Twisted Alexander polynomials and a partial order on the set of prime knots

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We give a survey of some recent papers by the authors and Masaaki Wada [6; 7; 8] relating the twisted Alexander polynomial with a partial order on the set of prime knots. We also give examples and pose open problems.

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Dedicated to Fred Cohen on the occasion of his 60th birthday

1 Introduction

The twisted Alexander polynomial was introduced by Lin [9] and Wada [18] independently. Lin defined it for a knot by using a regular Seifert surface and Wada defined it for a finitely presentable group by using Fox's free differential calculus. More generally, Jiang and Wang [3] studied twisted topological invariants for a 3–manifold by using representations of the fundamental group. Following this work, much research from many viewpoints has been carried out (or is currently being carried out) related to this invariant. See the bibliography for more precise details.

In this paper, we give a survey of some recent papers by the authors and Masaaki Wada [6; 7; 8] relating the twisted Alexander polynomial with a partial order on the set of prime knots. We also give examples and pose open problems.

In Section 2 we recall the definition of the twisted Alexander polynomial for a finitely presentable group as given by Wada. His definition is purely algebraic.

In Section 3 we state one of the fundamental results, which gives a relation between the existence of a surjection of groups and the twisted Alexander polynomial; see [8] for more details. This result gives us a new criterion for the non-existence of a surjective homomorphism between two groups. The corresponding fact about the (classical) Alexander polynomial is well known; see Crowell and Fox [1].

In Section 4 we explain through examples how to use twisted Alexander polynomials to determine the partial order on the set of prime knots defined by the existence of a

surjective homomorphism between the knot groups. Using the result of Section 3, we determined in [6] which pairs of prime knots from Rolfsen's knot table [15] have this relation. Rolfsen's knot table lists all the prime knots of 10 crossings or less.

In Section 5, we consider some problems which arise naturally from the results of Section 4. The existence problem of a surjective homomorphism between knot groups, or more generally 3–manifold groups, is related with the theory of degree one maps and the period of a knot.

In Section 6 we pose further problems for future study.

2 Definition of the twisted Alexander polynomial

In this section, we briefly recall the algebraic definition of the twisted Alexander polynomial, by using Fox's free differential calculus.

Let G be a finitely presentable group. Choose a presentation of G:

$$G = \langle x_1, \ldots, x_u \mid r_1, \ldots, r_v \rangle.$$

Suppose $\alpha\colon G \to \langle\, t_1, \ldots, t_l \mid t_i t_j = t_j t_i \,\rangle \cong \mathbb{Z}^l$ is a surjective homomorphism to the free abelian group with generators $t_1, \ldots, t_l$ and $\rho\colon G \to GL(n; R)$ a linear representation, where R is a unique factorization domain. These maps naturally induce ring homomorphisms $\widetilde{\rho}$ and $\widetilde{\alpha}$ from $\mathbb{Z}[G]$ to $\mathbb{Z}[GL(n; R)] = M(n; R)$ and $\mathbb{Z}[t_1{}^{\pm 1}, \ldots, t_l{}^{\pm 1}]$ respectively, where $M(n; R)$ denotes the matrix algebra of degree n over R. Then $\widetilde{\rho} \otimes \widetilde{\alpha}$ defines a ring homomorphism

$$\mathbb{Z}[G] \to M\left(n; R[t_1{}^{\pm 1}, \ldots, t_l{}^{\pm 1}]\right).$$

Let F_u be the free group on generators $x_1, \ldots, x_u$ and let

$$\Phi\colon \mathbb{Z}[F_u] \to M\left(n; R[t_1{}^{\pm 1}, \ldots, t_l{}^{\pm 1}]\right)$$

be the composite of the surjection $\mathbb{Z}[F_u] \to \mathbb{Z}[G]$ induced by the fixed presentation and the map $\widetilde{\rho} \otimes \widetilde{\alpha}\colon \mathbb{Z}[G] \to M(n; R[t_1{}^{\pm 1}, \ldots, t_l{}^{\pm 1}])$.

We define the $v \times u$ matrix M whose (i, j) component is the $n \times n$ matrix

$$\Phi\left(\frac{\partial r_i}{\partial x_j}\right) \in M\left(n; R[t_1{}^{\pm 1}, \ldots, t_l{}^{\pm 1}]\right),$$

where $\partial/\partial x$ denotes the Fox differential. This matrix M is called the Alexander matrix of the fixed presentation of G associated to the representation ρ.

