural homomorphism

$$\iota = \iota_{n_1,n_2}\colon A_{n_1} \times A_{n_2} \to A_n.$$

We denote by $\varpi_1\colon A_{n_1} \times A_{n_2} \to A_{n_1}$ and $\varpi_2\colon A_{n_1} \times A_{n_2} \to A_{n_2}$ the first and the second projections of the product $A_{n_1} \times A_{n_2}$, respectively, and by $H_{(n_1)}$, $H_{(n_2)}$ and $H_{(n-n_1-n_2)}$ the submodules of H spanned by $\{X_1, \dots, X_{n_1}\}$, $\{X_{n_1+1}, \dots, X_{n_1+n_2}\}$ and $\{X_{n_1+n_2+1}, \dots, X_n\}$, respectively. Then we have a direct-sum decomposition $H = H_{(n_1)} \oplus H_{(n_2)} \oplus H_{(n-n_1-n_2)}$, and can consider the map

$${\varpi_k}^*\colon H^*(A_{n_k}; H^*_{(n_k)} \otimes H_{(n_k)}^{\otimes(p+1)}) \to H^*(A_{n_1} \times A_{n_2}; H^* \otimes H^{\otimes(p+1)})$$

for $k = 1$ and 2. For any $p \ge 1$ we have:

Proposition 1.2

(1) $\iota^* h_p = {\varpi_1}^* h_p + {\varpi_2}^* h_p \in H^p(A_{n_1} \times A_{n_2}; H^* \otimes H^{\otimes(p+1)})$,

(2) $\iota^* \overline{h}_p = {\varpi_1}^* \overline{h}_p + {\varpi_2}^* \overline{h}_p \in H^p(A_{n_1} \times A_{n_2}; H^{\otimes p})$.

Proof Using the standard expansion std, we write simply

$$\tau^{(k)} := {\varpi_k}^* \tau_1^{\mathrm{std}} \in Z^1(A_{n_1} \times A_{n_2}; H^* \otimes H^{\otimes 2}).$$

Clearly we have $\mathrm{std}(\gamma_1) \in \prod_{p=0}^{\infty} {H_{(n_1)}}^{\otimes p} \subset \widehat{T}$ for any word γ_1 in the letters $x_1, \dots, x_{n_1}$. Similar conditions hold for any word γ_2 in the letters $x_{n_1+1}, \dots, x_{n_1+n_2}$ and any γ_3 in $x_{n_1+n_2+1}, \dots, x_n$. Hence, from the definition of τ_1^{θ} (1–5), we have

$$\iota^* \tau_1^{\mathrm{std}} = \tau^{(1)} + \tau^{(2)} \in Z^1(A_{n_1} \times A_{n_2}; H^* \otimes H^{\otimes 2}).$$

If we use the $\mathrm{GL}(H)$–homomorphism $\varsigma_2\colon (H^* \otimes H^{\otimes 2})^{\otimes 2} \to H^* \otimes H^{\otimes 3}$ in (1–6), then we have

$$\varsigma_{2*}(\tau^{(1)}\tau^{(2)}) = \varsigma_{2*}(\tau^{(2)}\tau^{(1)}) = 0 \in Z^2(A_{n_1} \times A_{n_2}; H^* \otimes H^{\otimes 3}). \tag{1–7}$$

In fact, $f(u) = 0$ for any $f \in H^*_{(n_1)}$ and $u \in H_{(n_2)}$ and vice versa. From Theorem 1.1 follows

$$\begin{aligned}\iota^* h_p &= \varsigma_{p*}(\iota^*[\tau_1^{\mathrm{std}}]^{\otimes p}) = \varsigma_{p*}((\tau^{(1)} + \tau^{(2)})^{\otimes p}) \\ &= \varsigma_{p*}((\tau^{(1)})^{\otimes p}) + \varsigma_{p*}((\tau^{(2)})^{\otimes p}) = {\varpi_1}^* h_p + {\varpi_2}^* h_p.\end{aligned}$$

Here ς_{p*} of each mixed term in $\tau^{(1)}$ and $\tau^{(2)}$ vanishes by (1–7). Applying r_{p*} to (1), we deduce (2). This completes the proof of the proposition. □

