On the Lusternik–Schnirelmann category of symmetric spaces of classical type

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We determine the Lusternik–Schnirelmann category of the irreducible, symmetric Riemann spaces $SU(n)/SO(n)$ and $SU(2n)/Sp(n)$ of type AI and AII respectively.

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1 Introduction

For a topological space X, the Lusternik–Schnirelmann category, L-S category for short and denoted by $\mathrm{cat}(X)$, is defined to be the least integer n such that there exists an open covering $\{A_1, \ldots, A_{n+1}\}$ of X with each A_i contractible in X. This homotopy invariant is known to be related to various problems; for instance, some geometric applications can be found in Korbaš and Szűcs [5].

First of all we recall a theorem due to Singhof [7]:

Theorem $\mathrm{cat}(SU(n)) = n-1$

The purpose of this note is to prove the following theorem along the line of idea of the proof of Singhof's theorem.

Theorem 1.1
(1) $\mathrm{cat}(SU(n)/SO(n)) = n-1$
(2) $\mathrm{cat}(SU(2n)/Sp(n)) = n-1$

One can prove the following theorem by the entirely similar method.

Theorem 1.1′
(1) $\mathrm{cat}(U(n)/O(n)) = n$
(2) $\mathrm{cat}(U(2n)/Sp(n)) = n$

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Observe that (1) of Theorem 1.1 for $n = 4$ improves the estimate of the L-S category of the oriented Grassmann manifold $\widetilde{G}_{6,3} = SO(6)/(SO(3) \times SO(3))$ given by Korbaš [4, Corollary C (a)].

Let $J = \begin{pmatrix} O & -E_n \\ E_n & O \end{pmatrix} \in SU(2n)$, where E_n denotes the $n \times n$ identity matrix.

We need the following lemma to give a proof of our result.

Lemma 1.2 *There are matrix representations*:
(1) $SU(n)/SO(n) = \{X \in SU(n) \mid {}^tX = X\}$
(2) $SU(2n)/Sp(n) = \{X \in SU(2n) \mid {}^tX = JX{}^tJ\}$

By Lemma 1.2, we can regard $SU(n)/SO(n)$ and $SU(2n)/Sp(n)$ as subspaces of $SU(n)$ and $SU(2n)$ respectively.

The paper is organized as follows. In Section 2 we will prove (1) of Theorem 1.1. In Section 3 we will prove (2) of Theorem 1.1. In Section 4 we study the L-S category of the irreducible symmetric Riemann spaces of classical type other than AI and AII. We will give a proof of Lemma 1.2, which may be a folklore, in the Appendix just for completeness.

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2 L-S category of $SU(n)/SO(n)$

In this section, we will prove (1) of Theorem 1.1. The mod 2 cohomology ring of $SU(n)/SO(n)$ is given as follows (see for example Mimura and Toda [6]):

$$H^*(SU(n)/SO(n); \mathbb{Z}/2) = \Lambda(x_2, x_3, \ldots, x_n),$$

where Λ denotes exterior algebra. Since the cup-length gives a lower bound of the L-S category (see for example Whitehead [8]), we have

$$n - 1 = \mathrm{cup}_{\mathbb{Z}/2}(SU(n)/SO(n)) \leq \mathrm{cat}(SU(n)/SO(n)).$$

Thus in order to determine $\mathrm{cat}(SU(n)/SO(n))$, it is sufficient to show the following proposition.

Proposition 2.1 $\mathrm{cat}(SU(n)/SO(n)) \leq n - 1$

Proof Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be different complex numbers with $|\lambda_r| = 1$ such that $\lambda_1 \lambda_2 \cdots \lambda_n \neq 1$. For $1 \leq r \leq n$ we define

$$A_r = \{X \in SU(n)/SO(n) \mid \lambda_r \text{ is not an eigenvalue of } X\}.$$

Observe here that we regard $X \in SU(n)/SO(n)$ as a matrix in $SU(n)$ by Lemma 1.2. Then the A_r's are clearly open sets, and form a covering of $SU(n)/SO(n)$, since the property $\lambda_1 \lambda_2 \cdots \lambda_n \neq 1$ implies that the λ_r's cannot all appear as the eigenvalues of any matrix in $SU(n)/SO(n)$.

