Twisted Morita–Mumford classes on braid groups

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Evaluating the twisted Morita–Mumford classes $\tilde{T}_p$ (Kawazumi [12]) on the Artin braid group $B_n$, we give the stable algebraic independence of the $\tilde{T}_p$'s on the automorphism group of the free group, $\text{Aut}(F_n)$. This is sharper than the results obtained by restricting them to the mapping class group (Kawazumi [9]).

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Introduction

In the cohomological study of the mapping class group for a surface, the Morita–Mumford classes, $e_i = (-1)^{i+1}k_i$, $i \geq 1$, [19; 17] play some important roles. As was proved by Miller [16] and Morita [17] independently, they are algebraically independent in the stable range $* < \frac{2}{3}g$. Madsen and Weiss [15] proved that the rational stable cohomology algebra of the mapping class groups, $H^*(M_\infty; \mathbb{Q})$, is generated by the Morita–Mumford classes. The Morita–Mumford classes have twisted variants, $m_{i,j} \in H^{2i+j-2}(M_{g,1}; \wedge^j H)$, $i, j \geq 0$, introduced by the author [11]. Here we denote by $\Sigma_{g,1}$ a 2–dimensional oriented compact connected $C^\infty$ manifold of genus $g$ with 1 boundary component, $M_{g,1}$ its mapping class group, $M_{g,1} := \pi_0\text{Diff}(\Sigma_{g,1}, \text{id on } \partial \Sigma_{g,1})$, and $H$ the integral first homology group of the surface $\Sigma_{g,1}$. The mapping class group $M_{g,1}$ acts on $H$ in an obvious way. The twisted variants also satisfy the algebraic independence. More precisely, the algebra $H^*(M_{g,1}; \wedge^* H) \otimes \mathbb{Q}$ is the polynomial algebra in the set $\{m_{i,j}; i \geq 0, j \geq 1, \text{ and } i + j \geq 2\}$ over the algebra $H^*(M_{g,1}; \mathbb{Q})$ in the range where the total degree $\leq \frac{2}{3}g$ (Kawazumi [9, Theorem 1.C].) Hence, from the theorem of Madsen and Weiss [15] stated above, the algebra $H^*(M_{g,1}; \wedge^* H) \otimes \mathbb{Q}$ is stably isomorphic to the polynomial algebra in the set $\{m_{i,j}; i \geq 0, j \geq 0, \text{ and } i + j \geq 2\}$ over $\mathbb{Q}$. Similar results hold for any other symplectic coefficients (Kawazumi [9, Theorem 1.B].) Furthermore all the cohomology classes on the mapping class group obtained by contracting the coefficients of the twisted ones using the intersection pairing $H^{\otimes 2} \to \mathbb{Z}$ are exactly the algebra generated by the (original) Morita–Mumford classes $e_i$’s (Morita [18], Kawazumi and Morita [13]).
Some of the twisted ones have the advantage over the original ones of being defined on the automorphism group of a free group, which has the mapping class group and the braid group as proper subgroups. Let \( n \geq 2 \) be an integer, \( F_n \) a free group of rank \( n \) with free basis \( x_1, x_2, \ldots, x_n \)

\[
F_n = \langle x_1, x_2, \ldots, x_n \rangle,
\]

and \( \text{Aut}(F_n) \) the automorphism group of the group \( F_n \). The Dehn–Nielsen theorem tells us the natural action of the group \( \mathcal{M}_{g,1} \) on the free group \( \pi_1(\Sigma_{g,1}) \) of rank 2\( g \) induces an injective homomorphism \( \mathcal{M}_{g,1} \to \text{Aut}(F_{2g}) \). In view of a theorem of Artin [2] the braid group \( B_n \) of \( n \) strings is embedded into the group \( \text{Aut}(F_n) \).

Now we denote by \( H \) and \( H^* \) the first integral homology and cohomology groups of the group \( F_n \)

\[
H := H_1(F_n; \mathbb{Z}) = F_n^\text{ab} = F_n/[F_n, F_n] \quad \text{and} \quad H^* := H^1(F_n; \mathbb{Z}) = \text{Hom}(H, \mathbb{Z}),
\]

respectively, on which the automorphism group \( \text{Aut}(F_n) \) acts in an obvious way. We write \([\gamma] := \gamma \mod [F_n, F_n] \in H\) for \( \gamma \in F_n \), and \( X_i := [x_i] \in H \) for \( i \), \( 1 \leq i \leq n \). In [12] we introduced cohomology classes

\[
h_p \in H^p(\text{Aut}(F_n); H^* \otimes H^\otimes(p+1)) \quad \text{and} \quad \overline{h}_p \in H^p(\text{Aut}(F_n); H^\otimes p)
\]

for \( p \geq 1 \). Restricted to the mapping class group \( \mathcal{M}_{g,1} \) they coincide with the twisted Morita–Mumford classes

\[
(p + 2)! h_p|_{\mathcal{M}_{g,1}} = m_{0,p+2} \in H^p(\mathcal{M}_{g,1}; H^\otimes(p+2)), \quad \text{and} \quad p! \overline{h}_p|_{\mathcal{M}_{g,1}} = -m_{1,p} \in H^p(\mathcal{M}_{g,1}; H^\otimes p).
\]

Here \( H \) and \( H^* \) are isomorphic to each other as \( \mathcal{M}_{g,1} \) modules because of the intersection pairing of the surface \( \Sigma_{g,1} \). The class \( p! \overline{h}_p \) can be regarded as an element in \( H^p(\text{Aut}(F_n); \bigwedge^p H) \).