It is easy to see that there is an integer $1 \leq j \leq u$ such that $\det \Phi(x_j - 1) \neq 0$. For such j, let us denote by M_j the $v \times (u-1)$ matrix obtained from M by removing the j-th column. We regard M_j as an $nv \times n(u-1)$ matrix with coefficients in $R[t_1^{\pm 1}, \ldots, t_l^{\pm 1}]$. Moreover, for an $n(u-1)$–tuple of indices

$$I = (i_1, i_2, \ldots, i_{n(u-1)}) \quad (1 \leq i_1 < i_2 < \cdots < i_{n(u-1)} \leq nv),$$

we denote by M_j^I the $n(u-1) \times n(u-1)$ square matrix consisting of the i_k-th rows of the matrix M_j, where $k = 1, 2, \ldots, n(u-1)$.

Then the twisted Alexander polynomial of a finitely presented group G for a representation $\rho \colon G \to GL(n; R)$ is defined to be a rational expression

$$\Delta_{G,\rho}(t_1, \ldots, t_l) = \frac{\gcd_I(\det M_j^I)}{\det \Phi(x_j - 1)}.$$

Here $\gcd_I(\det M_j^I)$ is the greatest common divisor of $\{\det M_j^I \mid I \subset \{1, \ldots, nv\}\}$. It is shown that $\Delta_{G,\rho}(t_1, \ldots, t_l)$ is independent of the choice of j such that $\det \Phi(x_j - 1) \neq 0$ up to a factor $\epsilon t_1^{\varepsilon_1} \cdots t_l^{\varepsilon_l}$, where $\epsilon \in R^\times$, $\varepsilon_i \in \mathbb{Z}$. Moreover, we can check that Tietze transformations on the presentation of G do not affect the twisted Alexander polynomial. Therefore the twisted Alexander polynomial $\Delta_{G,\rho}(t_1, \ldots, t_l)$ is independent of the choice of the presentation of G. See Wada [18] for details here.

3 Twisted Alexander polynomials and surjectivity of a group homomorphism

In this section, we present the following theorem from [8], which is one of the fundamental theorems, for applications [6; 7] of twisted Alexander polynomials.

Theorem 3.1 *Let G and G' be finitely presentable groups and α, α' surjective homomorphisms from G, G' to $\mathbb{Z}^l$ respectively. Suppose that there exists a surjective homomorphism $\varphi \colon G \to G'$ such that $\alpha = \alpha' \circ \varphi$. Then $\Delta_{G,\rho}$ is divisible by $\Delta_{G',\rho'}$ for any representation $\rho' \colon G' \to GL(n; R)$, where $\rho = \rho' \circ \varphi$. More precisely, the quotient of $\Delta_{G,\rho}$ by $\Delta_{G',\rho'}$ is a Laurent polynomial in $R[t_1^{\pm 1}, \ldots, t_l^{\pm}]$.*

The main motivation here is the following. Let $G(K)$ be the knot group $\pi_1(S^3 - K)$ of a knot K in S^3. For any knots K, K', if there exists a surjective homomorphism from $G(K)$ to $G(K')$, then the Alexander polynomial of K is divisible by that of K'. Murasugi [12] mentions that if there exists a surjective homomorphism from a knot group $G(K)$ to the trefoil knot group, then the twisted Alexander polynomial of $G(K)$

is divisible by that of the trefoil knot group. Theorem 3.1 is a generalization of his result.

Theorem 3.1 is proved by two different methods in [6]. The first one is purely algebraic using the definition of the twisted Alexander polynomial and linear algebra. The second one uses Reidemeister torsion. The twisted Alexander polynomial of a knot group may be interpreted in terms of Reidemeister torsion in the same way as the classical Alexander polynomial. These invariants of knots have been studied from the Reidemeister torsion viewpoint. For example, see Milnor [11], Kitano [5] or Kirk and Livingston [4].

We now consider an easy algebraic situation as an example.

Example 3.2 Let $G = \langle x_1, \ldots, x_u | r_1, \ldots, r_v \rangle$ be a finitely presented group with the abelianization $\alpha \colon G \to \mathbb{Z}^l$ and $\rho \colon G \to GL(n; R)$ a representation of G. Another group G' is defined by

$$G' = \langle x_1, \ldots, x_u | r_1, \ldots, r_v, s \rangle$$

where s is a word of $x_1, \ldots, x_u$. The natural projection $\pi \colon G \to G'$ is surjective. Suppose that the word s belongs to $\ker \alpha$ and $\ker \rho$, then there exists a surjective homomorphism $\alpha' \colon G' \to \mathbb{Z}^l$ and a representation $\rho' \colon G' \to GL(n; R)$ such that $\alpha = \alpha' \circ \pi$ and $\rho = \rho' \circ \pi$. By Theorem 3.1, $\Delta_{G,\rho}$ is divisible by $\Delta_{G',\rho'}$. Here we will verify the divisibility by the definition. Let M be the Alexander matrix of G associated to ρ. Then the Alexander matrix M' of G' associated to ρ' is obtained by adding n rows

