Determination of the order of the P–image by Toda brackets

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The present paper gives a proof of the author's paper [14] on the orders of Whitehead products of ι_n with $\alpha \in \pi^n_{n+k}$, $(n \geq k+2, k \leq 24)$ and improves and extends it. The method is to use composition methods in the homotopy groups of spheres and rotation groups.

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Introduction

This paper is a sequel to [5] by Golasiński and the author in the stable case. The methods are to use those of [5]. In particular, the EHP sequence, the method and result of Toda [18, Chapter 11] and the result of Nomura [15] are essentially used. Let π^n_{n+k} denote the 2 primary component of the homotopy group $\pi_{n+k}(S^n)$ of the n dimensional sphere S^n. Let ι_n be the identity class of S^n and $\alpha \in \pi^n_{n+k}$ for $n \geq k+2$. Then our result about the order of the Whitehead product $[\iota_n, \alpha] = P(E^{n-1}\alpha)$ is as follows:

Theorem 1 (Main Theorem) *Let $n \geq k+2$ and α be an element of π^n_{n+k}. Then, the order of the Whitehead product $[\iota_n, \alpha]$ for $n \equiv r \pmod 8$ with $0 \leq r \leq 7$ is as given in Tables 1 and 2 except as otherwise noted.*

1 Results from [5]

In this section, we shall collect the result of [5] that we need. We denote by $SO(n)$ the n-th rotation group and by $\Delta\colon \pi_k(S^n) \to \pi_{k-1}(SO(n))$ the connecting homomorphism. The notation $n \equiv i \pmod k$ is often written $n \equiv i\ (k)$. From the fact that $\pi_{4n+3}(SO(4n+3)) \cong \mathbb{Z}$ [7], we have $\Delta\eta_{4n+3} = 0$.

We recall $[\iota_n, \eta] = 0$ if and only if $n \equiv 3\ (4)$ or $n = 2, 6$;

$$[\iota_n, \eta^2] = 0 \text{ if and only if } n \equiv 2, 3\ (4) \text{ or } n = 5.$$

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Table 1

$\alpha\backslash r$	0	1	2	3	4	5	6	7
η	2	2	2	1	2	2	2	1
η^2	2	2	1	1	2	2	1	1
ν	8	2	4	2	8	$2, \neq 2^i-3$ $1, = 2^i-3$	4	1
ν^2	2	2	2	$2, \neq 2^i-5$ $1, = 2^i-5$	1	1	2	1
σ	16	2	16	2	16	2	16	$2, \ 7(16)$ $1, \ 15(16)$
$\eta\sigma$	2	2	2	1	2	2	$1, \not\equiv 22(32) \geq 54$ $2, \equiv 22(32) \geq 54$	1
ε	2	2	1	1	2	2	2	1
$\overline{\nu}$	2	2	2	1	2	2	2	1
$\eta^2\sigma$	2	$2, \neq 2^i-7$ $1, = 2^i-7$	1	1	2	$1, \not\equiv 53(64)$ $2, \equiv 53(64) \geq 117$	1	1
$\eta\varepsilon$	2	1	1	1	2	$1, \not\equiv 53(64)$ $2, \equiv 53(64) \geq 117$	1	1
ν^3	2	$2, \neq 2^i-7$ $1, = 2^i-7$	1	1	1	1	1	1
μ	2	2	2	1	2	2	2	1
$\eta\mu$	2	2	1	1	2	2	1	1
ζ	8	1	4	$1, \not\equiv 115(128)$ $2, \equiv 115(128) \geq 243$	8	1	4	1
σ^2	$2, 0(16)$	$2, \ 1(16)$ $1, \ 9(16)$	2	$2, \ 3(16)$ $1, \ 11(16)$	2	2	2	$1, 15(16)$
κ	2	2	2	2	2	2	2	1

For example, $\left\{ \begin{smallmatrix} 2, & \neq 2^i-3 \\ 1, & = 2^i-3 \end{smallmatrix} \right\}$, $\left\{ \begin{smallmatrix} 2, & 7(16) \\ 1, & 15(16) \end{smallmatrix} \right\}$ and $\{2, 0(16)\}$ mean $\left\{ \begin{smallmatrix} 2, & \text{for } n \neq 2^i-3 \geq 5 \\ 1, & \text{for } n = 2^i-3 \geq 5 \end{smallmatrix} \right\}$, $\left\{ \begin{smallmatrix} 2, & \text{for } n \equiv 7 \ (\text{mod } 16) \geq 23 \\ 1, & \text{for } n \equiv 15 \ (\text{mod } 16) \geq 15 \end{smallmatrix} \right\}$ and $\left\{ \begin{smallmatrix} 2, & \text{for } n \equiv 0 \ (\text{mod } 16) \geq 16 \\ \text{unsettled}, & \text{for } n \equiv 8 \ (\text{mod } 16) \geq 24 \end{smallmatrix} \right\}$, respectively.

Here η and η^2 mean exactly $\eta_n \in \pi^n_{n+1}$ and $\eta^2_n \in \pi^n_{n+2}$, respectively. Hereafter we deal with the 2 primary components. Denote by $\sharp\alpha$ the order of α in a group. We recall

$$\sharp[\iota_n, \nu] = \begin{cases} 8 & \text{if } n \equiv 0 \ (4) \geq 8, \ n \neq 12; \\ 4 & \text{if } n \equiv 2 \ (4) \geq 6, n = 4, 12; \\ 2 & \text{if } n \equiv 1, 3, 5 \ (8) \geq 9, \ n \neq 2^i - 3; \\ 1 & \text{if } n \equiv 7 \ (8), \ n = 2^i - 3 \geq 5. \end{cases}$$

We also recall

$$\Delta(\nu^2_{8n+k}) = 0 \text{ if } n \geq 0 \text{ and } k = 4, 5.$$

The following is one of the main results in [5]:

Theorem 1.1 $[\iota_n, \nu^2] = 0$ *if and only if* $n \equiv 4, 5, 7 \ (8)$ *or* $n = 2^i - 5$ *for* $i \geq 4$.

Table 2

$\alpha\backslash r$	0	1	2	3	4	5	6	7
$\eta\kappa$	2	1	1	1	2	2	1	1
ρ	32	2	32	2	32	2	32	a
$\eta\rho$	2	2	2	1	2	2	$1,\ \not\equiv 2^9-18(2^9)$ $2,\ \equiv 2^9-18(2^9)$ $\geq 2^{10}-18$	1
η^*	2	2	2	1	2	2	2,14(16)	1
$\eta\eta^*$	2	2	1	1	2	2, 13(16)	1	1
$\eta^2\rho$	2	2	1	1	2	$1,\ \not\equiv 2^{10}-19(2^{10})$ $2,\ \equiv 2^{10}-19(2^{10})$ $\geq 2^{11}-19$	1	1
$\nu\kappa$	2	1	2	2	2	1	1	1
$\overline{\mu}$	2	2	2	1	2	2	2	1
$\eta\overline{\mu}$	2	2	1	1	2	2	1	1
ν^*	8	2	4	2	8 or 4		4	1
$\overline{\zeta}$	8	1	4	$1,\ \not\equiv 2^{11}-21(2^{11})$ $2,\ \equiv 2^{11}-21(2^{11})$ $\geq 2^{12}-21$	8	1	4	1
$\overline{\sigma}$	2	2	2	2		1, 5(16)	1, 6(16)	1
$\overline{\kappa}$	8	2	8 or 4	2	4	2	4	1
σ^3	1, 8(16)	1, 9(16)	2	1, 11(16)	1	1	2	1
$\eta\overline{\kappa}$	2	2	2	1	2	2	1	1
$\eta^2\overline{\kappa}$	2	2	1	1	2	1	1	1
$\nu\overline{\sigma}$	$2^{\{*\}}$			1, 3(16)	1	1	1	1
$\eta^*\sigma$	2	2	1	1	2	2	1, 6(16)	1
$\nu\overline{\kappa}$	8 or 4		4	2	4	1	4	1
$\overline{\rho}$	16	2	16	2	16	2	16	b
$\eta\overline{\rho}$	2	2	2	1	2	2	$1,\ \not\equiv 2^{13}-26(2^{13})$ $2,\ \equiv 2^{13}-26(2^{13})$ $\geq 2^{14}-26$	1
$\eta\eta^*\sigma$	2	1	1	1	2	1, 5(16)	1	1
$\mu_{3,*}$	2	2	2	1	2	2	2	1
$\eta^2\overline{\rho}$	2	2	1	1	2	$1,\ \not\equiv 2^{14}-27(2^{14})$ $2,\ \equiv 2^{14}-27(2^{14})$ $\geq 2^{15}-27$	1	1
$\eta\mu_{3,*}$	2	2	1	1	2	2	1	1
$\nu^2\overline{\kappa}$		1	2	1	1	1	2	1
$\zeta_{3,*}$	8	1	4	$1,\ \not\equiv 2^{15}-29(2^{15})$ $2,\ \equiv 2^{15}-29(2^{15})$ $\geq 2^{16}-29$	8	1	4	1

$\{*\}$ The result holds if $\langle\overline{\nu},\sigma,\overline{\nu}\rangle=\eta\eta^*\sigma$.

$$a=\begin{cases}1, & n\not\equiv 2^8-17(2^8);\\ 2, & n\equiv 2^8-17(2^8)\geq 2^9-17,\end{cases}\qquad b=\begin{cases}1, & n\not\equiv 2^{12}-25(2^{12});\\ 2, & n\equiv 2^{12}-25(2^{12})\geq 2^{13}-25.\end{cases}$$

Let $n\equiv 7\ (16)\geq 23$. Then, there exists an element $\delta_{n-7}\in\pi^{n-7}_{2n-8}$ satisfying

$$[\iota_n,\iota]=E^7\delta_{n-7} \text{ and } H\delta_{n-7}=\sigma_{2n-15} \text{ if } n\equiv 7\ (16)\geq 23. \tag{1–1}$$

We recall

$$\sharp[\iota_n,\sigma]=\begin{cases}16 & \text{if } n\equiv 0\ (2)\geq 10;\\ 8 & \text{if } n=8;\\ 2 & \text{if } n\equiv 1\ (2)\geq 9,\ n\neq 11,\ n\not\equiv 15\ (16);\\ 1 & \text{if } n=11,\ n\equiv 15\ (16).\end{cases}$$

We also recall the elements $\tau_{2n} \in \pi^{2n}_{4n}$ and $\overline{\tau}_{4n} \in \pi^{4n}_{8n+2}$, which are the J images of the complex and symplectic characteristic elements, respectively. They satisfy the following.

Lemma 1.2

(1) $E\tau_{2n} = [\iota_{2n+1}, \iota], 2\tau_{4n+2} = [\iota_{4n+2}, \eta]$ *and* $H\tau_{2n} = (n+1)\eta_{4n-1}$;

(2) $E^2\overline{\tau}_{4n} = \tau_{4n+2}$ *and* $H\overline{\tau}_{4n} = \pm(n+1)\nu_{8n-1}$.

About the group structure of the stable k–stem π^s_k for $23 \le k \le 29$, we recall from [11] and [16] the following: $\pi^s_{23} = \{\overline{\rho}, \nu\overline{\kappa}, \eta^*\sigma\} \cong \mathbb{Z}_{16} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_2$; $\pi^s_{24} = \{\eta\overline{\rho}, \eta\eta^*\sigma\} \cong (\mathbb{Z}_2)^2$; $\pi^s_{25} = \{\eta^2\overline{\rho}, \mu_{3,*}\} \cong (\mathbb{Z}_2)^2$; $\pi^s_{26} = \{\eta\mu_{3,*}, \nu^2\overline{\kappa}\} \cong (\mathbb{Z}_2)^2$; $\pi^s_{27} = \{\zeta_{3,*}\} \cong \mathbb{Z}_8$; $\pi^s_{28} = \{\varepsilon\overline{\kappa}\} \cong \mathbb{Z}_2$; $\pi^s_{29} = 0$.

By Lemma 1.2(1) and the property of the Whitehead product,

$$[\iota_{4n+2}, \eta\alpha] = 0 \quad \text{if} \quad 2\alpha = 0.$$

Especially, for the elements $\beta = \nu, \zeta, \nu^*, \overline{\zeta}, \nu\overline{\kappa}, \zeta_{3,*}$, we know the relations $4\beta = \eta^3, \eta^2\mu, \eta^2\eta^*, \eta^2\overline{\mu}, \eta^3\overline{\kappa}, \eta^2\mu_{3,*}$. By the fact that $H[\iota_{4n+2}, 2\beta] = 4\beta$, we obtain

$$\sharp[\iota_{4n+2}, \beta] = 4 \ (\beta = \nu, \zeta, \nu^*, \overline{\zeta}, \nu\overline{\kappa}, \zeta_{3,*}). \tag{1–2}$$

Let $n \equiv 3 \ (4) \ge 7$. Then, by the fact that $\Delta\iota_n \circ \eta_{n-1} = \Delta\eta_n = 0$ and $2\eta_{n-1} = 0$, a Toda bracket $\{\Delta\iota_n, \eta_{n-1}, 2\iota\} \subset \pi_{n+1}(SO(n))$ is defined. The following result in [5] is useful to show the triviality of the Whitehead product $[\iota_n, \alpha]$:

Lemma 1.3 *Let* $n \equiv 3 \ (4) \ge 7$. *Then,*

(1) $\{\Delta\iota_n, \eta_{n-1}, 2\iota\} = 0$;

(2) $\Delta(E\{\eta_{n-1}, 2\iota_n, \alpha\}) = 0$, *if* $\alpha \in \pi_k(S^n)$ *is an element satisfying* $2\iota_n \circ \alpha = 0$.

By Lemma 1.3,

$$\Delta\alpha = 0 \text{ for } \alpha = \varepsilon_m, \mu_m, \overline{\mu}_m, \mu_{3,m} \ (m = 4n+3 \ge 3); \ \Delta\eta^*_{4n+3} = 0 \ (n \ge 4)$$

and so,

$$[\iota_{4n+3}, \alpha] = 0 \text{ for } \alpha = \varepsilon, \mu, \overline{\mu}, \mu_{3,*} \ (n \ge 0); \ [\iota_{4n+3}, \eta^*] = 0 \ (n \ge 4).$$

By [10],

$$\sharp[\iota_n, \mu] = \begin{cases} 2 & \text{if } n \equiv 0, 1, 2 \ (4) \ge 4; \\ 1 & \text{if } n \equiv 3 \ (4). \end{cases}$$

By [3], [4] and [10],

$$\sharp[\iota_n,\zeta]=\begin{cases}8 & \text{if } n\equiv 0\ (4)\geq 8;\\ 4 & \text{if } n\equiv 2\ (4)\geq 6;\\ 2 & \text{if } n\equiv 115\ (128)\geq 243;\\ 1 & \text{if } n\equiv 1\ (2)\geq 5,\ n\not\equiv 115\ (128).\end{cases}$$

The results for the other elemens in the J–image and μ–series are stated in the table.

2 Concerning Toda's results [18, Chapter 11]

We denote by P^n the real n dimensional projective space and set $\mathrm{P}^n_k=\mathrm{P}^n/\mathrm{P}^{k-1}$ for $k\leq n$. Let $i^{m,n}_k\colon \mathrm{P}^m_k\hookrightarrow \mathrm{P}^n_k$ and $p^n_{m,k}\colon \mathrm{P}^n_k\to \mathrm{P}^n_m$ for $0\leq k\leq m\leq n$ be the canonical inclusion and collapsing maps, respectively. We set $i^n_k=i^{n-1,n}_k$ and $p^n_k=p^n_{n-1,k}$ for $k\leq n-1$. We also set $i^{m,n}=i^{m,n}_1$, $p^n_m=p^n_{m,1}$. We write simply i for $i^{k,n}_k, i^n_k$ and p for p^n_k, unless otherwise stated.

