Quasi-polynomials and the Bethe Ansatz

E Mukhin
A Varchenko

We study solutions of the Bethe Ansatz equation related to the trigonometric Gaudin model associated to a simple Lie algebra $\mathfrak{g}$ and a tensor product of irreducible finite-dimensional representations. Having one solution, we describe a construction of new solutions. The collection of all solutions obtained from a given one is called a population. We show that the Weyl group of $\mathfrak{g}$ acts on the points of a population freely and transitively (under certain conditions).

To a solution of the Bethe Ansatz equation, one assigns a common eigenvector (called the Bethe vector) of the trigonometric Gaudin operators. The dynamical Weyl group projectively acts on the common eigenvectors of the trigonometric Gaudin operators. We conjecture that this action preserves the set of Bethe vectors and coincides with the action induced by the action on points of populations. We prove the conjecture for sl_2.

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1 Introduction

The Bethe Ansatz is a method to diagonalize a commuting family of linear operators, usually called Hamiltonians. The method is applied to Hamiltonians of numerous quantum integrable systems. Given a solution of a suitable system of equations (called the Bethe Ansatz equation), the Bethe Ansatz produces an eigenvector (called the Bethe vector). This paper is motivated by the Bethe Ansatz method applied to the trigonometric Gaudin model, see Markov, Schechtman and the second author [5; 15], and Section 5.

For the case of the trigonometric Gaudin model the Bethe Ansatz equation and the Bethe vectors depend on an additional parameter, a generic $\mathfrak{g}$–weight λ. The Bethe Ansatz equation has the form (3). The Bethe vectors have the form (15), see Proposition 5.1. The Bethe Ansatz equation (3) can be formulated as a system of suitable Wronskian equations for a tuple of polynomials $\boldsymbol{y} = (y_1, \dots, y_r)$ of one variable, where r is the rank of $\mathfrak{g}$ and the polynomials are labeled by simple roots of $\mathfrak{g}$, see Theorem 3.5.

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For example, let $\mathfrak{g} = sl_2$. The sl_2–weights can be identified with complex numbers. Consider the trigonometric Gaudin model associated to the tensor product of irreducible sl_2–modules with highest weights Λ_j, located respectively at points z_j. In this case, the Bethe Ansatz equation with parameter $\lambda \in \mathbb{C}$ is an equation on one polynomial y. The polynomial y satisfies the Bethe Ansatz equation, if and only if its roots are simple and there exists another polynomial $\widetilde{y}$ such that

$$y'(x^{\lambda+1}\widetilde{y}) \;-\; y(x^{\lambda+1}\widetilde{y})' \;=\; x^{\lambda}\prod_j (x-z_j)^{\Lambda_j}\;.$$

For a given y and a non-integer λ, the polynomial $\widetilde{y}$ is unique. One can show that for almost all λ, the roots of $\widetilde{y}$ are simple. Moreover, if the roots of $\widetilde{y}$ are simple, then the polynomial $\widetilde{y}$ also satisfy the Bethe Ansatz equation but with the new parameter $-\lambda-2$. Thus from one solution of the Bethe Ansatz equation with parameter λ (the polynomial y) we obtain another solution with parameter $-\lambda-2$ (the polynomial $\widetilde{y}$). We call this procedure the simple reproduction procedure.

For an arbitrary simple Lie algebra $\mathfrak{g}$, there is a similar simple reproduction procedure associated with every simple root of $\mathfrak{g}$. Consider an r–tuple $\boldsymbol{y} = (y_1, \dots, y_r)$ of polynomials forming a solution of the Bethe Ansatz equation associated with a generic $\mathfrak{g}$–weight λ. Then we have the i-th simple reproduction procedure for $i = 1, \dots, r$. The i-th simple reproduction procedure constructs a new tuple $\boldsymbol{y}^{(i)} = (y_1, \dots, y_{i-1}, \widetilde{y}_i, y_{i+1}, \dots, y_r)$ under certain conditions.

We call an r–tuple of polynomials $\boldsymbol{y} = (y_1, \dots, y_r)$ fertile with respect to λ if the i-th simple reproduction procedure is well-defined for $i = 1, \dots, r$. In particular, if $\boldsymbol{y}$ forms a solution of the Bethe Ansatz equation associated to λ, then $\boldsymbol{y}$ is fertile with respect to λ. Moreover, if the i-th simple reproduction procedure results in a generic (in an appropriate sense) r–tuple $\boldsymbol{y}^{(i)}$, then $\boldsymbol{y}^{(i)}$ also forms a solution of the Bethe Ansatz equation associated to the weight $s_i \cdot \lambda$, where s_i is the i-th elementary reflection in the Weyl group of $\mathfrak{g}$. It follows that the r–tuple $\boldsymbol{y}^{(i)}$ is fertile with respect to $s_i \cdot \lambda$.

We call an r–tuple of polynomials $\boldsymbol{y}$ super-fertile with respect to λ if all iterations of the simple reproduction procedures are well defined. We conjecture that if $\boldsymbol{y}$ forms a solution of the Bethe Ansatz equation then $\boldsymbol{y}$ is super-fertile. We prove the conjecture for simple Lie algebras of type A_r, B_r.

The set of all r–tuples obtained from a given super-fertile r–tuple by iterations of simple reproduction procedures is called a population.

For simple Lie algebras, we prove that the population obtained from a super-fertile r–tuple associated to a generic weight λ contains exactly one r–tuple associated to every weight of the form $w \cdot \lambda$, where w runs through the elements of the Weyl group of $\mathfrak{g}$. We also prove that the population does not contain any other r–tuples. This one-to-one correspondence between the tuples of the population and the weights of the form $w \cdot \lambda$, allows us to introduce a free and transitive action of the Weyl group on points of the population. Then the action of simple reflections is given by the simple reproduction procedures. The proof is based on the important fact that in the case of A_r, the populations are in one-to-one correspondence with certain spaces of quasi-polynomials, see Corollary 4.4.

If all elements of a population are generic and therefore correspond to solutions of the Bethe Ansatz equation, then we have an action of the Weyl group on the set of the solutions. In particular this defines an action of the Weyl group on the set of the associated Bethe vectors, considered up to proportionality.

On the other hand, the dynamical Weyl group commutes with the trigonometric Gaudin operators and projectively acts on eigenvectors of the trigonometric Gaudin operators, see Tarasov and the second author [18] and Lemma 5.5.

We conjecture that the action of the dynamical Weyl group maps the Bethe vectors to (scalar multiples of) the Bethe vectors and moreover, the two actions on the Bethe vectors coincide. We prove this conjecture for $\mathfrak{g} = sl_2$.

