Lie algebras of symplectic derivations and cycles on the moduli spaces

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We consider the Lie algebra consisting of all derivations on the free associative algebra, generated by the first homology group of a closed oriented surface, which kill the symplectic class. We find the first non-trivial abelianization of this Lie algebra and discuss its relation to unstable cohomology classes of the moduli space of curves via a theorem of Kontsevich.

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1 Introduction

In [16; 17], Kontsevich considered three infinite dimensional Lie algebras, denoted by $\mathfrak{c}_g, \mathfrak{a}_g, \mathfrak{l}_g$, and described the stable homology of them, as g tends to ∞, in terms of his *graph homology* for the case $\mathfrak{c}_\infty$, the rational cohomology of the moduli spaces of curves with unlabeled punctures for the case $\mathfrak{a}_\infty$, and the rational cohomology of the outer automorphism group O*ut* F_n of free groups F_n $(n \geq 2)$ for the case $\mathfrak{l}_\infty$, respectively. See also Conant and Vogtmann [2] for more details on this result.

In this paper, we consider the latter two Lie algebras. The ideal $\mathfrak{l}_g^+$ of the last Lie algebra $\mathfrak{l}_g$, consisting of derivations with *positive* degrees, appeared, before the work of Kontsevich cited above, in the study of Johnson homomorphisms which were introduced (by Johnson) in [11; 12]; see our papers [21; 22]. In [23], we constructed a large abelian quotient of the Lie algebra $\mathfrak{l}_g^+$ and by making use of this, we defined in [25] many homology classes of Out F_n via the above theorem of Kontsevich. These classes have been investigated by Conant and Vogtmann [3; 4] by interpreting them geometrically on the Outer Space of Culler and Vogtmann [5].

In this paper, we consider the associative case $\mathfrak{a}_g$. We show the first non-trivial abelian quotient of the ideal $\mathfrak{a}_g^+$. More precisely, we determine the weight 2 part of $H_1(\mathfrak{a}_g^+)$ where the grading by weights is induced by that of the graded Lie algebra $\mathfrak{a}_g^+$. We discuss possible application of this abelianization to the problem of finding unstable cohomology classes of the moduli space of curves. We also consider a generalization of our result to the case of the Lie algebra of derivations of a free associative alegbra *without* the symplectic constraint.

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2 Two Lie algebras of symplectic derivations

In this section, we recall the definition of the two Lie algebras $\mathfrak{a}_g, \mathfrak{l}_g$ mentioned in the previous section briefly.

Let Σ_g be a closed oriented surface of genus $g \geq 1$. We simply denote by H the first integral homology group $H_1(\Sigma_g;\mathbb{Z})$ and let $H_\mathbb{Q} = H \otimes \mathbb{Q} = H_1(\Sigma_g;\mathbb{Q})$. The intersection pairing

$$H \times H \ni (u,v) \longmapsto u \cdot v \in \mathbb{Z},$$

which is a non-degenerate skew symmetric pairing, induces the Poincaré duality isomorphisms

$$H \cong H^*, \quad H_\mathbb{Q} \cong H_\mathbb{Q}^*.$$

Let $\mathrm{Sp}(H_\mathbb{Q})$ denote the automorphism group of $H_\mathbb{Q}$ preserving the above pairing. If we choose a symplectic basis of $H_\mathbb{Q}$, then $\mathrm{Sp}(H_\mathbb{Q})$ can be identified with the algebraic group $\mathrm{Sp}(2g,\mathbb{Q})$. In short, we understand $H_\mathbb{Q}$ as a symplectic vector space of dimension $2g$.

Let $\mathcal{L}_g$ denote the free graded Lie algebra generated by H and let $\mathcal{L}_g^\mathbb{Q} = \mathcal{L}_g \otimes \mathbb{Q}$. We denote by $\mathcal{L}_g^\mathbb{Q}(k)$ the degree k part of $\mathcal{L}_g^\mathbb{Q}$. In particular, we have

$$\mathcal{L}_g^\mathbb{Q}(1) = H_\mathbb{Q}, \quad \mathcal{L}_g^\mathbb{Q}(2) \cong \Lambda^2 H_\mathbb{Q}, \quad \mathcal{L}_g^\mathbb{Q}(3) \cong H_\mathbb{Q} \otimes \Lambda^2 H_\mathbb{Q}/\Lambda^3 H_\mathbb{Q}$$

and so on. Next let $T(H_\mathbb{Q})$ denote the tensor algebra (without unit) on $H_\mathbb{Q}$. In other words, it is the free associative algebra without unit generated by $H_\mathbb{Q}$. It is a graded algebra whose degree k part is $H_\mathbb{Q}^{\otimes k}$. As is well known, we can consider $\mathcal{L}_g^\mathbb{Q}$ as a natural submodule of $T(H_\mathbb{Q})$ consisting of Lie elements. We denote by

$$\omega_0 \in \mathcal{L}_g^\mathbb{Q}(2) \cong \Lambda^2 H_\mathbb{Q} \subset H_\mathbb{Q}^{\otimes 2}$$

the symplectic class.

Definition 1 We define two Lie algebras $\mathfrak{a}_g$ and $\mathfrak{l}_g$ by setting

$$\mathfrak{a}_g = \{\text{derivation } D \text{ of the tensor algebra } T(H_\mathbb{Q}); D(\omega_0) = 0\}$$
$$\mathfrak{l}_g = \{\text{derivation } D \text{ of the free Lie algebra } \mathcal{L}_g^\mathbb{Q} : D(\omega_0) = 0\}.$$

These Lie algebras are naturally graded by the degrees of derivations. More precisely, we have direct sum decompositions

$$\mathfrak{a}_g = \bigoplus_{k=0}^{\infty} \mathfrak{a}_g(k), \quad \mathfrak{l}_g = \bigoplus_{k=0}^{\infty} \mathfrak{l}_g(k)$$

where $$\mathfrak{a}_g(k) = \{D \in \mathrm{Hom}\left(H_{\mathbb{Q}}, H_{\mathbb{Q}}^{\otimes(k+1)}\right); D(\omega_0) = 0\}$$
$$\mathfrak{l}_g(k) = \{D \in \mathrm{Hom}\left(H_{\mathbb{Q}}, \mathcal{L}_g^{\mathbb{Q}}(k+1)\right); D(\omega_0) = 0\}$$

denote the degree k summands.

It is easy to see that

$$\mathfrak{a}_g(0) = \mathfrak{l}_g(0) \cong \mathfrak{sp}(2g, \mathbb{Q}) \cong S^2 H_{\mathbb{Q}}$$

where $\mathfrak{sp}(2g, \mathbb{Q})$ denotes the Lie algebra of $\mathrm{Sp}(2g, \mathbb{Q})$ and $S^2 H_{\mathbb{Q}}$ denotes the second symmetric power of $H_{\mathbb{Q}}$. Also observe that $\mathfrak{l}_g(k)$ can be considered as a natural submodule of $\mathfrak{a}_g(k)$ so that $\mathfrak{l}_g$ is a Lie subalgebra of $\mathfrak{a}_g$.