$$\left(\Phi\left(\frac{\partial s}{\partial x_1}\right) \ \Phi\left(\frac{\partial s}{\partial x_2}\right) \ \cdots \ \Phi\left(\frac{\partial s}{\partial x_u}\right) \right)$$

to M. Since the numerator of $\Delta_{G',\rho'}$ is the greatest common divisor of $\det M'^{I'}_j$, $\gcd_{I'}(\det M'^{I'}_j)$ is a divisor of $\gcd_I(\det M^I_j)$. Furthermore, the denominators of $\Delta_{G,\rho}$ and $\Delta_{G',\rho'}$ are the same. Hence $\Delta_{G,\rho}$ is divisible by $\Delta_{G',\rho'}$.

4 A partial order in the knot table

In this section we consider prime knots. Let K be a prime knot and $G(K)$ its knot group. A partial order on the set of prime knots is defined as follows. For two prime knots K_1, K_2, we write $K_1 \geq K_2$ if there exists a surjective group homomorphism from $G(K_1)$ onto $G(K_2)$. It is known that the relation $\geq$ satisfies the condition of a partial order on the set of prime knots. The following result [6] can be obtained by applying Theorem 3.1 and using a computer.

Theorem 4.1 *The above partial order on the knots in Rolfsen's table is given as below:*

$8_5, 8_{10}, 8_{15}, 8_{18}, 8_{19}, 8_{20}, 8_{21}, 9_1, 9_6, 9_{16}, 9_{23}, 9_{24}, 9_{28}, 9_{40},$
$10_5, 10_9, 10_{32}, 10_{40}, 10_{61}, 10_{62}, 10_{63}, 10_{64}, 10_{65}, 10_{66}, 10_{76}, 10_{77}, 10_{78},$
$10_{82}, 10_{84}, 10_{85}, 10_{87}, 10_{98}, 10_{99}, 10_{103}, 10_{106}, 10_{112}, 10_{114}, 10_{139}, 10_{140},$ $\geq 3_1,$
$10_{141}, 10_{142}, 10_{143}, 10_{144}, 10_{159}, 10_{164}$

$$8_{18}, 9_{37}, 9_{40}, 10_{58}, 10_{59}, 10_{60}, 10_{122}, 10_{136}, 10_{137}, 10_{138} \geq 4_1,$$

$$10_{74}, 10_{120}, 10_{122} \geq 5_2.$$

First, we explain how to show the non-existence of surjective homomorphisms by using the twisted Alexander polynomials. Then, we construct explicitly a surjective homomorphism for each pair of knots which belongs to the above list.

4.1 Non-existence of surjective homomorphisms

We see how to apply the twisted Alexander polynomials of knots for the non-existence problem of surjective homomorphisms between knot groups.

For many pairs of knots which do not appear in Theorem 4.1, we can show easily that there exists no surjective homomorphism between their knot groups by using the classical Alexander polynomial. If the Alexander polynomial of $G(K)$ is not divisible by that of $G(K')$, then there exists no surjective homomorphism from $G(K)$ to $G(K')$. However, for some cases, it cannot be determined by using only the Alexander polynomial whether or not there exists a surjective homomorphism between knot groups.

Example 4.2 We explain how to find whether or not there exists a surjective homomorphism between $G(8_{11})$ and $G(3_1)$. The classical Alexander polynomials of them are respectively

$$\Delta_{8_{11}} = 2t^4 - 7t^3 + 9t^2 - 7t + 2,$$
$$\Delta_{3_1} = t^2 - t + 1.$$

Because $\Delta_{8_{11}}$ does not divide Δ_{3_1} clearly, there exists no surjective homomorphism from $G(3_1)$ to $G(8_{11})$. Therefore we can see

$$3_1 \not\geq 8_{11}.$$

However, since Δ_{3_1} divides $\Delta_{8_{11}}$, we cannot determine whether or not there exists a surjective homomorphism from $G(8_{11})$ to $G(3_1)$.

In cases that we can not determine the non-existence of a surjection, we apply the twisted Alexander polynomial to the knot groups.