In this note we confine ourselves to studying the behavior of \( \overline{h}_p \)'s restricted to the braid group \( B_n \), and consider the rational coefficients

\[
H_\mathbb{Q} := H \otimes \mathbb{Q} \quad \text{and} \quad H^*_\mathbb{Q} := H^* \otimes \mathbb{Q}.
\]

In this paper we prove the following result:

**Theorem 1.** The cohomology classes \( \overline{h}_p \)'s are algebraically independent in the algebra \( H^*(B_n; \bigwedge^* H_\mathbb{Q}) \) in the range where the total degree \( \leq n \).
Here the total degree of $\overline{h}_p$ is defined to be $2p$. Theorem 1 implies the algebraic independence on the automorphism group $\text{Aut}(F_n)$. This is sharper than that obtained by restricting them to the mapping class group $\mathcal{M}_{g,1}$ [9, Theorem 1.C], where the range is given by the inequality the total degree $\leq \frac{2g}{3} = \frac{1}{3}n$.

Theorem 1 was announced in [10]. Its proof given in Section 3 is based on some kind of primitiveness of the $h_p$’s (Proposition 1.2) and the evaluation of $h_n$ on the pure braid group of $n$ strings, $P_n$ (Lemma 2.4). In Section 4 we will give some remarks on the cohomology of the automorphism group $\text{Aut}(F_n)$.

1 Twisted Morita–Mumford classes on the automorphism group $\text{Aut}(F_n)$

Throughout this paper we denote by $C^*(G; M)$ the normalized standard complex of a group $G$ with values in a $G$–module $M$, and use the Alexander–Whitney cup product $\cup$: $C^*(G; M_1) \otimes C^*(G; M_2) \to C^*(G; M_1 \otimes M_2)$. Moreover we denote by $Z^p(G; M)$, $p \geq 0$, the $p$–cocycles in the cochain complex $C^*(G; M)$.

Now we recall the definition of the twisted cohomology classes $h_p$ and $\overline{h}_p$ on the automorphism group $\text{Aut}(F_n)$ for $p \geq 1$. The semi-direct product

$$\overline{A}_n := F_n \rtimes \text{Aut}(F_n)$$

admits an extension of groups

$$(1-1) \quad F_n \xrightarrow{\iota} \overline{A}_n \xrightarrow{\pi} \text{Aut}(F_n)$$

given by $\iota(\gamma) = (\gamma, 1)$ and $\pi(\gamma, \varphi) = \varphi$ for $\gamma \in F_n$ and $\varphi \in \text{Aut}(F_n)$. The map $k_0: \overline{A}_n \to H$, $(\gamma, \varphi) \mapsto [\gamma]$, satisfies the cocycle condition. We write also $k_0$ for the cohomology class $[k_0] \in H^1(\overline{A}_n; H)$. For each $p \geq 1$ we define $h_p$ by the image of the $(p+1)$-st power of the cohomology class $k_0$ under the Gysin map of the extension

$$(1-2) \quad h_p := \pi_\#(k_0 \otimes (p+1)) \in H^p(\text{Aut}(F_n); H^* \otimes H^{\otimes (p+1)})$$

[12]. Contracting the coefficients by the $\text{GL}(H)$–homomorphism

$$(1-3) \quad r_p: H^* \otimes H^{\otimes (p+1)} \to H^{\otimes p}, \quad f \otimes v_0 \otimes v_1 \otimes \cdots \otimes v_p \mapsto f(v_0)v_1 \otimes \cdots \otimes v_p,$$

we define

$$(1-4) \quad \overline{h}_p := r_p \circ (h_p) \in H^p(\text{Aut}(F_n); H^{\otimes p}).$$
The \( p \)-th exterior power \( k_0^p = p!k_0 \otimes p \) can be regarded as a cohomology class with coefficients in \( \wedge^p H \). Hence, if we consider the rational coefficients \( H_\mathbb{Q} \), we may regard \( h_p \) as a cohomology class in \( H^p(\text{Aut}(F_n); \wedge^p H_\mathbb{Q}) \).

A Magnus expansion \( \theta \) of the free group \( F_n \) gives an explicit cocycle representing the class \( h_p \). The completed tensor algebra generated by \( H \), \( \tilde{T} = \tilde{T}(H) := \prod_{m=0}^\infty H^\otimes m \), has a decreasing filtration of two-sided ideals \( \tilde{T}_p := \prod_{m \geq p} H^\otimes m \), \( p \geq 1 \). It should be remarked that the subset \( 1 + \tilde{T}_1 \) is a subgroup of the multiplicative group of the algebra \( \tilde{T} \). We call a map \( \theta : F_n \to 1 + \tilde{T}_1 \) a Magnus expansion of the free group \( F_n \), if \( \theta : F_n \to 1 + \tilde{T}_1 \) is a group homomorphism, and if \( \theta(\gamma) = 1 + [\gamma] \pmod{\tilde{T}_2} \) for any \( \gamma \in F_n \). We write \( \theta(\gamma) = \sum_{m=0}^\infty \theta_m(\gamma) \), \( \theta_m(\gamma) \in H^\otimes m \). The m-th component \( \theta_m : F_n \to H^\otimes m \) is a map, but not a group homomorphism. A Magnus expansion \( \text{std} : F_n \to 1 + \tilde{T}_1 \) is defined by \( \text{std}(x_i) := 1 + X_i \), \( 1 \leq i \leq n \). Here we denote \( X_i := [x_i] \in H \), the homology class of the generator \( x_i \). We call it the standard Magnus expansion. As is described in classical references, the value \( \text{std}(\gamma) \) for any word \( \gamma \in F_n \) is explicitly computed by means of Fox’ free differentials. All the results of this paper can be derived from the expansion \( \text{std} \).