2 Evaluation on the Artin braid groups

The n-th symmetric group $\mathfrak{S}_n$ acts on the space $\mathbf{C}^n$ by permuting the components. The open subset

$$Y_n := \{(z_1, z_2, \dots, z_n) \in \mathbf{C}^n;\ z_i \neq z_j \text{ for } i \neq j\}$$

is stable under the action of the group $\mathfrak{S}_n$. By definition, the Artin braid group of n strings, B_n, is the fundamental group of the quotient space $Y_n/\mathfrak{S}_n$, $B_n := \pi_1(Y_n/\mathfrak{S}_n)$. As was shown by Artin [2], the group B_n admits a presentation

$$\begin{aligned} &\text{generators:} && \sigma_i, \quad 1 \le i \le n-1, \\ &\text{relations:} && \sigma_i\sigma_j = \sigma_j\sigma_i, \quad \text{if } |i-j| \ge 2, \\ & && \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad \text{for } 1 \le i \le n-2. \end{aligned} \tag{2–1}$$

The pure braid group of n strings, P_n, is defined to be the fundamental group of the space Y_n, $P_n := \pi_1(Y_n)$. We have a natural extension of groups

$$P_n \to B_n \to \mathfrak{S}_n.$$

As is known, $A_{i,j}$, $1 \le i < j \le n$, given by

$$A_{i,j} := \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}{\sigma_i}^2{\sigma_{i+1}}^{-1}\cdots{\sigma_{j-2}}^{-1}{\sigma_{j-1}}^{-1}$$

can serve as a generating system of the group P_n. For details, see Birman [3].

The braid group B_n admits a natural homomorphism into the group $\mathrm{Aut}(F_n)$, $\xi\colon B_n \to \mathrm{Aut}(F_n)$. To recall how to construct it, we consider an action of the group $\mathfrak{S}_n$ on the space $Y_{n+1} \subset \mathbf{C}^{n+1} = \mathbf{C}^n \times \mathbf{C}$ given by

$$\rho(z_1, \dots, z_n, z_{n+1}) = (z_{\rho^{-1}(1)}, \dots, z_{\rho^{-1}(n)}, z_{n+1})$$

for $\rho \in \mathfrak{S}_n$. We denote by $\widehat{B_n}$ the fundamental group of the quotient space $Y_{n+1}/\mathfrak{S}_n$, $\widehat{B_n} := \pi_1(Y_{n+1}/\mathfrak{S}_n)$.

The forgetful map $Y_{n+1} \to Y_n$, $(z_1, \dots, z_n, z_{n+1}) \mapsto (z_1, \dots, z_n)$, induces a fibration

$$\mathbf{C} \setminus \{n \text{ points}\} \to Y_{n+1}/\mathfrak{S}_n \to Y_n/\mathfrak{S}_n$$

with a section $s\colon Y_n/\mathfrak{S}_n \to Y_{n+1}/\mathfrak{S}_n$ given by $(z_1, \dots, z_n) \mapsto (z_1, \dots, z_n, \frac{1}{n}\sum_{i=1}^n z_i + \sum_{j=1}^n |z_j - \frac{1}{n}\sum_{i=1}^n z_i|)$ (Arnol'd [1]). This fibration with the section s induces an extension of groups

$$F_n \xrightarrow{\iota} \widehat{B_n} \xrightarrow{\pi} B_n \tag{2–2}$$

with a split homomorphism $s\colon B_n \to \widehat{B_n}$. Thus we obtain a morphism of extensions of groups

$$\begin{tikzcd} F_n \arrow[r] \arrow[d, equal] & \widehat{B_n} \arrow[r] \arrow[d, "\widehat{\xi}"'] & B_n \arrow[d, "\xi"'] \\ F_n \arrow[r] & \overline{A_n} \arrow[r] & \mathrm{Aut}(F_n). \end{tikzcd} \tag{2–3}$$

The homomorphisms ξ and $\widehat{\xi}$ are explicitly given by

$$\iota(\xi(x)(\gamma)) = s(x)\gamma s(x)^{-1}$$
$$\widehat{\xi}(\iota(\gamma)s(x)) = (\gamma, \xi(x)) \in F_n \rtimes \mathrm{Aut}(F_n) = \overline{A_n}$$

for $x \in B_n$ and $\gamma \in F_n$. The group $\widehat{B_n}$ is embedded into B_{n+1} in an obvious way. Then the homomorphisms s and ι are described as

$$\begin{aligned} s(\sigma_i) &= \sigma_i \quad \text{for } 1 \le i \le n-1, \\ \iota(x_j) &= \sigma_n\sigma_{n-1}\cdots\sigma_{j+1}{\sigma_j}^2{\sigma_{j+1}}^{-1}\cdots{\sigma_{n-1}}^{-1}{\sigma_n}^{-1} \\ &= A_{j,n+1} \quad \text{for } 1 \le j \le n \end{aligned} \tag{2–4}$$

in terms of the presentation (2–1). So the homomorphism ξ is explicitly given by

$$\xi(\sigma_i)(x_j) = \begin{cases} x_{i+1}, & \text{if } j = i, \\ {x_{i+1}}^{-1}x_ix_{i+1}, & \text{if } j = i+1, \\ x_j, & \text{otherwise.} \end{cases} \tag{2–5}$$