Let $$\mathfrak{a}_g^+ = \bigoplus_{k=1}^{\infty} \mathfrak{a}_g(k), \quad \mathfrak{l}_g^+ = \bigoplus_{k=1}^{\infty} \mathfrak{l}_g(k)$$

denote the ideal of $\mathfrak{a}_g$ and $\mathfrak{l}_g$,respectively, consisting of elements with positive degrees. Then, as we already mentioned in the introduction, the latter one $\mathfrak{l}_g^+$ is the same as the Lie algebra $\mathfrak{h}_{g,1}^{\mathbb{Q}}$ considered in the study of Johnson homomorphisms (see [21; 22]) before the work of Kontsevich. In this paper, however, we use Kontsevich's notation $\mathfrak{l}_g^+$ instead of ours.

Proposition 2 *The Poincaré duality $H_{\mathbb{Q}} \cong H_{\mathbb{Q}}^*$ induces canonical isomorphisms*

$$\begin{aligned}\mathfrak{a}_g(k) &= \{D \in \mathrm{Hom}\left(H_{\mathbb{Q}}, H_{\mathbb{Q}}^{\otimes(k+1)}\right); D(\omega_0) = 0\} \\ &\cong \left(H_{\mathbb{Q}}^{\otimes(k+2)}\right)^{\mathbb{Z}/(k+2)\mathbb{Z}}\end{aligned}$$

for the associative case and

$$\begin{aligned}\mathfrak{l}_g(k) &= \{D \in \mathrm{Hom}\left(H_{\mathbb{Q}}, \mathcal{L}_g^{\mathbb{Q}}(k+1)\right); D(\omega_0) = 0\} \\ &\cong \mathrm{Ker}\left(H_{\mathbb{Q}} \otimes \mathcal{L}_g^{\mathbb{Q}}(k+1) \xrightarrow{[\ ,\ ]} \mathcal{L}_g^{\mathbb{Q}}(k+2)\right)\end{aligned}$$

for the Lie case, where for the former case the cyclic group $\mathbb{Z}/(k+2)\mathbb{Z}$ of order $k+2$ acts naturally on $H_{\mathbb{Q}}^{\otimes(k+2)}$ by cyclic permutations.

Proof The latter statement was proved in [23] and the former statement was mentioned already by Kontsevich [16; 17]. Here we give a proof of the former for completeness. Choose a symplectic basis $x_1, \cdots, x_g, y_1, \cdots, y_g$ of $H_{\mathbb{Q}}$. Then it can be checked that the isomorphism

$$\mathrm{Hom}\left(H_{\mathbb{Q}}, H_{\mathbb{Q}}^{\otimes(k+1)}\right) = H_{\mathbb{Q}}^* \otimes H_{\mathbb{Q}}^{\otimes(k+1)} \cong H_{\mathbb{Q}} \otimes H_{\mathbb{Q}}^{\otimes(k+1)}$$

induced by the Poincaré duality $H_{\mathbb{Q}}^* \cong H_{\mathbb{Q}}$ is given by the correspondence

$$\mathrm{Hom}\left(H_{\mathbb{Q}}, H_{\mathbb{Q}}^{\otimes(k+1)}\right) \ni D \longmapsto D^* \in H_{\mathbb{Q}} \otimes H_{\mathbb{Q}}^{\otimes(k+1)}$$

where

$$D^* = \sum_{i=1}^{g} \{x_i \otimes D(y_i) - y_i \otimes D(x_i)\}.$$

Now assume that D belongs to $\mathfrak{a}_g(k)$, namely $D(\omega_0) = 0$. Since D acts on $\omega_0 = \sum_i (x_i \otimes y_i - y_i \otimes x_i)$ by a derivation, we have

$$\begin{aligned} D(\omega_0) &= \sum_{i=1}^{g} \left\{D(x_i) \otimes y_i + x_i \otimes D(y_i) - D(y_i) \otimes x_i - y_i \otimes D(x_i)\right\} \\ &= D^* - \sigma_{k+2}(D^*) \end{aligned}$$

where σ_{k+2} is a generator of the cyclic group $\mathbb{Z}/(k+2)\mathbb{Z}$ acting cyclically on $H_{\mathbb{Q}} \otimes H_{\mathbb{Q}}^{\otimes(k+1)} = H_{\mathbb{Q}}^{\otimes(k+2)}$ by

$$\sigma_{k+2}(u_1 \otimes u_2 \otimes \cdots \otimes u_{k+2}) = u_2 \otimes \cdots \otimes u_{k+2} \otimes u_1.$$

It follows that $D(\omega_0) = 0$ if and only if D^* is invariant under the action of $\mathbb{Z}/(k+2)\mathbb{Z}$. This completes the proof. □

Example 3 If we apply the above proposition to the degree 1 cases, we see that

$$\mathfrak{a}_g(1) \cong \left(H_{\mathbb{Q}}^{\otimes 3}\right)^{\mathbb{Z}/3} \cong S^3 H_{\mathbb{Q}} \oplus \Lambda^3 H_{\mathbb{Q}}$$

and

$$\mathfrak{l}_g(1) \cong \mathrm{Ker}\left(H_{\mathbb{Q}} \otimes \mathcal{L}_g^{\mathbb{Q}}(2) \xrightarrow{[\ ,\ ]} \mathcal{L}_g^{\mathbb{Q}}(3)\right) \cong \Lambda^3 H_{\mathbb{Q}}$$

where $S^3 H_{\mathbb{Q}}$ denotes the third symmetric power of $H_{\mathbb{Q}}$.

In fact, as is well known, there is a canonical decomposition

$$H_{\mathbb{Q}}^{\otimes 3} \cong S^3 H_{\mathbb{Q}} \oplus \Lambda^3 H_{\mathbb{Q}} \oplus V \oplus V$$

where V denotes a certain $\mathrm{GL}(H_{\mathbb{Q}})$–irreducible representation. It is easy to see that both $S^3 H_{\mathbb{Q}}$ and $\Lambda^3 H_{\mathbb{Q}}$ are $\mathbb{Z}/3\mathbb{Z}$–invariant while there is no non-trivial $\mathbb{Z}/3\mathbb{Z}$–invariant subspace of $V \oplus V$. The latter statement follows from a well known fact that there exists a natural isomorphism

$$\mathcal{L}_g^{\mathbb{Q}}(3) \cong H_{\mathbb{Q}} \otimes \mathcal{L}_g^{\mathbb{Q}}(2) / \Lambda^3 H_{\mathbb{Q}}$$

where the factor $\Lambda^3 H_{\mathbb{Q}}$ corresponds precisely to the Jacobi identity of the free Lie algebra $\mathcal{L}_g^{\mathbb{Q}}$.

3 Main result

In our paper [23], we proved the following theorem.