Let $i\leq 4n+k-4$. We consider the exact sequence induced from a pair $(E^{n-1}\mathrm{P}^{n+k}_n, E^{n-1}\mathrm{P}^{n+k-1}_n)$ [18, (11.11)]:

$$\pi_i(E^{n-1}\mathrm{P}^{n+k-1}_n)\xrightarrow{i_*}\pi_i(E^{n-1}\mathrm{P}^{n+k}_n)\xrightarrow{I'_k}\pi_{i+k}(S^{2n+2k-1})\xrightarrow{\Delta_k}\pi_{i-1}(E^{n-1}\mathrm{P}^{n+k-1}_n),$$

where I'_k and Δ_k are defined by the following commutative diagram:

$$\begin{array}{ccccc}
\pi_i(E^{n-1}\mathrm{P}^{n+k}_n) & \xrightarrow{j_*} & \pi_i(E^{n-1}\mathrm{P}^{n+k}_n, E^{n-1}\mathrm{P}^{n+k-1}_n) & \xrightarrow{\partial} & \pi_{i-1}(E^{n-1}\mathrm{P}^{n+k-1}_n)\\
\downarrow = & & \downarrow p_*\cong & & \downarrow =\\
\pi_i(E^{n-1}\mathrm{P}^{n+k}_n) & \xrightarrow{p_*} & \pi_i(S^{2n+k-1}) & \xrightarrow{\Delta'} & \pi_{i-1}(E^{n-1}\mathrm{P}^{n+k-1}_n)\\
\downarrow = & & \downarrow E^k\cong & & \downarrow =\\
\pi_i(E^{n-1}\mathrm{P}^{n+k}_n) & \xrightarrow{I'_k} & \pi_{i+k}(S^{2n+2k-1}) & \xrightarrow{\Delta_k} & \pi_{i-1}(E^{n-1}\mathrm{P}^{n+k-1}_n).
\end{array}$$

We denote by

$$\gamma_{n,k}\colon S^n\to \mathrm{P}^n_k$$

the characteristic map of the $(n+1)$–cell $e^{n+1}=\mathrm{P}^{n+1}_k-\mathrm{P}^n_k$ for $k\leq n$. We set

$$\lambda_{n,k}=E^{n-1}\gamma_{n+k-1,n}.$$

By [18, Lemma 11.8],

$$\Delta_k(E^{k+1}\alpha) = \lambda_{n,k}\circ\alpha \ (\alpha \in \pi_{i-1}(S^{2n+k-2})) \quad \text{if} \quad i \le 4n+k-4.$$

We denote by $\phi(s) = \sharp\{1 \le i \le s \mid i \equiv 0,1,2,4\ (8)\}$. By use of [18, Lemma 11.8, Proposition 11.9], we obtain:

Proposition 2.1

(1) *Let $k \ge 1$ and $i \le 4n+k-4$. Assume that*

$$\lambda_{n,k}\circ\alpha = i_*\beta \quad in \quad \pi_{i-1}(E^{n-1}\mathrm{P}_n^{n+k-1})$$

for $\alpha \in \pi_{i-1}^{2n+k-2}$ and $\beta \in \pi_{i-1}^{2n-1}$. Then there exists an element $\delta \in \pi_{i+1}^{n+1}$ such that $P(E^{k+3}\alpha) = E^{k-1}\delta$ and $H\delta = \pm E^2\beta$.

(2) *Let $k \ge 2, l \ge 0, n \equiv l \pmod{2^{\phi(k)}}$ and $i \le 4n+k-4$. Assume that*

$$\lambda_{n,k}\circ\alpha = i_*\beta \quad in \quad \pi_{i-1}(E^{n-1}\mathrm{P}_n^{n+k-1})$$

for $\alpha \in \pi_{i-1}^{2n+k-2}$ and $\beta \in \pi_{i-1}^{2n-1}$. Then there exists an element $\delta \in \pi_{i+1}^{n+1}$ such that $P(E^{k+3}\alpha) = E^{k-1}\delta$ and $H\delta = \pm E^2\beta$.

Although (2) is a special case of (1), it is useful in the later arguments. Hereafter Proposition 2.1(2) is written Proposition 2.1[n;k,l]. We investigate the case $4 \le k \le 8$.

For $n \ge 2$, we set $M^n = E^{n-2}\mathrm{P}^2$. Let $\overline{\eta}_n \in [M^{n+2}, S^n] \cong \mathbb{Z}_4$ and $\widetilde{\eta}_n \in \pi_{n+2}(M^{n+1}) \cong \mathbb{Z}_4$ for $n \ge 3$ be an extension and a coextension of η_n, respectively. We know the following relations in the stable groups $\{\mathrm{P}^2, S^0\}$ and $\pi_3^s(\mathrm{P}^2)$: $2\overline{\eta} = \eta^2 p$ and $2\widetilde{\eta} = i\eta^2$. We use the relations

$$\overline{\eta}\widetilde{\eta} = \pm 2\nu = \langle \eta, 2\iota, \eta\rangle.$$

Toda brackets are often expressed as the stable forms.

From the fact that $E^2\mathrm{P}^3 = M^4 \vee S^5$, we take $E^2\gamma_3 = 2s_1 \pm (E^2 i^{2,3})\widetilde{\eta}_3$, where $s_1\colon S^5 \hookrightarrow E^2\mathrm{P}^3$ is the canonical inclusion. Since $E^2 p_3^4 \circ (E^2 i^{3,4}\circ s_1) = E^4 i^{1,2}$, we regard $E^2 i^{3,4}\circ s_1$ as a coextension of $E^3 i^{1,2} \in \pi_4(M^5) \cong \mathbb{Z}_2$. Set $\widetilde{\iota}_5 = E^2 i^{3,4}\circ s_1$. Then, by the relation

$$2(E^2 i^{3,4}\circ s_1) = \pm(E^2 i^{2,4})\widetilde{\eta}_3,$$

we obtain $\pi_5(E^2\mathrm{P}^4) = \{\widetilde{\iota}_5\} \cong \mathbb{Z}_8$, where $2\widetilde{\iota}_5 = \pm(E^2 i^{2,4})\widetilde{\eta}_3$ [13]. We set $\widetilde{\iota}_{n+3} = E^{n-2}\widetilde{\iota}_5 \in \pi_{n+3}(E^n\mathrm{P}^4) \cong \mathbb{Z}_8$ $(n \ge 2)$. We use the relation in the stable case:

(2–1) $$2\widetilde{\iota} = \pm i^{2,4}\widetilde{\eta}.$$

Notice that Proposition 2.1[n-2;2,l] for $l=2,3$ coincides with [18, Proposition 11.10] and Proposition 2.1[n-3;3,l] for $l=1,3$ does with [18, Proposition 11.11], respectively. In these cases, $\lambda_{n-k,k} \in \pi_{2n-k-2}(E^{n-k-1}\mathrm{P}^{n-1}_{n-k})$ is taken as follows:

$$\lambda_{n-2,2} = \begin{cases} i\eta + 2\iota & (n \equiv 0\ (4)); \\ i\eta & (n \equiv 1\ (4)); \end{cases}$$

$$\lambda_{n-3,3} = \begin{cases} 2s_1 \pm i^{2,3}\widetilde{\eta} & (n \equiv 0\ (4)); \\ \gamma_{5,3} \in \langle j, \eta, 2\iota\rangle & (n \equiv 2\ (4)), \end{cases}$$

where $i = E^{n-3}i^{n-1}_{n-2}$ and $j = E^{n-4}i^{n-3,n-1}_{n-3}$. By use the last part of this formula, we have $\lambda_{n-3,3}\circ\alpha = j_*\beta$ if $\beta \in \langle \eta, 2\iota, \alpha\rangle$. So, [18, Proposition 11.11.ii)] is exactly interpreted as follows:

Remark Let $i \le 4n-2$ and $n \equiv 3\ (4)$. Assume that $2\alpha = 0$ for $\alpha \in \pi^{2n}_{i-2}$ and $\{\eta_{2n+1}, 2\iota, E^2\alpha\} \ni \beta$, then $P(E^7\alpha) = E^2\beta$.

Hereafter we use [18, Proposition 11.11.ii)] in this version.

We use the cell structures

$$(\mathcal{P}^4) \qquad \mathrm{P}^4 = \mathrm{P}^2 \cup_{\widetilde{\eta}p} CM^3; \qquad (\mathcal{P}^4_2) \qquad \mathrm{P}^4_2 = S^2 \cup_{\eta p} CM^3.$$

By $(\mathcal{P}^4_2)$, we obtain $\pi^s_3(\mathrm{P}^4_2) = \{\widetilde{\iota}'\} \cong \mathbb{Z}_4$ and $\pi^s_4(\mathrm{P}^4_2) = \{\widetilde{\iota}'\eta\} \cong \mathbb{Z}_2$, where $\widetilde{\iota}' = p\widetilde{\iota}$ and $2\widetilde{\iota}' = i\eta$. Notice that $\gamma_4 = \widetilde{\iota}\eta$ and $\gamma_{4,2} = \widetilde{\iota}'\eta$.

Now, consider the case $k=4$. P^{n-1}_{n-4} has the following cell structures:

$$\mathrm{P}^{n-1}_{n-4} = \begin{cases} \mathrm{P}^3_0 = S^0 \vee \mathrm{P}^2 \vee S^3 & (n \equiv 0\ (4)); \\ \mathrm{P}^4 = \mathrm{P}^2 \cup_{\widetilde{\eta}p} CM^3 & (n \equiv 1\ (4)); \\ \mathrm{P}^5_2 = \mathrm{P}^4_2 \cup_{\widetilde{\iota}'\eta} e^5 & (n \equiv 2\ (4)); \\ \mathrm{P}^6_3 = \mathrm{P}^5_3 \cup_{\gamma_{5,3}} e^6 & (n \equiv 3\ (4)). \end{cases}$$

The following cell structure is also useful:

$$(\mathcal{P}^6_3) \qquad \mathrm{P}^6_3 = M^4 \cup_{i\widetilde{\eta}} CM^5.$$

In general, we have

$$\text{(2–2)} \qquad \gamma_{2n+1,k} \in \langle i, \gamma_{2n,k}, 2\iota\rangle.$$

We obtain the following:

$$\pi^s_3(\mathrm{P}^3_0; 2) = \{\iota, \widetilde{\eta}, \nu\} \cong \mathbb{Z}\oplus\mathbb{Z}_4\oplus\mathbb{Z}_8;\ \pi^s_4(\mathrm{P}^4) = \{\widetilde{\iota}\eta, i\nu\} \cong (\mathbb{Z}_2)^2;$$

$$\pi^s_5(\mathrm{P}^5) = \{\gamma_5\} \cong \mathbb{Z};\ \pi^s_5(\mathrm{P}^5_2) = \{\gamma_{5,2}, i\nu\} \cong \mathbb{Z}\oplus\mathbb{Z}_2;$$

$$\pi_5^s(\mathrm{P}_3^5) = \{\gamma_{5,3}, i'\widetilde{\eta}\} \cong \mathbb{Z} \oplus \mathbb{Z}_2$$

where

$$\gamma_5 \in \langle i^{4,5}\widetilde{\iota}, \eta, 2\iota\rangle, \tag{2–3}$$

$\gamma_{5,2} \in \langle i''\widetilde{\iota}', \eta, 2\iota\rangle$ and $\gamma_{5,3} \in \langle i'i, \eta, 2\iota\rangle$ $(i' = i_3^5, i'' = i_2^5)$. We also obtain

$$\pi_6^s(\mathrm{P}_3^6) = \{i_3^{4,6}\widetilde{\eta}\eta, i\nu\} \cong (\mathbb{Z}_2)^2.$$

Remark The indeterminacy of the bracket $\langle i'', \widetilde{\iota}'\eta, 2\iota\rangle$ is $\{i_2^{2,5}\nu\} + 2\pi_5^s(\mathrm{P}_2^5) \cong \mathbb{Z}_2 \oplus 2\mathbb{Z}$. Since the squaring operation $Sq^4\colon \widetilde{H}^2(\mathrm{P}_2^6;\mathbb{Z}_2) \to \widetilde{H}^6(\mathrm{P}_2^6;\mathbb{Z}_2)$ is trivial, we take simply $\gamma_{5,2} \in \langle i''\widetilde{\iota}', \eta, 2\iota\rangle$, whose indeterminacy is $2\pi_5^s(\mathrm{P}_2^5)$.

Notice that $\mathrm{P}_4^7 = S^4 \vee M^6 \vee S^7$. Let $s_2\colon S^7 \hookrightarrow \mathrm{P}_4^7$ and $t\colon M^6 \hookrightarrow \mathrm{P}_4^7$ are the canonical inclusions, respectively. The cell structure of P_{n-4}^n is given as follows:

$$(\mathcal{P}_4^8) \qquad \mathrm{P}_4^8 = \mathrm{P}_4^7 \cup_{\gamma_{7,4}} e^8 \ (n \equiv 0\ (8)),$$

where

$$\gamma_{7,4} = 2s_2 \pm t\widetilde{\eta} + i\nu; \tag{2–4}$$

$$(\mathcal{P}_5^9) \qquad \mathrm{P}_5^9 = \mathrm{P}_5^8 \cup_{\gamma_{8,5}} e^9 \ (\mathrm{P}_5^8 = E^4\mathrm{P}^4,\ n \equiv 1\ (8)),$$

where

$$\gamma_{8,5} = \widetilde{\iota}\eta + i\nu; \tag{2–5}$$

$$\mathrm{P}_6^{10} = \mathrm{P}_6^9 \cup_{\gamma_{5,2}+i\nu} e^{10}\ (\mathrm{P}_6^9 = E^4\mathrm{P}_2^5,\ n \equiv 2\ (8));$$
$$(\mathcal{P}_7^{11}) \qquad \mathrm{P}_7^{11} = \mathrm{P}_7^{10} \cup_{i\nu} e^{11}\ (\mathrm{P}_7^{10} = E^4\mathrm{P}_3^6,\ n \equiv 3\ (8));$$
$$\mathrm{P}_0^4\ (n \equiv 4\ (8));\ \mathrm{P}^5 = \mathrm{P}^4 \cup_{\widetilde{\iota}\eta} e^5\ (n \equiv 5\ (8));$$
$$\mathrm{P}_2^6 = \mathrm{P}_2^5 \cup_{\gamma_{5,2}} e^6\ (n \equiv 6\ (8));\ \mathrm{P}_3^7 = \mathrm{P}_3^6 \vee S^7\ (n \equiv 7\ (8)).$$

Notice that $(\mathcal{P}_7^{11})$ is obtained from the triviality of $\gamma_{10,8}\colon S^{10} \to \mathrm{P}_8^{10} = E^8\mathrm{P}_0^2$.

Let $x(n)$ be an integer such that it is odd or even according as n is even or odd. Then we can set

$$\lambda_{n-4,4} = \begin{cases} 2\iota \pm i_{n-3}^{n-2}\widetilde{\eta} + x(\frac{n}{4})i\nu & (n \equiv 0\ (4)); \\ \widetilde{\iota}\eta + x(\frac{n-1}{4})i\nu & (n \equiv 1\ (4)); \\ \gamma_{5,2} + x(\frac{n-2}{4})i\nu & (n \equiv 2\ (4)); \\ x(\frac{n-3}{4})i\nu & (n \equiv 3\ (4)). \end{cases}$$

Remark In the case $n \equiv 0\ (4)$, exactly,

$$\lambda_{n-4,4} = \begin{cases} 2s_2 \pm t\widetilde{\eta} + i\nu & (n \equiv 0\ (8)); \\ 2s_1 \pm i^{2,3}\widetilde{\eta} & (n \equiv 4\ (8)). \end{cases}$$

By Proposition 2.1, we obtain the following.

Proposition 2.2 *Let $i \leq 4n$ and $\alpha \in \pi_{i-1}^{2n+2}$.*

(1) *Let $n \equiv 0 \pmod 4$ and assume that $\widetilde{\eta}_{2n} \circ \alpha = 2\alpha = 0$. Then there exists an element $\delta \in \pi_{i+1}^{n+1}$ such that $P(E^7\alpha) = E^3\delta$ and $H\delta = x(\frac{n+4}{4})\nu_{2n+1}(E^2\alpha)$.*

(2) *Let $n \equiv 1 \pmod 4$ and assume that $\widetilde{\iota}_{2n+1}\eta_{2n+1} \circ \alpha = 0$. Then there exists an element $\delta \in \pi_{i+1}^{n+1}$ such that $P(E^7\alpha) = E^3\delta$ and $H\delta = x(\frac{n+3}{4})\nu_{2n+1}(E^2\alpha)$.*

(3) *Let $n \equiv 2 \pmod 4$ and assume that $E^{2n-3}\gamma_{5,2} \circ \alpha = 0$. Then there exists an element $\delta \in \pi_{i+1}^{n+1}$ such that $P(E^7\alpha) = E^3\delta$ and $H\delta = x(\frac{n+2}{4})\nu_{2n+1}(E^2\alpha)$.*

(4) *Let $n \equiv 3 \pmod 4$. Then there exists an element $\delta \in \pi_{i+1}^{n+1}$ such that $P(E^7\alpha) = E^3\delta$ and $H\delta = x(\frac{n+3}{4})\nu_{2n+1}(E^2\alpha)$.*

Notice the following: In Proposition 2.2(1),(3), the assumptions $\widetilde{\eta}_{2n}\alpha = 0$ and $E^{2n-3}\gamma_{5,2} \circ \alpha = 0$ imply the relations $\eta_{2n}\alpha' = 0$ and $2\iota_{2n+1} \circ \alpha' = 2\alpha' = 0$ respectively, where $E\alpha' = \alpha$.