We now evaluate the cohomology classes h_1 and $\overline{h}_{n-1}$ on the braid group B_n. Here we use the standard Magnus expansion std: $F_n \to 1 + \widehat{T}_1$ introduced in Section 1. For the rest of this section we write simply k_0, τ_1, h_p and $\overline{h}_p$ for $\widehat{\xi}^* k_0$, $\xi^*\tau_1^{\mathrm{std}}$, $\xi^* h_p$ and $\xi^*\overline{h}_p$, respectively. Let $\{l_i\}_{i=1}^n \subset H^*$ denote the dual basis of $\{X_i\}_{i=1}^n = \{[x_i]\}_{i=1}^n \subset H$.

Lemma 2.1

$$\tau_1(\sigma_i) = l_i \otimes (X_i \otimes X_{i+1} - X_{i+1} \otimes X_i) \in H^* \otimes H^{\otimes 2}$$

Proof From (1–5)

$$\begin{aligned} \tau_1(\sigma_i) &= \sum\nolimits_{j=1}^n l_j \otimes (\mathrm{std}_2(x_j) - |\sigma_i|^{\otimes 2}\mathrm{std}_2({\sigma_i}^{-1}(x_j))) \\ &= -l_i \otimes |\sigma_i|^{\otimes 2}\mathrm{std}_2({\sigma_i}^{-1}(x_i)) - l_{i+1} \otimes |\sigma_i|^{\otimes 2}\mathrm{std}_2({\sigma_i}^{-1}(x_{i+1})) \\ &= -l_i \otimes |\sigma_i|^{\otimes 2}\mathrm{std}_2(x_ix_{i+1}{x_i}^{-1}) - l_{i+1} \otimes |\sigma_i|^{\otimes 2}\mathrm{std}_2(x_i) \\ &= -l_i \otimes |\sigma_i|^{\otimes 2}\mathrm{std}_2(x_ix_{i+1}{x_i}^{-1}). \end{aligned}$$

On the other hand, we have

$$\mathrm{std}_2(x_i x_{i+1} {x_i}^{-1}) = X_i \otimes X_{i+1} - X_{i+1} \otimes X_i.$$

In fact, $X_i \otimes X_{i+1} = \mathrm{std}_2(x_i x_{i+1}) = \mathrm{std}_2(x_i x_{i+1} {x_i}^{-1} x_i) = \mathrm{std}_2(x_i x_{i+1} {x_i}^{-1}) + \mathrm{std}_2(x_i) + X_{i+1} \otimes X_i = \mathrm{std}_2(x_i x_{i+1} {x_i}^{-1}) + X_{i+1} \otimes X_i$. Therefore we obtain $\tau_1(\sigma_i) = -l_i \otimes |\sigma_i|^{\otimes 2}(X_i \otimes X_{i+1} - X_{i+1} \otimes X_i) = -l_i \otimes (X_{i+1} \otimes X_i - X_i \otimes X_{i+1})$, as was to be shown. □

The pure braid group P_n acts on the homology H trivially. Hence, from [12, Theorem 3.1], the restriction of τ_1 to P_n does not depend on the choice of Magnus expansions.

Lemma 2.2

$$\tau_1(A_{i,j}) = (l_i - l_j) \otimes (X_i \otimes X_j - X_j \otimes X_i)$$

Proof Recall the map τ_1 satisfies the cocycle condition on the automorphism group $\mathrm{Aut}(F_n)$. When we set $\gamma := \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}$, we have $A_{i,j} = \gamma {\sigma_i}^2 \gamma^{-1}$, so that