Theorem 4 [23] *There exists a surjective homomorphism*

$$\mathfrak{l}_g^+ \longrightarrow \Lambda^3 H_{\mathbb{Q}} \oplus \bigoplus_{k=1}^{\infty} S^{2k+1} H_{\mathbb{Q}}$$

of graded Lie algebras where $S^{2k+1} H_{\mathbb{Q}}$ denotes the $(2k+1)$-st symmetric power of $H_{\mathbb{Q}}$ and the target is considered as an abelian Lie algebra. In particular, the Lie algebra $\mathfrak{l}_g^+$ is not finitely generated.

In view of this result, we have the following important problem.

Problem 5 Determine whether the Lie algebra $\mathfrak{a}_g^+$ is finitely generated or not. In particular, is the abelianization $H_1(\mathfrak{a}_g^+)$ of $\mathfrak{a}_g^+$ finitely generated or not?

As the first step towards the solution of the above problem, we determined the weight 2 part of the abelianization of $\mathfrak{a}_g^+$ as follows. Recall here that the Lie algebra $\mathfrak{a}_g^+$ is graded so that its homology group $H_*(\mathfrak{a}_g^+)$ is bigraded. In particular, we have a direct sum decomposition

$$H_1(\mathfrak{a}_g^+) = \bigoplus_{m=1}^{\infty} H_1(\mathfrak{a}_g^+)_m$$

where $H_1(\mathfrak{a}_g^+)_m$ denotes the subspace generated by 1–cycles which are homogeneous of degree m. More precisely

$$H_1(\mathfrak{a}_g^+)_m = \text{quotient of } \mathfrak{a}_g(m) \text{ by } \sum_{i+j=m, i,j>0} [\mathfrak{a}_g(i), \mathfrak{a}_g(j)].$$

In particular, we have

$$H_1(\mathfrak{a}_g^+)_1 = \mathfrak{a}_g(1) \cong S^3 H_{\mathbb{Q}} \oplus \Lambda^3 H_{\mathbb{Q}}.$$

The following is the main theorem of this paper.

Theorem 6 *We have an isomorphism*

$$H_1(\mathfrak{a}_g^+)_2 \cong \Lambda^2 H_{\mathbb{Q}} / \mathbb{Q}(\omega_0)$$

of $\mathrm{Sp}(H_{\mathbb{Q}})$–modules. More precisely, the correspondence

$$H_{\mathbb{Q}}^{\otimes 4} \supset \mathfrak{a}_g(2) \xrightarrow{C_{13}} H_{\mathbb{Q}}^{\otimes 2} \longrightarrow \Lambda^2 H_{\mathbb{Q}} \bmod \mathbb{Q}(\omega_0) \tag{1}$$

induces the above isomorphism. Here

$$C_{13} : H_{\mathbb{Q}}^{\otimes 4} \longrightarrow H_{\mathbb{Q}}^{\otimes 2}$$

denotes the mapping given by

$$C_{13}(u_1 \otimes u_2 \otimes u_3 \otimes u_4) = (u_1 \cdot u_3)\, u_2 \otimes u_4 \quad (u_i \in H_{\mathbb{Q}})$$

and the last mapping in (1) *is the natural projection.*

4 Proof of the main result

In this section, we prove Theorem 6. We begin by determining the $\mathrm{Sp}(H_{\mathbb{Q}})$–irreducible decomposition of the degree 2 part $\mathfrak{a}_g(2)$ of the Lie algebra $\mathfrak{a}_g$. Here we use the terminology of [26] to describe irreducible representations of the algebraic group $\mathrm{Sp}(2g, \mathbb{Q})$ as well as their highest weight vectors (see also Fulton and Harris [6] for the generality of the representation theory of $\mathrm{Sp}(2g, \mathbb{Q})$). In particular, we use the symbol $[b_1 b_2 \cdots b_k]$ for expressing irreducible representations which are in one to one correspondence with Young diagrams. For example, the k-th symmetric power $S^k H_{\mathbb{Q}}$ is denoted by the symbol $[k]$ and $[1^2]$ denotes the irreducible representation $\Lambda^2 H_{\mathbb{Q}}/\mathbb{Q}(\omega_0)$. We denote by $\mathbb{Q}$ the one dimensional trivial representation which corresponds to the empty Young diagram. We also fix a symplectic basis $x_1, \cdots, x_g, y_1, \cdots, y_g$ of $H_{\mathbb{Q}}$ which we use to describe highest weight vectors. For example, the highest weight vector of $[1^2]$ is $x_1 \wedge x_2$.

Lemma 7 *The irreducible decomposition of the* $\mathrm{Sp}(H_{\mathbb{Q}})$*–module* $H_{\mathbb{Q}}^{\otimes 4}$ *is given by*

$$H_{\mathbb{Q}}^{\otimes 4} \cong 3\mathbb{Q} \oplus 6[1^2] \oplus 6[2] \oplus 2[2^2] \oplus 3[31] \oplus 3[21^2] \oplus [1^4] \oplus [4]$$

for all $g \geq 4$.

Proof This can be shown by a direct computation using the well known decomposition formula for tensor products of $\mathrm{Sp}(H_{\mathbb{Q}})$–irreducible representations (see eg [6]). □

Proposition 8 *We have a canonical decomposition*

$$\mathfrak{a}_g(2) \cong \mathbb{Q} \oplus 2[1^2] \oplus [2] \oplus [2^2] \oplus [21^2] \oplus [4]$$

for all $g \geq 4$.

Proof By Proposition 2, we have an isomorphism

$$\mathfrak{a}_g(2) \cong \left(H_{\mathbb{Q}}^{\otimes 4}\right)^{\mathbb{Z}/4\mathbb{Z}}.$$

Hence we have only to determine the $\mathbb{Z}/4\mathbb{Z}$–invariant part in the decomposition given in Lemma 7 above. We consider each irreducible component case by case.

Case (i) (the trivial representation $\mathbb{Q}$)

We define three $\mathrm{Sp}(H_{\mathbb{Q}})$–invariant elements of $H_{\mathbb{Q}}^{\otimes 4}$ by setting

$$\omega_{12} = \omega_0 \otimes \omega_0, \quad \omega_{13} = \omega_0 \otimes_{13} \omega_0, \quad \omega_{14} = \omega_0 \otimes_{14} \omega_0$$

where $\otimes_{13}$ and $\otimes_{14}$ are defined by

$$(u_1 \otimes u_2) \otimes_{13} (u_3 \otimes u_4) = u_1 \otimes u_3 \otimes u_2 \otimes u_4$$
$$(u_1 \otimes u_2) \otimes_{14} (u_3 \otimes u_4) = u_1 \otimes u_3 \otimes u_4 \otimes u_2.$$

It is easy to see that these three elements form a basis of the $\mathrm{Sp}(H_{\mathbb{Q}})$–trivial part of $H_{\mathbb{Q}}^{\otimes 4}$. Let $\sigma_4 \in \mathbb{Z}/4\mathbb{Z}$ be the generator as before. Then it acts on the above elements as

$$\sigma_4(\omega_{12}) = -\omega_{14}, \quad \sigma_4(\omega_{13}) = -\omega_{13}, \quad \sigma_4(\omega_{14}) = -\omega_{12}.$$

It follows that the dimension of the $\mathrm{Sp}(H_{\mathbb{Q}})$–trivial part of $\mathfrak{a}_g(2)$ is one and it is generated by the element $\omega_{12} - \omega_{14}$.