The reproduction procedure exists for solutions of the Bethe Ansatz equations associated with many quantum integrable models. In Sections 6, 7 we give two other examples of the situation in which the corresponding Bethe Ansatz equation admits a reproduction procedure and prove that the elements of the corresponding populations are also labeled by the elements of the Weyl group. These examples are related to the quasi-periodic Gaudin and XXX models. In joint work with Tarasov [7], we apply the results of Sections 6 and 7 to study the Bethe Ansatz of (gl_N, gl_M)–dual quasiperiodic Gaudin and XXX models.

The notions of the reproduction procedure and populations for the Bethe Ansatz equation of the Gaudin and XXX–type models were introduced in our papers [9; 13], see also [8; 10]. The populations in that situation are shown to be isomorphic to the flag variety of the Langlands dual algebra $\mathfrak{g}^\vee$ for all simple Lie algebras in the case of the Gaudin model by Frenkel [3] and the authors [11], and for the XXX–type model by the authors [10].

The paper is constructed as follows. Section 2 contains notation and definitions. In Section 3 we define the reproduction procedure and populations. In Section 4 we prove that for simple Lie algebras, the Weyl group acts freely and transitively on the elements of a population. In Section 5 we discuss two actions of the Weyl group on the Bethe vectors, the one given by the action of the dynamical Weyl groups and the one given by the reproduction procedure. We compare them in the case of sl_2. In Sections 6 and 7 we describe two more examples of the situation in which the Bethe Ansatz admit a reproduction procedure and the the Weyl group acts freely and transitively on the elements of a population.

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2 Master functions and critical points

2.1 Kac–Moody algebras

Let $A=(a_{ij})_{i,j=1}^r$ be a generalized Cartan matrix, $a_{ii}=2$, $a_{ij}=0$ if and only $a_{ji}=0$, $a_{ij}\in\mathbb{Z}_{\le 0}$ if $i\neq j$. We assume that A is symmetrizable, there is a diagonal matrix $D=\operatorname{diag}\{d_1,\dots,d_r\}$ with positive integers d_i such that $B=DA$ is symmetric.

Let $\mathfrak{g}=\mathfrak{g}(A)$ be the corresponding complex Kac–Moody Lie algebra (see Kac [4, Section 1.2]), $\mathfrak{h}\subset\mathfrak{g}$ the Cartan subalgebra. The associated scalar product is non-degenerate on $\mathfrak{h}^*$ and $\dim\mathfrak{h}=r+2d$, where d is the dimension of the kernel of the Cartan matrix A.

Let $\alpha_i\in\mathfrak{h}^*$, $\alpha_i^\vee\in\mathfrak{h}$, $i=1,\dots,r$, be the sets of simple roots, coroots, respectively. We have

$$\begin{aligned}&(\alpha_i,\alpha_j)=d_i\,a_{ij},\\&\langle\lambda,\alpha_i^\vee\rangle=2(\lambda,\alpha_i)/(\alpha_i,\alpha_i),\qquad \lambda\in\mathfrak{h}^*.\end{aligned}$$

Let $\mathcal{P}=\{\lambda\in\mathfrak{h}^*\mid\langle\lambda,\alpha_i^\vee\rangle\in\mathbb{Z}\}$ and $\mathcal{P}^+=\{\lambda\in\mathfrak{h}^*\mid\langle\lambda,\alpha_i^\vee\rangle\in\mathbb{Z}_{\ge 0}\}$ be the sets of integral and dominant integral weights.

Fix $\rho\in\mathfrak{h}^*$ such that $\langle\rho,\alpha_i^\vee\rangle=1$, $i=1,\dots,r$. We have $(\rho,\alpha_i)=(\alpha_i,\alpha_i)/2$.

The Weyl group $\mathcal{W}\in\operatorname{End}(\mathfrak{h}^*)$ is generated by reflections s_i, $i=1,\dots,r$,

$$s_i(\lambda)=\lambda-\langle\lambda,\alpha_i^\vee\rangle\alpha_i,\qquad \lambda\in\mathfrak{h}^*.$$

We use the notation

$$w\cdot\lambda=w(\lambda+\rho)-\rho,\qquad w\in\mathcal{W},\ \lambda\in\mathfrak{h}^*,$$

for the shifted action of the Weyl group.

2.2 The definition of master functions and critical points

We fix a Kac–Moody algebra $\mathfrak{g}=\mathfrak{g}(A)$. We fix $\boldsymbol{\Lambda}=(\Lambda_i)_{i=1}^n$, $\Lambda_i\in\mathcal{P}^+$; $\boldsymbol{z}=(z_i)_{i=1}^n\in\mathbb{C}^n$. We assume $z_i\neq 0$ and $z_i\neq z_j$ if $i\neq j$. The parameters $\boldsymbol{\Lambda},\boldsymbol{z}$ are always fixed and we often do not stress the dependence of our objects on these parameters.

In addition we choose $\lambda\in\mathfrak{h}^*$ and $\boldsymbol{l}=(l_1,\dots,l_r)\in\mathbb{Z}^r_{\geq 0}$. The choice of $\boldsymbol{l}$ is equivalent to the choice of the weight at infinity Λ_∞ defined by the formula:

$$\Lambda_\infty = \sum_{s=1}^n \Lambda_s - \sum_{i=1}^r l_i\alpha_i \in \mathcal{P}. \tag{1}$$

The *master function* $\Phi(\boldsymbol{t};\Lambda_\infty,\lambda)$ is defined as follows (see Felder, Schechtman and the second author [2; 15]):

$$\Phi(\boldsymbol{t};\Lambda_\infty,\lambda)=\prod_{i=1}^r\prod_{j=1}^{l_i}(t_j^{(i)})^{-(\lambda,\alpha_i)}\ \prod_{i=1}^r\prod_{j=1}^{l_i}\prod_{s=1}^n(t_j^{(i)}-z_s)^{-(\Lambda_s,\alpha_i)}\times \tag{2}$$

$$\prod_{i=1}^r\ \prod_{1\leq j<s\leq l_i}(t_j^{(i)}-t_s^{(i)})^{(\alpha_i,\alpha_i)}\ \prod_{1\leq i<j\leq r}\ \prod_{s=1}^{l_i}\prod_{k=1}^{l_j}(t_s^{(i)}-t_k^{(j)})^{(\alpha_i,\alpha_j)}.$$

The master function Φ is a function of variables $\boldsymbol{t}=(t_j^{(i)})_{i=1,\dots,r}^{j=1,\dots,l_i}$.

The master function Φ is symmetric with respect to permutations of variables with the same upper index.