$$\begin{aligned}
&\tau_1(A_{i,j}) \\
&= \tau_1(\gamma {\sigma_i}^2 \gamma^{-1}) = \tau_1(\gamma) + \gamma\tau_1({\sigma_i}^2) + \gamma{\sigma_i}^2\tau_1(\gamma^{-1}) \\
&= \tau_1(\gamma) + \gamma\tau_1({\sigma_i}^2) + \gamma\tau_1(\gamma^{-1}) = \tau_1(1) + \gamma\tau_1({\sigma_i}^2) = \gamma\tau_1({\sigma_i}^2) \\
&= \gamma(\tau_1(\sigma_i) + \sigma_i\tau_1(\sigma_i)) \\
&= \gamma(l_i \otimes (X_i \otimes X_{i+1} - X_{i+1} \otimes X_i)) + \gamma\sigma_i(l_i \otimes (X_i \otimes X_{i+1} - X_{i+1} \otimes X_i)) \\
&= \gamma((l_i - l_{i+1}) \otimes (X_i \otimes X_{i+1} - X_{i+1} \otimes X_i)) \\
&= (l_i - l_j) \otimes (X_i \otimes X_j - X_j \otimes X_i),
\end{aligned}$$

as was to be shown. □

To prove the nontriviality of $\overline{h}_{n-1}$ on the group B_n, we recall some basic facts on the cohomology of the pure braid group P_n. The space Y_n is an Eilenberg–MacLane space of type $(P_n, 1)$. The subspace $Y_n \cap \{z_1 + \cdots + z_n = 0\}$ is a deformation retract of the space Y_n and a Stein manifold of complex dimension $n-1$. Hence the cohomological dimension of the group P_n, $\mathrm{cd}\,P_n$, is not greater than $n-1$. Let $A^*(Y_n)$ be the algebra of all the complex-valued differential forms on the space Y_n. As was shown by Arnol'd [1], the $\mathbf{Z}$–subalgebra generated by the 1–forms

$$\omega_{i,j} := \frac{1}{2\pi\sqrt{-1}} \frac{dz_i - dz_j}{z_i - z_j}, \quad 1 \le i < j \le n,$$

is isomorphic to the cohomology algebra $H^*(Y_n; \mathbf{Z}) = H^*(P_n; \mathbf{Z})$. Especially in the case $* = 1$, $\{[\omega_{i,j}]\}_{1 \le i < j \le n}$ is a $\mathbf{Z}$–free basis of $H^1(P_n; \mathbf{Z})$, so that $\{[A_{i,j}]\}_{1 \le i < j \le n}$ is a $\mathbf{Z}$–free basis of $H_1(P_n; \mathbf{Z}) = {P_n}^{\mathrm{abel}}$.

Lemma 2.3

(1) ${k_0}^n \neq 0 \in H^n(Y_{n+1}; \bigwedge^n H_{\mathbf{Q}})$, *where* $P_{n+1} = \pi_1(Y_{n+1})$ *is regarded as a subgroup of* $\widehat{B_n} = \pi_1(Y_{n+1}/\mathfrak{S}_n)$.

(2) $h_{n-1} \neq 0 \in H^{n-1}(P_n; {H_{\mathbf{Q}}}^* \otimes \bigwedge^n H_{\mathbf{Q}})$.

Proof (1) From (2–3) and (2–4) we have

$$k_0(A_{i,j}) = \begin{cases} 0, & \text{if } i < j \leq n, \\ X_i, & \text{if } i < j = n+1, \end{cases}$$

that is

$$k_0 = \sum\nolimits_{i=1}^{n} \omega_{i,n+1} \otimes X_i \in H^1(Y_{n+1}; H).$$

If we restrict the n–form

$$\omega_{1,n+1}\omega_{2,n+1}\cdots\omega_{n,n+1} = (1/2\pi\sqrt{-1})^n \prod\nolimits_{i=1}^{n} (dz_i - dz_{n+1})/(z_i - z_{n+1})$$

to the subspace $Y_{n+1} \cap \{z_{n+1} = 0\}$, then we obtain the non-zero n–form $(1/2\pi\sqrt{-1})^n$ $\prod_{i=1}^{n}(dz_i/z_i)$. Hence the cohomology class

$${k_0}^n = n!\, \omega_{1,n+1}\omega_{2,n+1}\cdots\omega_{n,n+1} X_1 \wedge X_2 \wedge \cdots \wedge X_n \in H^n(Y_{n+1}; \bigwedge\nolimits^n H_{\mathbf{Q}})$$

does not vanish, as was to be shown.