Case (ii) (the representation $[1^2]$)

Let $x_1, \cdots, x_g, y_1, \cdots, y_g$ be the fixed symplectic basis of $H_{\mathbb{Q}}$. Then the element $x_1 \wedge x_2 = x_1 \otimes x_2 - x_2 \otimes x_1 \in H_{\mathbb{Q}}^{\otimes 2}$ is the highest weight vector of the unique summand $[1^2]$ in $H_{\mathbb{Q}}^{\otimes 2}$. We define six elements α_{ij} $(1 \leq i < j \leq 4)$ by setting

$$\alpha_{ij} = (x_1 \wedge x_2) \otimes_{ij} \omega_0$$

where

$$(u_1 \otimes u_2) \otimes_{ij} (u_3 \otimes u_4)$$

is defined to be the element obtained by applying a permutation on $u_1 \otimes u_2 \otimes u_3 \otimes u_4$ such that u_1 and u_2 go to the i-th and j-th places respectively, while u_3 and u_4 go to the k-th and ℓ-th places respectively $(k < \ell)$. For example

$$(u_1 \otimes u_2) \otimes_{23} (u_3 \otimes u_4) = u_3 \otimes u_1 \otimes u_2 \otimes u_4.$$

Then it is easy to see that the six elements α_{ij} $(i < j)$ form a basis of $6[1^2] \subset H_{\mathbb{Q}}^{\otimes 4}$. Now we compute the action of σ_4 on these elements. We find that

$$\alpha_{12}, \ \alpha_{13}, \ \alpha_{14}, \ \alpha_{23}, \ \alpha_{24}, \ \alpha_{34}$$

are transformed by σ_4 to

$$-\alpha_{14},\ -\alpha_{24},\ -\alpha_{34},\ -\alpha_{12},\ -\alpha_{13},\ -\alpha_{23}.$$

It follows that the $\mathbb{Z}/4\mathbb{Z}$–invariant part is generated by the following two elements

$$\alpha^{(1)} = \alpha_{13} - \alpha_{24}, \quad \alpha^{(2)} = \alpha_{12} - \alpha_{14} - \alpha_{23} + \alpha_{34}. \tag{2}$$

Case (iii) (the representation [2])

In this case, the element $x_1 \otimes x_1 \in H_{\mathbb{Q}}^{\otimes 2}$ is the highest weight vector of the unique summand [2] in $H_{\mathbb{Q}}^{\otimes 2}$. We define six elements β_{ij} $(1 \le i < j \le 4)$ by setting

$$\beta_{ij} = (x_1 \otimes x_1) \otimes_{ij} \omega_0.$$

Then it is easy to see that the six elements β_{ij} $(i < j)$ form a basis of $6[2] \subset H_{\mathbb{Q}}^{\otimes 4}$. Now we compute the action of σ_4 on these elements. We find that

$$\beta_{12},\ \beta_{13},\ \beta_{14},\ \beta_{23},\ \beta_{24},\ \beta_{34}$$

are transformed by σ_4 to

$$\beta_{14},\ \beta_{24},\ \beta_{34},\ -\beta_{12},\ -\beta_{13},\ -\beta_{23}.$$

It follows that the $\mathbb{Z}/4\mathbb{Z}$–invariant part is generated by the following single element

$$\beta_{12} + \beta_{14} - \beta_{23} + \beta_{34}.$$

Case (iv) (the representation $[2^2]$)

By Lemma 7, the multiplicity of $[2^2]$ in $H_{\mathbb{Q}}^{\otimes 4}$ is two. We have the following two linearly independent elements

$$\begin{aligned}
\gamma_1 &= (x_1 \wedge x_2) \otimes (x_1 \wedge x_2) \\
&= (x_1 \otimes x_2 - x_2 \otimes x_1) \otimes (x_1 \otimes x_2 - x_2 \otimes x_1) \\
\gamma_2 &= x_1 \otimes x_1 \otimes x_2 \otimes x_2 + x_2 \otimes x_2 \otimes x_1 \otimes x_1 \\
&\quad - x_1 \otimes x_2 \otimes x_1 \otimes x_2 - x_2 \otimes x_1 \otimes x_2 \otimes x_1
\end{aligned}$$

both of which are highest weight vectors. The action of σ_4 is given by

$$\sigma_4(\gamma_1) = -\gamma_2, \quad \sigma_4(\gamma_2) = -\gamma_1.$$

It follows that $\gamma_1 - \gamma_2$ generates the unique summand $[2^2]$ which is $\mathbb{Z}/4\mathbb{Z}$–invariant.

Case (v) (the representation [31])

By Lemma 7, the multiplicity of [31] in $H_{\mathbb{Q}}^{\otimes 4}$ is three. It is easy to see that the element

$$\begin{aligned}\delta_1 = 4x_2 \otimes x_1 \otimes x_1 \otimes x_1 - x_2 \otimes x_1 \otimes x_1 \otimes x_1 - x_1 \otimes x_2 \otimes x_1 \otimes x_1 \\ - x_1 \otimes x_1 \otimes x_2 \otimes x_1 - x_1 \otimes x_1 \otimes x_1 \otimes x_2\end{aligned}$$

is the highest weight vector of one particular summand $[31] \subset H_{\mathbb{Q}}^{\otimes 4}$. Now define

$$\delta_2 = \sigma_2(\delta_1), \quad \delta_3 = \sigma_3(\delta_1), \quad \delta_4 = \sigma_4(\delta_1).$$

Here σ_i acts on $H_{\mathbb{Q}}^{\otimes 4}$ by

$$\sigma_i(u_1 \otimes u_2 \otimes u_3 \otimes u_4) = u_i \otimes u_1 \otimes \cdots \otimes \widehat{u}_i \otimes \cdots \otimes u_4 \quad (i = 2, 3, 4)$$

(the symbol $\widehat{u}_i$ denotes we delete it). Then the four elements

$$\delta_1, \delta_2, \delta_3, \delta_4$$

are transformed by σ_4 to

$$\delta_4, \delta_1, \delta_2, \delta_3.$$

On the other hand, it is easy to see that $\delta_1 + \delta_2 + \delta_3 + \delta_4 = 0$ and also that $\delta_1, \delta_2, \delta_3$ are linearly independent. Hence there is no summand of type [31] which is $\mathbb{Z}/4\mathbb{Z}$–invariant.