A point $\boldsymbol{t}$ with complex coordinates is called a *critical point* associated to $(\boldsymbol{\Lambda},\boldsymbol{z};\Lambda_\infty,\lambda)$ (we often write just (Λ_∞,λ)) if the following system of algebraic equations is satisfied

$$-\frac{(\lambda,\alpha_i)}{t_j^{(i)}}-\sum_{s=1}^n\frac{(\Lambda_s,\alpha_i)}{t_j^{(i)}-z_s}+\sum_{s,\ s\neq i}\sum_{k=1}^{l_s}\frac{(\alpha_s,\alpha_i)}{t_j^{(i)}-t_k^{(s)}}+\sum_{s,\ s\neq j}\frac{(\alpha_i,\alpha_i)}{t_j^{(i)}-t_s^{(i)}}=0, \tag{3}$$

where $i=1,\dots,r$, $j=1,\dots,l_i$. In other words, a point $\boldsymbol{t}$ is a critical point if

$$\left(\Phi^{-1}\frac{\partial\Phi}{\partial t_j^{(i)}}\right)(\boldsymbol{t};\Lambda_\infty,\lambda)=0,\qquad i=1,\dots,r,\ j=1,\dots,l_i.$$

Note that the product of symmetric groups $S_{\boldsymbol{l}}=S_{l_1}\times\cdots\times S_{l_r}$ acts on the critical set of the master function permuting the coordinates with the same upper index. All orbits have the same cardinality $l_1!\dots l_r!$. We make no distinction between critical points in the same orbit.

The system of equations (3) coincides with the Bethe Ansatz equation of the non-homogeneous Gaudin model, see Reshetikhin and the second author [14] and Proposition 5.1.

Lemma 2.1 *For almost all values of λ, the set of critical points associated to $(\Lambda_\infty, \lambda)$ is finite.*

Proof The lemma follows from [9, Lemma 2.1]. □

3 Populations

3.1 The sl_2 populations

In the case of sl_2 the dominant integral weights are identified with non-negative integers and the system of equations (3) takes the form:

$$-\frac{\lambda}{t_j} - \sum_{s=1}^{n} \frac{\Lambda_s}{t_j - z_s} + \sum_{k,\, k\neq j}^{l} \frac{2}{t_j - t_k} = 0, \tag{4}$$

$j = 1, \dots, l$, where $\Lambda_s \in \mathbb{Z}_{>0}$ and $\lambda \in \mathbb{C}$. The weight Λ_∞ is given by $\Lambda_\infty = \sum_{s=1}^{n} \Lambda_s - 2l$.

We set

$$T(x) = \prod_{s=1}^{n} (x - z_s)^{\Lambda_s}, \qquad F = x \prod_{s=1}^{n} (x - z_s).$$

Given a tuple $\boldsymbol{t} = (t_1, \dots, t_l)$ we represent it by the polynomial $y(x) = \prod_{j=1}^{l} (x - t_j)$. We are interested in the zeros of $y(x)$ and therefore we make no distinction between $y(x)$ and $c y(x)$, where c is a non-zero constant.

A polynomial $y(x)$ is called *off-diagonal* with respect to $(\boldsymbol{\Lambda}, \boldsymbol{z})$ if $y(x)$ has only simple roots, $y(0) \neq 0$ and $y(z_s) \neq 0$ for all $s = 1, \dots, n$ such that $\Lambda_s \neq 0$.

Since $(\boldsymbol{\Lambda}, \boldsymbol{z})$ is fixed, we often call polynomials $y(x)$ off-diagonal with respect to $(\boldsymbol{\Lambda}, \boldsymbol{z})$ simply off-diagonal.

Lemma 3.1 (T J Stieltjes [17, Section 6.81]) *A polynomial y of degree l represents an sl_2 critical point associated to $(\Lambda_\infty, \lambda)$ if and only if y is off-diagonal and there exists a polynomial $C(x)$ such that*

$$F(x)\, y'' - F(x)\, \ln'(x^\lambda T(x))\, y'(x) + C(x)\, y(x) = 0. \tag{5}$$

Proof Equation (4) can be reformulated as the statement that the function $y'' - \ln'(x^\lambda T)y'$ equals zero at $x = t_j$ for all j. Therefore (4) is equivalent to the divisibility of the polynomial $Fy'' - F\ln'(x^\lambda T)y'$ by $y(x)$. $\square$

Note that the coefficients of y'', y', and y in (5) are polynomials of degree $n+1, n$, and at most $n-1$, respectively.

Theorem 3.2 *Let y represent an sl_2 critical point associated to $(\mathbf{\Lambda}, \mathbf{z}; \Lambda_\infty, \lambda)$. Then equation (5) has a solution of the form $x^{\lambda+1}\widetilde{y}(x)$ where $\widetilde{y}(x)$ is a polynomial. If λ is not a negative integer, then such a polynomial $\widetilde{y}(x)$ is unique.*

Moreover, there exists a finite set $C(\mathbf{\Lambda}, \mathbf{z}; \Lambda_\infty) \subset \mathbb{C}$ such that if λ is not a negative integer, $\lambda \notin C(\mathbf{\Lambda}, \mathbf{z}; \Lambda_\infty)$, and y represents an sl_2 critical point associated to $(\mathbf{\Lambda}, \mathbf{z}; \Lambda_\infty, \lambda)$ then polynomial $\widetilde{y}(x)$ represents an sl_2 critical point associated to $(\mathbf{\Lambda}, \mathbf{z}; -\Lambda_\infty, -\lambda-2)$.

We call $\widetilde{y}(x)$ *the immediate descendent of $y(x)$ with respect to λ.*

Proof Equation (5) is a Fuchsian differential equation, with singular points at $0, z_1, \dots, z_n, \infty$. At 0 the exponents of the equation are $0, \lambda+1$. Therefore, around 0 there is a solution of the form $u(x) = x^{\lambda+1}\widetilde{y}(x)$ where $\widetilde{y}(x)$ is a function holomorphic and non-vanishing at $x = 0$. Such a solution is unique if $\lambda + 1 \notin \mathbb{Z}_{\le 0}$.

At a point z_s the exponents are $(0, \Lambda_s + 1)$. Since $y(z_s) \neq 0$ and y is a polynomial solution, there is no monodromy around z_s, and thus $\widetilde{y}(x)$ is an entire function, cf Scherbak [16, Lemma 7]. The function $\widetilde{y}(x)$ is a polynomial since equation (5) is Fuchsian.

Denote
$$W(f,g) = f'g - fg'$$
the Wronskian of functions f and g. We have

$$W(y, x^{\lambda+1}\widetilde{y}(x)) = x^\lambda T(x). \tag{6}$$

Thus
$$\deg y + \deg \widetilde{y} = \sum_{s=1}^{n} \Lambda_s .$$

The polynomial $\widetilde{y}(x)$ satisfies the equation
$$F(x)\,\widetilde{y}'' - F(x)\,\ln'(x^{-\lambda-2}T(x))\,\widetilde{y}'(x) + \widetilde{C}(x)\,\widetilde{y}(x) = 0.$$

Thus if $\widetilde{y}(x)$ is off-diagonal, then the polynomial $\widetilde{y}(x)$ represents a critical point associated to $(\mathbf{\Lambda}, \mathbf{z}; -\Lambda_\infty, -\lambda-2)$.