We define a map \( \tau_1^\theta : \text{Aut}(F_n) \to H^* \otimes H^\otimes 2 \) by

\[
\tau_1^\theta(\varphi)[\gamma] = \theta_2(\gamma) - |\varphi| \otimes 2 \theta_2(\varphi^{-1}(\gamma)) \in H^\otimes 2
\]

for \( \gamma \in F_n \) and \( \varphi \in \text{Aut}(F_n) \). Here \( |\varphi| \in \text{GL}(H) \) is the automorphism of \( H = F_n^\text{abel} \) induced by \( \varphi \). This map \( \tau_1^\theta \) satisfies the cocycle condition [12, Lemma 2.1]. Now we introduce a \( \text{GL}(H) \)-homomorphism

\[
\zeta_p : (H^* \otimes H^\otimes 2)^\otimes p = \text{Hom}(H, H^\otimes 2)^\otimes p \to \text{Hom}(H, H^\otimes (p+1)) = H^* \otimes H^\otimes (p+1)
\]

for each \( p \geq 1 \). If \( p \geq 2 \), we define

\[
\zeta_p(u(1) \otimes u(2) \otimes \cdots \otimes u(p-1) \otimes u(p)) := (u(1) \otimes 1_H^\otimes (p-1)) \circ (u(2) \otimes 1_H^\otimes (p-2)) \circ \cdots \circ (u(p-1) \otimes 1_H) \circ u(p),
\]

where \( u(i) \in \text{Hom}(H, H^\otimes 2) = H^* \otimes H^\otimes 2 \), \( 1 \leq i \leq p \). In the case \( p = 1 \), we define \( \zeta_1 := 1_{H^* \otimes H^\otimes 2} \). Then we have:

**Theorem 1.1** [12, Theorem 4.1]

\[
h_p = \zeta_{p^*}([\tau_1^\theta] \in H^p(\text{Aut}(F_n); H^* \otimes H^\otimes (p+1))
\]

for any Magnus expansion \( \theta \) and each \( p \geq 1 \). In the case \( p = 1 \) we have \([\tau_1^\theta] = h_1 \in H^1(\text{Aut}(F_n); H^* \otimes H^\otimes 2)\).
Some kind of primitiveness of the cohomology classes \( h_p \) and \( \bar{h}_p \) follows from the theorem. We write simply \( A_n := \text{Aut}(F_n) \) for the remainder of the section. Suppose \( n_1 + n_2 \leq n \). Let \( A_{n_2} \) act on the words in the letters \( x_{n_1+1}, x_{n_1+2}, \ldots, x_{n_1+n_2} \) in an obvious way. Then we have a natural homomorphism

\[
i = i_{n_1,n_2} : A_{n_1} \times A_{n_2} \to A_n.
\]

We denote by \( \varpi_1 : A_{n_1} \times A_{n_2} \to A_{n_1} \) and \( \varpi_2 : A_{n_1} \times A_{n_2} \to A_{n_2} \) the first and the second projections of the product \( A_{n_1} \times A_{n_2} \), respectively, and by \( H_{(n_1)}, H_{(n_2)} \) and \( H_{(n-n_1-n_2)} \) the submodules of \( H \) spanned by \( \{X_1, \ldots, X_{n_1}\}, \{X_{n_1+1}, \ldots, X_{n_1+n_2}\} \) and \( \{X_{n_1+n_2+1}, \ldots, X_n\} \), respectively. Then we have a direct-sum decomposition \( H = H_{(n_1)} \oplus H_{(n_2)} \oplus H_{(n-n_1-n_2)} \), and can consider the map

\[
\varpi_k : H^*(A_{nk}; H_{(nk)}^* \otimes H_{(nk)}^{\otimes(p+1)}) \to H^*(A_{n_1} \times A_{n_2}; H^* \otimes H^{\otimes(p+1)}),
\]

for \( k = 1 \) and \( 2 \). For any \( p \geq 1 \) we have:

**Proposition 1.2**

1. \( i^*h_p = \varpi_1^*h_p + \varpi_2^*h_p \in H^p(A_{n_1} \times A_{n_2}; H^* \otimes H^{\otimes(p+1)}) \).
2. \( i^*\bar{h}_p = \varpi_1^*\bar{h}_p + \varpi_2^*\bar{h}_p \in H^p(A_{n_1} \times A_{n_2}; H^{\otimes p}) \).

**Proof** Using the standard expansion \( \text{std} \), we write simply

\[
i^{(k)} := \varpi_k^*\tau_{1}^{\text{std}} \in Z^1(A_{n_1} \times A_{n_2}; H^* \otimes H^{\otimes 2}).
\]

Clearly we have \( \text{std}(\gamma_1) \in \prod_{p=0}^{\infty} H_{(n_1)}^* \otimes p \subset \bar{T} \) for any word \( \gamma_1 \) in the letters \( x_1, \ldots, x_{n_1} \). Similar conditions hold for any word \( \gamma_2 \) in the letters \( x_{n_1+1}, \ldots, x_{n_1+n_2} \) and any \( \gamma_3 \) in \( x_{n_1+n_2+1}, \ldots, x_n \). Hence, from the definition of \( \tau_{1}^{\theta} \) (1–5), we have

\[
i^*\tau_{1}^{\text{std}} = \tau^{(1)} + \tau^{(2)} \in Z^1(A_{n_1} \times A_{n_2}; H^* \otimes H^{\otimes 2}).
\]