For the case $k = 8$, we obtain:

Proposition 2.3 *Let $n \equiv l \pmod 8$ and $i \leq 4n+4$. Let $\alpha \in \pi_{i-1}^{2n+6}$.*

(1) *Assume that $\pi_{2n+6}(E^{n-1}\mathrm{P}_n^{n+7}) \circ \alpha = 0$. Then, $P(E^{11}\alpha)$ desuspends eight dimensions.*

(2) *Assume that $(\pi_{2n+6}(E^{n-1}\mathrm{P}_n^{n+7}) - \{i \circ \sigma\}) \circ \alpha = 0$ for $\alpha \in \pi_{i-1}^{2n+6}$. Then there exists an element $\delta \in \pi_{i+1}^{n+1}$ such that $P(E^{11}\alpha) = E^7\delta$ and $H\delta = x\sigma_{2n+1}(E^2\alpha)$, where x is even or odd according as $n \equiv l \pmod{16}$ or $n \equiv l + 8 \pmod{16}$.*

Hereafter Proposition 2.3(2) is written Proposition 2.3[[n;8,r]] for $r = l$ or $l + 8$. We introduce some notation. If $[\iota_n, \alpha]$ for $\alpha \in \pi_m^n$ desuspends k dimensions with Hopf invariant $\theta \in \pi^{2n-2k-1}_{n+m-k-1}$, that is, if there exists an element $\delta \in \pi^{n-k}_{n+m-k-1}$ satisfying $E^k\delta = [\iota_n, \alpha]$ and $H\delta = \theta$, we write

$$H(E^{-k}[\iota_n, \alpha]) = \theta.$$

Then, immediately we obtain $P\theta = [\iota_{n-k-1}, E^{-(n-k)}\theta] = 0$. δ is written

$$\delta = \delta(\theta) = E^{-k}[\iota_n, \alpha].$$

By the fact that $\sharp[\iota_n, [\iota, \iota]] = 2 + (-1)^n$ $(n \geq 3)$ and [2, Corollary 7.4], $[\iota_n, \alpha \circ \beta] = [\iota_n, \alpha] \circ E^{n-1}\beta$ for $\beta \in \pi^m_l$ and so,

$$H(E^{-k}[\iota_n, \alpha \circ \beta]) = H(E^{-k}[\iota_n, \alpha]) \circ E^{n-k-1}\beta. \tag{2–6}$$

If $[\iota_n, \alpha] \neq 0$, we write

$$H(E^{-k}[\iota_n, \alpha]_{\neq 0}) = \theta.$$

By Lemma 1.2 and by abuse of notation for α, we obtain

Example 2.4

(1) $H(E^{-1}[\iota_{2n+1}, \alpha]) = (n+1)\eta_{4n-1}\alpha$ $(\delta = \tau_{2n}\alpha), [\iota_{4n-1}, \eta\alpha] = 0$.

(2) $H(E^{-3}[\iota_{4n+3}, \alpha]) = \pm(n+1)\nu_{8n-1}\alpha$ $(\delta = \overline{\tau}_{4n}\alpha)$, $[\iota_{8n-1}, \nu\alpha] = 0$.

Notice that Example 2.4(1) induces [18, Proposition 11.10.ii)] and Example 2.4(2) does Proposition 2.2(4).

First of all, we write up the results obtained from [18, Proposition 11.10].

Proposition 2.5

(1) *Let* $n \equiv 0, 1\ (4)$. *Then,* $H(E^{-1}[\iota_n, \alpha_1]_{\neq 0}) = \eta\alpha_1$ *for* $\alpha_1 = \eta, \eta\sigma, \overline{\nu}, \varepsilon, \mu, \kappa, \eta\rho, \eta^*, \overline{\mu}, \eta\overline{\kappa}, \eta^*\sigma, \mu_{3,*}$ *and* $H(E^{-1}[\iota_n, \alpha_2]) = 0$ *for* $\alpha_2 = \eta\varepsilon, \eta^2\sigma, \sigma^2, \eta\kappa, \eta^2\rho, \overline{\sigma}, \nu\overline{\sigma}, \eta\eta^*\sigma, \eta^2\overline{\rho}$.

(2) $H(E^{-1}[\iota_{4n}, \beta]_{\neq 0}) = \eta\beta$ *for* $\beta = \eta^2, \eta\mu, \eta\eta^*, \eta\overline{\mu}, \eta^2\overline{\kappa}, \eta\mu_{3,*}$.

(3) $H(E^{-1}[\iota_{4n+1}, \delta_1]_{\neq 0}) = \eta\delta_1$ *for* $\delta_1 = \sigma, \rho, \overline{\kappa}, \overline{\rho}$ *and* $H(E^{-1}[\iota_{4n+1}, \delta_2]) = 0$ *for* $\delta_2 = \nu, \zeta, \nu^*, \overline{\zeta}, \nu\overline{\kappa}, \zeta_{3,*}$.

(4) *If* $\sharp[\iota_{4n}, \nu^*] = 8$, *then* $[\iota_{4n+1}, \eta\eta^*] \neq 0$.

Proof We prove (1) for κ. By [18, Proposition 11.10], $H(E^{-1}[\iota_n,\kappa]) = \eta\kappa$. Assume that $[\iota_n,\kappa] = 0$. Then, by the EHP sequence, $\delta \in P\pi^{2n-1}_{2n+14} = \{[\iota_{n-1},\eta\kappa],[\iota_{n-1},\rho]\}$ for $\delta = E^{-1}[\iota_n,\kappa]$. Applying the Hopf homomorphism $H\colon \pi^{n-1}_{2n+12} \to \pi^{2n-3}_{2n+12}$ to this relation implies $\eta\kappa = 0$ for $n \equiv 0 \pmod 4$ and $\eta\kappa \in \{2\rho\}$ for $n \equiv 1 \pmod 4$. This is a contradiction.

Next, we prove (2) for $\eta\eta^*$. Let $n \equiv 0\ (4)$. By [18, Proposition 11.11], $H(E^{-1}[\iota_n,\eta\eta^*]) = \eta^2\eta^* = 4\nu^*$. The assumption $[\iota_n,\eta\eta^*] = 0$ induces $\delta \in P\pi^{2n-1}_{2n-19}$ and a contradictory relation $4\nu^* = 0$ for $\delta = E^{-1}[\iota_n,\eta\eta^*]$. The proof of (3) is similarly obtained.

Finally, we show (4). Assume that $[\iota_{4n+1},\eta\eta^*] = 0$. From the fact that $[\iota_{4n+1},\eta\eta^*] = E(\tau_{4n}\eta\eta^*)$ and the assumption $\sharp[\iota_{4n},\nu^*] = 8$, we have $\tau_{4n}\eta\eta^* \in \{4[\iota_{4n},\nu^*],[\iota_{4n},\eta\overline{\mu}]\}$. This implies a contradictory relation $4\nu^* = 0$, and hence (4) follows. □

Hereafter, "the assumption $[\iota_n,\alpha] = 0$" is written "$\mathcal{ASM}[\alpha]$" and "a contradictory relation $\beta \in B$" is written "$\mathcal{CDR}[\beta \in B]$". As an application of [18, Proposition 11.11], we show:

Proposition 2.6

(1) $H(E^{-2}[\iota_{4n+2},\alpha]) \in \langle \eta, 2\iota, \alpha\rangle$ *if* $2\alpha = 0$,
$H(E^{-2}[\iota_{4n+2},\alpha_1]_{\neq 0}) \in \langle \eta, 2\iota, \alpha_1\rangle$ *for* $\alpha_1 = \nu^2, 8\sigma, \sigma^2, 16\rho, \sigma^3, 8\overline{\rho}, \nu^2\overline{\kappa}$ *and*
$H(E^{-2}[\iota_{4n+2},\alpha_2]) = 0$ *for* $\alpha_2 = \eta\sigma, \overline{\nu}, \varepsilon, \nu^3, \eta\rho, \overline{\sigma}, \eta\overline{\rho}$.

(2) $H(E^{-2}[\iota_{4n},\beta_1]_{\neq 0}) \in \langle 2\iota, \eta, \beta_1\rangle$ *for* $\beta_1 = \eta\kappa, \eta^2\rho, \eta\eta^*\sigma$.

(3) $H(E^{-2}[\iota_{4n},\beta_2]) = 0$ *for* $\beta_2 = 4\nu, 8\sigma, 4\zeta, \sigma^2, 16\rho, 4\overline{\zeta}, \overline{\sigma}, 4\overline{\kappa}, 4\nu\overline{\kappa}, 8\overline{\rho}, 4\zeta_{3,*}$.

Proof Let $n \equiv 2\ (4)$. The first part of (1) is a direct consequence of [18, Proposition 11.11.ii)]. By the fact that $\langle \eta, 2\iota, \sigma^2\rangle \ni \eta^* \pmod{\eta\rho}$ and [18, Proposition 11.11.ii)],

$$H(E^{-2}[\iota_n,\sigma^2]) = \eta^*.$$

$\mathcal{ASM}[\sigma^2]$ induces $E\delta \in P\pi^{2n-1}_{2n+14} = \{[\iota_{n-1},\alpha]\} = \{E(\tau_{n-2}\alpha)\}$ (Lemma 1.2(1)) and $\delta \pmod{\tau_{n-2}\rho, \tau_{n-2}\eta\kappa} \in P\pi^{2n-3}_{2n+13}$, where $\delta = E^{-2}[\iota_n,\sigma^2]$ and $\alpha = \rho, \eta\kappa$. Hence, $\mathcal{CDR}[\eta^* \pmod{\eta\rho} = 0]$ and the second part of (1) for σ^2 follows. Next we prove the second part of (1) for $\nu^2\overline{\kappa}$. By the fact that $\langle \eta, 2\iota, \nu^2\rangle \ni \varepsilon \pmod{\eta\sigma}$ and [18, Proposition 11.11.ii)], $H(E^{-2}[\iota_n,\nu^2]) = \varepsilon$ and $H(E^{-2}[\iota_n,\nu^2\overline{\kappa}]) = \varepsilon\overline{\kappa}$ by (2–6). $\mathcal{ASM}[\nu^2\overline{\kappa}]$ induces $E(\delta\overline{\kappa}) \in \{[\iota_{n-1},\zeta_{3,*}]\}$ and $\delta\overline{\kappa} \pmod{\tau_{n-2}\zeta_{3,*}} \in P\pi^{2n-3}_{2n+25}$, where $\delta = E^{-2}[\iota_n,\nu^2]$. By the relation $\eta\zeta_{3,*} = 0$, we obtain $\mathcal{CDR}[\varepsilon\overline{\kappa} = 0]$.

The third part of (1) follows from [18, Proposition 11.11.ii)] and the fact that $\langle \eta, 2\iota, \alpha_2 \rangle \ni 0$. By [18, Proposition 11.11.i)],

$$(\diamond) \qquad H(E^{-2}[\iota_{4n}, \eta\kappa]) = \langle 2\iota, \eta, \eta\kappa \rangle = \nu\kappa.$$

$\mathcal{A}SM[\eta\kappa]$ implies $E\delta \in P\pi^{2n-1}_{2n+15} = \{E(\tau_{n-2}\eta\rho), E(\tau_{n-2}\eta^*)\}$ and $\delta \pmod{\tau_{n-2}\eta\rho, \tau_{n-2}\eta^*} \in P\pi^{2n-3}_{2n+14}$, where $\delta = E^{-2}[\iota_n, \eta\kappa]$. Hence, $\mathcal{C}DR[\nu\kappa \pmod{\eta^2\rho, \eta\eta^*} = 0]$ and the first part of (2) follows. By the parallel argument, the rest of the assertion follows. We use the following facts: $\langle 2\iota, \eta, \beta_2 \rangle = 0$; $\langle \eta, 2\iota, 16\rho \rangle \ni \bar{\mu} \pmod{\eta^2\rho, \eta\eta^*}$; $\langle \eta, 2\iota, \sigma^3 \rangle \ni \eta^*\sigma \pmod{\eta^3\bar{\kappa}}$; $\langle 2\iota, \eta, \eta^2\rho \rangle \ni \bar{\zeta} \pmod{2\bar{\zeta}}$; $\langle 2\iota, \eta, \eta\eta^*\sigma \rangle = \nu^2\bar{\kappa}$ [6]. □

By Proposition 2.6(2), we obtain

$$\text{(2–7)} \qquad [\iota_{4n+1}, \nu\kappa] = 0$$

$$\text{and} \qquad [\iota_{4n+1}, \nu^2\bar{\kappa}] = 0.$$

Here we summarize Toda brackets in $\pi^s_*(\mathrm{P}^2)$ needed in the subsequent arguments. Since $\pi^s_7(\mathrm{P}^2) = \{i\nu^2\} \cong \mathbb{Z}_2$ and $\pi^s_5(\mathrm{P}^2) = \{\widetilde{\eta}\eta^2\} \cong \mathbb{Z}_2$, the indeterminacy of the bracket $\langle i\bar{\eta}, \widetilde{\eta}, \nu \rangle \subset \pi^s_8(\mathrm{P}^2)$ is $i\bar{\eta} \circ \pi^s_7(\mathrm{P}^2) + \pi^s_5(\mathrm{P}^2) \circ \nu = 0$. We set $\widetilde{\nu^2} = \langle i\bar{\eta}, \widetilde{\eta}, \nu \rangle$, which is a coextension of ν^2. Let $\widetilde{\sigma^2} \in \langle i, 2\iota, \sigma^2 \rangle \subset \pi^s_{16}(\mathrm{P}^2)$ be a coextension of σ^2 and $\overline{i\nu} \in \{M^5, \mathrm{P}^2\}$ an extension of $i\nu \in \pi^s_4(\mathrm{P}^2)$. Then, we show:

Lemma 2.7

(1) $\langle i\bar{\eta}, \widetilde{\eta}, \nu^* \rangle \ni \widetilde{\sigma^2}\sigma \pmod{i\eta^2\bar{\kappa}, i\nu\bar{\sigma}}$.

(2) $\langle i\nu, 2\iota, \sigma^2 \rangle = i\nu^*$.

(3) $\langle i\nu, 2\iota, 16\rho \rangle = i\bar{\zeta}$.

(4) $\langle i\nu, 2\iota, \eta^* \rangle = 0$.

(5) $\langle \overline{i\nu}, \widetilde{\eta}, 4\iota \rangle = \pi^s_7(\mathrm{P}^2)$.

(6) $\langle \widetilde{\eta}p, \widetilde{\eta}\eta^2, \eta \rangle = 0$.

(7) $\langle \widetilde{\eta}p, \widetilde{\eta}\eta^2, \sigma^2 \rangle \ni 0 \pmod{\widetilde{\eta}\eta\bar{\mu}}$.

(8) $\langle i\eta\bar{\eta}, \widetilde{\eta}, \nu \rangle = \widetilde{\nu^2}\eta = i\varepsilon$, $\widetilde{\nu^2}\sigma = 0$ *and* $\widetilde{\nu^2} = \langle \widetilde{\eta}, \nu, \eta \rangle$.

(9) $\langle \widetilde{\eta}, \nu, \nu^3 \rangle = i\eta\kappa$.

(10) $\widetilde{\nu^2}\eta\eta^* = i\eta\eta^*\sigma$ *and* $\langle \widetilde{\eta}p, \widetilde{\eta}\eta^2, \nu^* \rangle \ni i\eta\eta^*\sigma \pmod{\widetilde{\eta}\eta^2\bar{\kappa}}$.

Proof Since $\langle p, i\overline{\eta}, \widetilde{\eta}\rangle = \pm\nu$ and $\nu\nu^* = \sigma^3$, we have $p\circ\langle i\overline{\eta}, \widetilde{\eta}, \nu^*\rangle = \sigma^3$. This leads to (1). By the fact that $\nu^* \in \langle \nu, 2\sigma, \sigma\rangle$ and $\nu\circ\pi^s_{15} = 0$, we see that

$$\langle i\nu, 2\iota, \sigma^2\rangle \subset \langle i\nu, 2\sigma, \sigma\rangle \ni i\nu^* \ (\mathrm{mod}\ i\nu\circ\pi^s_{15} + \pi^s_{12}(\mathrm{P}^2)\circ\sigma = \{\widetilde{\eta}\mu\sigma\}).$$

We have $p\circ\langle i\nu, 2\iota, \sigma^2\rangle = \langle p, i\nu, 2\iota\rangle\circ\sigma^2 \subset \pi^s_3\circ\sigma^2 = 0$, $p(i\nu^*) = 0$ and $p(\widetilde{\eta}\mu\sigma) = \eta\mu\sigma = \eta^2\rho$. This leads to (2).

We obtain

$$\langle i\nu, 2\iota, 16\rho\rangle \subset \langle i\nu, 8\iota, 4\rho\rangle \supset i\circ\langle \nu, 8\iota, 4\rho\rangle \ni i\overline{\zeta}$$
$$(\mathrm{mod}\ i\nu\circ\pi^s_{16} + \pi^s_5(\mathrm{P}^2)\circ 4\rho = 0).$$

We get that

$$\langle i\nu, 2\iota, \eta^*\rangle \subset \langle i, 2\nu, \eta^*\rangle \supset \langle i, 2\iota, 0\rangle \ni 0 \ (\mathrm{mod}\ i_*\pi^s_{20} + \pi^s_5(\mathrm{P}^2)\circ\eta^*).$$

Since $\widetilde{\eta}\eta^2\eta^* = 4\widetilde{\eta}\nu^* = 0$, the indeterminacy is $\{i\overline{\kappa}\}$. Hence, (4) follows from the fact that $\langle \overline{\eta}, i\nu, 2\iota\rangle \subset \pi^s_5 = 0$ and $\overline{\eta}\circ i\overline{\kappa} = \eta\overline{\kappa}$.

The indeterminacy of $\langle \overline{i\nu}, \widetilde{\eta}, 4\iota\rangle$ contains $\overline{i\nu}\circ\pi^s_4(\mathrm{P}^2) = \{i\nu^2\} = \pi^s_7(\mathrm{P}^2)$.