Finally, we prove that for all but finitely many λ, the polynomial $\tilde{y}(x)$ is off-diagonal. If $\tilde{y}(x)$ is not off-diagonal, then $\tilde{y}(x)$ has zero of order $\Lambda_s + 1$ at at least one of z_s. We show that such a pair $(y(x), \tilde{y}(x))$ is possible for at most finitely many λ.

Consider a family of polynomials $y_\lambda(x) = \prod_j^l (x - t_{j,\lambda})$ which algebraically depends on λ and such that $y_\lambda(x)$ represents a critical point corresponding to $(\Lambda_\infty, \lambda)$ for all but finitely many λ.

We have a finite number of such families and for all but finitely many λ every polynomial representing a critical point belongs to such a family.

Let $\tilde{y}_\lambda(x) = \prod_{j=1}^{\tilde{l}} (x - \tilde{t}_{j,\lambda})$ be the descendent polynomial of $y_\lambda(x)$.

From (6) we get

$$\frac{\lambda+1}{x} + \sum_{i=1}^{\tilde{l}} \frac{1}{x - \tilde{t}_{j,\lambda}} - \sum_{i=1}^{l} \frac{1}{x - t_{j,\lambda}} = \frac{(\lambda + \tilde{l} - l)\ \prod_{j=1}^n (x - z_j)^{\Lambda_j}}{x \prod_{i=1}^{\tilde{l}} (x - \tilde{t}_{j,\lambda})\ \prod_{i=1}^{l} (x - t_{j,\lambda})}.$$

Let λ tend to infinity. Comparing the main terms of asymptotics of the left and right hand sides , we conclude that the limit of $y_\lambda \tilde{y}_\lambda$ is T. The polynomial T has zero of order Λ_s at z_s and therefore $\tilde{y}_\lambda$ cannot have a zero of order $\Lambda_s + 1$ at z_s for all but finitely many λ. It follows that $\tilde{y}_\lambda$ has a zero at z_s for only finitely many values of λ. □

Corollary 3.3 *A polynomial $y(x)$ represents an sl_2 critical point associated to $(\Lambda_\infty, \lambda)$ if and only if $\deg y(x) = l$, $y(x)$ is off-diagonal with respect to $(\boldsymbol{\Lambda}, \boldsymbol{z})$ and there exists a polynomial $\tilde{y}(x)$ such that $W(y, x^{\lambda+1}\tilde{y}) = x^\lambda T(x)$.*

Note that if $y(x)$ represents a critical point associated to $(\Lambda_\infty, \lambda)$ and if the descendent polynomial $\tilde{y}(x)$ is off-diagonal then $\tilde{y}(x)$ represents a critical point associated to $(w\Lambda_\infty, w \cdot \lambda)$, where $w \neq \mathrm{id}$ is the generator of the sl_2 Weyl group.

The polynomials $y(x)$ and $\tilde{y}(x)$ may coincide (up to a multiplicative constant). For example, if there are no z_s, if $l = 0$ and $y(x) = 1$, then $\tilde{y}(x) = 1/(\lambda + 1)$. The polynomials $y(x)$ and $\tilde{y}(x)$ are constant multiples of each other, but they represent critical points associated to different weights $(\Lambda_\infty = 0,\ \lambda)$ and $(\Lambda_\infty = 0,\ -\lambda - 2)$, respectively.

Assume that $\lambda \in \mathbb{C}$ is not an integer. Then the unordered pair $\{\,(y(x), \lambda),\ \ (\tilde{y}(x), -\lambda - 2)\,\}$, is called *the sl_2 population originated at* $(y(x), \lambda)$. Here $y(x), \tilde{y}(x)$ are considered up to a multiplicative constant.

Lemma 3.4 *Let λ be not an integer. Let $P = \{ (y(x), \lambda), (\tilde{y}(x), -\lambda - 2) \}$ be the sl_2 population originated at $(y(x), \lambda)$. Assume that $\tilde{y}(x)$ is off-diagonal. Then $y(x)$ is a descendent polynomial of $\tilde{y}(x)$ with respect to $-\lambda - 2$ and the population originated at $(\tilde{y}(x), -\lambda - 2)$ coincides with P.*

Proof We have

$$x^{-\lambda-2} T = (x^{-\lambda-1})^2 x^{\lambda} T = (x^{-\lambda-1})^2 W(y, x^{\lambda+1} \tilde{y}) = W(\tilde{y}, -x^{-\lambda-1} y),$$

cf the Wronskian identities in [9, Lemma 9.2]. □

3.2 The populations in the case of a general Kac–Moody algebra

Recall that we fixed a Kac–Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$ of rank r, a non-negative integer $n \in \mathbb{Z}_{\geq 0}$, an n–tuple of $\mathfrak{g}$–weights $\mathbf{\Lambda} = (\Lambda_i)_{i=1}^n$, $\Lambda_i \in \mathcal{P}^+$, and an n–tuple $\boldsymbol{z} = (z_i)_{i=1}^n \in \mathbb{C}^n$ such that $z_i \neq 0$ and $z_i \neq z_j$ if $i \neq j$.

For $i = 1, \dots, r$, we set

$$T_i(x) = \prod_{s=1}^{n} (x - z_s)^{\langle \Lambda_s, \alpha_i^\vee \rangle}. \tag{7}$$

Given a set of numbers $\boldsymbol{t} = (t_j^{(i)})_{i=1,\dots,r}^{j=1,\dots,l_i}$, we represent it by the r–tuple of polynomials $\boldsymbol{y} = (y_1, \dots, y_r)$, where $y_i(x) = \prod_{j=1}^{l_i} (x - t_j^{(i)})$, $i = 1, \dots, r$. We are interested only in the roots of the polynomials y_i, therefore we make no distinction between tuples $(y_1, \dots, y_r)$ and $(c_1 y_1, \dots, c_r y_r)$, where c_i are non-zero constants.

An r–tuple of polynomials $\boldsymbol{y}$ is called *off-diagonal with respect to* $(\mathbf{\Lambda}, \boldsymbol{z})$ if its roots do not belong to the union of singular hyperplanes in (3). Namely $\boldsymbol{y}$ is off-diagonal if for $i = 1, \dots, n$, all roots of polynomial y_i are simple, non-zero, different from the roots of polynomials y_j for all j such that $(\alpha_j, \alpha_i) \neq 0$ and different from the roots of polynomial T_i.

Since $\mathbf{\Lambda}, \boldsymbol{z}$ are fixed, we often call r–tuples of polynomials $\boldsymbol{y}$ which are off-diagonal with respect to $(\mathbf{\Lambda}, \boldsymbol{z})$ simply off-diagonal.