Now we fix A_r and let B be a connected component of A_r. In order to show that A_r is contractible in $SU(n)/SO(n)$, it is sufficient to show that B is so, since $SU(n)/SO(n)$ is pathwise connected.

Next, let $\mathfrak{u}(n) = \{X \in M(n, \mathbb{C}) \mid X^* = -X\}$, and we will define a map $\log\colon B \to \mathfrak{u}(n)$ as follows. Let $X \in B \subset A_r$ and $\lambda_r = e^{i\alpha}$, where $0 \leq \alpha < 2\pi$. Then X can be diagonalized by a suitable matrix $P \in U(n)$ as $X = PD(e^{i\theta_1}, \ldots, e^{i\theta_n})P^*$, where $D(a_1, \ldots, a_n)$ denotes a diagonal matrix defined by

$$D(a_1, \ldots, a_n) = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix},$$

and we may take $\alpha < \theta_j < \alpha + 2\pi$ for each j, since X does not have $\lambda_r = e^{i\alpha}$ as its eigenvalue. We define a function $\log\colon B \to \mathfrak{u}(n)$ by

$$\log X = PD(i\theta_1, \ldots, i\theta_n)P^*,$$

where it is easy to see that the definition does not depend on the choice of P, and the function log is clearly continuous. Since $X = \exp(\log X)$ by definition, we have

$$1 = \det X = \det(\exp(\log X)) = \exp(\operatorname{tr}(\log X)).$$

Since the maps $\operatorname{tr}\colon M(n, \mathbb{C}) \to \mathbb{C}$, which is the trace function, and $\log\colon B \to \mathfrak{u}(n)$ are continuous and since B is connected, there exists an integer k such that $\operatorname{tr}(\log X) = 2\pi i k$ for all $X \in B$.

Now we define a constant matrix X_0 in $SU(n)/SO(n)$ by

$$X_0 = \exp\left(\frac{2\pi i k}{n}\right) \cdot E_n,$$

and we show that B is contractible to X_0. In order to define a contracting homotopy, we use the fact that $\mathfrak{u}(n)$ is a vector space, which allows us to construct linear homotopies.

We define a homotopy $F\colon B\times[0,1]\to SU(n)/SO(n)$ by

$$F(X,s)=\exp\left((1-s)\log X+s\frac{2\pi ik}{n}E_n\right).$$

Clearly, the function F is continuous such that $F(X,0)=\exp(\log X)=X$ and $F(X,1)=X_0$ for all $X\in B$. Here we need to check that $F(X,s)\in SU(n)/SO(n)$ for all $X\in B$ and $s\in[0,1]$. Since $\mathfrak{u}(n)$ is the Lie algebra of $U(n)$, we have $F(X,s)\in U(n)$. Hence it is sufficient to show that $\det(F(X,s))=1$ and that ${}^tF(X,s)=F(X,s)$. The former equality can be seen as follows:

$$\begin{aligned}\det(F(X,s))&=\det\left(\exp\left((1-s)\log X+s\frac{2\pi ik}{n}E_n\right)\right)\\&=\exp\left(\operatorname{tr}\left((1-s)\log X+s\frac{2\pi ik}{n}E_n\right)\right)\\&=\exp\left((1-s)\operatorname{tr}(\log X)+s\frac{2\pi ik}{n}\operatorname{tr}(E_n)\right)\\&=\exp\left(2\pi ik(1-s)+2\pi iks\right)\\&=\exp\left(2\pi ik\right)\\&=1.\end{aligned}$$

The latter equality can be seen as follows:

$$\begin{aligned}{}^tF(X,s)&={}^t\exp\left((1-s)\log X+s\frac{2\pi ik}{n}E_n\right)\\&=\exp\left((1-s)\log{}^tX+s\frac{2\pi ik}{n}{}^tE_n\right)\\&=\exp\left((1-s)\log X+s\frac{2\pi ik}{n}E_n\right)\\&=F(X,s).\end{aligned}$$

□

3 L-S category of $SU(2n)/Sp(n)$

In this section, we will prove (2) of Theorem 1.1. The integral cohomology ring of $SU(2n)/Sp(n)$ is given as follows (see for example [6]):

$$H^*(SU(2n)/Sp(n);\mathbb{Z})=\Lambda(x_5,x_9,\dots,x_{4n-3}),$$

where Λ denotes exterior algebra. Since the cup-length gives a lower bound of the L-S category, we have

$$n-1=\mathrm{cup}_{\mathbb{Z}}(SU(2n)/Sp(n))\leq \mathrm{cat}(SU(2n)/Sp(n)).$$

Thus in order to determine $\mathrm{cat}(SU(2n)/Sp(n))$, it is sufficient to show the following proposition.