We take a Wirtinger presentation of any knot group $G(K)$ as follows:

$$G(K) = \langle x_1, x_2, \ldots, x_u \mid r_1, r_2, \ldots, r_{u-1} \rangle.$$

Here we note that generators are conjugate with each other. The abelianization

$$\alpha \colon G(K) \to \mathbb{Z} \cong \langle t \rangle$$

is obtained by mapping each generator x_i to t. We consider the twisted Alexander polynomial for a 2–dimensional unimodular representation over a finite field. Now we fix a prime integer p and take a representation

$$\rho \colon G(K) \to SL(2; \mathbb{F}_p).$$

Here $\mathbb{F}_p$ is the finite prime field $\mathbb{Z}/p\mathbb{Z}$. Then we obtain the Alexander matrix M associated to ρ in $M((u-1) \times u; M(2; \mathbb{F}_p[t, t^{-1}]))$. Further it always holds that $\det \Phi(x_1 - 1) \neq 0$ because

$$\begin{aligned}
\Phi(x_1 - 1) &= \alpha(x_1) \otimes \rho(x_1) - E \\
&= t\rho(x_1) - E
\end{aligned}$$

where E is the identity matrix of degree 2. We write M_1 for the $(u-1) \times (u-1)$ matrix obtained from M by removing the first column. Here $\Delta_{K,\rho}^N(t)$ denotes the determinant of M_1 and $\Delta_{K,\rho}^D(t)$ the determinant of $\Phi(x_1 - 1)$.

In this case, the twisted Alexander polynomial of $G(K)$ for a representation $\rho \colon G(K) \to SL(2; \mathbb{F}_p)$ is defined to be

$$\Delta_{K,\rho}(t) = \frac{\Delta_{K,\rho}^N(t)}{\Delta_{K,\rho}^D(t)} = \frac{\det M_1}{\det \Phi(x_1 - 1)}.$$

Here we can prove the non-existence of a surjective homomorphism between the groups of any two knots except for the pairs listed in Theorem 4.1. A criterion for the non-existence is obtained by applying Theorem 3.1 for knot groups.

Corollary 4.3 *Let K_1 and K_2 be two knots. If there exists a representation $\rho_2 \colon G(K_2) \to SL(2; \mathbb{F}_p)$ such that $\Delta_{K_1,\rho_1}^N(t)$ is not divisible by $\Delta_{K_2,\rho_2}^N(t)$ or $\Delta_{K_1,\rho_1}^D(t) \neq \Delta_{K_2,\rho_2}^D(t)$ for any representation $\rho_1 \colon G(K_1) \to SL(2; \mathbb{F}_p)$, then there exists no surjective homomorphism from $G(K_1)$ onto $G(K_2)$.*

By applying Corollary 4.3 with the aid of a computer, we can prove that there exists no surjective homomorphism between the remaining pairs of knots. All the twisted Alexander polynomials which we use to check the non-existence of surjections are listed in [6].

Example 4.4 We can show the non-existence of a surjective homomorphism from $G(8_{11})$ to $G(3_1)$ by using the twisted Alexander polynomials of $SL(2; \mathbb{F}_5)$–representations as follows.

First, we compute that the numerators and denominators of the twisted Alexander polynomials of $G(8_{11})$ associated to all $SL(2; \mathbb{F}_5)$–representations. We obtain the pairs of $(\Delta^N_{8_{11},\rho_i}(t), \Delta^D_{8_{11},\rho_i}(t))$ $(i = 1, 2, \ldots, 10)$ as follows:

$$
\begin{aligned}
&(\Delta^N_{8_{11},\rho_i}(t), \Delta^D_{8_{11},\rho_i}(t)) = \\
&(2t^6 + 2t^5 + t^4 + 4t^3 + t^2 + 2t + 2, t^2 + t + 1), \\
&(3t^6 + 2t^5 + 4t^4 + 4t^3 + 4t^2 + 2t + 3, t^2 + 4t + 1), \\
&(t^8 + t^6 + t^2 + 1, t^2 + 1), \\
&(2t^8 + 2t^7 + 4t^6 + t^5 + 4t^4 + t^3 + 4t^2 + 2t + 2, t^2 + 4t + 1), \\
&(2t^8 + 3t^7 + 4t^6 + 4t^5 + 4t^4 + 4t^3 + 4t^2 + 3t + 2, t^2 + t + 1), \\
&(4t^8 + 3t^6 + t^4 + 3t^2 + 4, t^2 + 1), \\
&(4t^8 + t^7 + t^6 + 3t^4 + t^2 + t + 4, t^2 + 4t + 1), \\
&(4t^8 + 2t^7 + t^5 + 2t^4 + t^3 + 2t + 4, t^2 + 3t + 1), \\
&(4t^8 + 3t^7 + 4t^5 + 2t^4 + 4t^3 + 3t + 4, t^2 + 2t + 1), \\
&(4t^8 + 4t^7 + t^6 + 3t^4 + t^2 + 4t + 4, t^2 + t + 1).
\end{aligned}
$$