An r–tuple of polynomials $\boldsymbol{y}$ is called *fertile in the i-th direction, $i \in \{1, \dots, r\}$ with respect to* λ if there exists a polynomial $\tilde{y}_i$ such that

$$W(y_i, x^{\langle \lambda + \rho, \alpha_i^\vee \rangle} \tilde{y}_i) = x^{\langle \lambda, \alpha_i^\vee \rangle} \, T_i \prod_{j,\, j \neq i} y_j^{-a_{ij}}. \tag{8}$$

Then the tuple of polynomials $\boldsymbol{y}^{(i)} := (y_1, \dots, \tilde{y}_i, \dots, y_r)$ is called *an immediate descendent of $\boldsymbol{y}$ in the i-th direction with respect to λ.*

Recall that $s_i \in \mathcal{W}$ are elementary reflections in the Weyl group of $\mathfrak{g}$.

We call an r–tuple of polynomials $\boldsymbol{y}^{(i_1,i_2,\dots,i_m)}$, where $i_k \in \{1,\dots,r\}$, $k=1,\dots,m$, *a descendent of* $\boldsymbol{y}$ *with respect to* λ *in the directions* $(i_1,\dots,i_m)$ if there exist r–tuples of polynomials $\boldsymbol{y}^{(i_1,i_2,\dots,i_k)}$, $k=1,\dots,m-1$, such that for $k=1,\dots,m$, the r–tuple $\boldsymbol{y}^{(i_1,i_2,\dots,i_k)}$ is an immediate descendent of $\boldsymbol{y}^{(i_1,i_2,\dots,i_{k-1})}$ in the i_k-th direction with respect to $(s_{i_{k-1}}\dots s_{i_2}s_{i_1})\cdot\lambda$.

An r–tuple of polynomials $\boldsymbol{y}$ is called *fertile with respect to* λ if it is fertile in all directions $i=1,\dots,r$.

An r–tuple of polynomials $\boldsymbol{y}$ is called *super-fertile with respect to* λ if it is fertile with respect to λ and all descendents $\boldsymbol{y}^{(i_1,i_2,\dots,i_k)}$ of $\boldsymbol{y}$ with respect to λ are fertile with respect to $(s_{i_k}\dots s_{i_1})\cdot\lambda$.

Theorem 3.5 *Let* $\mathfrak{g}$ *be a Kac–Moody algebra. An* r*–tuple* $\boldsymbol{y}$ *represents a* $\mathfrak{g}$ *critical point associated to* $(\boldsymbol{\Lambda},\boldsymbol{z};\Lambda_\infty,\lambda)$ *if and only if* $\deg y_i = l_i$, $i=1,\dots,r$, $\boldsymbol{y}$ *is off-diagonal with respect to* $(\boldsymbol{\Lambda},\boldsymbol{z})$ *and fertile with respect to* λ. *Moreover, if an immediate descendent of* $\boldsymbol{y}$ *in the direction* i, $\boldsymbol{y}^{(i)}=(y_1,\dots,\tilde{y}_i,\dots,y_r)$, *is off-diagonal with respect to* $(\boldsymbol{\Lambda},\boldsymbol{z})$ *then it represents a* $\mathfrak{g}$ *critical point associated to* $(\boldsymbol{\Lambda},\boldsymbol{z};s_i\Lambda_\infty,s_i\cdot\lambda)$.

Proof The first part of the theorem follows immediately from the case of sl_2, see Corollary 3.3.

To show the second part we show that roots of $\boldsymbol{y}^{(i)}$ satisfy system (3), where λ is changed to $s_i\cdot\lambda$. Let $\tilde{t}_j^{(i)}$ denote the roots of $\tilde{y}_i$.

The equations of system (3) corresponding to coordinates $\tilde{t}_j^{(i)}$ are satisfied by Theorem 3.2.

The equations of system (3) corresponding to coordinates $\tilde{t}_j^{(k)}$ such that $a_{ki}=0$, are satisfied because these equations are the same for $\tilde{\boldsymbol{y}}$ and $\boldsymbol{y}$.

For any k, such that $k\neq i$ and $a_{ik}\neq 0$, choose a root $t_j^{(k)}$ of the polynomial y_k. Setting $x=t_j^{(k)}$ in the i-th equation of (8), we get

$$\sum_s \frac{1}{t_j^{(k)}-t_s^{(i)}} = \sum_s \frac{1}{t_j^{(k)}-\tilde{t}_s^{(i)}} + \frac{\langle\lambda+\rho,\alpha_i^\vee\rangle}{t_j^{(k)}}.$$

This implies that the equation of system (3) corresponding to the coordinate $\tilde{t}_j^{(k)}$ is satisfied as well. □

Note that if the tuple $y^{(i)}$ is off-diagonal, then it is again fertile and we can find the r–tuple of polynomials $y^{(i,j)}$. However, in general, we do not know if the tuple $y^{(i)}$ is off-diagonal. It is true in the case of sl_2 and almost all non-integral weights λ, see Theorem 3.2. We have the following conjecture.

Conjecture 3.6 *An off-diagonal fertile tuple is super-fertile.*

We prove this conjecture for the simple Lie algebras of types A_r, B_r, see Theorems 4.5 and 4.8.

For an r–tuple of polynomials y and a $\mathfrak{g}$ weight λ, let $P(y,\lambda)$ be the set of all pairs of the form $(y^{(i_1,i_2,\dots,i_m)}, (s_{i_m}\dots s_{i_2}s_{i_1})\cdot\lambda)$, where $i_k\in\{1,\dots,r\}$, $m\in\mathbb{Z}_{\geq 0}$, $k=1,\dots,m$, and $y^{(i_1,i_2,\dots,i_m)}$ is a descendent of y with respect to λ in directions $(i_1,\dots,i_m)$.

We call the set $P(y,\lambda)$ *the prepopulation originated at* (y,λ). If y is a super-fertile r–tuple with respect to λ then we call the set $P(y,\lambda)$ *the population originated at* (y,λ).

Lemma 3.7 *Let y be super-fertile with respect to λ and let P be the population originated at (y,λ). Let $(\tilde{y},\tilde{\lambda})\in P$. Then $\tilde{y}$ is super-fertile with respect to $\tilde{\lambda}$ and the population originated at $(\tilde{y},\tilde{\lambda})$ is also P. In particular, different populations do not intersect.*

Proof By Lemma 3.4 we obtain that if $y^{(i)}$ is an immediate descendent of y in the direction i, then y is also an immediate descendent of $y^{(i)}$ in the direction i. The lemma follows. □

We call a weight λ *strongly non-integral* if for any element of the Weyl group $s\in\mathcal{W}$ and any $i\in\{1,\dots,r\}$ the number $\langle s\cdot\lambda,\alpha_i^\vee\rangle$ is not an integer.