We obtain

$$\langle \widetilde{\eta}p, \widetilde{\eta}\eta^2, \eta\rangle \subset \langle \widetilde{\eta}, 4\nu, \eta\rangle \supset \langle 0, \nu, \eta\rangle \ni 0 \ (\mathrm{mod}\ \widetilde{\eta}\circ\pi^s_5 + \pi^s_7(\mathrm{P}^2)\circ\eta = 0).$$

We see that

$$\langle \widetilde{\eta}p, \widetilde{\eta}\eta^2, \sigma^2\rangle \subset \langle \widetilde{\eta}, 4\nu, \sigma^2\rangle \ni 0 \ (\mathrm{mod}\ \widetilde{\eta}\circ\pi^s_{18} + \pi^s_7(\mathrm{P}^2)\circ\sigma^2),$$

where $\pi^s_7(\mathrm{P}^2)\circ\sigma^2 = 0$ and $\widetilde{\eta}\nu^* = 0$ because $\langle 2\iota, \eta, \nu^*\rangle \subset \{2\overline{\kappa}\}$. This leads to (7).

By the equality $\langle \eta\overline{\eta}, \widetilde{\eta}, \nu\rangle = \varepsilon$ [5, Lemma 4.2], $i\overline{\eta}\widetilde{\nu^2} \in i\langle \eta\overline{\eta}, \widetilde{\eta}, \nu\rangle = i\varepsilon$. This implies $\overline{\eta}\widetilde{\nu^2} = \varepsilon$. We have $i\varepsilon \in \langle i\eta\overline{\eta}, \widetilde{\eta}, \nu\rangle$ (mod $i\eta\overline{\eta}\circ\pi^s_7(\mathrm{P}^2) + \pi^s_6(\mathrm{P}^2)\circ\nu = 0$) and $\widetilde{\nu^2}\eta \in i\langle 2\iota, \nu^2, \eta\rangle \ni i\varepsilon$ (mod $i\eta\sigma$). Composing $\overline{\eta}$ on the left to this relation yields $\widetilde{\nu^2}\eta = i\varepsilon$. We have $\widetilde{\nu^2}\sigma = \langle i\overline{\eta}, \widetilde{\eta}, \nu\rangle\circ\sigma = i\overline{\eta}\circ\langle \widetilde{\eta}, \nu, \sigma\rangle = 0$. Since $p\circ\langle \widetilde{\eta}, \nu, \eta\rangle = \nu^2$, we can set $\langle \widetilde{\eta}, \nu, \eta\rangle = \widetilde{\nu^2} + ai\sigma$ for $a \in \{0, 1\}$. By the fact that $\eta\overline{\eta}\circ\langle \widetilde{\eta}, \nu, \eta\rangle = \langle \eta\overline{\eta}, \widetilde{\eta}, \nu\rangle\circ\eta = \eta\varepsilon$ and $\eta\overline{\eta}(\widetilde{\nu^2} + ai\sigma) = \eta\varepsilon + a\eta^2\sigma$, we have $a = 0$.

By the relations $\nu^3 = \eta\overline{\nu}$, $\langle 2\iota, \nu^2, \overline{\nu}\rangle \ni \eta\kappa$ (mod 2ρ) and (8),

$$\langle \widetilde{\eta}, \nu, \nu^3\rangle \supset \langle \widetilde{\eta}, \nu, \eta\rangle\circ\overline{\nu} = \widetilde{\nu^2}\overline{\nu} \in i\langle 2\iota, \nu^2, \overline{\nu}\rangle = i\eta\kappa$$
$$(\mathrm{mod}\ \widetilde{\eta}\circ\pi^s_{13} + \pi^s_7(\mathrm{P}^2)\circ\nu^3 = 0).$$

By (8) and [11, (6.3)], $\widetilde{\nu^2}\eta\eta^* = i\varepsilon\eta^* = i\eta\eta^*\sigma$. By the fact that $2\widetilde{\eta}\overline{\eta} = \widetilde{\eta}\eta^2 p = i\eta\overline{\eta}\circ i\overline{\eta}$, $p_*\pi^s_{24}(\mathrm{P}^2) = \pi^s_{22} = \{\eta^2\overline{\kappa}, \nu\overline{\sigma}\} \cong (\mathbb{Z}_2)^2$ and (1),

$$\langle \widetilde{\eta}p, \widetilde{\eta}\eta^2, \nu^*\rangle \supset \langle \widetilde{\eta}\eta^2 p, \widetilde{\eta}, \nu^*\rangle \supset i\eta\overline{\eta}\circ\langle i\overline{\eta}, \widetilde{\eta}, \nu^*\rangle \ni i\eta\eta^*\sigma$$

$$(\mathrm{mod}\ \widetilde{\eta} p \circ \pi_{24}^s(\mathrm{P}^2) + \pi_7^s(\mathrm{P}^2) \circ \nu^* = \{\widetilde{\eta}\eta^2\overline{\kappa}\}).$$

This leads to (10). □

We recall from [12] that $\{\mathrm{P}^4, S^0\} = \{\overline{\eta}'\} \cong \mathbb{Z}_8$ and

(2–8) $$\overline{\eta}'\widetilde{\iota} = \nu, \quad \text{where} \quad \overline{\eta}' \in \langle \overline{\eta}, \widetilde{\eta} p, p_{4,2} \rangle.$$

We obtain the following.

Lemma 2.8

(1) $\pi_7^s(\mathrm{P}^4) = \{\widetilde{\widetilde{\eta}\eta^2}, i\nu^2\} \cong (\mathbb{Z}_2)^2$ *and* $\pi_7^s(\mathrm{P}^6) = \{\widetilde{\eta}', i\nu^2\} \cong \mathbb{Z}_8 \oplus \mathbb{Z}_2$*, where* $\widetilde{\widetilde{\eta}\eta^2} \in \langle i^{2,4}, \widetilde{\eta} p, \widetilde{\eta}\eta^2 \rangle$, $\widetilde{\eta}' \in \langle i^{4,6}, \widetilde{\iota}\overline{\eta}, \widetilde{\eta} \rangle$ *and* $4\widetilde{\eta}' \equiv i^{4,6}\widetilde{\widetilde{\eta}\eta^2}\ (\mathrm{mod}\ i\nu^2)$.

(2) $\pi_7^s(\mathrm{P}_3^6) = \{\widetilde{\eta}''\} \cong \mathbb{Z}_8$*, where* $\widetilde{\eta}'' = p_3^6\widetilde{\eta}'$.

(3) $\pi_7^s(\mathrm{P}^4) \circ \eta = \pi_7^s(\mathrm{P}^4) \circ \sigma^2 = 0$ *and* $\pi_7^s(\mathrm{P}^6) \circ \sigma^2 = \pi_7^s(\mathrm{P}_3^6) \circ \sigma^2 = 0$.

Proof (1) is just [12, Proposition 4.1]. (2) is obtained by use of the cell structure $(\mathcal{P}_3^6)$ and (1). The first two equalities in (3) are obtained by Lemma 2.7(6),(7) and the relation $i^{2,4}\widetilde{\eta} p = 0 \in \{M^3, \mathrm{P}^4\}$. To show the next two equalities in (3), it suffices to prove $\langle \widetilde{\iota}\overline{\eta}, \widetilde{\eta}, \sigma^2 \rangle \ni 0$. By (2–1), the relation $\langle \widetilde{\eta}, \nu, \sigma \rangle = 0$ and the second equality in (3),

$$\langle \widetilde{\iota}\overline{\eta}, \widetilde{\eta}, \sigma^2 \rangle \subset \langle \widetilde{\iota}, 2\nu, \sigma^2 \rangle \supset \langle i^{2,4}\widetilde{\eta}, \nu, \sigma \rangle \circ \sigma \ni 0\ (\mathrm{mod}\ \widetilde{\iota} \circ \pi_{18}^s).$$

We have $2\widetilde{\iota}\nu^* = i^{2,4}\widetilde{\eta}\nu^* = 0$. By the fact that $\{M^6, S^0\} = \{\nu^2 p\} \cong \mathbb{Z}_2$, (2–8) and (1), $\overline{\eta}' \circ \widetilde{\iota}\nu^* = \sigma^3$, $\overline{\eta}' \circ \langle \widetilde{\iota}\overline{\eta}, \widetilde{\eta}, \sigma^2 \rangle = \langle \overline{\eta}', \widetilde{\iota}\overline{\eta}, \widetilde{\eta} \rangle \circ \sigma^2$ and $8\langle \overline{\eta}', \widetilde{\iota}\overline{\eta}, \widetilde{\eta} \rangle = \langle 8\iota, \overline{\eta}', \widetilde{\iota}\overline{\eta} \rangle \circ \widetilde{\eta} \subset \{M^6, S^0\} \circ \widetilde{\eta} = 0$. This implies $\langle \overline{\eta}', \widetilde{\iota}\overline{\eta}, \widetilde{\eta} \rangle \subset 2\pi_7^s$ and $\overline{\eta}' \circ \langle \widetilde{\iota}\overline{\eta}, \widetilde{\eta}, \sigma^2 \rangle = 0$. □

We show:

Lemma 2.9

(1) $H(E^{-3}[\iota_{4n}, \alpha]) = \frac{1+(-1)^n}{2}\nu\alpha$ *for* $\alpha = 4\nu, \nu^2, 8\sigma, \nu^3, 4\zeta, 16\rho, \nu\kappa, 4\nu^*, 4\overline{\zeta}, \overline{\sigma}, 4\overline{\kappa}, 4\nu\overline{\kappa}, 8\overline{\rho}$*. In particular,* $H(E^{-3}[\iota_{8n}, \alpha]) = \nu\alpha$ *for* $\alpha = \nu^2, \nu\kappa, \overline{\sigma}, 4\overline{\kappa}$.

(2) $H(E^{-7}[\iota_{8n}, \beta]) = 0$ *for* $\beta = 8\sigma, 16\rho, 8\overline{\rho}$.

(3) $H(E^{-7}[\iota_{8n}, \sigma^2]) = 0$ *or* σ^3.

Proof (1) is a direct consequence of Proposition 2.2(1). Let $n \equiv 0\ (8)$. We have $\mathrm{P}_{n-8}^{n-1} = E^{n-8}\mathrm{P}_0^7$ and $\gamma_{n-1,n-8} \in 2\pi_7^s(S^7) \oplus \pi_7^s(\mathrm{P}^6) \oplus \pi_7^s$. By Lemma 2.8(1), $\lambda_{n-8,8} \circ \beta = 0$. Hence, by Proposition 2.3[[n-8;8,0]], $[\iota_n, \beta]$ desuspends eight dimensions. Similarly, by Lemma 2.8(3) and Proposition 2.3[[n-8;8,0]], $\lambda_{n-8,8} \circ \sigma^2 = 0$ or $i\sigma^3$. □

By Lemma 2.9(1),

(2–9) $$[\iota_{8n+4}, \nu\bar{\sigma}] = 0.$$

We need the following.

Lemma 2.10 *$H(E^{-5}[\iota_{8n+6}, \alpha]) = 0$ for $\alpha = \eta, \varepsilon, \bar{\nu}, \mu, \kappa, \eta^*, \nu\kappa, \bar{\mu}, \bar{\sigma}, \eta\bar{\kappa}, \nu\bar{\sigma}$, $\mu_{3,*}$.*

Proof We show the assertion for $\alpha = \eta, \varepsilon, \mu, \kappa, \eta^*, \bar{\sigma}$. Let $n \equiv 6\ (8)$. In Proposition 2.1[n-6;6,0], $\mathrm{P}^{n-1}_{n-6} = E^{n-6}\mathrm{P}^5_0$. We take $\lambda_{n-6,6} = \gamma_5$. By (2–3) and (2–1), $\gamma_5\eta = \pm i^{2,5}\widetilde{\eta}\nu = 0$. We obtain $\gamma_5\varepsilon = 0$, because $\langle \eta, 2\iota, \varepsilon\rangle = \{\eta\varepsilon\}$. By the fact that $\langle \eta, 2\iota, \mu\rangle = \pm 2\zeta$ and $\langle 2\iota, \eta, \zeta\rangle = 0$,

$$\gamma_5\mu \in i^{4,5}\widetilde{\iota} \circ \langle \eta, 2\iota, \mu\rangle = i^{2,5}\widetilde{\eta}\zeta = 0.$$

By the relation $\langle \eta, 2\iota, \eta^*\rangle \ni \pm 2\nu^*\ (\mathrm{mod}\ \eta\bar{\mu})$, we have $\gamma_5\eta^* = i^{2,5}\widetilde{\eta}\nu^* = 0$. By the fact that $\langle \eta, 2\iota, \kappa\rangle \ni 0\ (\mathrm{mod}\ \eta\rho)$ and $\langle \eta, 2\iota, \bar{\sigma}\rangle \ni 0\ (\mathrm{mod}\ \eta\bar{\kappa})$, we obtain $\gamma_5\kappa = \gamma_5\bar{\sigma} = 0$. By the parallel argument and (2–6), the assertion holds for the other elements. □

Immediately,

(2–10) $$P\pi^{16n+13}_{16n+29} \subset E^6\pi^{8n}_{16n+21}.$$

Hereafter we use the following convention.

Convention

In the EHP sequence arguments:

(1) Higher suspended elements in a relation are omitted. For example, in a relation $E^k\delta \in \{[\iota_{n-1}, \beta], [\iota_{n-1}, \gamma]\}$, if $[\iota_{n-1}, \gamma] = E^l\gamma'$ for some element γ' and $l \geq k+1$, then $[\iota_{n-1}, \gamma]$ is omitted.

(2) Elements of order 2 having independent Hopf invariants in a relation are omitted, if other elements are suspended. For example, in a relation $E^k\delta\ (\mathrm{mod}\ \delta_1) \in \{[\iota_n, \beta]\}$ $(k \geq 1)$, if $2\delta_1 = 0$, $H\delta_1 \neq 0$ and $H[\iota_n, \beta] = 0$, then δ_1 disappears in the relation.

Now, we show the following:

Proposition 2.11 (1) *$H(E^{-3}[\iota_{8n+3}, \alpha]) = 0$ if $\nu\alpha = 0$.*

(2) *$H(E^{-3}[\iota_{8n+3}, \beta]_{\neq 0}) = \nu\beta$ for $\beta = \kappa, \nu^*, \bar{\sigma}, \bar{\kappa}, \nu\bar{\kappa}$.*

(3) *$H(E^{-3}[\iota_{8n+3}, \nu\kappa]_{\neq 0}) = 4\bar{\kappa}$ if $\sharp[\iota_{8n}, \bar{\kappa}] = 8$.*

Proof By Example 2.4(2), it suffices to prove the non-triviality in (2) and (3). We show it for ν^*. Let $n \equiv 3\ (8)$. By Lemma 1.2(2), $[\iota_n, \nu^*] = E^3(\overline{\tau}_{n-3}\nu^*)$. $\mathcal{ASM}[\nu^*]$ and (1–2) for $\overline{\zeta}$ induce $E^2(\overline{\tau}_{n-3}\nu^*) \in \{[\iota_{n-1}, \overline{\sigma}]\} \subset E^3\pi^{n-4}_{2n+13}$ (Proposition 2.6(1)), $E(\overline{\tau}_{n-3}\nu^*) \in P\pi^{2n-3}_{2n+17} = \{E(\tau_{n-3}\overline{\kappa})\}$, $\overline{\tau}_{n-3}\nu^*$ (mod $\tau_{n-3}\overline{\kappa}) \in P\pi^{2n-5}_{2n+16}$ and hence, $\mathcal{CDR}[\sigma^3$ (mod $\eta\overline{\kappa}) = 0]$. By the parallel argument, (2) for the other elements follows. We show (3). Assume that $E^3(\overline{\tau}_{8n}\nu\kappa) = [\iota_{8n+3}, \nu\kappa] = 0$. Then, $E^2(\overline{\tau}_{8n}\nu\kappa) \in \{[\iota_{8n+2}, 4\nu^*], [\iota_{8n+2}, \eta\overline{\mu}]\} = 0$ and $E(\overline{\tau}_{8n}\nu\kappa) \in \{[\iota_{8n+1}, \overline{\zeta}], [\iota_{8n+1}, \overline{\sigma}]\} = \{E(\tau_{8n}\overline{\zeta}), E(\tau_{8n}\overline{\sigma})\}$. This and the assumption $\sharp[\iota_{8n}, \overline{\kappa}] = 8$ imply $\overline{\tau}_{8n}\nu\kappa + a\tau_{8n}\overline{\zeta} + b\tau_{8n}\overline{\sigma} \in \{4[\iota_{8n}, \overline{\kappa}]\}$. Since $\eta\overline{\zeta} = \eta\overline{\sigma} = 0$, we get $\mathcal{CDR}[\nu^2\kappa = 0]$. □

Immediately,

$$[\iota_{8n+7}, \sigma^3] = 0.$$

By Proposition 2.2(3), we have:

Lemma 2.12 *$H(E^{-3}[\iota_{4n+2}, \alpha]) = \frac{1+(-1)^n}{2}\nu\alpha$ for $\alpha = \overline{\nu}, \varepsilon, \kappa, \nu\kappa, \nu^2\kappa, \overline{\sigma}, \nu\overline{\sigma}$. In particular, $H(E^{-3}[\iota_{8n+2}, \alpha]) = \nu\alpha$ for $\alpha = \kappa, \nu\kappa, \nu^2\kappa, \overline{\sigma}$.*

Immediately,

(2–11) $$[\iota_{8n+6}, \nu\kappa] = 0$$

and $$[\iota_{8n+6}, \nu\overline{\sigma}] = 0.$$

We need the following:

Lemma 2.13

(1) $H(E^{-3}[\iota_{4n+1}, \alpha]) = \frac{1+(-1)^n}{2}\nu\alpha$ *for* $\alpha = \nu, \nu^2, \nu\kappa, \nu^*, \overline{\sigma}, \nu^2\kappa$, $\nu\overline{\sigma}, \nu\overline{\kappa}$ *and* $H(E^{-3}[\iota_{4n+1}, \beta_1]) = H(E^{-3}[\iota_{4n+1}, \eta\beta_2]) = 0$ *for* $\beta_1 = \zeta, \overline{\zeta}, \zeta_{3,*}$; $\beta_2 = \mu, \overline{\mu}, \mu_{3,*}$. *In particular,* $H(E^{-3}[\iota_{8n+1}, \alpha]) = \nu\alpha$ *for* $\alpha = \nu, \nu^2, \nu\kappa, \nu^*, \overline{\sigma}$, $\nu^2\kappa, \nu\overline{\kappa}$.