Case (vi) (the representation $[21^2]$)

By Lemma 7, the multiplicity of $[21^2]$ in $H_{\mathbb{Q}}^{\otimes 4}$ is three. It is easy to see that the element

$$\begin{aligned}\epsilon_1 &= x_1 \otimes (x_1 \wedge x_2 \wedge x_3) \\ &= x_1 \otimes \sum_{\tau \in \mathfrak{S}_3} \operatorname{sgn} \tau \, x_{\tau(1)} \otimes x_{\tau(2)} \otimes x_{\tau(3)}\end{aligned}$$

is the highest weight vector of a certain summand $[21^2] \subset H_{\mathbb{Q}}^{\otimes 4}$. Define

$$\epsilon_2 = \sigma_2(\epsilon_1), \quad \epsilon_3 = \sigma_3(\epsilon_1), \quad \epsilon_4 = \sigma_4(\epsilon_1).$$

Then the four elements

$$\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$$

are transformed by σ_4 to

$$\epsilon_4, \epsilon_1, \epsilon_2, \epsilon_3.$$

On the other hand, it is easy to check that $\epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4 = 0$ and also that $\epsilon_1, \epsilon_2, \epsilon_3$ are linearly independent. We can now deduce that the element

$$\epsilon_1 + \epsilon_3 = \epsilon_2 + \epsilon_4$$

generates a unique summand of type $[21^2]$ which is $\mathbb{Z}/4\mathbb{Z}$–invariant.

Case (vii) (the representation $[1^4]$)

In this case, there is a unique $[1^4]$ in $H_{\mathbb{Q}}^{\otimes 4}$ whose highest weight vector is

$$x_1 \wedge x_2 \wedge x_3 \wedge x_4 = \sum_{\tau \in \mathfrak{S}_4} \operatorname{sgn} \tau \, x_{\tau(1)} \otimes x_{\tau(2)} \otimes x_{\tau(3)} \otimes x_{\tau(4)}.$$

This element is not cyclically invariant. Hence $[1^4]$ does not appear in $\mathfrak{a}_g(2)$. In general, $[1^{k+2}]$ appears in $\mathfrak{a}_g(k)$ if and only if k is odd.

Case (viii) (the representation [4])

In this case, clearly $[4] = S^4 H_{\mathbb{Q}}$ is $\mathbb{Z}/4\mathbb{Z}$–invariant so that it appears in $\mathfrak{a}_g(2)$. In general, $[k+2] = S^{k+2} H_{\mathbb{Q}}$ appears in $\mathfrak{a}_g(k)$ for any k.

This completes the proof. □

Now we prove the main theorem of this paper.

Proof of Theorem 6 Our task is to prove the exactness of the sequence

$$\Lambda^2 \mathfrak{a}_g(1) \xrightarrow{[\ ,\ ]} \mathfrak{a}_g(2) \xrightarrow{C_{13}} \Lambda^2 H_{\mathbb{Q}}/\mathbb{Q}(\omega_0) \cong [1^2] \longrightarrow 0 \tag{3}$$

which implies the required isomorphism

$$H_1(\mathfrak{a}_g^+)_2 \cong \Lambda^2 H_{\mathbb{Q}}/\mathbb{Q}(\omega_0).$$

We begin by showing that the homomorphism C_{13} vanishes identically on commutators. For this, we consider $\mathfrak{a}_g(1)$ to be a submodule of $H_{\mathbb{Q}}^{\otimes 3}$ by Proposition 2. Let

$$\xi = \sum_i u_1^i \otimes u_2^i \otimes u_3^i \in \mathfrak{a}_g(1) \subset H_{\mathbb{Q}}^{\otimes 3}$$
$$\eta = \sum_j v_1^j \otimes v_2^j \otimes v_3^j \in \mathfrak{a}_g(1) \subset H_{\mathbb{Q}}^{\otimes 3}$$

be any two elements of $\mathfrak{a}_g(1)$. Then the bracket $[\xi, \eta] \in \mathfrak{a}_g(2)$ is given by $[\xi, \eta] = \sum_{i,j} [\xi, \eta]_{ij}$ where

$$\begin{aligned}
&[\xi, \eta]_{ij} \\
=&(u_1^i \cdot v_2^j)\, v_1^j \otimes u_2^i \otimes u_3^i \otimes v_3^j + (u_1^i \cdot v_3^j)\, v_1^j \otimes v_2^j \otimes u_2^i \otimes u_3^i \\
&-(v_1^j \cdot u_2^i)\, u_1^i \otimes v_2^j \otimes v_3^j \otimes u_3^i - (v_1^j \cdot u_3^i)\, u_1^i \otimes u_2^i \otimes v_2^j \otimes v_3^j.
\end{aligned}$$

It follows that

$$\begin{aligned} &C_{13}([\xi,\eta]_{ij}) \\ &=(u_1^i\cdot v_2^j)(v_1^j\cdot u_3^i)\,u_2^i\otimes v_3^j+(u_1^i\cdot v_3^j)(v_1^j\cdot u_2^i)\,v_2^j\otimes u_3^i \\ &-(v_1^j\cdot u_2^i)(u_1^i\cdot v_3^j)\,v_2^j\otimes u_3^i-(v_1^j\cdot u_3^i)(u_1^i\cdot v_2^j)\,u_2^i\otimes v_3^j \\ &=0 \end{aligned}$$

and hence $C_{13}([\xi,\eta])=0$ as required. We will later observe that the vanishing of C_{13} on commutators is valid in a broader context (see Proposition 15).

Next we compute the value of C_{13} on the component of type $[1^2]$ in $\mathfrak{a}_g(2)$. In the notation of Case (ii) of Proposition 8, we have

$$\alpha_{12}=\sum_{i=1}^{g}(x_1\otimes x_2-x_2\otimes x_1)\otimes(x_i\otimes y_i-y_i\otimes x_i)$$

so that

$$C_{13}(\alpha_{12})=-x_2\otimes x_1+x_1\otimes x_2.$$

Similarly, we have

$$\begin{aligned} &C_{13}(\alpha_{13})=0,\\ &C_{13}(\alpha_{14})=x_1\otimes x_2-x_2\otimes x_1,\quad C_{13}(\alpha_{23})=x_1\otimes x_2-x_2\otimes x_1,\\ &C_{13}(\alpha_{24})=2g\,(x_1\otimes x_2-x_2\otimes x_1),\quad C_{13}(\alpha_{34})=x_1\otimes x_2-x_2\otimes x_1. \end{aligned}$$

It follows that the values of C_{13} on the two elements $\alpha^{(1)},\alpha^{(2)}$ (see (2) in Proposition 8) are given by

$$C_{13}(\alpha^{(1)})=-2g\,(x_1\otimes x_2-x_2\otimes x_1),\quad C_{13}(\alpha^{(2)})=0.$$

Thus we have proved the exactness at the last factor of the sequence (3), namely the surjectivity of the homomorphism

$$C_{13}\colon\ \mathfrak{a}_g(2)\longrightarrow\Lambda^2 H_{\mathbb{Q}}/\mathbb{Q}(\omega_0).$$

It remains to prove that all the irreducible components appearing in Proposition 8 other than the one generated by the element $\alpha^{(1)}$ above can be represented as commutators. Consider the degree two part $\mathfrak{l}_g(2)$ of the graded Lie algebra $\mathfrak{h}_{g,1}^{\mathbb{Q}}$ which is a graded Lie subalgebra of $\mathfrak{a}_g^+$. It is a consequence of a fundamental paper [8] by Hain, suitably adapted to the case with one boundary component as in [24], that the bracket operation

$$\Lambda^2\mathfrak{l}_g(1)\xrightarrow{[\,\cdot\,]}\mathfrak{l}_g(2)\cong\mathbb{Q}\oplus[1^2]\oplus[2^2]$$

is surjective (see also [26] for a related result).