Proposition 3.1 $\mathrm{cat}(SU(2n)/Sp(n))\leq n-1$

In order to prove Proposition 3.1, we need some lemmas.

Lemma 3.2 *Let X be any matrix in $SU(2n)/Sp(n)$. If λ is an eigenvalue of X, then*

$$\dim W_\lambda \geq 2,$$

where $W_\lambda\subset\mathbb{C}^{2n}$ denotes the corresponding eigenspace.

Proof There exists an eigenvector $v\neq 0$ in $\mathbb{C}^{2n}$ such that $Xv=\lambda v$. Since X satisfies $XX^*=E_{2n}$ and ${}^tX=JX^tJ$, it follows by an easy calculation that $X(J\overline{v})=\lambda(J\overline{v})$. Consequently we have that if v is an eigenvector of λ, so is $J\overline{v}$. Hence it is sufficient to prove that v and $J\overline{v}$ are linearly independent. If $av+bJ\overline{v}=0$ $(a,b\in\mathbb{C})$, we have $\overline{av}+\overline{b}Jv=0$, and by solving the simultaneous equations, we see $(|a|^2+|b|^2)v=0$, which implies $a=b=0$. □

Let $\lambda_1,\lambda_2,\dots,\lambda_n$ be different complex numbers with $|\lambda_r|=1$ such that $\lambda_1^2\lambda_2^2\cdots\lambda_n^2\neq 1$. For $1\leq r\leq n$ we define

$$A_r=\{X\in SU(2n)/Sp(n)\mid \lambda_r \text{ is not an eigenvalue of } X\}.$$

Lemma 3.3 *A family $\{A_r\}_{1\leq r\leq n}$ forms an open covering of $SU(2n)/Sp(n)$:*

$$SU(2n)/Sp(n)=\bigcup_{r=1}^{n}A_r.$$

Proof Let $X\in(SU(2n)/Sp(n))\setminus\bigcup_{r=1}^n A_r=\bigcap_{r=1}^n\{(SU(2n)/Sp(n))\setminus A_r\}$. Then X has λ_r as its eigenvalue. Furthermore we see by Lemma 3.2 that the multiplicity of the eigenvalue λ_r is 2 for each r. Consequently, X can be diagonalized by a suitable matrix $P\in U(2n)$:

$$X=PD(\lambda_1,\lambda_1,\dots,\lambda_n,\lambda_n)P^*.$$

Therefore we have that $\det X=\lambda_1^2\lambda_2^2\cdots\lambda_n^2\neq 1$ which contradicts the fact $X\in SU(2n)$. □

Proof of Proposition 3.1 By Lemma 3.3, it is sufficient to show that each A_r is contractible in $SU(2n)/Sp(n)$; but since $SU(2n)/Sp(n)$ is pathwise connected, it is sufficient to show that any connected component of A_r is contractible in $SU(2n)/Sp(n)$. Now we fix A_r and let B be a connected component of A_r. We will show that B is contractible in $SU(2n)/Sp(n)$.

In a similar way to that in Section 2, we can define a continuous function $\log\colon B \to \mathfrak{u}(2n)$ such that $\exp(\mathrm{tr}(\log X)) = 1$ for $X \in B$. Then, as was seen before, there exists an integer k such that $\mathrm{tr}(\log X) = 2\pi i k$ for all $X \in B$.

Now define a constant matrix X_0 in $SU(2n)/Sp(n)$ by

$$X_0 = \exp\left(\frac{\pi i k}{n}\right) \cdot E_{2n}$$

and a contracting homotopy $F\colon B \times [0,1] \to SU(2n)/Sp(n)$ by

$$F(X,s) = \exp\left((1-s)\log X + s\frac{\pi i k}{n}E_{2n}\right).$$

Clearly, F is continuous such that $F(X,0) = \exp(\log X) = X$ and $F(X,1) = X_0$ for all $X \in B$. Here we need to check that $F(X,s) \in SU(2n)/Sp(n)$ for all $X \in B$ and $s \in [0,1]$. Since $\mathfrak{u}(2n)$ is the Lie algebra of $U(2n)$, we have $F(X,s) \in U(2n)$. Hence it is sufficient to show that $\det(F(X,s)) = 1$ and that ${}^tF(X,s) = JF(X,s){}^tJ$. The former equality can be seen as follows:

$$\begin{aligned}
\det(F(X,s)) &= \det\left(\exp\left((1-s)\log X + s\frac{\pi i k}{n}E_{2n}\right)\right)\\
&= \exp\left(\mathrm{tr}\left((1-s)\log X + s\frac{\pi i k}{n}E_{2n}\right)\right)\\
&= \exp\left((1-s)\,\mathrm{tr}(\log X) + s\frac{\pi i k}{n}\,\mathrm{tr}(E_{2n})\right)\\
&= \exp\left(2\pi i k(1-s) + 2\pi i k s\right)\\
&= \exp\left(2\pi i k\right)\\
&= 1.
\end{aligned}$$

The latter equality can be seen as follows:

$$\begin{aligned}
{}^tF(X,s) &= {}^t\exp\left((1-s)\log X + s\frac{\pi i k}{n}E_{2n}\right)\\
&= \exp\left((1-s)\log {}^tX + s\frac{\pi i k}{n}\,{}^tE_{2n}\right)\\
&= \exp\left((1-s)\log(JX{}^tJ) + s\frac{\pi i k}{n}E_{2n}\right)\\
&= \exp\left((1-s)J(\log X){}^tJ + s\frac{\pi i k}{n}E_{2n}\right)\\
&= \exp\left(J\left((1-s)\log X + s\frac{\pi i k}{n}E_{2n}\right){}^tJ\right)\\
&= J\exp\left((1-s)\log X + s\frac{\pi i k}{n}E_{2n}\right){}^tJ\\
&= JF(X,s){}^tJ.
\end{aligned}$$

□

4 Lusternik–Schnirelmann category of the irreducible symmetric Riemann space of classical type

First let us recall a theorem due to Ganea (for a proof see [1]):

Proposition 4.1 *If X is an $(r-1)$–connected CW–complex for $r \geq 1$, then*

$$\mathrm{cat}(X) \leq \dim(X)/r.$$

We show the following

Proposition 4.2 *If V is a simply connected, complex d–manifold which admits a Kähler metric, then*

$$\mathrm{cat}(V) = d.$$

In fact, following James [3], it is proved as follows; we have $\mathrm{cat}(V) \leq d$ by Proposition 4.1, since V is simply connected. But with any Kähler metric, there exists a closed 2–form on V whose dth power is the volume element and so cannot be cohomologous to zero. Hence we have $\mathrm{cat}(V) \geq d$, since the cup-length gives a lower bound of the L-S category.

According to Helgason [2, page 518], the irreducible symmetric Riemann spaces of classical type which has a Hermitian structure are known to be of type

$$\mathrm{A\,III},\ \mathrm{BD\,I}\,(q=2),\ \mathrm{BD\,II}\,(n=2),\ \mathrm{D\,III},\ \mathrm{C\,I}.$$

Now we also recall from Proposition 4.1 of [2] the following

Proposition 4.3 *The Hermitian structure of a Hermitian symmetric space is Kählerian.*

It follows from this proposition that the spaces of above type have Kähler metric. Hence we see the L-S category of these spaces by Proposition 4.2, since a Hermitian symmetric space is a complex manifold by definition (see [2, page 372]). Thus, with Theorem 1.1, the L-S category of the irreducible symmetric Riemann space of classical type, except that of type BD I $(q \neq 2)$, is determined as follows:

	G/K	Kähler	dimension	$\mathrm{cat}(G/K)$
A I	$SU(n)/SO(n)$ $(n>2)$	no	$(n-1)(n+2)/2$	$n-1$
A II	$SU(2n)/Sp(n)$ $(n>1)$	no	$(n-1)(2n+1)$	$n-1$
A III	$U(p+q)/(U(p)\times U(q))$ $(p\geq q\geq 1)$	yes	$2pq$	pq
BD I	$SO(p+q)/(SO(p)\times SO(q))$ $(p\geq q\geq 2,\ p+q\neq 4)$	yes $(q=2)$ no $(q\neq 2)$	pq	p $(q=2)$? $(q\neq 2)$
BD II	$SO(n+1)/SO(n)$ $(n\geq 2)$	yes $(n=2)$ no $(n\neq 2)$	n	1
D III	$SO(2l)/U(l)$ $(l\geq 4)$	yes	$l(l-1)$	$l(l-1)/2$
C I	$Sp(n)/U(n)$ $(n\geq 3)$	yes	$n(n+1)$	$n(n+1)/2$
C II	$Sp(p+q)/(Sp(p)\times Sp(q))$ $(p\geq q\geq 1)$	no	$4pq$	pq