(2) $H(E^{-4}[\iota_{8n+5}, \delta_1]) = 0$ *for* $\delta_1 = \eta^2, \nu, \eta^2\sigma, \eta\varepsilon, \eta^2\rho, \nu\kappa, \eta\mu, \eta\eta^*, \eta\overline{\mu}$.

(3) $H(E^{-4}[\iota_{8n+1}, \delta_2]) = 0$ *for* $\delta_2 = \nu^3, \eta^2\sigma, \sigma^2, \eta^2\rho, \nu\overline{\sigma}, \eta\eta^*\sigma, \eta^2\overline{\rho}, \nu^2\overline{\kappa}$.

(4) $H(E^{-6}[\iota_{8n+5}, \eta^2\delta_3]) = 0$ *for* $\delta_3 = \rho, \overline{\rho}$.

Proof (1) is a direct consequence of Proposition 2.2(2). Let $n \equiv 5\ (8)$. Then, $\mathrm{P}^{n-1}_{n-5} = E^{n-5}\mathrm{P}^4_0$ and we can take $\lambda_{n-5,5} = \widetilde{\iota}\eta$. By the relations $\eta\delta_1 = 0, 4\widetilde{\iota} = i\eta^2$ (2–1) and $\eta\delta = 0$ ($\delta = \nu, \zeta, \nu^*, \overline{\zeta}$), we have $\lambda_{n-5,5} \circ \delta_1 = 0$. Hence, Proposition 2.1[n-8;5,0] leads to (2).

In Proposition [n-5;5,4], $\mathrm{P}^{n-1}_{n-5} = E^{n-9}\mathrm{P}^8_4$ for $n \equiv 1\ (8)$. By $(\mathcal{P}^8_4)$, we have

$$(*) \qquad \pi^s_8(\mathrm{P}^8_4) = i''_*\pi^s_8(\mathrm{P}^7_4) = \{i''s_2\eta, i''ti\nu\} \cong (\mathbb{Z}_2)^2 \ (i = i^{4,7}_4, i'' = i^8_4).$$

So, we take

$$\gamma_{8,4} = i''(s_2\eta + ti\nu) \tag{2–12}$$

and $\lambda_{n-5,5} \circ \delta_2 = 0$.

In Proposition 2.1[n-7;7,6], $\mathrm{P}^{n-1}_{n-7} = E^{n-13}\mathrm{P}^{12}_6$ for $n \equiv 5\ (8)$. Since $\mathrm{P}^{12}_6/\mathrm{P}^7_6 = \mathrm{P}^{12}_8 = E^8\mathrm{P}^4_0$, we have $p^{12}_{8,6*}(\lambda_{n-7,7} \circ \eta^2) \in \pi^s_4(\mathrm{P}^4) \circ \eta^2 = 0$ and $\lambda_{n-7,7} \circ \eta^2 \in i^{7,12}_{6\ \ *}\pi^s_{14}(\mathrm{P}^7_6)$. Hence, by the fact that $\pi^s_{14}(\mathrm{P}^7_6) \cong \pi^s_8 \oplus \pi^s_7$ and $\pi^s_8 \circ \delta_3 = \pi^s_7 \circ \delta_3 = 0$, we obtain $\lambda_{n-7,7} \circ \eta^2\delta_3 = 0$. □

By Lemma 2.13(1), we obtain

$$[\iota_{8n+5}, \sigma^3] = 0, \tag{2–13}$$

$$[\iota_{8n+5}, \nu\bar{\sigma}] = 0$$

and

$$P\pi^{8n+3}_{8n+21+k} \subset E^3\pi^{4n-2}_{8n+16+k} \ (k = 0, 1). \tag{2–14}$$

We also note the following.

Remark $H(E^{-3}[\iota_{8n+1}, \nu\kappa]) = \nu^2\kappa$, while $[\iota_{8n+1}, \nu\kappa] = 0$ (2–7).

3 Concerning Nomura's results [15]

In this section, we recollect Nomura's results [15], prove a part of them by using Proposition 2.1 and add results needed in the next section. By use of the cell structures of P^{n-1}_{n-k}, we determine some group structures of $\pi^s_{n-1}(\mathrm{P}^{n-1}_{n-k})$ for $4 \le k \le 8$, which overlap with [17, Section 3]. First we show the result including the known one [15, 4.10;18].

Lemma 3.1 *$H(E^{-7}[\iota_{16n+3}, \sigma]) = \sigma^2$ and $H(E^{-7}[\iota_{16n+k}, \sigma^2]) = \sigma^3$ for $k = 0, 1, 3, 7$.*

Proof Let $n \equiv 0\ (16)$. By (1–1), $[\iota_n, \sigma^2] = \sigma_n \circ [\iota_{n+7}, \iota] = E^7(\sigma_{n-7}\delta_n)$ and $H(\sigma_{n-7}\delta_n) = \sigma^3_{2n-15}$. Let $n \equiv 7\ (16)$. By (1–1), $[\iota_n, \sigma^2] = E^7(\delta_{n-7}\sigma)$ and

$H(\delta_{n-7}\sigma) = \sigma^3_{2n-15}$. Let $n \equiv 1$ (16). We have $P^{n-1}_{n-8} = E^{n-17}P^{16}_9$ and $P^{n-2}_{n-8} = E^{n-17}P^{15}_9 = E^{n-9}P^7 = E^{n-9}P^6 \vee S^{n-2}$. By inspecting [12, Proposition 4.3],

$$\pi^s_8(P^6) = \{\widetilde{\overline{\eta}}'\eta, \widetilde{i\nu}, i^{2,6}\widetilde{\nu^2}, i\sigma\} \cong (\mathbb{Z}_2)^4,$$

where $\widetilde{i\nu} \in \langle i^{4,6}\widetilde{\iota}, \overline{\eta}, i\nu\rangle$ and $\widetilde{i\nu} \circ \sigma \in \langle i^{4,6}\widetilde{\iota}, \overline{\eta}, i\nu\rangle \circ \sigma = i^{4,6}\widetilde{\iota} \circ \langle \overline{\eta}, i\nu, \sigma\rangle = 0$. So, by Lemma 2.7(8), $\pi^s_8(P^6) \circ \sigma^2 = \{i\sigma^3\}$. Since $p_*\colon \pi^s_{16}(P^{16}_9) \to \pi^s_{16}(S^{16})$ is trivial, $\pi^s_{16}(P^{16}_9) = i_*\pi^s_{16}(P^{15}_9)$. This implies $(\pi^s_{16}(P^{16}_9) - \{i\sigma\}) \circ \sigma^2 = 0$ and hence, by Proposition 2.3[[n-8;8,9]], the assertion follows.

Next, let $n \equiv 3$ (16). In Proposition 2.3[[n-8;8,11]], $P^{n-1}_{n-8} = E^{n-11}P^{10}_3$. Since $\{EP^4, P^2\} = \{i\overline{\eta}', \widetilde{\overline{\eta}}\eta p^4_3, i\nu p\} \cong (\mathbb{Z}_2)^3$, $P^7_3 = P^6_3 \vee S^7$ and Sq^4 is trivial on $\widetilde{H}^3(P^8_3;\mathbb{Z}_2)$, we can take $P^8_3 = M^4 \cup_{i\overline{\eta}'} C(E^3P^4)$. From the relations $\overline{\eta}'\widetilde{\iota}\eta = 0$ and $\overline{\eta}'\widetilde{\iota}\nu = \nu^2$ (2–8), we obtain $\pi^s_8(P^8_3) = \{\widetilde{\widetilde{\iota}\eta}, \widetilde{i\nu}'\}$ and $\pi^s_9(P^8_3) = \{\widetilde{\widetilde{\iota}\eta}\eta\} \cong \mathbb{Z}_2$, where

$$\widetilde{\widetilde{\iota}\eta} \in \langle i', \overline{\eta}', \widetilde{\iota}\eta\rangle \quad \text{and} \quad \widetilde{i\nu}' \in \langle i', \overline{\eta}', i\nu\rangle \ (i' = i^{3,8}_3).$$

By (2–5), we obtain

$$\gamma_{8,3} = \widetilde{\widetilde{\iota}\eta} + \widetilde{i\nu}'.$$

By the fact that $\pi^s_6(P^4) = \{\widetilde{\iota}\nu\} \cong \mathbb{Z}_2$ and (2–8), we obtain $\langle \overline{\eta}', i\nu, \eta\rangle = \pi^s_6 = \{\nu^2\}$ and $\widetilde{i\nu}'\eta \in i' \circ \langle \overline{\eta}', i\nu, \eta\rangle = \{i'\nu^2\} = 0$. Hence,

$$\gamma_{8,3}\eta = \widetilde{\widetilde{\iota}\eta}\eta$$

and $\pi^s_{10}(P^{10}_3) = i'''_*\pi^s_{10}(P^8_3) = i'''_*i''_*\pi^s_{10}(M^4) = \{i'''i''\widetilde{\nu^2}, i\sigma\}$ $(i''' = i^{8,10}_3, i'' = i^{4,8}_3)$. Therefore, by Lemma 2.7(8), $(\pi^s_{10}(P^{10}_3) - \{i\sigma\}) \circ \sigma = 0$. This implies $H(E^{-7}[\iota_n, \sigma]) = \sigma^2$ and $H(E^{-7}[\iota_n, \sigma^2]) = \sigma^3$ (2–6). □

Immediately, $$[\iota_{16n+11}, \sigma^2] = 0,\ [\iota_{16n+k}, \sigma^3] = 0\ (k = 8, 11, 15)$$

and

(3–1) $$[\iota_{16n+9}, \sigma^3] = 0.$$

Next, we show the following [15, Table 2, 4.15;16].

Lemma 3.2

(1) $H(E^{-4}[\iota_{8n+4}, 16\rho]) = \overline{\zeta}$.

(2) $H(E^{-5}[\iota_{8n+3}, \nu\kappa]) = \eta^2\overline{\kappa}$.

(3) $H(E^{-6}[\iota_{8n}, \nu^3]) = \eta\kappa$.

Proof Let $n \equiv 4\ (8)$. In Proposition 2.1[n-5;5,7], $\mathrm{P}^{n-1}_{n-5} = E^{n-12}\mathrm{P}^{11}_7$. Let $\widetilde{2\iota} \in \langle i', i\nu, 2\iota\rangle$ $(i = i^{7,10}_7, i' = i^{11}_7)$ be a coextension of 2ι in $(\mathcal{P}^{11}_7)$. By Lemma 2.8, we can take

$$\gamma_{11,7} = \widetilde{2\iota} + i^{11}_7\widetilde{\eta}''. \tag{3–2}$$

By Lemma 2.7(3), $\lambda_{n-5,5} \circ 16\rho \in i' \circ \langle i\nu, 2\iota, 16\rho\rangle = i\overline{\zeta}$.

In Proposition 2.1[n-6;6,1], $\mathrm{P}^{n-1}_{n-6} = E^{n-11}\mathrm{P}^{10}_5$ for $n \equiv 3\ (8)$. By the cell structure $(\mathcal{P}^4)$, we obtain $\{M^5, \mathrm{P}^4\} = \{\widetilde{\iota}\overline{\eta}, i^{2,4}\overline{i\nu}\} \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2$, where $2\widetilde{\iota}\overline{\eta} = \widetilde{\iota}\eta^2 p$. Since Sq^k on $\widetilde{H}^{9-k}(\mathrm{P}^{10}_5; \mathbb{Z}_2)$ is non-trivial for $k = 2, 4$,

$$(\mathcal{P}^{10}_5) \qquad \mathrm{P}^{10}_5 = \mathrm{P}^8_5 \cup_{\widetilde{\iota}\overline{\eta}+i^{2,4}\overline{i\nu}} CM^9 \ (\mathrm{P}^8_5 = E^4\mathrm{P}^4).$$

From the natural isomorphisms $\pi^s_{10}(\mathrm{P}^{10}_5) \cong \pi^s_{10}(\mathrm{P}^8_5) \cong \pi^s_6(\mathrm{P}^4) = \{\widetilde{\iota}\nu\} \cong \mathbb{Z}_2$, we obtain

$$\pi^s_{10}(\mathrm{P}^{10}_5) = \{i'\widetilde{\iota}\nu\} \cong \mathbb{Z}_2 \ (i' = i^{8,10}_5),$$

$$\gamma_{10,5} = i'\widetilde{\iota}\nu \tag{3–3}$$

and

$$(\mathcal{P}^{11}_5) \qquad \mathrm{P}^{11}_5 = \mathrm{P}^{10}_5 \cup_{i'\widetilde{\iota}\nu} e^{11}.$$

Hence, by the relation $4\overline{\kappa} = \nu^2\kappa$ and (2–1), $\lambda_{n-6,6} \circ \nu\kappa = 4i'\widetilde{\iota}\overline{\kappa} = i\eta^2\overline{\kappa}$.

In Proposition 2.1[n-7;7,1], $\mathrm{P}^{n-1}_{n-7} = E^{n-8}\mathrm{P}^7$ for $n \equiv 0\ (8)$. Let $s_3\colon S^7 \hookrightarrow \mathrm{P}^7 = \mathrm{P}^6 \vee S^7$ be the canonical inclusion. Then, we take

$$\gamma_7 = 2s_3 + i^{6,7}\widetilde{\eta}'. \tag{3–4}$$

By Lemma 2.8(1), $\widetilde{\eta}' \circ \nu^3 \in i_{4,6} \circ \langle \widetilde{\iota}\overline{\eta}, \widetilde{\eta}, \nu^3\rangle$. By (2–1) and Lemmas 2.7(6),(9), 2.8(3),

$$\langle \widetilde{\iota}\overline{\eta}, \widetilde{\eta}, \nu^3\rangle \subset \langle \widetilde{\iota}, 2\nu, \nu^3\rangle \supset i^{2,4} \circ \langle \widetilde{\eta}, \nu, \nu^3\rangle = i\eta\kappa$$

$$(\mathrm{mod}\ \widetilde{\iota} \circ \pi^s_{13} + \pi^s_7(\mathrm{P}^4) \circ \eta\overline{\nu} = 0).$$

Hence, $\lambda_{n-7,7} \circ \nu^3 = i\eta\kappa$. □

Immediately,

$$[\iota_{8n+7}, \overline{\zeta}] = [\iota_{8n+5}, \eta^2\overline{\kappa}] = [\iota_{8n+1}, \eta\kappa] = 0.$$

By the way, the argument in [5, Section 4] implies that $\Delta_{\mathbb{H}}\colon \pi_{8n+10}(S^{8n+7}) \to \pi_{8n+9}(Sp(2n+1))$ is trivial on the the 2 primary component and

$$\Delta(\eta^2_{8n+5}\overline{\kappa}) = 4i_*\Delta_{\mathbb{H}}(\overline{\kappa}_{8n+7}) = i_*\Delta_{\mathbb{H}}(\nu_{8n+7})\nu\kappa = 0,$$

where $\Delta_{\mathbb{H}}$ is the the symplectic connecting map and $i\colon Sp(2n+1) \hookrightarrow SO(8n+7)$ the canonical inclusion.

The non-triviality of $[\iota_{8n}, \nu^3]$ is proved in [5].

Now we show the following result overlapping with [15, 4.12].

Lemma 3.3 $H(E^{-4}[\iota_{8n+4}, \sigma^2]) = \nu^*$ *and* $H(E^{-5}[\iota_{8n+5}, \sigma^2]) = \overline{\sigma}$.

Proof In Proposition 2.1[n-5;5,7], $\mathrm{P}^{n-1}_{n-5} = E^{n-12}\mathrm{P}^{11}_7$ for $n \equiv 4\ (8)$. By Lemmas 2.7(2), 2.8(3) and (3–2), $\lambda_{n-5,5} \circ \sigma^2 = i'\langle i\nu, 2\iota, \sigma^2\rangle = i\nu^*$.