Thus we have only to prove that the three components $[2], [21^2], [4]$ appearing in the decomposition of $\mathfrak{a}_g(2)$ belong to the commutator ideal of $\mathfrak{a}_g^+$. Recall from Example 3 that $\mathfrak{a}_g(1) \cong \Lambda^3 H_{\mathbb{Q}} \oplus S^3 H_{\mathbb{Q}}$. We consider case by case.

Case of the representation [2] We consider two elements

$$\begin{aligned}
\xi_1 &= x_2 \wedge y_2 \wedge x_3 \in \Lambda^3 H_{\mathbb{Q}} \subset \mathfrak{a}_g(1) \\
\eta_1 &= x_1 \otimes x_1 \otimes y_3 + x_1 \otimes y_3 \otimes x_1 + y_3 \otimes x_1 \otimes x_1 \in S^3 H_{\mathbb{Q}} \subset \mathfrak{a}_g(1).
\end{aligned}$$

Then explicit computation implies

$$\begin{aligned}
&[\xi_1, \eta_1] \\
=&- x_1 \otimes x_1 \otimes (x_2 \otimes y_2 - y_2 \otimes x_2) - (x_2 \otimes y_2 - y_2 \otimes x_2) \otimes x_1 \otimes x_1 \\
&- x_1 \otimes (x_2 \otimes y_2 - y_2 \otimes x_2) \otimes x_1 \\
&+ x_2 \otimes x_1 \otimes x_1 \otimes y_2 - y_2 \otimes x_1 \otimes x_1 \otimes x_2.
\end{aligned}$$

If we apply the contraction $C_{11}\colon H_{\mathbb{Q}}^{\otimes 4} \to H_{\mathbb{Q}}^{\otimes 2}$ defined by

$$C_{11}(u_1 \otimes u_2 \otimes u_3 \otimes u_4) = (u_1 \cdot u_2)\, u_3 \otimes u_4$$

to the above element $[\xi_1, \eta_1]$, we obtain

$$-2x_1 \otimes x_1$$

which is the highest weight vector of $[2] \subset H_{\mathbb{Q}}^{\otimes 2}$. It follows that the summand [2] is a commutator.

Case of the representation $[21^2]$ We consider two elements

$$\begin{aligned}
\xi_2 &= x_1 \wedge x_2 \wedge x_3 \in \Lambda^3 H_{\mathbb{Q}} \subset \mathfrak{a}_g(1) \\
\eta_2 &= x_1 \otimes x_1 \otimes y_1 + x_1 \otimes y_1 \otimes x_1 + y_1 \otimes x_1 \otimes x_1 \in S^3 H_{\mathbb{Q}} \subset \mathfrak{a}_g(1).
\end{aligned}$$

Then explicit computation implies

$$\begin{aligned}
&[\xi_2, \eta_2] \\
=&x_1 \otimes (x_1 \otimes (x_2 \otimes x_3 - x_3 \otimes x_2) - (x_2 \otimes x_3 - x_3 \otimes x_2) \otimes x_1) \\
&-x_2 \otimes (x_1 \otimes x_1 \otimes x_3 - x_3 \otimes x_1 \otimes x_1) \\
&+x_3 \otimes (x_1 \otimes x_1 \otimes x_2 - x_2 \otimes x_1 \otimes x_1).
\end{aligned}$$

This is the highest weight vector of the summand $[21^2] \subset \mathfrak{a}_g(2)$ because if we replace x_3 with x_2 in the above tensor, we obtain 0 and the same is true if we replace x_2 with x_1.

Case of the representation [4] We consider the element

$$\xi_3 = x_1 \otimes x_1 \otimes x_1 \in S^3 H_{\mathbb{Q}} \subset \mathfrak{a}_g(1)$$

and the above element $\eta_2 \in S^3 H_{\mathbb{Q}} \subset \mathfrak{a}_g(1)$. Then explicit computation shows

$$[\xi_3, \eta_2] = 4x_1 \otimes x_1 \otimes x_1 \otimes x_1$$

which is the highest weight vector of the summand $[4] \subset \mathfrak{a}_g(2)$.

This completes the proof of Theorem 6. □

5 Cycles on the moduli spaces

There are several method of constructing explicit (co)cycles of the mapping class group as well as the moduli space of curves. They include group cohomological construction given in [24], combinatorial construction using the natural cell structures on the moduli spaces by Witten and Kontsevich, Kontsevich's construction [16; 17] using A_∞ algebras with scalar products, and more recently Penner and the author [28] obtained another combinatorial method. However all the classes obtained in these ways turn out to be expressed as polynomials in the Mumford–Morita–Miller tautological classes as shown in the work of Kawazumi and the author [14; 15] for the group cohomological cocycles and Igusa [10] and Mondello [20] for the Witten and Kontsevich cycles.

Recently Madsen and Weiss [19] proved that the stable cohomology of the mapping class group is the polynomial algebra generated by the Mumford–Morita–Miller classes. Hence we can say that there are many ways of constructing cocycles for any stable cohomology class. On the other hand, the works of Harer and Zagier [9] and Penner [29] determining the orbifold Euler characteristics of the moduli space imply that there should exist huge amount of unstable classes. However there are very few works on the unstable classes (see Looijenga [18] and Tommasi [30]). It would be worthwhile to try to use the following (suitably modified) theorem of Kontsevich to produce unstable homology classes of the moduli space.

Theorem 9 (Kontsevich [16; 17]) *There is an isomorphism*

$$PH^k(\mathfrak{a}_\infty^+)_{2n}^{\mathrm{Sp}} \cong \bigoplus_{2g-2+m=n>0,\, m>0} H_{2n-k}(\mathbf{M}_g^m; \mathbb{Q})_{\mathfrak{S}_m}.$$

Here P denotes the *primitive part* of $H^*(\mathfrak{a}_\infty^+)^{\mathrm{Sp}}$ which has a natural structure of Hopf algebra, the subscript $2n$ denotes the weight $2n$ part and $\mathbf{M}_g^m$ denotes the moduli space of genus g smooth curves with m punctures.