If we use the \( \text{GL}(H) \)-homomorphism \( \zeta_2 : (H^* \otimes H^{\otimes 2}) \otimes 2 \to H^* \otimes H^{\otimes 3} \) in (1–6), then we have

\[
(1-7) \quad \zeta_2^*(\tau^{(1)} \tau^{(2)}) = \zeta_2^*(\tau^{(2)} \tau^{(1)}) = 0 \in Z^2(A_{n_1} \times A_{n_2}; H^* \otimes H^{\otimes 3}).
\]

In fact, \( f(u) = 0 \) for any \( f \in H^*_{(n_1)} \) and \( u \in H_{(n_2)} \) and vice versa. From **Theorem 1.1** follows

\[
i^*h_p = \zeta_{p,*}(i^*[\tau_1^{\text{std}}] \otimes p) = \zeta_{p,*}(\tau^{(1)} \otimes p)
\]

\[
= \zeta_{p,*}(\tau^{(1)} \otimes p) + \zeta_{p,*}(\tau^{(2)} \otimes p) = \varpi_1^*h_p + \varpi_2^*h_p.
\]

Here \( \zeta_{p,*} \) of each mixed term in \( \tau^{(1)} \) and \( \tau^{(2)} \) vanishes by (1–7). Applying \( r_{p,*} \) to (1), we deduce (2). This completes the proof of the proposition. \( \square \)
2 Evaluation on the Artin braid groups

The \( n \)-th symmetric group \( S_n \) acts on the space \( \mathbb{C}^n \) by permuting the components. The open subset

\[
Y_n := \{ (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n; \ z_i \neq z_j \text{ for } i \neq j \}
\]

is stable under the action of the group \( S_n \). By definition, the Artin braid group of \( n \) strings, \( B_n \), is the fundamental group of the quotient space \( Y_n/S_n \), \( B_n := \pi_1(Y_n/S_n) \). As was shown by Artin [2], the group \( B_n \) admits a presentation

\[
\text{generators: } \sigma_i, \quad 1 \leq i \leq n-1,
\]

\[
\text{relations: } \sigma_i \sigma_j = \sigma_j \sigma_i, \quad \text{if } |i - j| \geq 2,
\]

\[
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \text{for } 1 \leq i \leq n-2.
\]

The pure braid group of \( n \) strings, \( P_n \), is defined to be the fundamental group of the space \( Y_n \), \( P_n := \pi_1(Y_n) \). We have a natural extension of groups

\[
P_n \to B_n \to \mathfrak{S}_n.
\]

As is known, \( A_{i,j}, 1 \leq i < j \leq n \), given by

\[
A_{i,j} := \sigma_{j-1} \sigma_{j-2} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-2}^{-1} \sigma_{j-1}^{-1}
\]

can serve as a generating system of the group \( P_n \). For details, see Birman [3].

The braid group \( B_n \) admits a natural homomorphism into the group \( \text{Aut}(\mathbb{F}_n) \), \( \xi: B_n \to \text{Aut}(\mathbb{F}_n) \). To recall how to construct it, we consider an action of the group \( \mathfrak{S}_n \) on the space \( Y_n = \mathbb{C}^n \), \( \mathfrak{S}_n ! Y_n = \mathbb{C}^n \times \mathbb{C} \) given by

\[
\rho(z_1, \ldots, z_n, z_{n+1}) = (z_{\rho^{-1}(1)}, \ldots, z_{\rho^{-1}(n)}, z_{n+1})
\]

for \( \rho \in \mathfrak{S}_n \). We denote by \( \hat{B}_n \) the fundamental group of the quotient space \( Y_{n+1}/\mathfrak{S}_n \), \( \hat{B}_n := \pi_1(Y_{n+1}/\mathfrak{S}_n) \).

The forgetful map \( Y_{n+1} \to Y_n \), \( (z_1, \ldots, z_n, z_{n+1}) \mapsto (z_1, \ldots, z_n) \), induces a fibration

\[
\mathbb{C} \setminus \{ \text{points} \} \to Y_{n+1}/\mathfrak{S}_n \to Y_n/\mathfrak{S}_n
\]

with a section \( s: Y_n/\mathfrak{S}_n \to Y_{n+1}/\mathfrak{S}_n \) given by \( (z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_n, \frac{1}{n} \sum_{i=1}^n z_i + \sum_{j=1}^n (z_j - \frac{1}{n} \sum_{i=1}^n z_i) ) \) (Arnol’d [1]). This fibration with the section \( s \) induces an extension of groups

\[
F_n \to \hat{B}_n \to B_n
\]
with a split homomorphism \( s \colon B_n \to \widehat{B}_n \). Thus we obtain a morphism of extensions of groups

\[
\begin{array}{ccc}
F_n & \longrightarrow & \widehat{B}_n \\
\downarrow \xi & & \downarrow \hat{\xi} \\
F_n & \longrightarrow & \widehat{A}_n \\ & \longrightarrow & \text{Aut}(F_n).
\end{array}
\]