In Proposition 2.1[n-6;6,7], $\mathrm{P}^{n-1}_{n-6} = E^{n-13}\mathrm{P}^{12}_7$ for $n \equiv 5\ (8)$. We see that $\{M^7, \mathrm{P}^6_3\} = \{i'\overline{i\nu}, i'\widetilde{\eta}\eta, \widetilde{\eta}'' p\} \cong (\mathbb{Z}_2)^3$ $(i' = i^{4,6}_3)$. By $(\mathcal{P}^{11}_7)$, we have

$$(\mathcal{P}^{12}_7) \qquad\qquad \mathrm{P}^{12}_7 = \mathrm{P}^{10}_7 \cup_{i'\overline{i\nu}+\widetilde{\eta}'' p} CM^{11}$$

and $\pi^s_{12}(\mathrm{P}^{12}_7) = \{\widetilde{i\eta}, i''\widetilde{i\nu}\} \cong (\mathbb{Z}_2)^2$, where $\widetilde{i\eta} \in \langle i'', i'\overline{i\nu} + \widetilde{\eta}'' p, i\eta\rangle$ $(i'' = i^{10,12}_7)$ and $\widetilde{i\nu} \in \langle i', \overline{\eta}, i\nu\rangle \in \pi^s_{12}(\mathrm{P}^{10}_7)$. Since $\langle i'', i'\overline{i\nu} + \widetilde{\eta}'' p, i\eta\rangle \supset \langle i'', (i'\overline{i\nu} + \widetilde{\eta}'' p)\circ i, \eta\rangle = \langle i'', i'i\nu, \eta\rangle \supset \langle i''i'i, \nu, \eta\rangle$, we can choose $\widetilde{i\eta}$ such that

$$\widetilde{i\eta} \in \langle i''', \nu, \eta\rangle \ (i''' = i''i'i = i^{7,12}_7). \tag{3–5}$$

From the fact that Sq^4 is trivial on $\widetilde{H}^9(\mathrm{P}^{13}_7; \mathbb{Z}_2)$, we take $\gamma_{12,7} = \widetilde{i\eta}$ and

$$\widetilde{i\eta} \circ \sigma^2 \in i''' \circ \langle \nu, \eta, \sigma^2\rangle = i'''\overline{\sigma} \ (\mathrm{mod}\ 0).$$

This implies $\lambda_{n-6,6} \circ \sigma^2 = i'''\overline{\sigma}$. □

Immediately,

$$[\iota_{8n+7}, \nu^*] = [\iota_{8n+7}, \overline{\sigma}] = 0.$$

Next, we prove the following [15, 4.13;14;16,Table 2].

Lemma 3.4

(1) $H(E^{-5}[\iota_{8n+2}, \eta]) = \nu^2$.

(2) $H(E^{-6}[\iota_{8n+1}, \eta^2]) = \varepsilon$.

(3) $H(E^{-5}[\iota_{8n+2}, \eta^*]) = \sigma^3$.

(4) $H(E^{-6}[\iota_{8n+1}, \eta\eta^*]) = \eta^*\sigma$.

(5) $H(E^{-6}[\iota_{8n+6}, \kappa]) = \overline{\kappa}$.

Proof In Proposition 2.1[n-6;6,6], $\mathrm{P}^{n-1}_{n-6} = E^{n-10}\mathrm{P}^9_4$ for $n \equiv 2\ (8)$, $\pi^s_9(\mathrm{P}^9_4) \cong \pi^s_9(\mathrm{P}^9_5) \cong \mathbb{Z}\ (\mathcal{P}^9_5)$ and $\gamma_{9,4}\circ\eta^* \in i'''\circ\langle\gamma_{8,4}, 2\iota, \eta^*\rangle$ $(i''' = i^9_4)$ (2–2). By the relations $\langle p, i, 2\iota\rangle = \pm\iota$, $\langle i\nu, 2\iota, \eta\rangle = 0$, (2–4) and (2–12), we have $2i''s_2 = i''(i\nu \pm t\widetilde{\eta})$ $(i'' = i^8_4)$ and

$$\langle\gamma_{8,4}, 2\iota, \eta\rangle \subset \langle i''s_2\eta, 2\iota, \eta\rangle + \langle i''ti\nu, 2\iota, \eta\rangle \ni \pm 2i''s_2\nu = i\nu^2$$
$$(\mathrm{mod}\ i''(s_2\eta + ti\nu)\circ\pi^s_2 + \pi^s_9(\mathrm{P}^8_4)\circ\eta).$$

The indeterminacy is trivial, because $\pi^s_9(\mathrm{P}^8_4) = \{i''s_2\eta^2\} \cong \mathbb{Z}_2$ and $i''s_2\eta^3 = 4i''s_2\nu = 0$. This implies $\lambda_{n-6,6}\circ\eta = i\nu^2$.

In Proposition 2.1[n-7;7,2], $\mathrm{P}^{n-1}_{n-7} = E^{n-9}\mathrm{P}^8_2$ for $n \equiv 1\ (8)$. We obtain $\{M^5, \mathrm{P}^4_2\} = \{\widetilde{\iota}'\overline{\eta}, i\nu\} \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2$, $\pi^s_6(\mathrm{P}^6_2) = \{i'\widetilde{\iota}'\nu\} \cong \mathbb{Z}_2$ $(i' = i^{4,6}_2)$ and $\pi^s_7(\mathrm{P}^6_2) = \{\widetilde{\eta}'''\} \cong \mathbb{Z}_8$, where $\widetilde{\eta}''' \in \langle i', \widetilde{\iota}'\overline{\eta}, \widetilde{\eta}\rangle$ and $2\widetilde{\eta}''' \in \langle i', i\eta\overline{\eta}, \widetilde{\eta}\rangle$ $(i = i^{2,4}_2)$. We also obtain $\{M^7, \mathrm{P}^6_2\} = \{i'\overline{\widetilde{\iota}'\nu}, \widetilde{\eta}'''p\} \cong (\mathbb{Z}_2)^2$. By the cell structures

$$\mathrm{P}^6_2 = \mathrm{P}^4_2 \cup_{\widetilde{\iota}'\overline{\eta}} CM^5 \quad\text{and}\quad \mathrm{P}^8_2 = \mathrm{P}^6_2 \cup_{\widetilde{\eta}'''p} CM^7,$$

we have $\pi^s_8(\mathrm{P}^6_2) = \{\widetilde{\eta}'''\eta, \widetilde{i\nu}'', i'\widetilde{\iota}\nu^2\} \cong (\mathbb{Z}_2)^3$ and $\pi^s_8(\mathrm{P}^8_2) = \{\widetilde{\iota}'''\eta\} \oplus i''_*\pi^s_8(\mathrm{P}^6_2)$, where $\widetilde{i\nu}'' \in \langle i'\widetilde{\iota}', \overline{\eta}, i\nu\rangle$, $\widetilde{\iota}''' \in \langle i'', \widetilde{\eta}'''p, i\rangle$ $(i'' = i^{6,8}_2)$ and $2\widetilde{\iota}''' = i''\widetilde{\eta}'''$ [12, Proposition 4.2]. We can take $\gamma_{8,2} \equiv \widetilde{\iota}'''\eta\ (\mathrm{mod}\ i''_*\pi^s_8(\mathrm{P}^6_2))$. Since $\widetilde{i\nu}''\circ\eta \in i'\widetilde{\iota}'\circ\langle\overline{\eta}, i\nu, \eta\rangle = i'\widetilde{\iota}'\nu^2$, we obtain $\widetilde{i\nu}''\circ\eta^2 = 0$. By Lemma 2.7(8), $\gamma_{8,2}\circ\eta^2 = 2i''\widetilde{\eta}'''\nu \in i''i'\circ\langle i\eta\overline{\eta}, \widetilde{\eta}, \nu\rangle = i\varepsilon$.

By the same argument as (1) and by Lemma 2.7(4), $\lambda_{n-6,6}\circ\eta^* = i\sigma^3$. By the same argument as (2) and by Lemma 2.7(1), $\gamma_{8,2}\eta\eta^* = i\eta^*\sigma$.

Since $\langle\eta, 2\iota, \kappa\rangle \ni 0$, we can choose a coextension $\widetilde{\kappa} \in \pi^s_{16}(\mathrm{P}^2)$ satisfying $\overline{\eta}\widetilde{\kappa} = 0$. Notice that $\langle\nu, \eta, \eta\kappa\rangle = \pm 2\overline{\kappa}$ and $\langle\nu, \overline{\eta}, \widetilde{\kappa}\rangle = \pm\overline{\kappa}$. In Proposition 2.1[n-7;7,7], $\mathrm{P}^{n-1}_{n-7} = E^{n-14}\mathrm{P}^{13}_7$ for $n \equiv 6\ (8)$. By use of $(\mathcal{P}^{12}_7)$, we get that

$$\pi^s_{13}(\mathrm{P}^{12}_7) = \{\widetilde{i\eta}\eta, i'\widetilde{\eta}''\eta^2, i\nu^2\} \cong (\mathbb{Z}_2)^3\ (i' = i^{10,12}_7).$$

We obtain $\widetilde{i\eta}\eta\kappa \in i'''\circ\langle\nu, \eta, \eta\kappa\rangle = 2i\overline{\kappa} = 0$. By (3–5), there exists an extension $\widetilde{i\overline{\eta}} \in \langle i''', \nu, \overline{\eta}\rangle$ of $\gamma_{12,7} = \widetilde{i\eta}$. By (2–2), we obtain $\gamma_{13,7}\circ\kappa \in i^{13}_7\circ\langle\gamma_{12,7}, 2\iota, \kappa\rangle \ni i^{13}_7\widetilde{i\overline{\eta}}\widetilde{\kappa}$ $(\mathrm{mod}\ i^{13}_{7\,*}\pi^s_{13}(\mathrm{P}^{12}_7)\circ\kappa = 0)$. We obtain $\widetilde{i\overline{\eta}}\widetilde{\kappa} \in i'''\circ\langle\nu, \overline{\eta}, \widetilde{\kappa}\rangle = i\overline{\kappa}$ $(\mathrm{mod}\ i''\circ\{M^6, S^0\}\circ\widetilde{\kappa} = \{i''\nu^2\kappa\} = 0)$ and hence, $\lambda_{n-7,7}\circ\kappa = i\overline{\kappa}$. □

Immediately,

$$[\iota_{8n+4}, \sigma^3] = [\iota_{8n+2}, \eta^*\sigma] = [\iota_{8n+7}, \overline{\kappa}] = 0.$$

Given an element $\alpha \in \pi_k(S^n)$, a lift $[\alpha] \in \pi_k(SO(n+1))$ of α is an element satisfying $p_{n+1}(\mathbb{R})[\alpha] = \alpha$, where $p_{n+1}(\mathbb{R})$: $SO(n+1) \to S^n$ is the projection. A lift $[\alpha]$ exists if and only if $\Delta\alpha = 0 \in \pi_{k-1}(SO(n))$. Let $n \equiv 7$ (8). We know $\Delta\nu_n = 0$ [9]. Note the fact that $\Delta\kappa_n = 0$ [5, Section 5] is obtained by constructing a lift of κ_n is given by

$$[\kappa_n] \in \{[\nu_n], \overline{\eta}, \widetilde{\overline{\nu}}\} \subset \pi_{n+14}(SO(n+1)) \; (\widetilde{\overline{\nu}} : \text{ a coextension of } \overline{\nu}).$$

By the parallel argument, lifts of $\overline{\sigma}_n$ and $\overline{\kappa}_n$ are taken as follows:

$$[\overline{\sigma}_n] \in \{[\nu_n], \eta, \sigma^2\} \subset \pi_{n+19}(SO(n+1));$$

$$[\overline{\kappa}_n] \in \{[\nu_n], \overline{\eta}, \widetilde{\kappa}\} \subset \pi_{n+20}(SO(n+1)).$$

Hence,

$$\Delta\overline{\sigma}_{8n+7} = \Delta\overline{\kappa}_{8n+7} = 0.$$

We need the following result overlapping with [15, 4.14].

Lemma 3.5

(1) $H(E^{-6}[\iota_{8n+3}, \alpha]) = 0$ *if* $\nu\alpha = 0$.

(2) $H(E^{-6}[\iota_{8n+4k}, 4\nu^*]) = \eta\eta^*\sigma$ *or* 0 *according as* $k = 0$ *or* 1.

Proof In Proposition 2.1[n-7;7,4], $\mathrm{P}^{n-1}_{n-7} = E^{n-11}\mathrm{P}^{10}_4$ for $n \equiv 3$ (8). We have $\{\mathrm{P}^4, S^1\} = \{\eta\overline{\eta}p^4_3, \nu p\} \cong (\mathbb{Z}_2)^2$ $(p = p^4_4)$, $\eta\overline{\eta}p^4_3 \circ (\overline{i\eta} + i^{2,4}\overline{i\nu}) = \eta^2\overline{\eta}$ and $p \circ (\overline{i\eta} + (i^{2,4})\overline{i\nu}) = 0$. So, by the fact that $\{M^5, S^0\} = 0$ and $(\mathcal{P}^{10}_5)$, p is extendible on $\overline{p} \in \{\mathrm{P}^{10}_5, S^8\}$ and $\{\mathrm{P}^{10}_5, S^5\} = \{\nu\overline{p}\} \cong \mathbb{Z}_2$. Hence,

$$E\mathrm{P}^{10}_4 = S^5 \cup_{\nu\overline{p}} C\mathrm{P}^{10}_5.$$

Since $(\overline{i\eta} + (i^{2,4})\overline{i\nu}) \circ i\nu = i\nu^2$, we have $i'i\nu^2 = 0$ in $\pi^s_{11}(\mathrm{P}^{10}_5)$ $(i' = i^{8,10}_5)$. By Lemma 2.7(5), $\langle i^{2,4}\overline{i\nu}, \widetilde{\eta}, 4\iota\rangle \supset i^{2,4} \circ \langle\overline{i\nu}, \widetilde{\eta}, 4\iota\rangle = \{i\nu^2\}$. So, by $(\mathcal{P}^{10}_5)$ and Lemma 2.8(1), $\pi^s_{11}(\mathrm{P}^{10}_5) = \{\widetilde{\eta}^{IV}\} \cong \mathbb{Z}_8$, where $\widetilde{\eta}^{IV} \in \langle i', \overline{i\eta} + i^{2,4}\overline{i\nu}, \widetilde{\eta}\rangle$ and $4\widetilde{\eta}^{IV} = i'\widetilde{\widetilde{\eta}\eta^2}$. By the fact that $\langle p', i\overline{\eta}, \widetilde{\eta}\rangle = \pm\nu$ $(p' = p^2_2)$ and

$$\langle p, i^{2,4}\overline{i\nu}, \widetilde{\eta}\rangle \subset \langle p', 0, \widetilde{\eta}\rangle \ni 0 \; (\bmod\; p' \circ \pi^s_5(\mathrm{P}^2) + \{\mathrm{P}^2, S^0\} \circ \widetilde{\eta} = \{2\nu\}),$$

we obtain $\overline{p} \circ \widetilde{\eta}^{IV} = \pm\nu$. So, by (3–3) and the relation $\overline{p} \circ i'\widetilde{\iota} = 0$ $(i'\widetilde{\iota} \in \pi^s_7(\mathrm{P}^{10}_5))$, we conclude that $\pi^s_{10}(\mathrm{P}^{10}_4) = \{\widetilde{i'\widetilde{\iota}}\nu\} \cong \mathbb{Z}_2$ and $\gamma_{10,4} = \widetilde{i'\widetilde{\iota}}\nu$, where $\widetilde{i'\widetilde{\iota}} \in \pi^s_7(\mathrm{P}^{10}_4)$ is a coextension of $i'\widetilde{\iota}$. This leads to (1).

In Proposition 2.1[n-7;7,1], $\mathrm{P}^{n-1}_{n-7} = E^{n-8}\mathrm{P}^7$ for $n \equiv 0$ (8). By (3–4) and Lemma 2.7(10), $\lambda_{n-7,7} \circ 4\nu^* = i^{4,7}\widetilde{\widetilde{\eta}\eta^2}\nu^* = i^{2,7} \circ \langle\widetilde{\eta}p, \widetilde{\eta}\eta^2, \nu^*\rangle = i\eta\eta^*\sigma$. Hence, $\lambda_{n-7,7} \circ 4\nu^* = i\eta\eta^*\sigma$.

In Proposition 2.1[n-7;7,5], $P^{n-1}_{n-7} = E^{n-12}P^{11}_5$ for $n \equiv 4 \pmod 8$. By use of $(\mathcal{P}^{11}_5)$, we can take

$$\text{(3–6)} \qquad \gamma_{11,5} = i''\tilde{\eta}^{IV} + \widetilde{2\iota}, \text{ where } \widetilde{2\iota} \in \langle i''', \tilde{\iota}\nu, 2\iota\rangle \ (i'' = i^{11}_5, i''' = i^{8,11}_5).$$

By Lemma 2.7(10), $\tilde{\eta}^{IV} \circ 4\nu^* = i'\widetilde{\tilde{\eta}\eta^2} \circ \nu^* = i\eta\eta^*\sigma$. By the relation $\widetilde{2\iota} \circ \eta \in i''' \circ \langle \tilde{\iota}\nu, 2\iota, \eta\rangle$ and Lemmas 2.7(8),2.8(3),

$$\langle \tilde{\iota}\nu, 2\iota, \eta\rangle \subset \langle \tilde{\iota}, 2\nu, \eta\rangle \supset \langle i'\tilde{\eta}, \nu, \eta\rangle \ni i^{2,4}\widetilde{\nu^2} \ (\mathrm{mod}\ \pi^s_7(\mathrm{P}^4) \circ \eta = 0).$$

Hence, $\widetilde{2\iota} \circ \eta = i'''i^{2,4}\widetilde{\nu^2}$ and $\widetilde{2\iota} \circ 4\nu^* = i'''i^{2,4}\widetilde{\nu^2}\eta\eta^* = i\eta\eta^*\sigma$ by Lemma 2.7(10). Thus, by (3–6), $\lambda_{n-7,7} \circ 4\nu^* = 0$. This leads to (2). □

Immediately,

$$[\iota_{8n+1}, \eta\eta^*\sigma] = 0.$$

Finally, we need the following [15, 4.8;9;10;11;16;17;18].