If we apply the above theorem to Tommasi's unstable cohomology class in $H^5(\mathbf{M}_4;\mathbb{Q})$, for example, it should yield certain elements in $PH^k(\mathfrak{a}_\infty^+)_{14}^{\mathrm{Sp}}$ for $k=7,9$.

Now if we consider the abelianization $\mathfrak{a}_g^+ \to H_1(\mathfrak{a}_g^+)$ of the Lie algebra $\mathfrak{a}_g^+$ and apply the above theorem of Kontsevich, we obtain a homomorphism

$$\lim_{g\to\infty} PH^k\left(H_1(\mathfrak{a}_g^+)\right)_{2n}^{\mathrm{Sp}} \longrightarrow \bigoplus_{2g-2+m=n>0,\, m>0} H_{2n-k}(\mathbf{M}_g^m;\mathbb{Q})_{\mathfrak{S}_m}.$$

We can then ask which part of the cohomology of the moduli spaces can be obtained in this way. In particular, if the abelianization of $\mathfrak{a}_g^+$ was large like in the case of $\mathfrak{l}_g^+$, then we would obtain many candidates for unstable classes of the moduli spaces.

Now our main result (Theorem 6) together with Example 3 gives rise to a surjective homomorphism

$$\mathfrak{a}_g^+ \longrightarrow S^3 H_{\mathbb{Q}} \oplus \Lambda^3 H_{\mathbb{Q}} \oplus \Lambda^2 H_{\mathbb{Q}}/\mathbb{Q}(\omega_0)$$

and hence

$$\lim_{g\to\infty} PH^k(S^3 H_{\mathbb{Q}} \oplus \Lambda^3 H_{\mathbb{Q}} \oplus \Lambda^2 H_{\mathbb{Q}}/\mathbb{Q}(\omega_0))^{\mathrm{Sp}} \longrightarrow \bigoplus_{2g-2+m=n>0,\, m>0} H_{2n-k}(\mathbf{M}_g^m;\mathbb{Q})_{\mathfrak{S}_m}.$$

It can be shown that the source of the above homomorphism contain many cohomology classes. By restricting further to the weight 2 part, we obtain

$$\lim_{g\to\infty} PH^k(\Lambda^2 H_{\mathbb{Q}}/\mathbb{Q}(\omega_0))^{\mathrm{Sp}} \longrightarrow \bigoplus_{k=2g-2+m>0,\, m>0} H_k(\mathbf{M}_g^m;\mathbb{Q})_{\mathfrak{S}_m}$$

whose left hand side can be determined as follows.

Proposition 10 *The* Sp*–trivial part* $H^*\left(\Lambda^2 H_{\mathbb{Q}}/\mathbb{Q}(\omega_0)\right)^{\mathrm{Sp}}$ *of the cohomology group of the irreducible representation* $\Lambda^2 H_{\mathbb{Q}}/\mathbb{Q}(\omega_0) \cong [1^2]$ *is isomorphic to the cohomology of the product* $S^5 \times S^9 \times S^{13} \times \cdots$ *as* g *tends to* ∞ *so that we have*

$$\lim_{g\to\infty} PH^k\left(\Lambda^2 H_{\mathbb{Q}}/\mathbb{Q}(\omega_0)\right)^{\mathrm{Sp}} \cong \begin{cases} \mathbb{Q} & (k \equiv 1 \bmod 4,\, k \neq 1) \\ 0 & (\text{otherwise}). \end{cases}$$

Proof By the classical invariant theory of Weyl, any Sp–invariant cohomology class in degree k of the Sp–module $\Lambda^2 H_{\mathbb{Q}}/\mathbb{Q}(\omega_0)$ can be expressed as a linear combination of certain contractions

$$C\colon \Lambda^k(\Lambda^2 H_{\mathbb{Q}}/\mathbb{Q}(\omega_0))\to\mathbb{Q}$$

each of which is a various multiples of the intersection pairing on $H_{\mathbb{Q}}$. Such contractions can be enumerated by bivalent graphs and primitivity in cohomology corresponds to connectedness of graphs. There is only one connected bivalent graph with k vertices, namely a k–gon, for each $k \geq 2$. Let C_k be the corresponding contraction. Now we apply C_k on the following particular element

$$\lambda_k = \sum_{\tau\in\mathfrak{S}_k} \operatorname{sgn}\tau\, \lambda_k^{(\tau(1))} \otimes\cdots\otimes \lambda_k^{(\tau(k))}$$

of $\Lambda^k(\Lambda^2 H_{\mathbb{Q}}/\mathbb{Q}(\omega_0))$, where

$$\lambda_k^{(1)} = x_1\wedge y_2,\ \lambda_k^{(2)} = x_2\wedge y_3,\ \cdots,$$
$$\lambda_k^{(k-1)} = x_{k-1}\wedge y_k,\ \lambda_k^{(k)} = x_k\wedge y_1.$$

By explicit computations, we see that

$$C_k(\lambda_k) \neq 0$$

if and only if $k \equiv 1 \mod 4$ $(k \neq 1)$. On the other hand, it can be checked easily that any contraction corresponding to *disconnected* graph vanishes on λ_k. The result follows from these facts. □

Remark 11 If we replace $\Lambda^2 H_{\mathbb{Q}}/\mathbb{Q}(\omega_0)$ by the second symmetric power $S^2 H_{\mathbb{Q}}$ in the above argument, we obtain similar result where $k \equiv 1 \bmod 4$ is replaced by $k \equiv 3 \bmod 4$. This can be also obtained by making use of the fact that $S^2 H_{\mathbb{Q}}$ is isomorphic to the Lie algebra of $\mathrm{Sp}(2g,\mathbb{Q})$.

Thus we obtain infinitely many unstable *odd* dimensional homology classes of the moduli spaces.

Remark 12 If we apply Theorem 9 to Proposition 10, we obtain homomorphisms

$$\lim_{g\to\infty} PH^k\left(\Lambda^2 H_{\mathbb{Q}}/\mathbb{Q}(\omega_0)\right)^{\mathrm{Sp}} \longrightarrow H_{4k+1}(\mathbf{M}_1^{4k+1})_{\mathfrak{S}_{4k+1}}$$

for all $k \geq 1$. Recently, Conant [1] proved that the cycles of the genus one moduli spaces obtained in this way are all non-trivial. It would be interesting to compare this result with that of Getzler [7] where the rational cohomology of genus one moduli spaces are determined.