Note that if λ is strongly non-integral, then, in particular, the weights $s\cdot\lambda$, $s\in\mathcal{W}$, do not belong to the reflection hyperplanes. Therefore the map $\mathcal{W}\to\mathcal{W}\cdot\lambda$ mapping $w\in\mathcal{W}$ to $w\cdot\lambda$ is bijective.

Lemma 3.8 *Let λ be strongly non-integral and let r–tuples y_1, y_2 be descendents of an r–tuple y with respect to λ in the directions $(i_1,\dots,i_m)$. Then the r–tuples y_1, y_2 coincide.*

Proof Lemma follows from the corresponding sl_2 statement, see Theorem 3.2. □

Let λ be strongly non-integral. Let $\boldsymbol{y}$ be super-fertile with respect to λ. Let P be the population originated at $(\boldsymbol{y},\lambda)$. For $i \in \{1,\dots,r\}$, let $a_i:\ P\to P$ be the map of the simple reproduction in the i-th direction which maps $(\tilde{\boldsymbol{y}},\tilde\lambda)$ to $(\tilde{\boldsymbol{y}}^{(i)}, s_i\cdot\tilde\lambda)$. According to Lemma 3.8, the map a_i is well defined. By Lemma 3.4 we have $a_i^2 = id$. In particular a_i are invertible.

Let $\mathcal{A}$ be the subgroup of the group of all permutations of the elements in P generated by a_i, $i=1,\dots,r$.

Conjecture 3.9 *There is an isomorphism of groups $\mathcal{A}\to\mathcal{W}$ which maps a_i to s_i.*

If $(\tilde{\boldsymbol{y}},\tilde\lambda)\in P$ then $\tilde{\boldsymbol{y}}$ is a descendent of $\boldsymbol{y}$ in some directions $(i_1,\dots,i_m)$ and we have $\tilde\lambda = w\cdot\lambda$ for some $w\in\mathcal{W}$. Since λ is strongly non-integral, such w is unique and we have $w = s_{i_m}\dots s_{i_1}$. This defines a map

$$\begin{aligned} \tau:\qquad P\quad &\to \mathcal{W},\\ (\tilde{\boldsymbol{y}}, w\cdot\lambda) &\mapsto w. \end{aligned} \tag{9}$$

Since $\boldsymbol{y}$ is super-fertile, τ is a surjective map. Conjecture 3.9 is true if and only if the map τ is a bijection for all populations P.

Note that the map $\tau=\tau(\boldsymbol{y},\lambda)$ depends on the choice of the element $(\boldsymbol{y},\lambda)\in P$. However, if it is bijective for one element of the population, then it is clearly bijective for all elements of this population.

Conjecture 3.9 for the case of sl_2 is proved in Lemma 3.4.

Below we prove Conjecture 3.9 for simple Lie algebras.

4 Proof of Conjecture 3.9 for simple Lie algebras

4.1 The case of sl_{N+1}

We have roots $\alpha_1,\dots,\alpha_N$ with scalar products $(\alpha_i,\alpha_i)=2$, $(\alpha_i,\alpha_{i\pm1})=-1$ and 0 otherwise.

We fix weights $\boldsymbol{\Lambda}=(\Lambda_1,\dots,\Lambda_n),\Lambda_\infty$, points $\boldsymbol{z}=(z_1,\dots,z_n)$. The weights Λ_s, $s=1,\dots,n$, are dominant integral sl_{N+1} weights and the points z_s, $s=1,\dots,n$, are non-zero, pairwise different complex numbers. We define polynomials T_i as in (7).

We also fix a strongly non-integral sl_{N+1} weight λ.

For any N–tuple of functions $\boldsymbol{y} = (y_1, \dots, y_N)$ and sl_{N+1} weight λ, we set $y_{N+1} = 1$ and define the linear differential operator of order $N+1$:

$$D(\boldsymbol{y},\lambda) = (\partial - \ln'(\frac{\prod_{s=1}^{N} x^{(\lambda,\alpha_s)}T_s}{y_N}))\ \dots (\partial - \ln'(\frac{y_2 T_1 x^{(\lambda,\alpha_1)}}{y_1}))\ (\partial - \ln'(y_1))$$

$$= \prod_{i}^{0\to N} \left(\partial \ - \ \ln' \left(\frac{y_{N+1-i}\prod_{s=1}^{N-i} x^{(\lambda,\alpha_s)}T_s}{y_{N-i}}\right)\right). \tag{10}$$

For an N–tuple of polynomials $\boldsymbol{y}$ and an sl_{N+1} weight λ, let as before the prepopulation $P = P(\boldsymbol{y},\lambda)$ be the set of all descendents of $\boldsymbol{y}$ paired with the corresponding weight.

Lemma 4.1 *Let $\boldsymbol{y}$ represent an sl_{N+1} critical point associated to $(\Lambda_\infty, \lambda)$. Then the prepopulation P contains the elements*

$$(\boldsymbol{y}^{(i,i-1,\dots,1)}, (s_i s_{i-1} \dots s_1)\cdot\lambda),$$

where $i = 0, \dots, N$.

Proof Since $\boldsymbol{y}$ represents a critical point, it is fertile and none of y_j has multiple roots. Moreover we have the Bethe Ansatz equation (3) for each root of each polynomial y_j.

In particular there exist polynomials $\widetilde{y}_i$ such that

$$W(y_i, x^{(\lambda+\rho,\alpha_i)}\widetilde{y}_i) = x^{(\lambda,\alpha_i)} y_{i-1} y_{i+1} T_i.$$

Note that if $t_j^{(i-1)}$ is a root of y_{i-1} then either $\widetilde{y}_i(t_j^{(i-1)}) \neq 0$ or $\widetilde{y}_i$ vanishes at $t_j^{(i-1)}$ to order 2. In the former case, in the same way as in Theorem 3.5, we see that the Bethe Ansatz equation for the root $t_j^{(i-1)}$ of y_{i-1} in the N–tuple $\boldsymbol{y}^{(i)}$ is still valid. In addition the Bethe Ansatz equations for roots of $y_1, \dots, y_{i-2}$ in the N–tuple $\boldsymbol{y}^{(i)}$ are also satisfied since they are exactly the same as in the N–tuple $\boldsymbol{y}$.