On the other hand, for a certain $SL(2; \mathbb{F}_5)$–representation ρ_0, the numerators and denominators of the twisted Alexander polynomials of $G(3_1)$ is given by

$$
\Delta^N_{3_1,\rho_0}(t) = t^4 + 2t^3 + 2t^2 + 2t + 1, \quad \Delta^D_{3_1,\rho_0}(t) = t^2 + 2t + 1.
$$

For any i, it is seen that $\Delta^N_{8_{11},\rho_i}(t)$ is not divisible by $\Delta^N_{3_1,\rho_0}(t)$ or $\Delta^D_{8_{11},\rho_i}(t) \neq \Delta^D_{3_1,\rho_0}(t)$. Then there exists no surjective homomorphism from $G(8_{11})$ onto $G(3_1)$. Therefore we obtain

$$
8_{11} \not\geq 3_1.
$$

Using Alexander polynomials, we cannot determine the non-existence of surjections between knot groups for 201 pairs of knots. However, we can prove the non-existence of surjections by using the twisted Alexander polynomials. To prove

it, we take $2, 3, 5, 7, 11, 17$ as a prime integer p of $SL(2; \mathbb{F}_p)$–representation for $26, 50, 81, 33, 10, 1$ cases respectively. All the data to check them are shown in [6].

4.2　Construction of surjective homomorphisms

In this subsection, we explain how to construct a surjective homomorphism between the groups of each pair of knots which appears in the list of Theorem 4.1.

Example 4.5　We show that there exist surjective homomorphisms $G(8_5) \to G(3_1)$ and $G(8_{18}) \to G(3_1)$. The knot group of 8_5, 8_{18} and 3_1 admit Wirtinger presentations as follows:

$$G(8_5) = \left\langle \begin{array}{l} y_1, y_2, y_3, y_4, \\ y_5, y_6, y_7, y_8 \end{array} \middle| \begin{array}{l} y_7 y_2 \bar{y}_7 \bar{y}_1, \ y_8 y_3 \bar{y}_8 \bar{y}_2, \ y_6 y_4 \bar{y}_6 \bar{y}_3, \ y_1 y_5 \bar{y}_1 \bar{y}_4, \\ y_3 y_6 \bar{y}_3 \bar{y}_5, \ y_4 y_7 \bar{y}_4 \bar{y}_6, \ y_2 y_8 \bar{y}_2 \bar{y}_7 \end{array} \right\rangle,$$

$$G(8_{18}) = \left\langle \begin{array}{l} y_1, y_2, y_3, y_4, \\ y_5, y_6, y_7, y_8 \end{array} \middle| \begin{array}{l} y_4 y_1 \bar{y}_4 \bar{y}_2, \ y_5 y_3 \bar{y}_5 \bar{y}_2, \ y_6 y_3 \bar{y}_6 \bar{y}_4, \ y_7 y_5 \bar{y}_7 \bar{y}_4, \\ y_8 y_5 \bar{y}_8 \bar{y}_6, \ y_1 y_7 \bar{y}_1 \bar{y}_6, \ y_5 y_8 \bar{y}_5 \bar{y}_7 \end{array} \right\rangle,$$

$$G(3_1) = \langle x_1, x_2, x_3 \mid x_3 x_1 \bar{x}_3 \bar{x}_2, \ x_1 x_2 \bar{x}_1 \bar{x}_3 \rangle,$$

where $\bar{x} = x^{-1}, \bar{y} = y^{-1}$. We define a map φ from $G(8_5)$ to $G(3_1)$ as follows:

$$\varphi(y_1) = x_3, \ \varphi(y_2) = x_2, \ \varphi(y_3) = x_1, \ \varphi(y_4) = x_3,$$
$$\varphi(y_5) = x_3, \ \varphi(y_6) = x_2, \ \varphi(y_7) = x_1, \ \varphi(y_8) = x_3.$$

It is easy to check that this map φ gives a homomorphism by computing the images of the relators. Moreover, it is clear that φ is surjective by its definition. Then we obtain

$$8_5 \geq 3_1.$$

Next, we define another map φ' from $G(8_{18})$ to $G(3_1)$ as follows:

$$\varphi'(y_1) = x_1, \quad \varphi'(y_2) = x_2, \quad \varphi'(y_3) = x_1, \ \varphi'(y_4) = x_3,$$
$$\varphi'(y_5) = x_3, \ \varphi'(y_6) = x_1 x_3 \bar{x}_1, \ \varphi'(y_7) = x_3, \ \varphi'(y_8) = x_1.$$

Similarly it can be seen that the map φ' is also a surjective group homomorphism. Then we get

$$8_{18} \geq 3_1.$$

In [6], we constructed surjective group homomorphisms for all pairs of knots in Theorem 4.1 explicitly. This completes the proof of Theorem 4.1.