(2) Since $\operatorname{cd} P_n \leq n-1$, the Gysin map of the extension

$$F_n \xrightarrow{\iota} P_{n+1} \xrightarrow{\pi} P_n$$

gives an isomorphism

$$\pi_\sharp\colon H^n(P_{n+1}; M) \xrightarrow{\cong} H^{n-1}(P_n; H^* \otimes M)$$

for any P_n–module M. Hence $h_{n-1} = \pi_\sharp {k_0}^n \neq 0$ by (1). □

The map $r_n\colon {H_{\mathbf{Q}}}^* \otimes \bigwedge^n H_{\mathbf{Q}} \to \bigwedge^{n-1} H_{\mathbf{Q}}$ is an isomorphism because $\dim_{\mathbf{Q}} H_{\mathbf{Q}} = n$. Hence we obtain:

Lemma 2.4

$$\overline{h}_{n-1} \neq 0 \in H^{n-1}(P_n; \bigwedge\nolimits^{n-1} H_{\mathbf{Q}}).$$

3 Proof of Theorem 1

Our proof of Theorem 1 is based on Proposition 1.2 and Lemma 2.4. For $q \leq n$ we denote by $\mathcal{P}_{n-q}(q)$ the set of all the non-negative partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-q} \geq 0)$ of q into $n-q$ parts. For $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-q} \geq 0) \in \mathcal{P}_{n-q}(q)$ we introduce a cohomology class $\overline{h}_\lambda$ and a subgroup $P_\lambda \subset P_n$ by

$$\overline{h}_\lambda := \overline{h}_{\lambda_1}\overline{h}_{\lambda_2}\cdots\overline{h}_{\lambda_{n-q}} \in H^q(B_n; \textstyle\bigwedge^q H_{\mathbf{Q}}) \subset H^q(P_n; \bigwedge^q H_{\mathbf{Q}}), \quad \text{and}$$
$$P_\lambda := P_{\lambda_1+1} \times P_{\lambda_2+1} \times \cdots \times P_{\lambda_{n-q}+1} \subset P_n,$$

respectively. Here $P_{0+1} = P_1$ is the trivial group $\{1\}$. Denote by $\iota_\lambda\colon P_\lambda \hookrightarrow P_n$ the obvious inclusion map and $\varpi_k\colon P_\lambda \to P_{\lambda_k+1}$ the obvious projection. Theorem 1 follows from:

Theorem 3.1 *The cohomology classes $\{\overline{h}_\lambda;\ \lambda \in \mathcal{P}_{n-q}(q)\}$ are linearly independent in $H^q(P_n; \bigwedge^q H_{\mathbf{Q}})$.*

In fact, when $q \leq n/2$, the set of all the non-negative partitions of q into $n-q$ parts does not depend on n.

Endow the partitions $\mathcal{P}_{n-q}(q)$ with the lexicographic order. For example, $(q \geq 0 \geq \cdots \geq 0)$ is the maximal partition. Theorem 3.1 is reduced to the following

Assertions For any λ and $\mu \in \mathcal{P}_{n-q}(q)$ we have:

(A) $\iota_\lambda{}^*\overline{h}_\lambda \neq 0 \in H^q(P_\lambda; \bigwedge^q H_{\mathbf{Q}})$

(B) If $\mu \gneqq \lambda$, then $\iota_\lambda{}^*\overline{h}_\mu = 0 \in H^q(P_\lambda; \bigwedge^q H_{\mathbf{Q}})$.

In fact, assume we have a nontrivial linear relation

$$\sum_{\lambda \in \mathcal{P}_{n-q}(q)} c_\lambda \overline{h}_\lambda = 0 \in H^q(P_n; \textstyle\bigwedge^q H_{\mathbf{Q}}).$$

Choose the minimum λ satisfying $c_\lambda \neq 0$. Applying $\iota_\lambda{}^*$ to the relation, we obtain $c_\lambda \iota_\lambda{}^*\overline{h}_\lambda = 0$ from Assertion (B). Assertion (A) implies $c_\lambda = 0$, which contradicts the choice of λ.