The homomorphisms \( \xi \) and \( \hat{\xi} \) are explicitly given by

\[
\xi(x)(y) = s(x)ys(x)^{-1}
\]

\[
\hat{\xi}(\gamma s(x)) = (\gamma, \xi(x)) \in F_n \times \text{Aut}(F_n) = \widehat{A}_n
\]

for \( x \in B_n \) and \( \gamma \in F_n \). The group \( \widehat{B}_n \) is embedded into \( B_{n+1} \) in an obvious way. Then the homomorphisms \( s \) and \( \iota \) are described as

\[
\iota(x_j) = \sigma_n \sigma_{n-1} \cdots \sigma_{j+1} \sigma_j \sigma_{j+1}^{-1} \cdots \sigma_{n-1}^{-1} \sigma_n^{-1}
\]

\[
= A_{j,n+1} \quad \text{for } 1 \leq j \leq n
\]

in terms of the presentation (2–1). So the homomorphism \( \xi \) is explicitly given by

\[
\xi(\sigma_i)(x_j) = \begin{cases} 
  x_{i+1}, & \text{if } j = i, \\
  x_{i+1}^{-1}x_ix_{i+1}, & \text{if } j = i+1, \\
  x_j, & \text{otherwise}.
\end{cases}
\]

We now evaluate the cohomology classes \( h_1 \) and \( \overline{h}_{n-1} \) on the braid group \( B_n \). Here we use the standard Magnus expansion \( \text{std} \colon F_n \to 1 + \widehat{T}_1 \) introduced in Section 1. For the rest of this section we write simply \( k_0, \tau_1, h_p \) and \( \overline{h}_p \) for \( \xi^* k_0, \xi^* \tau_1^\text{std}, \xi^* h_p \) and \( \xi^* \overline{h}_p \), respectively. Let \( \{X_i\}_{i=1}^n \subset H^* \) denote the dual basis of \( \{X_i\}_{i=1}^n = \{[x_i]_{i=1}^n \subset H \}

Lemma 2.1

\[
\tau_1(\sigma_i) = l_i \otimes (X_i \otimes X_{i+1} - X_{i+1} \otimes X_i) \in H^* \otimes H^\otimes 2
\]

Proof From (1–5)

\[
\tau_1(\sigma_i) = \sum_{j=1}^n l_j \otimes (\text{std}_2(x_j) - |\sigma_i|^\otimes 2 \text{std}_2(\sigma_i^{-1}(x_j)))
\]

\[
= -l_i \otimes |\sigma_i|^\otimes 2 \text{std}_2(\sigma_i^{-1}(x_i)) - l_{i+1} \otimes |\sigma_i|^\otimes 2 \text{std}_2(\sigma_i^{-1}(x_{i+1}))
\]

\[
= -l_i \otimes |\sigma_i|^\otimes 2 \text{std}_2(x_ix_{i+1}x_i^{-1}) - l_{i+1} \otimes |\sigma_i|^\otimes 2 \text{std}_2(x_i)
\]

\[
= -l_i \otimes |\sigma_i|^\otimes 2 \text{std}_2(x_ix_{i+1}x_i^{-1}).
\]
On the other hand, we have
\[ \text{std}_2(x_i x_{i+1} x_i^{-1}) = X_i \otimes X_{i+1} - X_{i+1} \otimes X_i. \]
In fact, \( X_i \otimes X_{i+1} = \text{std}_2(x_i x_{i+1}) = \text{std}_2(x_i x_{i+1} x_i^{-1} x_i) = \text{std}_2(x_i x_{i+1} x_i^{-1}) + \text{std}_2(x_i) + X_{i+1} \otimes X_i = \text{std}_2(x_i x_{i+1} x_i^{-1}) + X_{i+1} \otimes X_i. \) Therefore we obtain \( \tau_1(\sigma_j) = -i \otimes |\sigma_j|^2 (x_i \otimes X_{i+1} - X_{i+1} \otimes X_i) = -i \otimes (X_{i+1} \otimes X_i - X_i \otimes X_{i+1}), \) as was to be shown.

The pure braid group \( \mathcal{P}_n \) acts on the homology \( H \) trivially. Hence, from [12, Theorem 3.1], the restriction of \( \tau_1 \) to \( \mathcal{P}_n \) does not depend on the choice of Magnus expansions.

**Lemma 2.2**
\[ \tau_1(A_{i,j}) = (l_i - l_j) \otimes (X_i \otimes X_j - X_j \otimes X_i) \]

**Proof** Recall the map \( \tau_1 \) satisfies the cocycle condition on the automorphism group \( \text{Aut}(\mathcal{P}_n) \). When we set \( \gamma := \sigma_{j-1} \sigma_{j-2} \cdots \sigma_{i+1} \), we have \( A_{i,j} = \gamma \sigma_i \cdot \gamma^{-1} \), so that
\[
\begin{align*}
\tau_1(A_{i,j}) &= \tau_1(\gamma \sigma_i \cdot \gamma^{-1}) = \tau_1(\gamma) + \gamma \tau_1(\sigma_i) + \gamma \sigma_i \cdot \tau_1(\gamma^{-1}) \\
&= \tau_1(\gamma) + \gamma \tau_1(\sigma_i) + \gamma \tau_1(\gamma^{-1}) = \tau_1(\gamma) + \gamma \tau_1(\sigma_i) = \gamma \tau_1(\sigma_i) \\
&= \gamma(l_i \otimes (X_i \otimes X_{i+1} - X_{i+1} \otimes X_i)) + \gamma \sigma_i(l_i \otimes (X_i \otimes X_{i+1} - X_{i+1} \otimes X_i)) \\
&= \gamma((l_i - l_{i+1}) \otimes (X_i \otimes X_{i+1} - X_{i+1} \otimes X_i)) \\
&= (l_i - l_j) \otimes (X_i \otimes X_j - X_j \otimes X_i),
\end{align*}
\]
as was to be shown.