Consider the next equation for $\widetilde{y}_{i-1}$:

$$W(y_{i-1}, x^{(s_i\cdot\lambda+\rho,\alpha_{i-1})}\widetilde{y}_{i-1}) = x^{(s_i\cdot\lambda,\alpha_{i-1})} y_{i-2}\widetilde{y}_i T_{i-1}.$$

We have

$$x^{(s_i\cdot\lambda+\rho,\alpha_{i-1})}\widetilde{y}_{i-1} = y_{i-1}\int \frac{x^{(s_i\cdot\lambda,\alpha_{i-1})} y_{i-2}\widetilde{y}_i T_{i-1}}{y_{i-1}^2} dx.$$

We claim that the integrand does not have residues. Indeed, the residues could occur only at the roots $t_j^{(i-1)}$ of y_{i-1}. If $\widetilde{y}_i(t_j^{(i-1)}) = 0$ then the integrand is holomorphic at $x = t_j^{(i-1)}$. If $\widetilde{y}_i(t_j^{(i-1)}) \neq 0$ then the absence of the residue is equivalent to the

Bethe Ansatz equation corresponding to $t_j^{(i-1)}$ which is satisfied. Therefore $\tilde{y}_{i-1}$ is a polynomial and there exists a descendent $y^{(i,i-1)}$.

Similarly, we prove that $\boldsymbol{y}^{(i,i-1,\dots,i-m)}$ is a well-defined N–tuple of polynomials for $m = 2, \dots, i-1$. □

If $(\tilde{\boldsymbol{y}}, \tilde{\lambda}) \in P$ then we write

$$\tilde{\lambda} = w(\tilde{\lambda}) \cdot \lambda = \lambda - \sum_{i=1}^{N} a_i(\tilde{\lambda})\alpha_i,$$

where $w(\tilde{\lambda}) \in \mathcal{W}$ and $a_i(\tilde{\lambda}) \in \mathbb{R}$. Define *the shifted prepopulation* $\bar{P}(\boldsymbol{y}, \lambda)$ as the following set of N–tuples of functions:

$$\bar{P} = \{(x^{a_1(\tilde{\lambda})}\tilde{y}_1, x^{a_2(\tilde{\lambda})}\tilde{y}_2, \dots, x^{a_N(\tilde{\lambda})}\tilde{y}_N) \mid (\tilde{\boldsymbol{y}}, \tilde{\lambda}) \in P\}.$$

Note that the shifted prepopulation $\bar{P}(\boldsymbol{y}, \lambda)$ depends on the choice of an element $(\boldsymbol{y}, \lambda) \in P$. However, the difference is not very essential: if $(\tilde{y}, \tilde{\lambda}) \in P$ and $\bar{P}(\tilde{y}, \tilde{\lambda})$ is the corresponding shifted prepopulation then there exists $a_i \in \mathbb{R}$ such that the i-th function in any N–tuples in $\bar{P}(\tilde{y}, \tilde{\lambda})$ is obtained via multiplication of the i-th function of the corresponding N–tuple in $\bar{P}(\boldsymbol{y}, \lambda)$ by x^{a_i}.

Lemma 4.2 *Let $\tilde{\boldsymbol{y}} \in \bar{P}(\boldsymbol{y}, \lambda)$. Then $D(\tilde{\boldsymbol{y}}, \lambda) = D(\boldsymbol{y}, \lambda)$.*

Proof Let $(\boldsymbol{v}, \mu) \in P$ and let $(v^{(i)}, s_i \cdot \mu) \in P$ be the immediate descendent of $\boldsymbol{v}$ with respect to μ in the direction i. Then we have $v_k^{(i)} = v_k$ for $k \neq i$ and $W(v_i, v_i^{(i)} x^{(\mu+\rho,\alpha_i)}) = x^{(\mu,\alpha_i)} v_{i-1} v_{i+1} T_i$. The last relation can be rewritten as

$$W(x^{a_i(\mu)} v_i, v_i^{(i)} x^{a_i(s_i \cdot \mu)}) = x^{(\lambda,\alpha_i)}\; v_{i-1} x^{a_{i-1}(\mu)}\; v_{i+1} x^{a_{i+1}(\mu)} T_i. \tag{11}$$

Let

$$\begin{aligned}(\bar{\boldsymbol{v}}, \mu) &= ((x^{a_1(\mu)} v_1, \dots, x^{a_N(\mu)} v_N), \mu) \in \bar{P}(\boldsymbol{y}, \lambda),\\ (\bar{\boldsymbol{v}}^{(i)}, s_i \cdot \mu) &= ((x^{a_1(s_i\cdot\mu)} v_1^{(i)}, \dots, x^{a_N(s_i\cdot\mu)} v_N^{(i)}), s_i \cdot \mu) \in \bar{P}(\boldsymbol{y}, \lambda).\end{aligned}$$

Identity (11) reads $W(\bar{v}_i, \bar{v}_i^{(i)}) = x^{(\lambda,\alpha_i)} \bar{v}_{i-1} \bar{v}_{i+1} T_i$ and therefore

$$\begin{aligned}&-\ln''(\bar{v}_i) + \ln'(\bar{v}_i)(\ln'(T_i x^{(\lambda,\alpha_i)} \bar{v}_{i-1}\bar{v}_{i+1}) - \ln'(\bar{v}_i)) =\\ &-\ln''(\bar{v}_i^{(i)}) + \ln'(\bar{v}_i^{(i)})(\ln'(T_i x^{(\lambda,\alpha_i)} \bar{v}_{i-1}^{(i)} \bar{v}_{i+1}^{(i)}) - \ln'(\bar{v}_i^{(i)})).\end{aligned} \tag{12}$$

Compare $D(\bar{\boldsymbol{v}})$ with $D(\bar{\boldsymbol{v}}^{(i)})$. All factors but two successive ones in these operators are the same. The products of the two middle factors are the same by (12) □

We call $D(\boldsymbol{y},\lambda)=D$ *the operator associated to the shifted prepopulation* $\bar{P}(\boldsymbol{y},\lambda)$. It follows from Lemma 4.2 if $\tilde{\boldsymbol{y}}\in\bar{P}(\boldsymbol{y},\lambda)$ then $D\tilde{y}_1=0$.

We use the following notation for Wronskians and *divided Wronskians*:

$$W(u_1,\dots,u_i)=\det(u_k^{(j-1)})_{k,j=1}^{i},$$
$$W^\dagger(u_1,\dots,u_i)=\frac{W(u_1,\dots,u_i)}{(x^{(\lambda,\alpha_1)}T_1)^{i-1}(x^{(\lambda,\alpha_2)}T_2)^{i-2}\dots(x^{(\lambda,\alpha_{i-1})}T_{i-1})},$$

$i=1,\dots,N+1$, where $u_k^{(j-1)}$ denotes the $(j-1)st$ derivative of u_k with respect to variable x.