We remark that we could find many surjective homomorphisms by using computer. Once they are found by computer, it is easy to check by hand that they are surjective homomorphisms.

5 Further results related to Theorem 4.1

In the previous section, we determined the partial order "$\geq$" in Rolfsen's knot table. These results lead to the following problems:

(1) Are three knots 3_1, 4_1 and 5_2 minimal elements?

(2) Which relation can a pair of a periodic knot and its quotient knot realize?

(3) Which relation can a degree one map between knot exteriors realize?

In this section, we give partial answers to these problems. First, we prove that 3_1 and 4_1 are minimal elements. To do this, we study a surjective homomorphism from a fibered knot to another one. Next, we determine which relation a pair of a periodic knot and its quotient knot realizes. Finally, we study which relation given in Theorem 4.1 can be induced by a degree one map. We do not know the complete answer of the realizing problem for a degree one map. Here we describe which surjection constructed in the previous section is realized by a degree one map.

5.1 Surjection between fibered knots

In this subsection, we prove the minimality of 3_1 and 4_1.

Theorem 5.1 3_1 *and* 4_1 *are minimal elements under this partial ordering.*

To prove this theorem, we start to study a surjective homomorphism from a fibered knot group to another knot group. Let K_1 and K_2 be knots in S^3. We assume that K_1 is a fibered knot of genus g_1. Here we obtain the following.

Proposition 5.2   *If there exists a surjective homomorphism* $\varphi\colon G(K_1) \to G(K_2)$, *then* K_2 *is also a fibered knot. Further the genus* g_2 *of* K_2 *is less than or equal to* g_1.

Proof   By the result of Neuwirth [13] and Stallings [17], the commutator subgroup $[G(K_1), G(K_1)]$ of $G(K_1)$ is a free group of rank $2g_1$, because K_1 is a fibered knot of genus g_1. It is isomorphic to the fundamental group of its fiber surface. Restricting this surjection φ on $[G(K_1), G(K_1)]$, we have a surjection

$$\varphi|_{[G(K_1),G(K_1)]}\colon [G(K_1), G(K_1)] \to [G(K_2), G(K_2)].$$

Here it is clear that $[G(K_2), G(K_2)]$ is also a finitely generated group, because it is the image of $[G(K_1), G(K_1)]$ by a surjection φ. Hence the commutator subgroup $[G(K_2), G(K_2)]$ of $G(K_2)$ is also a free group of a finite rank. By applying the result of [13] and [17] again, it is seen that K_2 is also a fibered knot. The genus of K_2 is denoted by g_2. Since φ is a surjection between free groups, it is clear that g_2 is less than or equal to g_1. This completes the proof.   □

As a corollary, we can prove Theorem 5.1 as follows.

Proof of Theorem 5.1 We put $K = 3_1$ or 4_1. They are the fibered knots of genus 1. Now we assume that K' is a non-trivial prime knot such that $K \geq K'$. Here there exists a surjective homomorphism $\varphi\colon G(K) \to G(K')$ from the above assumption. By the above proposition, K' is a fibered knot of genus 1, too. Since any genus 1 fibered knot is 3_1 or 4_1, K' is 3_1 or 4_1. It means that 3_1 and 4_1 are minimal elements. □

We remark that Silver and Whitten studied the same result in [16, Proposition 3.11]

5.2 Period of a knot

The periods of knots with up to 10 crossings are listed in [7]. To supplement Theorem 4.1, we have the following theorem.

Theorem 5.3 *The following relations are realized by pairs of a periodic knot and its quotient knot:*

$$\left. \begin{array}{l} 8_5, 8_{15}, 8_{19}, 8_{21}, 9_1, 9_{16}, 9_{28}, 9_{40}, \\ 10_{61}, 10_{63}, 10_{64}, 10_{66}, 10_{76}, 10_{78}, 10_{98}, 10_{139}, 10_{141}, 10_{142}, 10_{144} \end{array} \right\} \geq 3_1,$$

$$8_{18}, 10_{58}, 10_{60}, 10_{122}, 10_{136}, 10_{138} \geq 4_1,$$

$$10_{120} \geq 5_2.$$

5.3 Degree one maps

Earlier we constructed surjections to prove Theorem 4.1. In this subsection, we study which surjection is induced by a degree one map.