To prove the nontriviality of \( \mathcal{H}_n \) on the group \( B_n \), we recall some basic facts on the cohomology of the pure braid group \( \mathcal{P}_n \). The space \( Y_n \) is an Eilenberg–MacLane space of type \( (\mathcal{P}_n, 1) \). The subspace \( Y_n \cap \{z_1 + \cdots + z_n = 0\} \) is a deformation retract of the space \( Y_n \) and a Stein manifold of complex dimension \( n - 1 \). Hence the cohomological dimension of the group \( \mathcal{P}_n \), \( \text{cd} \mathcal{P}_n \), is not greater than \( n - 1 \). Let \( A^*(Y_n) \) be the algebra of all the complex-valued differential forms on the space \( Y_n \). As was shown by Arnol’d [1], the \( \mathbb{Z} \)–subalgebra generated by the 1–forms
\[
\omega_{i,j} := \frac{1}{2\pi \sqrt{-1}} \frac{dz_i - dz_j}{z_i - z_j}, \quad 1 \leq i < j \leq n,
\]
is isomorphic to the cohomology algebra \( H^*(Y_n; \mathbb{Z}) = H^*(\mathcal{P}_n; \mathbb{Z}) \). Especially in the case \( * = 1 \), \( \{\omega_{i,j}\}_{1 \leq i < j \leq n} \) is a \( \mathbb{Z} \)–free basis of \( H^1(\mathcal{P}_n; \mathbb{Z}) \), so that \( \{A_{i,j}\}_{1 \leq i < j \leq n} \) is a \( \mathbb{Z} \)–free basis of \( H_1(\mathcal{P}_n; \mathbb{Z}) = \mathcal{P}_n \).
Lemma 2.3

(1) \( k_0^n \neq 0 \in H^n(Y_{n+1}; \bigwedge^n H_Q) \), where \( P_{n+1} = \pi_1(Y_{n+1}) \) is regarded as a subgroup of \( \widehat{\mathbb{S}}_n = \pi_1(Y_{n+1}/\mathbb{S}_n) \).

(2) \( h_{n-1} \neq 0 \in H^{n-1}(P_n; H^*_Q \otimes \bigwedge^n H_Q) \).

Proof (1) From (2–3) and (2–4) we have

\[
k_0(A_{i,j}) = \begin{cases} 
0, & \text{if } i < j \leq n, \\
X_i, & \text{if } i < j = n+1,
\end{cases}
\]

that is

\[
k_0 = \sum_{i=1}^{n} \omega_{i,n+1} \otimes X_i \in H^1(Y_{n+1}; H).
\]

If we restrict the \( n \)-form

\[
\omega_{1,n+1} \omega_{2,n+1} \cdots \omega_{n,n+1} = (1/2\pi \sqrt{-1})^n \prod_{i=1}^{n} (dz_i - dz_{n+1})/(z_i - z_{n+1})
\]

to the subspace \( Y_{n+1} \cap \{z_{n+1} = 0\} \), then we obtain the non-zero \( n \)-form \((1/2\pi \sqrt{-1})^n \prod_{i=1}^{n} (dz_i/z_i) \). Hence the cohomology class

\[
k_0^n = n! \omega_{1,n+1} \omega_{2,n+1} \cdots \omega_{n,n+1} X_1 \wedge X_2 \wedge \cdots \wedge X_n \in H^n(Y_{n+1}; \bigwedge^n H_Q)
\]

does not vanish, as was to be shown.

(2) Since \( \text{cd} P_n \leq n - 1 \), the Gysin map of the extension

\[
F_n \xrightarrow{i} P_{n+1} \xrightarrow{\pi} P_n
\]

gives an isomorphism

\[
\pi_*: H^n(P_{n+1}; M) \cong H^{n-1}(P_n; H^* \otimes M)
\]

for any \( P_n \)-module \( M \). Hence \( h_{n-1} = \pi_* k_0^n \neq 0 \) by (1). \( \square \)

The map \( r_n: H^*_Q \otimes \bigwedge^n H_Q \to \bigwedge^{n-1} H_Q \) is an isomorphism because \( \dim_Q H_Q = n \).

Hence we obtain:

Lemma 2.4

\[
\overline{h}_{n-1} \neq 0 \in H^{n-1}(P_n; \bigwedge^{n-1} H_Q).
\]
3 Proof of Theorem 1

Our proof of Theorem 1 is based on Proposition 1.2 and Lemma 2.4. For $q \leq n$ we denote by $\mathcal{P}_{n-q}(q)$ the set of all the non-negative partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-q} \geq 0)$ of $q$ into $n-q$ parts. For $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-q} \geq 0) \in \mathcal{P}_{n-q}(q)$ we introduce a cohomology class $\overline{h}_\lambda$ and a subgroup $P_\lambda \subset P_n$ by

$$
\overline{h}_\lambda := \overline{h}_{\lambda_1} \overline{h}_{\lambda_2} \cdots \overline{h}_{\lambda_{n-q}} \in H^q(B_n; \bigwedge^q H_Q) \subset H^q(P_n; \bigwedge^q H_Q), \quad \text{and} \quad P_\lambda := P_{\lambda_1+1} \times P_{\lambda_2+1} \times \cdots \times P_{\lambda_{n-q}+1} \subset P_n.
$$

respectively. Here $P_{0+1} = P_1$ is the trivial group $\{1\}$. Denote by $\iota_\lambda : P_\lambda \hookrightarrow P_n$ the obvious inclusion map and $\sigma_k : P_\lambda \to P_{\lambda_k+1}$ the obvious projection. Theorem 1 follows from:

**Theorem 3.1** The cohomology classes $\{\overline{h}_\lambda : \lambda \in \mathcal{P}_{n-q}(q)\}$ are linearly independent in $H^q(P_n; \bigwedge^q H_Q)$.