Lemma 3.6

(1) $H(E^{-6}[\iota_{8n+5}, \eta\kappa]) = \eta\bar{\kappa}$.

(2) $H(E^{-6}[\iota_{8n+4}, \nu\kappa]) = \nu\bar{\kappa}$.

(3) $H(E^{-6}[\iota_{8n+2}, 4\bar{\kappa}]) = \nu^2\bar{\kappa}$.

(4) $H(E^{-7}[\iota_{16n+14}, \eta^*]) = \eta^*\sigma$ *and* $H(E^{-7}[\iota_{16n+13}, \eta\eta^*]) = \eta\eta^*\sigma$.

(5) $H(E^{-11}([\iota_{16n+5}, \nu]) = \sigma^2$.

(6) $H(E^{-13}[\iota_{16n+3}, \nu^2]) = \bar{\sigma}$, $H(E^{-11}[\iota_{16n+2}, \eta\sigma]) = \bar{\sigma}$ *and* $H(E^{-13}[\iota_{16n+1}, \nu^3]) = \nu\bar{\sigma}$.

Immediately,

$$[\iota_{8n+6}, \eta\bar{\kappa}] = [\iota_{8n+5}, \nu\bar{\kappa}] = [\iota_{8n+3}, \nu^2\bar{\kappa}] = [\iota_{16n+9}, \sigma^2] = 0,$$

$$[\iota_{16n+5}, \bar{\sigma}] = [\iota_{16n+6}, \bar{\sigma}] = [\iota_{16n+3}, \nu\bar{\sigma}] = [\iota_{16n+6}, \eta^*\sigma] = [\iota_{16n+5}, \eta\eta^*\sigma] = 0.$$

4 Completion of the proof of Theorem 1

First we show:

Proposition 4.1 $[\iota_n, \sigma^2] \neq 0$ *for* $n \equiv 4, 5\ (8)$ *or* $n \equiv 0, 1, 3\ (16)$.

Proof By Lemma 3.1, we can set $[\iota_n, \sigma^2] = E^7\delta$ and $H\delta = \sigma^3$. Let $n \equiv 0\ (16)$. By [1], $[\iota_{n-1}, \iota]$ desuspends seven dimensions. So, $\mathcal{ASM}[\sigma^2]$ implies $E^5\delta \in P\pi^{2n-3}_{2n+13} \subset E^6\pi^{n-8}_{2n+5}$ (2–10) and $E^4\delta \in P\pi^{2n-5}_{2n+12}$. By Lemma 2.13(2), $[\iota_{n-3}, \alpha]$ for $\alpha = \eta\eta^*, \eta^2\rho, \nu\kappa$ desuspends five dimensions. Hence, by the relation $H(E^{-1}[\iota_{n-3}, \overline{\mu}]) = \eta\overline{\mu}$, we have $E^3\delta \in \{[\iota_{n-4}, 4\nu^*], [\iota_{n-4}, \eta\overline{\mu}]\}$. By Lemma 2.9(1), $[\iota_{n-4}, 4\nu^*]$ desuspends four dimensions. Therefore, by the relation $H(E^{-1}[\iota_{n-4}, \eta\overline{\mu}]) = 4\overline{\zeta}$ (Proposition 2.5(2)), $E^2\delta \in \{[\iota_{n-5}, \overline{\zeta}], [\iota_{n-5}, \overline{\sigma}]\} \subset E^3\pi^{n-8}_{2n+5}$ (Proposition 2.11(2)). Hence, $E\delta \in \{[\iota_{n-6}, 4\overline{\kappa}]\} \subset E^2\pi^{n-8}_{2n+5}$ (Proposition 2.6(1)), $\delta \in P\pi^{2n-13}_{2n+8}$ and $\mathcal{CDR}[\sigma^3 = 0]$.

Let $n \equiv 1\ (16)$. $\mathcal{ASM}[\sigma^2]$ implies $E^6\delta \in \{[\iota_{n-1}, \eta\kappa], [\iota_{n-1}, 16\rho]\}$. By Lemma 2.9(2), $[\iota_{n-1}, 16\rho]$ desuspends eight dimensions and $E^5\delta \pmod{E\beta} \in P\pi^{2n-3}_{2n+13} = 0$ for $\beta = E^{-2}[\iota_{n-1}, \eta\kappa]$. So, by ($\diamond$) $E^4\delta \in P\pi^{2n-5}_{2n+12} \subset E^6\pi^{n-9}_{2n+4}$ (Lemma 2.10), $E^3\delta \in P\pi^{2n-7}_{2n+11} \subset E^4\pi^{n-8}_{2n+5}$ (Lemma 2.13(1)), $E^2\delta \in \{[\iota_{n-5}, 4\overline{\zeta}], [\iota_{n-5}, \overline{\sigma}]\} \subset E^3\pi^{n-8}_{2n+5}$ (Proposition 2.6(3)), $E\delta \in \{[\iota_{n-6}, \overline{\kappa}]\} \subset E^3\pi^{n-9}_{2n+4}$ (Proposition 2.11(2)) and hence, $\mathcal{CDR}[\delta \in P\pi^{2n-13}_{2n+8}]$.

Let $n \equiv 3\ (16)$. $\mathcal{ASM}[\sigma^2]$ implies $E^6\delta \in \{[\iota_{n-1}, 16\rho]\}$. Since $H(E^{-2}[\iota_{n-1}, 16\rho]) = \overline{\mu}$ by Proposition 2.6(1), $E^5\delta \pmod{E\beta_1} \in P\pi^{2n-3}_{2n+13} = \{E(\tau_{n-3}\eta\rho), E(\tau_{n-3}\eta^*)\}$ for $\beta_1 = E^{-2}[\iota_{n-1}, 16\rho]$. So, $E^4\delta \in \{[\iota_{n-3}, \alpha]\}$ for $\alpha \in \pi^s_{17}$. We obtain $H(E^{-1}[\iota_{n-3}, \overline{\mu}]) = \eta\overline{\mu}$ (Proposition 2.5(1)), $H(E^{-1}[\iota_{n-3}, \eta\eta^*]) = \eta^2\eta^*$ (Proposition 2.5(2)), $H(E^{-2}[\iota_{n-3}, \eta^2\rho]) = x\overline{\zeta}$ (x : odd) (Proposition 2.6(2)) and $H(E^{-3}[\iota_{n-3}, \nu\kappa]) = \nu^2\kappa$ (Lemma 2.9(1)). This induces $E^3\delta \pmod{E\beta_2, E^2\delta_1} \in P\pi^{2n-7}_{2n+11} = 0$ for $\beta_2 = E^{-2}[\iota_{n-3}, \eta^2\rho]$ and $\delta_1 = E^{-3}[\iota_{n-3}, \nu\kappa]$. Hence, by (1–2) for $\overline{\zeta}$, $E^2\delta \pmod{E\delta_1} \in \{[\iota_{n-5}, \overline{\sigma}]\} \subset E^6\pi^{n-11}_{2n+2}$ (Lemma 2.10), $E\delta \in \{E(\tau_{n-7}\overline{\kappa})\}$ and $\mathcal{CDR}[\delta \pmod{\tau_{n-7}\overline{\kappa}} \in P\pi^{2n-13}_{2n+8}]$.

Let $n \equiv 4\ (8)$. Lemma 3.3 and $\mathcal{ASM}[\sigma^2]$ imply $E^3\delta \in P\pi^{2n-1}_{2n+14} = \{[\iota_{n-1}, \rho]\} \subset E^4\pi^{n-5}_{2n+8}$ (Proposition 2.11(1)), for $\delta = \delta(\nu^*) = E^{-4}[\iota_n, \sigma^2]$. By Proposition 2.6(1), $H(E^{-2}[\iota_{n-2}, \eta^*]) = 2\nu^*$ and $[\iota_{n-2}, \eta\rho]$ desuspends three dimensions. This induces $E\delta \pmod{E\delta_1} \in \{[\iota_{n-3}, \alpha]\}$, where $\delta_1 = \delta(2\nu^*) = E^{-2}[\iota_{n-2}, \eta^*]$ and $\alpha = \eta^2\rho, \eta\eta^*, \nu\kappa, \overline{\mu}$. Hence, $\delta \pmod{\delta_1, \tau_{n-4}\alpha} \in \{[\iota_{n-4}, \nu^*], [\iota_{n-4}, \eta\overline{\mu}]\}$ and $\mathcal{CDR}[\nu^* \pmod{\eta\overline{\mu}} \in \{2\nu^*\}]$.

Let $n \equiv 5\ (8)$. Lemma 3.3 and $\mathcal{ASM}[\sigma^2]$ induce $E^4\delta \in \{[\iota_{n-1}, \eta\kappa], [\iota_{n-1}, 16\rho]\}$, where $\delta = \delta(\overline{\sigma}) = E^{-5}[\iota_n, \sigma^2]$. By ($\diamond$) and Lemma 3.2(1), $E^3\delta \pmod{E\delta_1, E^3\delta_2} \in P\pi^{2n-3}_{2n+13} = 0$ and $E^2\delta \pmod{E^2\delta_2} \in \{[\iota_{n-3}, \nu\kappa], [\iota_{n-3}, \overline{\mu}]\}$, where $\delta_1 = \delta(\nu\kappa) = E^{-2}[\iota_{n-1}, \eta\kappa]$ and $\delta_2 = \delta(\overline{\zeta}) = E^{-4}[\iota_{n-1}, 16\rho]$. By Proposition 2.6(1), $[\iota_{n-3}, \nu\kappa]$ desuspends three dimensions and $H(E^{-2}[\iota_{n-3}, \overline{\mu}]) = 2\overline{\zeta}$. Hence, for $\delta_3 = \delta(2\overline{\zeta}) = E^{-2}[\iota_{n-3}, \overline{\mu}]$, we have $E\delta \pmod{E\delta_2, E\delta_3} \in P\pi^{2n-7}_{2n+11} = \{E(\tau_{n-5}\nu^*), E(\tau_{n-5}\eta\overline{\mu})\}$, $\delta \pmod{\delta_2, \delta_3, \tau_{n-5}\nu^*, \tau_{n-5}\eta\overline{\mu}} \in P\pi^{2n-9}_{2n+10}$ and $\mathcal{CDR}[\overline{\sigma} \pmod{\overline{\zeta}} \in \{2\overline{\zeta}\}]$. □

Next we show the following:

Proposition 4.2 *$H(E^{-3}[\iota_{8n}, \nu^2\kappa]_{\neq 0}) = 4\nu\bar{\kappa}$ and $H(E^{-3}[\iota_{8n+2}, \nu\kappa]_{\neq 0}) = 4\bar{\kappa}$.*

Proof Let $n \equiv 0$ (8). By Lemma 2.9(1) and (2–6), $H(E^{-3}[\iota_n, \nu^2\kappa]) = \nu^3\kappa = 4\nu\bar{\kappa}$ ($\delta\kappa = E^{-3}[\iota_n, \nu^2\kappa]$) for $\delta = E^{-3}[\iota_n, \nu^2]$. Then, $\mathcal{ASM}[\nu^2\kappa]$ induces $E^2(\delta\kappa) \in P\pi^{2n-1}_{2n+20} = 0$, $E(\delta\kappa) \in P\pi^{2n-3}_{2n+19} = \{[\iota_{n-2}, \nu\bar{\sigma}]\} \subset E^6\pi^{n-8}_{2n+11}$ (Lemma 2.10), and hence $\delta\kappa \in P\pi^{2n-5}_{2n+18}$ and $\mathcal{CDR}[4\nu\bar{\kappa} = 0]$.

Next, let $n \equiv 2$ (8). By Lemma 2.12, there exists an element $\delta \in \pi^{n-3}_{2n+13}$ such that $[\iota_n, \nu\kappa] = E^3\delta$ and $H\delta = \nu^2\kappa$. Hence, $\mathcal{ASM}[\nu\kappa]$ and (2–14) induce $E\delta \in \{[\iota_{n-2}, 4\bar{\zeta}], [\iota_{n-2}, \bar{\sigma}]\} \subset E^2\pi^{n-4}_{2n+12}$ (Proposition 2.6(3)) and $\mathcal{CDR}[\delta \in P\pi^{2n-5}_{2n+15} = 0]$. □

By Propositions 2.11(3), 4.2 and the properties of Whitehead products,

$$\sharp[\iota_{8n}, \bar{\kappa}] = 8 \quad \text{and} \quad \sharp[\iota_{8n}, \nu\kappa] = \sharp[\iota_{8n+3}, \nu\kappa] = \sharp[\iota_{8n+2}, \kappa] = 2.$$

We show:

Proposition 4.3 *$\sharp[\iota_{8n+6}, \kappa] = \sharp[\iota_{8n+5}, \eta\kappa] = \sharp[\iota_{8n+4}, \nu\kappa] = 2$.*

Proof Let $n \equiv 6$ (8). Lemma 3.4(5) and $\mathcal{ASM}[\kappa]$ imply $E^5\delta \in \{[\iota_{n-1}, \rho], [\iota_{n-1}, \eta\kappa]\}$ for $\delta = \delta(\bar{\kappa}) = E^{-6}[\iota_n, \kappa]$. By the relation $H(\tau_{n-2}\rho) = \eta\rho$ and Lemma 3.6(1), $E^4\delta \in \{[\iota_{n-2}, \eta\rho], [\iota_{n-2}, \eta^*]\}$. By Proposition 2.5(1),

$$(\star) \qquad H(E^{-1}[\iota_{n-2}, \eta\rho]) = \eta^2\rho; \; H(E^{-1}[\iota_{n-2}, \eta^*]) = \eta\eta^*.$$

Therefore, $E^3\delta \in P\pi^{2n-5}_{2n+12} = \{E^3(\bar{\tau}_{n-6}\nu\kappa)\}$. By the fact that $[\iota_{n-4}, \eta\bar{\mu}] = [\iota_{n-4}, \eta^2\eta^*] = 0$ and (2–14), $E^2\delta \pmod{E^2(\bar{\tau}_{n-6}\nu\kappa)} = 0$, $E\delta \pmod{E(\bar{\tau}_{n-6}\nu\kappa)} \in P\pi^{2n-9}_{2n+10} \subset E^3\pi^{n-8}_{2n+5}$, $\delta \pmod{\bar{\tau}_{n-6}\nu\kappa} \in \{[\iota_{n-6}, \bar{\kappa}]\}$ and hence, $\mathcal{CDR}[\bar{\kappa} \in \{2\bar{\kappa}\}]$.

Let $n \equiv 5$ (8). Lemma 3.6(1) and $\mathcal{ASM}[\eta\kappa]$ imply $E^5\delta \in \{[\iota_{n-1}, \eta\rho], [\iota_{n-1}, \eta^*]\}$ for $\delta = \delta(\eta\bar{\kappa}) = E^{-6}[\iota_n, \eta\kappa]$. By ($\star$), $E^4\delta \in P\pi^{2n-3}_{2n+14} = \{[\iota_{n-2}, \nu\kappa]\} \subset E^5\pi^{n-7}_{2n+7}$ (Lemma 3.2(2)), $E^3\delta \in \{[\iota_{n-3}, 4\nu^*], [\iota_{n-3}, \eta\bar{\mu}]\} = 0$, $E^2\delta \in P\pi^{2n-7}_{2n+12} \subset E^3\pi^{n-7}_{2n+7}$ (2–14) and $E\delta \in \{[\iota_{n-5}, 4\bar{\kappa}]\} \subset E^3\pi^{n-10}_{2n+9}$ (Proposition 2.6(3)). Hence, $\mathcal{CDR}[\delta \in P\pi^{2n-11}_{2n+10} = 0]$.