First, we recall the definition of a degree one map in the case of a knot exterior. Let $E(K_i)$ be oriented knot exteriors of K_i in S^3 for $i = 1, 2$. A continuous map

$$f\colon (E(K_1), \partial E(K_1)) \to (E(K_2), \partial E(K_2))$$

is called a degree one map if its induced map

$$f_*\colon H_3(E(K_1), \partial E(K_1); \mathbb{Z}) \to H_3(E(K_2), \partial E(K_2); \mathbb{Z})$$

has a degree one. It is known that a degree one map $f\colon (E(K_1), \partial E(K_1)) \to (E(K_2), \partial E(K_2))$ induces a surjective homomorphism $f_*\colon G(K_1) \to G(K_2)$. See Hempel [2] for this fact.

Let $f\colon (E(K_1), \partial E(K_1)) \to (E(K_2), \partial E(K_2))$ be a degree one map. By the definition of degree one map, the induced map

$$f_*\colon H_3(E(K_1), \partial E(K_1); \mathbb{Z}) \to H_3(E(K_2), \partial E(K_2); \mathbb{Z})$$

is an isomorphism. From the homology long exact sequence, we have

$$0 \to H_3(E(K_i), \partial E(K_i); \mathbb{Z}) \xrightarrow{\partial} H_2(\partial E(K_i); \mathbb{Z}) \to H_2(E(K_i); \mathbb{Z}) \to \cdots$$

Since each knot exterior $E(K_i)$ is homologically a circle, then $H_2(E(K_i); \mathbb{Z})$ is vanishing. Hence the above boundary map

$$\partial\colon H_3(E(K_i), \partial E(K_i); \mathbb{Z}) \to H_2(\partial E(K_i); \mathbb{Z})$$

is an isomorphism. By the naturality, it is seen that

$$f\colon (E(K_1), \partial E(K_1)) \to (E(K_2), \partial E(K_2))$$

is a degree one map if and only if $f|_{\partial E(K_1)}\colon \partial E(K_1) \to \partial E(K_2)$ is a degree one map. Since $\partial E(K_i)$ is a 2–dimensional torus, the degree of f is determined by the determinant of $f_*\colon H_1(\partial E(K_1); \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \to H_1(\partial E(K_2); \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Now we fix a basis of $H_1(\partial E(K_i); \mathbb{Z})$ so that the first element is the meridian m_i of K_i and the second one is a longitude l_i of K_i. By such fixing bases of $H_1(\partial E(K_i); \mathbb{Z})$, we can represent f_* as a 2×2–matrix. The determinant of this matrix is the degree of $f\colon \partial E(K_1) \to \partial E(K_2)$. In our examples in [6], any f_* maps the meridian of K_1 to that of K_2. Then we have to compute only the image $f_*(l_1)$, which can be written $f_*(l_1) = am_2 + bl_2$. Under this setting, f_* on the boundary can represented by the matrix $\begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}$. Hence, it is clear that f is a degree one map if and only if $b = \pm 1$.

By using the above argument, we can check which surjective homomorphism in [6] is induced by a degree one map.

Example 5.4 Here we consider $G(8_5)$, $G(8_{18})$ and $G(3_1)$ of Example 4.5. We check whether or not the surjective homomorphisms which are constructed in Example 4.5 are induced by a degree one map.

First, we check whether or not the map $\varphi\colon G(8_5) \to G(3_1)$ of Example 4.5 is induced by a degree one map. Pairs of the meridians and the longitudes of $G(8_5)$ and $G(3_1)$ with respect to the presentations of Example 4.5 are chosen as follows respectively:

$$(y_1, y_7 y_8 y_6 \bar{y}_1 y_3 y_4 y_2 \bar{y}_5 \bar{y}_1 \bar{y}_1 y_1 \bar{y}_1), \quad (x_3, \bar{x}_2 \bar{x}_3 \bar{x}_1 x_3 x_3 x_3).$$

Clearly, the image of the meridian of $G(8_5)$ under φ is the meridian of $G(3_1)$. On the other hand, the image of the longitude of $G(8_5)$ under φ is

$$\varphi(\text{longitude of } 8_5) = \varphi(y_7 y_8 y_6 \bar{y}_1 y_3 y_4 y_2 \bar{y}_5 \bar{y}_1 \bar{y}_1 \bar{y}_1 \bar{y}_1)$$
$$= x_1 x_3 x_2 \bar{x}_3 x_1 x_3 x_2 \bar{x}_3 \bar{x}_3 \bar{x}_3 \bar{x}_3 \bar{x}_3$$
$$= \bar{x}_3 \bar{x}_3 \bar{x}_3 x_1 x_3 x_2 \bar{x}_3 \bar{x}_3 \bar{x}_3 x_1 x_3 x_2.$$

Then the image of the longitude of 8_5 is equal to the (-2) times of the longitude of 3_1 in the first homology group. Then this surjective homomorphism φ is not induced by a degree one map.