As for the remaining cases;

Firstly, the space of type BD II, the real Stiefel manifold $SO(n+1)/SO(n)$, is homeomorphic to S^n, and hence we have $\mathrm{cat}(SO(n+1)/SO(n))=1$.

Secondly, it is known that the space of type C II, the symplectic Grassmann manifold $Sp(p+q)/(Sp(p)\times Sp(q))$, is 3–connected. Hence by Proposition 4.1, we obtain an upper bound $\mathrm{cat}(Sp(p+q)/(Sp(p)\times Sp(q)))\leq 4pq/4=pq$. It is also known that the cohomology ring of the symplectic Grassmann manifold $Sp(p+q)/(Sp(p)\times Sp(q))$ is similar to that of the complex Grassmann manifold $U(p+q)/(U(p)\times U(q))$ (see for example [6]), so we have that $\mathrm{cup}(Sp(p+q)/(Sp(p)\times Sp(q))) = \mathrm{cup}(U(p+q)/(U(p)\times U(q)))$, which is given by pq, since the cup-length of $U(p+q)/(U(p)\times U(q))$ is equal to the L-S category of it. Hence we obtain a lower bound $\mathrm{cat}(Sp(p+q)/(Sp(p)\times Sp(q))) \geq pq$.

Concluding remark: the mod 2 cohomology of type BD I ($q \neq 2$), $SO(p+q)/(SO(p)\times SO(q))$, $p \geq q > 2$, is not known yet.

Appendix A

Proof of Lemma 1.2

(1) Let $K_n = \{X \in SU(n) \mid {}^tX = X\}$ and define an action of $P \in SU(n)$ on K_n by

$$P{\cdot}X = PX{}^tP \quad (X \in K_n).$$

We will show that $X \in K_n$ is represented as follows:

$$X = P{}^tP = PE_n{}^tP \quad (P \in SU(n)).$$

Let $X \in K_n$. Since $X+\bar{X}$ and $i(X-\bar{X})$ are real symmetric matrices which commute with each other, they can be diagonalized by a suitable matrix $B \in SO(n)$:

$${}^tB(X+\bar{X})B = D(a_1,\dots,a_n), \quad {}^tBi(X-\bar{X})B = D(b_1,\dots,b_n).$$

Then we have

$${}^tBXB = D((a_1-ib_1)/2,\dots,(a_n-ib_n)/2),$$

where $|(a_k-ib_k)/2| = 1$ for $1 \leq k \leq n$, since ${}^tBXB \in SU(n)$. Now we can take complex numbers $c_1,\dots,c_n$ such that ${c_k}^2 = (a_k-ib_k)/2$ and $c_1\cdots c_n = 1$. Then we have ${}^tBXB = CC = C{}^tC$, where $C = D(c_1,\dots,c_n) \in SU(n)$. By taking $P = BC$, we have

$$X = BC{}^tC{}^tB = P{}^tP \quad (P \in SU(n)),$$

which implies that the action is transitive.

On the other hand, the isotropy group at E_n is given by

$$\{P \in SU(n) \mid P{}^tP = E_n\} = \{P \in SU(n) \mid \bar{P} = P\} = SO(n).$$

Since $SU(n)$ is compact, we obtain

$$SU(n)/SO(n) = \{X \in SU(n) \mid {}^tX = X\}.$$

(2) There is an embedding $c'\colon Sp(n) \to SU(2n)$ defined by

$$c'(X) = \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix} \quad (X = A + jB)$$

such that $Sp(n) = \{X \in SU(2n) \mid XJ{}^tX = J\}$.