In fact, when $q \leq n/2$, the set of all the non-negative partitions of $q$ into $n-q$ parts does not depend on $n$.

Endow the partitions $\mathcal{P}_{n-q}(q)$ with the lexicographic order. For example, $(q \geq 0 \geq \cdots \geq 0)$ is the maximal partition. Theorem 3.1 is reduced to the following

**Assertions** For any $\lambda$ and $\mu \in \mathcal{P}_{n-q}(q)$ we have:

(A) $\iota_\lambda^* \overline{h}_\lambda \neq 0 \in H^q(P_\lambda; \bigwedge^q H_Q)$

(B) If $\mu \geq \lambda$, then $\iota_\lambda^* \overline{h}_\mu = 0 \in H^q(P_\lambda; \bigwedge^q H_Q)$.

In fact, assume we have a nontrivial linear relation

$$
\sum_{\lambda \in \mathcal{P}_{n-q}(q)} c_\lambda \overline{h}_\lambda = 0 \in H^q(P_n; \bigwedge^q H_Q).
$$

Choose the minimum $\lambda$ satisfying $c_\lambda \neq 0$. Applying $\iota_\lambda^*$ to the relation, we obtain $c_\lambda \iota_\lambda^* \overline{h}_\lambda = 0$ from Assertion (B). Assertion (A) implies $c_\lambda = 0$, which contradicts the choice of $\lambda$.

**Proof of Assertion (A)** Let $b_1 \geq b_2 \geq \cdots \geq b_{\lambda_1} > b_{\lambda_1+1} = 0$ be the dual partition of $\lambda$. The number of $\lambda_k$’s equal to $p$ is $b_p - b_{p+1}$. We abbreviate $\overline{h}_{p,k} := \sigma_k^* \overline{h}_p$. Since $c d P_{\lambda_k+1} \leq \lambda_k$, we have $\overline{h}_{p,k} = 0$ if $p > \lambda_k$, or equivalently, $k > b_p$. Moreover
we have $\bar{h}_{\lambda,k} \cdot \bar{h}_{p,k} = 0$ for any $p \geq 1$ since $H^{\lambda_k + p}(P_{\lambda + 1}; \bigwedge^{\lambda_k + p} H_Q) = 0$. From Proposition 1.2 we have

$$\iota_{\lambda}^* h_p = \sum_{k=1}^{\mu - q} \bar{h}_{p,k} \in H^p(P_{\lambda}; \bigwedge^p H),$$

so that

$$\iota_{\lambda}^* h_{\lambda} = \prod_{p=1}^{\mu - q} (\bar{h}_{p,1} + \bar{h}_{p,2} + \cdots + \bar{h}_{p,n-q})^{b_p - b_{p+1}}$$

$$= \prod_{p=1}^{\lambda_1} (\bar{h}_{p,1} + \bar{h}_{p,2} + \cdots + \bar{h}_{p,p})^{b_p - b_{p+1}} = \prod_{p=1}^{\lambda_1} (\bar{h}_{p,b_{p+1}} + \cdots + \bar{h}_{p,p})^{b_p - b_{p+1}}$$

$$\prod_{p=1}^{\lambda_1} \left( (b_p - b_{p+1})! \bar{h}_{p,b_{p+1}} \cdots \bar{h}_{p,p} \right) = \left( \prod_{p=1}^{\lambda_1} (b_p - b_{p+1})! \right) \bar{h}_{\lambda_1,1} \bar{h}_{\lambda_2,2} \cdots \bar{h}_{\lambda_{n-q},n-q}.$$

Here the fifth equal sign comes from the equation $\bar{h}_{\lambda,k} \cdot \bar{h}_{p,k} = 0$. Clearly $r_\lambda := \prod_{p=1}^{\lambda_1} (b_p - b_{p+1})!$ is a positive integer. From Lemma 2.4 and the Künneth formula $\bar{h}_{\lambda_1,1} \bar{h}_{\lambda_2,2} \cdots \bar{h}_{\lambda_{n-q},n-q} \neq 0 \in H^q(P_{\lambda}; \bigwedge^q H_Q).$ This proves Assertion (A).

**Proof of Assertion (B)** Suppose $\mu > \lambda$ with respect to the lexicographic order, namely, $\mu_1 = \lambda_1 \geq \mu_2 = \lambda_2 \geq \cdots \geq \mu_h = \lambda_h \geq \mu_{h+1} > \lambda_{h+1}$ for some $\lambda, 0 \leq h < n-q$. Let $v := (v_1 \geq v_2 \geq \cdots \geq v_h)$ be the (truncated) partition of $q := \lambda_1 + \lambda_2 + \cdots + \lambda_h$ defined by $v_k := \lambda_k = \mu_k, k \leq h$. From Assertion (A)

$$\iota_{\lambda}^* (\bar{h}_{\mu_1} \bar{h}_{\mu_2} \cdots \bar{h}_{\mu_h}) = r_v \bar{h}_{\mu_1,1} \bar{h}_{\mu_2,2} \cdots \bar{h}_{\mu_h,h} \in H^{q'}(P_{\lambda}; \bigwedge^{q'} H).$$

In fact, from $\mu_h > \lambda_{h+1}$, we have $\bar{h}_{\mu_i,j} = 0$ if $i < j$. Since $\mu_{h+1} \geq \lambda_k$ for any $k \geq h + 1$, we have

$$\iota_{\lambda}^* (\bar{h}_{\mu_1} \cdots \bar{h}_{\mu_h}) = r_v \bar{h}_{\mu_1,1} \cdots \bar{h}_{\mu_h,h}(\bar{h}_{\mu_{h+1},1} + \cdots + \bar{h}_{\mu_{h+1},h}) = 0$$

Hence $\iota_{\lambda}^* (\bar{h}_{\mu}) = 0$, as was to be shown.