Next, we check whether or not the map $\varphi': G(8_{18}) \to G(3_1)$ of Example 4.5 is induced by a degree one map. Similarly, we fix pairs of the meridians and the longitudes of $G(8_{18})$ and $G(3_1)$ with respect to the presentations of Example 4.5:

$$(y_1, \bar{y}_4 y_5 \bar{y}_6 y_7 \bar{y}_8 y_1 \bar{y}_2 y_3), \quad (x_1, \bar{x}_3 \bar{x}_1 \bar{x}_2 x_1 x_1 x_1).$$

Similarly, the image of the meridian of $G(8_{18})$ under φ' is the meridian of $G(3_1)$. Moreover, the image of the longitude of $G(8_{18})$ under φ' is

$$\varphi'(\text{longitude of } 8_{18}) = \varphi'(\bar{y}_4 y_5 \bar{y}_6 y_7 \bar{y}_8 y_1 \bar{y}_2 y_3)$$
$$= \bar{x}_3 x_3 x_1 \bar{x}_3 \bar{x}_1 x_3 \bar{x}_1 x_1 \bar{x}_2 x_1$$
$$= x_1 \bar{x}_3 \bar{x}_1 x_3 \bar{x}_2 x_1.$$

Then it is seen that the image of the longitude of 8_{18} is equal to the longitude of 3_1. Therefore this surjective homomorphism φ' is induced by a degree one map.

We can check which surjection of Theorem 4.1 is induced by a degree one map. Finally, we obtain the following as the result.

Theorem 5.5 *The following relations are realized by degree one maps:*

$$8_{18}, 10_5, 10_9, 10_{32}, 10_{40}, 10_{103}, 10_{106}, 10_{112}, 10_{114}, 10_{159}, 10_{164} \geq 3_1,$$

$$9_{37}, 9_{40} \geq 4_1,$$

$$10_{74}, 10_{122} \geq 5_2.$$

We do not know whether or not other relations are realized by degree one maps.

6 Problems

In this section, we present other problems connected with the partial order $\geq$.

In Subsection 5.1, we proved that 3_1 and 4_1 are minimal. However, it remains open whether or not 5_2 is minimal. So the first problem is the following.

(1) Characterize and determine the minimal knots under this partial order.

As we see in Subsection 5.1, if there exists a surjective homomorphism from the knot group of a fibered knot onto another knot group, then its target knot is fibered. We can restrict the partial order to the set of prime fibered knot.

(2) Determine the partial order on the set of fibered knots.

The next problem is to study the relation between this partial ordering and knot invariants. There are some invariants to measure and classify the complexity of knots. The most fundamental invariant is the crossing number. So the following problem arises naturally.

(3) If there exists a surjective homomorphism from $G(K_1)$ to $G(K_2)$, then is the crossing number of K_1 greater than that of K_2?

By Theorem 4.1, the answer is positive in case the crossing number is smaller than or equal to 10. Moreover, we can check that there exists no surjective homomorphism from the knot groups of knots with up to 10 crossings to that of alternating knots with 11 crossings. It still remains open for higher crossing cases.

The next problem also arises naturally.

(4) If there exists a surjective homomorphism from $G(K_1)$ to $G(K_2)$, then is the bridge number of K_1 greater than or equal to that of K_2?

The answer is also positive in case the crossing number is smaller than or equal to 10.

(5) If there exists a surjective homomorphism from $G(K_1)$ to $G(K_2)$ and both of K_1 and K_2 are hyperbolic knots, then is the volume of $S^3 - K_1$ greater than that of $S^3 - K_2$?

We can calculate the hyperbolic volume for a given knot by SnapPea [19], or find its value on the web page KnotInfo [10]. Thus we can make sure that every pair of knots in the list of Theorem 4.1 satisfies this inequality.

In Subsection 5.3, we considered which surjective homomorphisms are induced by degree one map. However, this was restricted to the surjective homomorphisms which are constructed in [6]. We do not know whether or not other pairs $K_1 \geq K_2$ can be realized by degree one maps. There may exist another surjective homomorphism other

than we constructed, which is induced by degree one map. Therefore we should study the following problem.

(6)   Decide all pairs of knot in Rolfsen's table which admit degree one maps.

We could not determine all geometric interpretations for the existences of surjective homomorphisms. However, Ohtsuki, Riley and Sakuma gave a systematic construction of surjective homomorphisms between 2–bridge link groups [14].

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