6 Further results

It seems worthwhile to consider the Lie algebra consisting of derivations on free associative algebras *without* the symplectic constraint. For this, let H_n denote a vector space over $\mathbb{Q}$ of dimension $n \geq 2$. Also let $T(H_n)$ denote the tensor algebra without unit generated by H_n. Now define

$$\mathrm{Der}(T(H_n)) = \{\text{derivation } D \text{ of the free associative algebra } T(H_n)\}.$$

It has a natural structure of a graded Lie algebra

$$\mathrm{Der}(T(H_n)) = \bigoplus_{k=0}^{\infty} \mathrm{Der}(T(H_n))(k)$$

where $\quad \mathrm{Der}(T(H_n)(k) = \mathrm{Hom}\left(H_n, H_n^{\otimes(k+1)}\right) \cong H_n^* \otimes H_n^{\otimes(k+1)}.$

Let $$\mathrm{Der}^+(T(H_n)) = \bigoplus_{k=1}^{\infty} H_n^* \otimes H_n^{\otimes(k+1)}$$

be the ideal of $\mathrm{Der}(T(H_n))$ consisting of derivations with positive degrees. By definition, $\mathfrak{a}_g^+$ is clearly a Lie subalgebra of $\mathrm{Der}^+(T(H_{2g}))$ and we are interested in the structure of $\mathrm{Der}^+(T(H_n))$ as well. In particular, we have the following problem.

Problem 13 Determine whether the Lie algebra $\mathrm{Der}^+(T(H_n))$ is finitely generated or not. In particular, is the abelianization

$$H_1(\mathrm{Der}^+(T(H_n)))$$

of $\mathrm{Der}^+(T(H_n))$ finitely generated or not?

Remark 14 We can also consider the free Lie algebra $\mathcal{L}(H_n)$ generated by H_n and also the Lie algebra $\mathrm{Der}(\mathcal{L}(H_n))$ consisting of derivations of it. This Lie algebra can be naturally considered as a Lie subalgebra of $\mathrm{Der}(T(H_n))$. In this case, however, we know that $\mathrm{Der}^+(\mathcal{L}(H_n))$ is *not* finitely generated because the surjective homomorphisms $\mathrm{trace}(k)\colon \mathrm{Der}(\mathcal{L}(H_n))(k) \to S^k H_n$ defined in [23] vanish on the commutator ideal $[\mathrm{Der}^+(\mathcal{L}(H_n)), \mathrm{Der}^+(\mathcal{L}(H_n))]$ for any $k \geq 2$ (see [27]).

The Lie algebra $\mathrm{Der}^+(T(H_n))$ is graded. It follows that the abelianization $H_1(\mathrm{Der}^+(T(H_n)))$ is bigraded and we have a direct sum decomposition

$$H_1(\mathrm{Der}^+(T(H_n))) = \bigoplus_{m=1}^{\infty} H_1(\mathrm{Der}^+(T(H_n)))_m.$$

Clearly we have

$$H_1(\mathrm{Der}^+(T(H_n)))_1 = H_n^* \otimes H_n^{\otimes 2}.$$

Now we show that the next summand, namely the weight 2 part of the abelianization of $\mathrm{Der}^+(T(H_n))$ is non-trivial. For this, consider the contraction

$$C_{13}\colon \mathrm{Der}(T(H_n))(2) \cong H_n^* \otimes H_n^{\otimes 3} \longrightarrow H_n^{\otimes 2}$$

defined by

$$C_{13}(f \otimes u_1 \otimes u_2 \otimes u_3) = f(u_2)\, u_1 \otimes u_3$$

where $f \in H_n^*, u_i \in H_n$. It is clearly surjective.

Proposition 15 *The composition*

$$\Lambda^2\mathrm{Der}(T(H_n))(1) \xrightarrow{[\,,\,]} \mathrm{Der}(T(H_n))(2) \xrightarrow{C_{13}} H_n^{\otimes 2}$$

is trivial. In other words, the homomorphism C_{13} is trivial on the commutator ideal $[\mathrm{Der}^+(T(H_n)), \mathrm{Der}^+(T(H_n))]$ of the Lie algebra $\mathrm{Der}^+(T(H_n))$ so that it induces a surjection

$$H_1(\mathrm{Der}^+(T(H_n)))_2 \longrightarrow H_n^{\otimes 2}.$$

Proof The bracket operation on $\Lambda^2\mathrm{Der}(T(H_n))(1)$ is given by

$$\begin{aligned}
&[f \otimes u_1 \otimes u_2, h \otimes v_1 \otimes v_2] \\
&\quad = f(v_1)\, h \otimes u_1 \otimes u_2 \otimes v_2 + f(v_2)\, h \otimes v_1 \otimes u_1 \otimes u_2 \\
&\quad\ \ -h(u_1)\, f \otimes v_1 \otimes v_2 \otimes u_2 - h(u_2)\, f \otimes u_1 \otimes v_1 \otimes v_2
\end{aligned}$$

where $f, h \in H_n^*, u_i, v_i \in H_n$. It follows that

$$\begin{aligned}
&C_{13}([f \otimes u_1 \otimes u_2, h \otimes v_1 \otimes v_2]) \\
&\quad = f(v_1)h(u_2)\, u_1 \otimes v_2 + f(v_2)h(u_1)\, v_1 \otimes u_2 \\
&\quad\ \ -h(u_1)f(v_2)\, v_1 \otimes u_2 - h(u_2)f(v_1)\, u_1 \otimes v_2 \\
&\quad = 0.
\end{aligned}$$

This completes the proof. □

Remark 16 There is a close connection between the contraction C_{13} and the trace map. In fact, the equality $C_{13} = -2\,\mathrm{trace}(2)$ holds so that the following diagram is commutative (up to a non-zero scalar)

$$\begin{array}{ccc} \mathrm{Der}(\mathcal{L}(H_n))(2) & \xrightarrow{\mathrm{trace}(2)} & S^2 H_n \\ \downarrow & & \downarrow \\ \mathrm{Der}(T(H_n))(2) & \xrightarrow{C_{13}} & H_n^{\otimes 2}. \end{array}$$

Conjecture 17 The contraction $C_{13}\colon \mathrm{Der}(T(H_n))(2) \to H_n^{\otimes 2}$ induces an isomorphism

$$H_1(\mathrm{Der}^+(T(H_n)))_2 \cong H_n^{\otimes 2}.$$

By an explicit computation using a computer, we have obtained the following result.

Proposition 18 *The above conjecture is true for the case $n = 2$, namely we have an isomorphism*

$$H_1(\mathrm{Der}^+(T(H_2)))_2 \cong H_2^{\otimes 2}.$$

Furthermore, the homomorphism

$$\mathrm{Der}(T(H_2))(1) \otimes \mathrm{Der}(T(H_2))(k) \xrightarrow{[\,,\,]} \mathrm{Der}(T(H_2))(k+1)$$

induced by the bracket operation is surjective for $k = 2, 3$ so that

$$H_1(\mathrm{Der}^+(T(H_2)))_m = 0$$

for $m = 3, 4$.

Remark 19 At present, we have no idea about the (non)-triviality of the abelianization $H_1(\mathrm{Der}^+(T(H_n)))_m$ for $n \geq 3$ and $m \geq 3$. In the case of $\mathrm{Der}^+(\mathcal{L}(H_n))$, Kassabov [13] has determined the abelianization of it in a certain stable range (cf [27]).

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