Proof of Assertion (A) Let $b_1 \geq b_2 \geq \cdots \geq b_{\lambda_1} > b_{\lambda_1+1} = 0$ be the dual partition of λ. The number of λ_k's equal to p is $b_p - b_{p+1}$. We abbreviate $\overline{h}_{p,k} := \varpi_k{}^*\overline{h}_p$. Since $\operatorname{cd} P_{\lambda_k+1} \leq \lambda_k$, we have $\overline{h}_{p,k} = 0$ if $p > \lambda_k$, or equivalently, $k > b_p$. Moreover we have $\overline{h}_{\lambda_k,k}\overline{h}_{p,k} = 0$ for any $p \geq 1$ since $H^{\lambda_k+p}(P_{\lambda_k+1}; \bigwedge^{\lambda_k+p} H_{\mathbf{Q}}) = 0$. From Proposition 1.2 we have

$$\iota_\lambda{}^*\overline{h}_p = \sum\nolimits_{k=1}^{n-q} \overline{h}_{p,k} \in H^p(P_\lambda; \bigwedge\nolimits^p H),$$

so that

$$\begin{aligned}
&\iota_\lambda{}^*\overline{h}_\lambda = \prod_{k=1}^{n-q} \iota_\lambda{}^*\overline{h}_{\lambda_k} = \prod_{p=1}^{\lambda_1} (\iota_\lambda{}^*\overline{h}_p)^{b_p-b_{p+1}} \\
&= \prod_{p=1}^{\lambda_1} (\overline{h}_{p,1} + \overline{h}_{p,2} + \cdots + \overline{h}_{p,n-q})^{b_p-b_{p+1}} \\
&= \prod_{p=1}^{\lambda_1} (\overline{h}_{p,1} + \overline{h}_{p,2} + \cdots + \overline{h}_{p,b_p})^{b_p-b_{p+1}} = \prod_{p=1}^{\lambda_1} (\overline{h}_{p,b_{p+1}+1} + \cdots + \overline{h}_{p,b_p})^{b_p-b_{p+1}} \\
&= \prod_{p=1}^{\lambda_1} (b_p - b_{p+1})!\, \overline{h}_{p,b_{p+1}+1} \cdots \overline{h}_{p,b_p} \\
&= \left(\prod_{p=1}^{\lambda_1} (b_p - b_{p+1})!\right) \overline{h}_{\lambda_1,1}\overline{h}_{\lambda_2,2} \cdots \overline{h}_{\lambda_{n-q},n-q}.
\end{aligned}$$

Here the fifth equal sign comes from the equation $\overline{h}_{\lambda_k,k}\overline{h}_{p,k} = 0$. Clearly $r_\lambda := \prod_{p=1}^{\lambda_1}(b_p - b_{p+1})!$ is a positive integer. From Lemma 2.4 and the Künneth formula $\overline{h}_{\lambda_1,1}\overline{h}_{\lambda_2,2} \cdots \overline{h}_{\lambda_{n-q},n-q} \neq 0 \in H^q(P_\lambda; \bigwedge^q H_{\mathbf{Q}})$. This proves Assertion (A). □

Proof of Assertion (B) Suppose $\mu > \lambda$ with respect to the lexicographic order, namely, $\mu_1 = \lambda_1 \geq \mu_2 = \lambda_2 \geq \cdots \geq \mu_h = \lambda_h \geq \mu_{h+1} > \lambda_{h+1}$ for some h, $0 \leq h < n-q$. Let $\nu := (\nu_1 \geq \nu_2 \geq \cdots \geq \nu_h)$ be the (truncated) partition of $q' := \lambda_1 + \lambda_2 + \cdots + \lambda_h$ defined by $\nu_k := \lambda_k = \mu_k$, $k \leq h$. From Assertion (A)

$$\iota_\lambda{}^*(\overline{h}_{\mu_1}\overline{h}_{\mu_2} \cdots \overline{h}_{\mu_h}) = r_\nu \overline{h}_{\mu_1,1}\overline{h}_{\mu_2,2} \cdots \overline{h}_{\mu_h,h} \in H^{q'}(P_\lambda; \bigwedge\nolimits^{q'} H).$$

In fact, from $\mu_h > \lambda_{h+1}$, we have $\overline{h}_{\mu_i,j} = 0$ if $i < j$. Since $\mu_{h+1} \gneqq \lambda_k$ for any $k \geq h+1$, we have

$$\iota_\lambda{}^*(\overline{h}_{\mu_1} \cdots \overline{h}_{\mu_h}\overline{h}_{\mu_{h+1}}) = r_\nu \overline{h}_{\mu_1,1} \cdots \overline{h}_{\mu_h,h}(\overline{h}_{\mu_{h+1},1} + \cdots + \overline{h}_{\mu_{h+1},h}) = 0$$

Hence $\iota_\lambda{}^*(\overline{h}_\mu) = 0$, as was to be shown. □

This completes the proof of Theorem 3.1 and Theorem 1.

4 Concluding remarks

We conclude this note by giving some remarks on the twisted cohomology of the automorphism group $\mathrm{Aut}(F_n)$ and the braid group B_n.