Let $L_{2n} = \{X \in SU(2n) \mid {}^tX = -X\}$ and define an action of $P \in SU(2n)$ on L_{2n} by

$$P{\cdot}X = PX{}^tP \quad (X \in L_{2n}).$$

We will show that $X \in L_{2n}$ is represented as follows:

$$X = PJ{}^tP \quad (P \in SU(2n)).$$

Let λ be an eigenvalue of $X \in L_{2n}$. Here observe that $|\lambda| = 1$, since $X \in SU(2n)$. There exists an eigenvector $v \in \mathbb{C}^{2n}$ such that $Xv = \lambda v$ and $|v| = 1$. Since X satisfies $XX^* = E_{2n}$ and ${}^tX = -X$, it follows by an easy calculation that $X\overline{v} = -\lambda\overline{v}$. Let W be the 2–dimensional subspace of $\mathbb{C}^{2n}$ spanned by v and $\overline{v}$. By repeating this procedure to the orthogonal complement $W^{\perp}$ of W, we can take consequently an orthonormal basis $\{v_1, \overline{v}_1, \dots, v_n, \overline{v}_n\}$ in $\mathbb{C}^{2n}$ such that

$$Xv_k = \lambda_k v_k, \quad X\overline{v}_k = -\lambda_k\overline{v}_k \quad (k = 1, \dots, n),$$

where $\lambda_1, \dots, \lambda_n$ are eigenvalues of X. Put

$$w_k = \frac{1}{\sqrt{2}}(v_k + \overline{v}_k), \quad w_k{}' = \frac{-i}{\sqrt{2}}(v_k - \overline{v}_k) \quad (k = 1, \dots, n).$$

Then $\{w_1, w_1{}', \dots, w_n, w_n{}'\}$ forms an orthonormal basis in $\mathbb{R}^{2n}$ such that

$$Xw_k = i\lambda_k w_k{}', \quad Xw_k{}' = -i\lambda_k w_k \quad (k = 1, \dots, n).$$

Thus we have the following:

$${}^tBXB = \begin{pmatrix} O & -D(i\lambda_1, \dots, i\lambda_n) \\ D(i\lambda_1, \dots, i\lambda_n) & O \end{pmatrix},$$

where $B = (w_1, w_1{}', \dots, w_n, w_n{}') \in O(2n)$. Observe that we can choose B in $SO(2n)$ by replacing λ_1 with $-\lambda_1$, if necessary.

Now we take complex numbers $c_1, \dots, c_n$ such that $c_k{}^2 = i\lambda_k$ for each k, and let $C = D(c_1, \dots, c_n, c_1, \dots, c_n) \in U(2n)$. Then we have

$$\text{(A.1)} \qquad {}^tBXB = CJC = CJ{}^tC.$$

We can choose C in $SU(2n)$. In fact, since ${}^tBXB \in SU(2n)$, we have

$$\det({}^tBXB) = i^{2n}\lambda_1^2 \cdots \lambda_n^2 = (i^n\lambda_1 \cdots \lambda_n)^2 = 1,$$

so $i^n\lambda_1 \cdots \lambda_n = \pm 1$. Hence $\det(C) = c_1^2 \cdots c_n^2 = i^n\lambda_1 \cdots \lambda_n = \pm 1$. If $\det(C) = -1$, then replacing C with that multiplied by $\begin{pmatrix} D(0,1,\dots,1) & D(1,0,\dots,0) \\ D(1,0,\dots,0) & D(0,1,\dots,1) \end{pmatrix}$, we have $\det(C) = 1$.

By taking $P = BC$, we deduce by (A.1) that

$$X = BCJ{}^tC{}^tB = PJ{}^tP \quad (P \in SU(2n)),$$

which implies that the action is transitive.

On the other hand, the isotropy group at J is given by

$$\{P \in SU(2n) \mid PJ{}^tP = J\} = Sp(n).$$

Since $SU(2n)$ is compact, we obtain

$$SU(2n)/Sp(n) = \{X \in SU(2n) \mid {}^tX = -X\}.$$

Further multiplying by J, we obtain

$$\begin{aligned} SU(2n)/Sp(n) &= \{JX \in SU(2n) \mid {}^tX = -X\} \\ &= \{X \in SU(2n) \mid {}^t({}^tJX) = -{}^tJX\} \\ &= \{X \in SU(2n) \mid {}^tXJ = JX\} \\ &= \{X \in SU(2n) \mid {}^tX = JX{}^tJ\}. \end{aligned}$$

□

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