Let $n \equiv 4$ (8). $E^5\delta \in \{E^3(\bar{\tau}_{n-4}\nu^*)\}$ for $\delta = \delta(\nu\bar{\kappa}) = E^{-6}[\iota_n, \nu\kappa]$. By the relation $H(E^{-3}[\iota_{n-2}, \bar{\sigma}]) = \nu\bar{\sigma}$ (Lemma 2.12) and (1–2) for $\bar{\zeta}$, $E^4\delta \pmod{E^2(\bar{\tau}_{n-4}\nu^*)} \in \{E^3\delta_1\}$ and $E^3\delta \pmod{E(\bar{\tau}_{n-4}\nu^*), E^2\delta_1} \in \{E(\tau_{n-4}\bar{\kappa})\}$, where $\delta_1 = \delta(\nu\bar{\sigma}) = E^{-3}[\iota_{n-2}, \bar{\sigma}]$. From the relations $H(\bar{\tau}_{n-4}\nu^*) = \sigma^3$, $H(\tau_{n-4}\bar{\kappa}) = \eta\bar{\kappa}$ and $H(E^{-1}[\iota_{n-4}, \eta\bar{\kappa}]) = \eta^2\bar{\kappa}$ (Proposition 2.5(1)), we obtain $E^2\delta \pmod{E\delta_1} \in \{[\iota_{n-4}, \sigma^3]\} \subset E^7\pi^{n-11}_{2n+5}$ (Lemma 2.9(3)), $E\delta \in P\pi^{2n-9}_{2n+13} = 0$, $\delta \in P\pi^{2n-11}_{2n+12}$ and hence, $\mathcal{CDR}[\nu\bar{\kappa} \in 2\pi^s_{23}]$. □

Since $[\iota_{8n+4}, \nu^2] = 0$, $[\iota_{8n+6}, \nu\kappa] = 0$ (2–11) and $H[\iota_{2n}, \overline{\kappa}] = \pm 2\overline{\kappa}$, we have

$$\sharp[\iota_{8n+k}, \overline{\kappa}] = 4 \text{ for } k = 4, 6.$$

Similarly, $$\sharp[\iota_{8n+4}, \nu\overline{\kappa}] = 4.$$

Now, we show:

Proposition 4.4 $\sharp[\iota_{8n+2}, \eta^*] = \sharp[\iota_{8n+1}, \nu^*] = \sharp[\iota_{8n}, 4\nu^*] = 2.$

Proof Let $n \equiv 2$ (8). By (2–7) and Lemma 3.4(2);(3), $[\iota_{n-1}, \alpha] \in E^6\pi^{n-7}_{2n+8}$ for $\alpha = \nu\kappa, \eta^2\rho$ and $\eta\eta^*$. So, $\mathcal{ASM}[\eta^*]$ induces $E^4\delta \in \{E(\tau_{n-2}\overline{\mu})\}$ and $E^3\delta \in \{[\iota_{n-2}, 4\nu^*], [\iota_{n-2}, \eta\overline{\mu}]\}$ for $\delta = \delta(\sigma^3) = E^{-5}[\iota_n, \eta^*]$. By the fact that $H(E^{-1}[\iota_{n-2}, \eta\overline{\mu}]) = 4\overline{\zeta}$ (Proposition 2.5(2)) and $[\iota_{n-2}, 4\nu^*] \in E^6\pi^{n-8}_{2n+7}$ (Lemma 3.5(2)), $E^2\delta \in P\pi^{2n-7}_{2n+12} = 0$, $E\delta \in \{[\iota_{n-4}, 4\overline{\kappa}]\} = 0$ and $\mathcal{CDR}[\delta \in P\pi^{2n-9}_{2n+12}]$.

Let $n \equiv 1$ (8). Lemma 2.13(1) and $\mathcal{ASM}[\nu^*]$ imply $E^2\delta \in \{[\iota_{n-1}, 4\overline{\zeta}], [\iota_{n-1}, \overline{\sigma}]\} \subset E^4\pi^{n-5}_{2n+12}$ (Lemma 2.9(1)), where $\delta = \delta(\sigma^3) = E^{-3}[\iota_n, \nu^*]$. Hence, $E\delta \in P\pi^{2n-3}_{2n+17} = 0$ and $\mathcal{CDR}[\delta \in P\pi^{2n-7}_{2n+14}]$.

Let $n \equiv 0$ (8). Lemma 3.5(2) and $\mathcal{ASM}[4\nu^*]$ imply $E^5\delta \in P\pi^{2n-1}_{2n+18} = 0$ and $E^4\delta \in \{[\iota_{n-2}, 4\overline{\kappa}]\} = 0$ (2–11) for $\delta = \delta(\eta\eta^*\sigma) = E^{-6}[\iota_n, 4\nu^*]$. Therefore, by (2–13), $E^3\delta \in \{E(\tau_{n-4}\eta\overline{\kappa})\}$ and $E^2\delta \in \{[\iota_{n-4}, \eta^2\overline{\kappa}], [\iota_{n-4}, \nu\overline{\sigma}]\}$. By the relation $H(E^{-1}[\iota_{n-4}, \eta^2\overline{\kappa}]) = 4\nu\overline{\kappa}$ (Proposition 2.5(2)) and (2–9), $E\delta \in \{[\iota_{n-5}, \nu\overline{\kappa}], [\iota_{n-5}, \overline{\rho}]\} \subset E^3\pi^{n-8}_{2n+9}$, $\delta \in P\pi^{2n-11}_{2n+13}$ and $\mathcal{CDR}[\eta\eta^*\sigma = 0]$. □

By Propositions 2.5(4) and 4.4,

$$[\iota_{8n+1}, \eta\eta^*] \neq 0.$$

We show:

Proposition 4.5 $\sharp[\iota_{16n+14}, \eta^*] = \sharp[\iota_{16n+13}, \eta\eta^*] = 2.$

Proof We use Lemma 3.6(4). Let $n \equiv 14$ (16). By Lemma 2.13(4), $[\iota_{n-1}, \eta^2\rho]$ desuspends seven dimensions. So, by the relation $[\iota_{n-1}, \overline{\mu}] = E(\tau_{n-2}\overline{\mu})$, (2–7) and $\mathcal{ASM}[\eta^*]$, $E^5\delta \in \{[\iota_{n-2}, 4\nu^*], [\iota_{n-2}, \eta\overline{\mu}]\}$ for $\delta = \delta(\eta^*\sigma) = E^{-7}[\iota_n, \eta^*]$. By the relation $H(E^{-1}[\iota_{n-2}, \eta\overline{\mu}]) = 4\overline{\zeta}$ and Lemma 3.5(2), $E^4\delta \in \{[\iota_{n-3}, \overline{\zeta}], [\iota_{n-3}, \overline{\sigma}]\}$. By the relation $\nu\overline{\zeta} = 0$ and Lemma 3.5(1), $E^3\delta \pmod{E^2(\overline{\tau}_{n-6}\overline{\sigma})} \in \{[\iota_{n-4}, 4\overline{\kappa}]\}$. By (3–1), $E^2\delta \pmod{E(\overline{\tau}_{n-6}\overline{\sigma}), E^2\delta_1} \in \{E(\tau_{n-6}\eta\overline{\kappa})\}$, where $\delta_1 = \delta(4\nu\overline{\kappa}) = [\iota_{n-4}, 4\overline{\kappa}]$. This induces $E\delta \pmod{E\delta_1} \in P\pi^{2n-11}_{2n+11} = \{E\delta_2, [\iota_{n-6}, \nu\overline{\sigma}]\}$, where $E\delta_2 = [\iota_{n-6}, \eta^2\overline{\kappa}]$, $H\delta_2 = 4\nu\overline{\kappa}$ and $[\iota_{n-6}, \nu\overline{\sigma}] \subset E^2\pi^{n-8}_{2n+7}$ (Proposition 2.5(1)). Hence, $\delta \pmod{\delta_1, \delta_2} \in P\pi^{2n-13}_{2n+10}$ and $\mathcal{CDR}[\eta^*\sigma \in 2\pi^s_{23}]$.

Next, let $n \equiv 13\ (16)$. $\mathcal{ASM}[\eta\eta^*]$ implies $E^6\delta \in \{[\iota_{n-1}, 4\nu^*], [\iota_{n-1}, \eta\overline{\mu}]\}$ for $\delta = \delta(\eta\eta^*\sigma) = E^{-7}[\iota_n, \eta\eta^*]$. By the relation $H(E^{-1}[\iota_{n-1}, \eta\overline{\mu}]) = \eta^2\overline{\mu}$ and Lemma 3.5(2), $E^5\delta \in \{[\iota_{n-2}, \overline{\zeta}], [\iota_{n-2}, \overline{\sigma}]\}$. By Lemmas 3.5(1) and 3.6(3), $E^4\delta \pmod{E^2(\overline{\tau}_{n-5}\overline{\sigma})} \in \{[\iota_{n-3}, 4\overline{\kappa}]\} \subset E^6\pi^{n-9}_{2n+7}$ and $E^3\delta \pmod{E(\overline{\tau}_{n-5}\overline{\sigma})} \in \{[\iota_{n-4}, \eta\overline{\kappa}], [\iota_{n-4}, \sigma^3]\}$. From the fact that $[\iota_{n-4}, \eta\overline{\kappa}] = E(\tau_{n-5}\eta\overline{\kappa})$ and (3–1), $E^2\delta \in \{[\iota_{n-5}, \eta^2\overline{\kappa}], [\iota_{n-5}, \nu\overline{\sigma}]\}$. Since $H(E^{-1}[\iota_{n-5}, \eta^2\overline{\kappa}]) = 4\nu\overline{\kappa}$ and $H(E^{-3}[\iota_{n-5}, \nu\overline{\sigma}]) = \nu^2\overline{\sigma} = 0$ (Lemma 2.9(1),[16]), $E\delta \in P\pi^{2n-11}_{2n+12} \subset E^7\pi^{n-13}_{2n+3}$ (1–1), $\delta \in P\pi^{2n-13}_{2n+11}$ and $\mathcal{CDR}[\eta\eta^*\sigma = 0]$. □

We show the following:

Proposition 4.6 $H(E^{-3}[\iota_{8n+k}, \overline{\sigma}]_{\neq 0}) = \nu\overline{\sigma}$ *for* $k = 0, 1, 2$.

Proof Let $n \equiv 0\ (8)$. By Lemmas 2.9(1), 2.12 and 2.13, there exists an element $\delta(k) \in \pi^{n+k-3}_{2n+2k+15}$ such that $[\iota_{n+k}, \overline{\sigma}] = E^3\delta(k)$ and $H\delta(k) = \nu\overline{\sigma}$. For $k = 0$, $\mathcal{ASM}[\overline{\sigma}]$ induces $E^2\delta(0) \in P\pi^{2n-1}_{2n+19} = 0$, $E\delta(0) \in P\pi^{2n-3}_{2n+18} \subset E^2\pi^{n-4}_{2n+14}$ (Proposition 2.6(1)) and $\mathcal{CDR}[\delta(0) \in P\pi^{2n-5}_{2n+17}]$. By the parallel argument to Proposition 4.4 for ν^*, the assertion follows for $k = 1$. For $k = 2$, $\mathcal{ASM}[\overline{\sigma}]$ induces $E^2\delta(2) \in \{E(\tau_n\overline{\kappa})\}$ and $E\delta(2) \in \{[\iota_n, \eta\overline{\kappa}], [\iota_n, \sigma^3]\}$. Since $[\iota_n, \sigma^3] \subset E^7\pi^{n-7}_{2n+13}$ (Lemma 2.9(3)), we obtain $\delta(2) \pmod{\beta} = 0$ and $\mathcal{CDR}[\nu\overline{\sigma} \pmod{\eta^2\overline{\kappa}} = 0]$, where $\beta = \delta(\eta^2\overline{\kappa}) = E^{-1}[\iota_n, \eta\overline{\kappa}]$ (Proposition 2.5(1)). □

We show the following:

Proposition 4.7 $H(E^{-5}[\iota_{8n+2}, \eta\overline{\kappa}]_{\neq 0}) = \nu^2\overline{\kappa}$ *and* $H(E^{-6}[\iota_{8n+1}, \eta^2\overline{\kappa}]_{\neq 0}) = \varepsilon\overline{\kappa}$.

Proof Let $n \equiv 2\ (8)$. By Lemma 3.4(1) and (2–6), $H(E^{-5}[\iota_n, \eta\overline{\kappa}]) = \nu^2\overline{\kappa}$. We set $\delta = \delta(\nu^2) = E^{-5}[\iota_n, \eta]$. $\mathcal{ASM}[\eta\overline{\kappa}]$ induces $E^4(\delta\overline{\kappa}) \in P\pi^{2n-1}_{2n+21} \subset E^5\pi^{n-6}_{2n+14}$ (Lemmas 2.13(3),3.4(2)) and $E^3(\delta\overline{\kappa}) \in \{[\iota_{n-2}, 4\nu\overline{\kappa}], [\iota_{n-2}, 8\overline{\rho}], [\iota_{n-2}, \eta^*\sigma]\}$. By Lemma 2.9(1), the first two Whitehead products desuspend four dimensions, respectively. Hence, by the relation $H(E^{-1}[\iota_{n-2}, \eta^*\sigma]) = \eta\eta^*\sigma$, we obtain $E^2(\delta\overline{\kappa}) = 0$, $E(\delta\overline{\kappa}) \in P\pi^{2n-7}_{2n+18} \subset E^2\pi^{n-6}_{2n+14}$ (Proposition 2.6(1)), $\delta\overline{\kappa} \in P\pi^{2n-9}_{2n+17}$ and $\mathcal{CDR}[\nu^2\overline{\kappa} = 0]$.

Next, let $n \equiv 1\ (8)$. By Lemma 3.4(2), $H(E^{-6}[\iota_n, \eta^2\overline{\kappa}]) = \varepsilon\overline{\kappa}$. $\mathcal{ASM}[\eta^2\overline{\kappa}]$ implies $E^5(\delta\overline{\kappa}) \in \{[\iota_{n-1}, 4\nu\overline{\kappa}], [\iota_{n-1}, 8\overline{\rho}], [\iota_{n-1}, \eta^*\sigma]\}$ for $\delta = \delta(\varepsilon) = E^{-6}[\iota_n, \eta^2]$. By Lemma 2.9(2), $[\iota_{n-1}, 8\overline{\rho}]$ desuspends eight dimensions. By Lemma 3.2(3), $[\iota_{n-1}, 4\nu\overline{\kappa}] = [\iota_{n-1}, \nu^3]\kappa$ desuspends six dimensions. So, by the relation $H(E^{-1}[\iota_{n-1}, \eta^*\sigma]) = \eta\eta^*\sigma$, we have $E^4(\delta\overline{\kappa}) \in P\pi^{2n-3}_{2n-27} = 0$, $E^3(\delta\overline{\kappa}) \in \{[\iota_{n-3}, \mu_{3,*}]\} \subset E^6\pi^{n-9}_{2n+12}$ (Lemma 2.10), $E^2(\delta\overline{\kappa}) \in \{[\iota_{n-4}, \eta\mu_{3,*}]\} \subset E^4\pi^{n-8}_{2n+13}$ (Lemma 2.13(1)), $E(\delta\overline{\kappa}) \in \{[\iota_{n-5}, 4\zeta_{3,*}]\} \subset E^3\pi^{n-8}_{2n+7}$ (Proposition 2.6(3)), $\delta\overline{\kappa} \in P\pi^{2n-11}_{2n+17}$ and hence, $\mathcal{CDR}[\varepsilon\overline{\kappa} = 0]$. □

According to Mahowald [8], the following seems to be true.

Conjecture 4.8 $\langle \nu, \eta, \overline{\sigma} \rangle = \langle \overline{\nu}, \sigma, \overline{\nu} \rangle = \eta\eta^*\sigma$.

By use of the Jacobi identity for Toda brackets, Conjecture 4.8 and the relations $\langle \eta, \nu, \eta \rangle = \nu^2, \sigma\overline{\sigma} = 0$ [16], we obtain

$$\langle 2\iota, \nu^2, \overline{\sigma} \rangle = \langle 2\iota, \eta, \eta\eta^*\sigma \rangle = \nu^2\overline{\kappa}.$$

By this fact, we can show

$$[\iota_{8n}, \nu\overline{\sigma}] \neq 0.$$

Proof Let $n \equiv 0$ (8). In Proposition 2.1[n-5;5,3], $\mathrm{P}^{n-1}_{n-5} = E^{n-8}\mathrm{P}^7_3$ and $\gamma_{7,3} = 2s_4 + i^7_3\widetilde{\eta}''$, where $s_4 = p^7_3 s_3$ (3–4). By Lemma 2.7(8),

$$\widetilde{\eta}'' \circ \nu\overline{\sigma} \in i^{4,6}_3 \circ \langle i\overline{\eta}, \widetilde{\eta}, \nu \rangle \circ \overline{\sigma} = i \circ \langle 2\iota, \nu^2, \overline{\sigma} \rangle = i\nu^2\overline{\kappa}.$$

This shows

$$H(E^{-4}[\iota_n, \nu\overline{\sigma}]) = \nu^2\overline{\kappa}.$$

For $\delta = \delta(\nu^2\overline{\kappa}) = E^{-4}[\iota_n, \nu\overline{\sigma}]$, $\mathcal{ASM}[\nu\overline{\sigma}]$ implies $E^3\delta = 0$ and $E^2\delta \in P\pi^{2n-3}_{2n+21}$ $\subset E^3\pi^{n-5}_{2n+16}$ (Proposition 2.6(1)), $E\delta \in \{[\iota_{n-3}, \eta^2\overline{\rho}], [\iota_{n-3}, \mu_{3,*}]\}$, $\delta \pmod{\tau_{n-4}\eta^2\overline{\rho}, \tau_{n-4}\mu_{3,*}} \in P\pi^{2n-7}_{2n+19}$ and hence, $\mathcal{CDR}[\nu^2\overline{\kappa} \pmod{\eta\mu_{3,*}} = 0]$. □

Finally, by Proposition 2.6(1) and Lemma 2.13(1), we note the following.

Remark $H(E^{-2}[\iota_{8n+2}, 4\overline{\kappa}]) = \varepsilon\kappa = \eta^2\overline{\kappa}$ and $H(E^{-3}[\iota_{8n+1}, \nu\overline{\kappa}]) = \nu^2\overline{\kappa}$.

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