The IA–automorphism group IA_n is defined to be the kernel of the action of the group $\mathrm{Aut}(F_n)$ on the homology group $H = {F_n}^{\mathrm{abel}}$. We have an extension of groups $IA_n \to \mathrm{Aut}(F_n) \to \mathrm{GL}(H)$. The map τ_1^θ restricted to IA_n gives an isomorphism of the abelianization of the group IA_n onto the module $H^* \otimes \bigwedge^2 H$

$$\tau_1 \colon {IA_n}^{\mathrm{abel}} \overset{\cong}{\to} H^* \otimes \bigwedge^2 H$$

(Cohen and Pakianathan [5], Farb [6], Kawazumi [12]). Here we embed $\bigwedge^2 H$ into $H^{\otimes 2}$ by $X_i \wedge X_j \mapsto X_i \otimes X_j - X_j \otimes X_i$ for $1 \le i, j \le n$. Lemma 2.2 implies $\xi^* \colon H^1(IA_n; \mathbf{Z}) \to H^1(P_n; \mathbf{Z})$ is surjective. From the result of Arnol'd [1] quoted in Section 2, the cohomology algebra $H^*(P_n; \mathbf{Z})$ is generated by the first cohomology classes. Hence we obtain:

Corollary 4.1 *The algebra homomorphism*

$$\xi^* \colon H^*(IA_n; \mathbf{Z}) \to H^*(P_n; \mathbf{Z})$$

induced by the homomorphism $\xi \colon P_n \to IA_n$ is surjective.

It should be remarked that it does *not* imply that the map $\xi^* \colon H^*(\mathrm{Aut}(F_n); M) \to H^*(B_n; M)$ is surjective for a $\mathbf{Q}[\mathrm{GL}(H)]$–module M. In fact, the quotient groups $\mathrm{Aut}(F_n)/IA_n = \mathrm{GL}(H)$ and $B_n/P_n = \mathfrak{S}_n$ differ from each other.

Fred Cohen [4, Lemma 7.2, page 261] described the action of the symmetric group $\mathfrak{S}_n$ on the integral cohomology of the group P_n, $H^*(P_n; \mathbf{Z})$. Later Lehrer and Solomon [14] gave another explicit description of the $\mathbf{Q}[\mathfrak{S}_n]$–module $H^*(P_n; \mathbf{Q})$. Moreover Cohen [4, Theorem 3.1, page 225] computed the twisted cohomology $H^*(B_n; H^{\otimes m} \otimes \mathbb{F})$ for any field $\mathbb{F}$ and any $m \ge 0$. It would be interesting if one could describe the submodule of $H^*(B_n; M)$ generated by all the possible algebraic combinations coming from the twisted Morita–Mumford classes h_p's in an explicit manner. Here we should remark the $\mathfrak{S}_n$–invariant inner product $\cdot \colon H \otimes H \to \mathbf{Z}$ defined by $X_i \cdot X_j = \delta_{i,j}$, $1 \le i, j \le n$, gives a B_n–isomorphism $H \cong H^*$.

As was stated in Introduction, the algebra $H^*(\mathcal{M}_{g,1}; \bigwedge^* H_{\mathbf{Q}})$ is stably isomorphic to the polynomial algebra in the twisted Morita–Mumford classes $m_{i,j}$'s. The intersection pairing of the surface $\Sigma_{g,1}$, $H^{\otimes 2} \to \mathbf{Z}$, gives an isomorphism $H \cong H^*$ of $\mathcal{M}_{g,1}$–modules, so that the cocycle τ_1^θ restricted to $\mathcal{M}_{g,1}$ can be regarded as a cocycle

$\tau_1^\theta\colon \mathcal{M}_{g,1} \to H^{\otimes 3}$. As was proved by Kawazumi and Morita in [13], for any twisted Morita–Mumford class $m_{i,j}$ we have an $\mathcal{M}_{g,1}$–homomorphism $C\colon (H^{\otimes 3})^{\otimes(2i+j-2)} \to \mathbf{Z}$ obtained from the intersection pairing such that $C_*[\tau_1^\theta]^{2i+j-2} = m_{i,j}$. In other words, the natural map

$$((\bigwedge\nolimits^* H^1(\mathcal{I}_{g,1};\mathbf{Q})) \otimes M)^{\mathrm{Sp}(H)} \to H^*(\mathcal{M}_{g,1}; M)$$

is stably surjective for any finite dimensional $\mathbf{Q}[\mathrm{Sp}(H)]$–module M. Here $\mathcal{I}_{g,1}$ is the Torelli group, i.e, the kernel of the action of $\mathcal{M}_{g,1}$ on the homology H.