This completes the proof of Theorem 3.1 and Theorem 1.
4 Concluding remarks

We conclude this note by giving some remarks on the twisted cohomology of the automorphism group $\text{Aut}(F_n)$ and the braid group $B_n$.

The IA–automorphism group $IA_n$ is defined to be the kernel of the action of the group $\text{Aut}(F_n)$ on the homology group $H = F_n^{\text{abel}}$. We have an extension of groups $IA_n \to \text{Aut}(F_n) \to \text{GL}(H)$. The map $\tau^0_1$ restricted to $IA_n$ gives an isomorphism of the abelianization of the group $IA_n$ onto the module $H^* \otimes \wedge^2 H$

(Cohen and Pakianathan [5], Farb [6], Kawazumi [12]). Here we embed $\wedge^2 H$ into $H^\otimes 2$ by $X_i \wedge X_j \mapsto X_i \otimes X_j - X_j \otimes X_i$ for $1 \leq i, j \leq n$. Lemma 2.2 implies $\xi^*: H^1(IA_n; \mathbb{Z}) \to H^1(P_n; \mathbb{Z})$ is surjective. From the result of Arnol’d [1] quoted in Section 2, the cohomology algebra $H^*(P_n; \mathbb{Z})$ is generated by the first cohomology classes. Hence we obtain:

**Corollary 4.1** The algebra homomorphism

$$\xi^*: H^*(IA_n; \mathbb{Z}) \to H^*(P_n; \mathbb{Z})$$

induced by the homomorphism $\xi: P_n \to IA_n$ is surjective.

It should be remarked that it does not imply that the map $\xi^*: H^*(\text{Aut}(F_n); M) \to H^*(B_n; M)$ is surjective for a $\mathbb{Q}[\text{GL}(H)]$–module $M$. In fact, the quotient groups $\text{Aut}(F_n)/IA_n = \text{GL}(H)$ and $B_n/P_n = \mathbb{S}_n$ differ from each other.

Fred Cohen [4, Lemma 7.2, page 261] described the action of the symmetric group $\mathbb{S}_n$ on the integral cohomology of the group $P_n$, $H^*(P_n; \mathbb{Z})$. Later Lehrer and Solomon [14] gave another explicit description of the $\mathbb{Q}[\mathbb{S}_n]$–module $H^*(P_n; \mathbb{Q})$. Moreover Cohen [4, Theorem 3.1, page 225] computed the twisted cohomology $H^*(B_n; H^\otimes m \otimes \mathbb{F})$ for any field $\mathbb{F}$ and any $m \geq 0$. It would be interesting if one could describe the submodule of $H^*(B_n; M)$ generated by all the possible algebraic combinations coming from the twisted Morita–Mumford classes $h_p$’s in an explicit manner. Here we should remark the $\mathbb{S}_n$–invariant inner product $\cdot : H \otimes H \to \mathbb{Z}$ defined by $X_i \cdot X_j = \delta_{i,j}$, $1 \leq i, j \leq n$, gives a $B_n$–isomorphism $H \cong H^*$.

As was stated in Introduction, the algebra $H^*(\mathcal{M}_{g,1}; \wedge^* H\mathbb{Q})$ is stably isomorphic to the polynomial algebra in the twisted Morita–Mumford classes $m_{i,j}$’s. The intersection pairing of the surface $\Sigma_{g,1}$, $H^\otimes 2 \to \mathbb{Z}$, gives an isomorphism $H \cong H^*$ of $\mathcal{M}_{g,1}$–modules, so that the cocycle $\tau^0_1$ restricted to $\mathcal{M}_{g,1}$ can be regarded as a cocycle.
Twisted Morita–Mumford classes on braid groups

As was proved by Kawazumi and Morita in [13], for any twisted Morita–Mumford class $m_{i,j}$ we have an $\mathcal{M}_{g,1}$–homomorphism $C: (H^3)^{(2i+j-2)} \to \mathbb{Z}$ obtained from the intersection pairing such that $C[M_{g,1}]^2 = m_{i,j}$. In other words, the natural map

$$(\bigwedge^* H^1(I_{g,1};\mathbb{Q})) \otimes M^{\text{Sp}(H)} \to H^*(\mathcal{M}_{g,1};M)$$

is stably surjective for any finite dimensional $\mathbb{Q}[\text{Sp}(H)]$–module $M$. Here $I_{g,1}$ is the Torelli group, i.e, the kernel of the action of $\mathcal{M}_{g,1}$ on the homology $H$.

Recently Galatius [7] proved the rational reduced cohomology $\tilde{H}^*(\text{Aut}(F_n);\mathbb{Q})$ vanishes in a stable range. It would be very interesting to know whether a similar result holds also for twisted coefficients.

**Expectation 4.2** For a finite dimensional $\mathbb{Q}[\text{GL}(H)]$–module $M$, the natural map

$$(\bigwedge^* H^1(I_{A_n};\mathbb{Q})) \otimes M^{\text{GL}(H)} \to H^*(\text{Aut}(F_n);M)$$

is surjective in some stable range.

In the case $M$ is the trivial module $\mathbb{Q}$, this expectation is exactly the fact that $\tilde{H}^*(\text{Aut}(F_n);\mathbb{Q})$ vanishes in some stable range, which Galatius [7] proved. A result of Hatcher and Wahl [8] tells us it holds also for $M = (H^*)^{\otimes m}$ for any $m \geq 1$.

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