Recently Galatius [7] proved the rational reduced cohomology $\widetilde{H}^*(\mathrm{Aut}(F_n);\mathbf{Q})$ vanishes in a stable range. It would be very interesting to know whether a similar result holds also for twisted coefficients.

Expectation 4.2 *For a finite dimensional* $\mathbf{Q}[\mathrm{GL}(H)]$*–module* M*, the natural map*

$$((\bigwedge\nolimits^* H^1(IA_n;\mathbf{Q})) \otimes M)^{\mathrm{GL}(H)} \to H^*(\mathit{Aut}(F_n); M)$$

is surjective in some stable range.

In the case M is the trivial module $\mathbf{Q}$, this expectation is exactly the fact that $\widetilde{H}^*(\mathrm{Aut}(F_n);\mathbf{Q})$ vanishes in some stable range, which Galatius [7] proved. A result of Hatcher and Wahl [8] tells us it holds also for $M = (H^*)^{\otimes m}$ for any $m \geq 1$.

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References

[1] **V I Arnol'd**, *The cohomology ring of the group of dyed braids*, Mat. Zametki 5 (1969) 227–231 MR0242196

[2] **E Artin**, *Theorie der Zopfe*, Hamburg Abh. 4 (1925) 47–72

[3] **J S Birman**, *Braids, links, and mapping class groups*, Annals of Mathematics Studies, Princeton University Press (1974) MR0375281

[4] **F R Cohen**, **T J Lada**, **J P May**, *The homology of iterated loop spaces*, Lecture Notes in Mathematics 533, Springer, Berlin (1976) MR0436146

[5] **F Cohen**, **J Pakianathan**, *On automorphism groups of free groups, and their nilpotent quotients*, in preparation

[6] **B Farb**, *The Johnson homomorphisms for* $\mathrm{Aut}(F_n)$, in preparation

[7] **S Galatius**, *Stable homology of automorpism groups of free groups* arXiv:math.AT/0610216

[8] **A Hatcher**, **N Wahl**, *Stabilization for the automorphisms of free groups with boundaries*, Geom. Topol. 9 (2005) 1295–1336 MR2174267

[9] **N Kawazumi**, *On the stable cohomology algebra of extended mapping class groups for surfaces*, Hokkaido University Preprint Series in Mathematics, 311 (1995) Available at http://eprints.math.sci.hokudai.ac.jp/archive/00000473/

[10] **N Kawazumi**, *Certain cohomology classes on the automorphism groups of free groups*, Sūrikaisekikenkyūsho Kōkyūroku (1997) 35–42 MR1643725Analysis of discrete groups, II (Kyoto, 1996)

[11] **N Kawazumi**, *A generalization of the Morita-Mumford classes to extended mapping class groups for surfaces*, Invent. Math. 131 (1998) 137–149 MR1489896

[12] **N Kawazumi**, *Cohomological aspects of Magnus expansions* arXiv:math.GT/0505497

[13] **N Kawazumi**, **S Morita**, *The primary approximation to the cohomology of the moduli space of curves and cocycles for the stable characteristic classes*, Math. Res. Lett. 3 (1996) 629–641 MR1418577

[14] **G I Lehrer**, **L Solomon**, *On the action of the symmetric group on the cohomology of the complement of its reflecting hyperplanes*, J. Algebra 104 (1986) 410–424 MR866785

[15] **I Madsen**, **M Weiss**, *The stable moduli space of Riemann surfaces: Mumford's conjecture* arXiv:math.AT/0212321

[16] **E Y Miller**, *The homology of the mapping class group*, J. Differential Geom. 24 (1986) 1–14 MR857372

[17] **S Morita**, *Characteristic classes of surface bundles*, Invent. Math. 90 (1987) 551–577 MR914849

[18] **S Morita**, *A linear representation of the mapping class group of orientable surfaces and characteristic classes of surface bundles*, from: "Topology and Teichmüller spaces (Katinkulta, 1995)", World Sci. Publ. River Edge, NJ (1996) 159–186 MR1659679

[19] **D Mumford**, *Towards an enumerative geometry of the moduli space of curves*, from: "Arithmetic and geometry, Vol. II", Progr. Math. 36, Birkhäuser, Boston (1983) 271–328 MR717614

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