Lemma 4.3 *Let $\boldsymbol{y}$ either represent an sl_{N+1} critical point associated to (Λ_∞,λ) or be super-fertile with respect to λ. Then there exist functions $u_1,\dots,u_{N+1}$ such that $Du_i=0$, $y_i=W^\dagger(u_1,\dots,u_i)$ for $i=1,\dots,N+1$ and $u_i x^{-(\lambda+\rho,\alpha_1+\dots+\alpha_{i-1})}$ are polynomials for $i=1,\dots,N+1$.*

Proof Let u_i to be the first coordinate of the N–tuple in the element of $\bar{P}(\boldsymbol{y},\lambda)$ corresponding to $\boldsymbol{y}^{(i-1,i-2,\dots,1)}$ which is a descendent of $\boldsymbol{y}$ with respect to λ in directions $(i-1,i-2,\dots,1)$. If $\boldsymbol{y}$ represents an sl_{N+1} critical point with respect to λ then such an N–tuple exists by Lemma 4.1.

We have $Du_i=0$ by Lemma 4.2.

The condition $W^\dagger(u_1,\dots,u_i)=y_i$ follows from the standard Wronskian identities, cf [9, proof of Lemma 5.5]. □

Corollary 4.4 *Let $\boldsymbol{y}$ be an N–tuple of polynomials and $l_i=\deg y_i$, $i=1,\dots,N$. Let $\boldsymbol{\Lambda}_\infty$ be given by (1). Let $\boldsymbol{y}$ represent an sl_{N+1} critical point associated to (Λ_∞,λ) or let $\boldsymbol{y}$ be super-fertile with respect to λ. Then the kernel of the operator $D(\boldsymbol{y},\lambda)$ is spanned by functions of the form*

$$p_0,\ p_1x^{(\lambda+\rho,\alpha_1)},\dots,p_Nx^{(\lambda+\rho,\alpha_1+\dots+\alpha_N)}, \tag{13}$$

where p_i is a polynomial of degree $\deg y_1+(\Lambda_\infty,\alpha_1+\dots+\alpha_i)$, $p_i(0)\neq0$. The only singular points of the operator $D(\boldsymbol{y},\lambda)$ in $\mathbb{C}^*$ are regular singular points located at $z_1,\dots,z_n$, and the exponents at z_i, $i=1,\dots,n$, are

$$z_i:0,(\Lambda_i+\rho,\alpha_1),(\Lambda_i+\rho,\alpha_1+\alpha_2),\dots,(\Lambda_i+\rho,\alpha_1+\dots+\alpha_N), \tag{14}$$

Conversely, if a linear differential operator D of order $N+1$ has the kernel spanned by functions of the form (13) and the only non-zero singular points of D in $\mathbb{C}^*$ are regular *singular points at* $z_1, \dots, z_n$ *with the exponents given by* (14)*, then the* N*–tuple* $\boldsymbol{y}$ *given by the divided Wronskians*

$$y_i = W^\dagger(p_0, p_1 x^{(\lambda+\rho,\alpha_1)}, \dots, p_{i-1} x^{(\lambda+\rho,\sum_{j=1}^{i-1}\alpha_j)})$$

is super-fertile with respect to λ *and satisfies* $\deg y_i = l_i$, $i = 1, \dots, N$.

Proof For $i = 0, \dots, N$, we set

$$p_i = u_{i+1} x^{-(\lambda+\rho,\alpha_1+\dots+\alpha_i)},$$

where $u_1, \dots, u_{N+1}$ are as in Lemma 4.3. By Lemma 4.3, $p_0, \dots, p_N$ are polynomials.

Now, the first part of Corollary 4.4 follows from Lemma 4.3 by standard Wronskian identities, cf [9, Lemmas 5.8 and 5.10].

Conversely, let V be the kernel of the operator D. We have $(N+1)!$ distinguished full flags in V such that the divided Wronskians of all spaces which form the flags are of the form $x^a p(x)$ where $p(x)$ is a polynomial. Namely, for a permutation $w \in S_{N+1}$ of the set $\{0, 1, \dots, N\}$ we have a full flag $\mathcal{F}_w$ such that the space of dimension i is spanned by $p_{w(0)} x^{(\lambda+\rho),\alpha_{w(0)}} \dots, p_{w(i-1)} x^{(\lambda+\rho),\alpha_{w(i-1)}}$.

For each such flag $\mathcal{F}_w$, we have the corresponding element p_w in $\overline{P}(\boldsymbol{y}, \lambda)$. Each element p_w is obviously fertile and the immediate descendents of p_w in the i-th direction is $p_{(i,i+1)w}$. Therefore, the N–tuple $\boldsymbol{y}$ is fertile with respect to λ. □

Theorem 4.5 *Conjectures 3.6, 3.9 hold for the case of* sl_{N+1}.

Proof Conjecture 3.6 follows from Theorem 3.5 and Corollary 4.4.

Conjecture 3.9 in the case of sl_{N+1} follows from the proof of the converse statement of Corollary 4.4. □

4.2 The case of B_2

In the case of B_2 we have two roots α_1, α_2 such that $(\alpha_1, \alpha_1) = 4$, $(\alpha_2, \alpha_2) = 2$, $(\alpha_1, \alpha_2) = -2$.

The key observation is that B_2 populations can be embedded in sl_4 populations.

Given a B_2 weight Λ, define the sl_4 weight Λ^A by

$$\langle \Lambda^A, (\alpha_1^A)^\vee \rangle = \langle \Lambda^A, (\alpha_3^A)^\vee \rangle = \langle \Lambda, \alpha_1^\vee \rangle, \qquad \langle \Lambda^A, (\alpha_2^A)^\vee \rangle = \langle \Lambda, \alpha_2^\vee \rangle,$$

where α_i^A are roots of sl_4.

Note that if Λ is strongly non-integral then Λ^A is strongly non-integral.

Lemma 4.6 *A pair* $\boldsymbol{y} = (y_1, y_2)$ *represents a* B_2 *critical point associated to* $(\boldsymbol{z}, \boldsymbol{\Lambda}; \Lambda_\infty, \lambda)$ *if and only if the triple* $\boldsymbol{y}^A = (y_1, y_2, y_1)$ *represents an* sl_4 *critical point associated to* $(\boldsymbol{z}, \boldsymbol{\Lambda}^A; \Lambda_\infty^A, \lambda^A)$*. Moreover there is an embedding* $P(\boldsymbol{y}, \lambda) \to P^A(\boldsymbol{y}^A, \lambda^A)$ *which sends* $((\widetilde{y}_1, \widetilde{y}_2), \widetilde{\lambda}) \in P(\boldsymbol{y}, \lambda)$ *to* $((\widetilde{y}_1, \widetilde{y}_2, \widetilde{y}_1), \widetilde{\lambda}^A) \in P^A(\boldsymbol{y}^A, \lambda^A)$*.*

Proof Follows immediately from the definitions. □

Theorem 4.7 *Conjecture 3.9 holds in the case of root system* B_2*.*

Proof Recall the surjective maps $\tau : P \to \mathcal{W}$ and $\tau^A : P^A \to \mathcal{W}^A$, where $\mathcal{W}$ and $\mathcal{W}^A$ are the B_2 and A_3 Weyl groups, see (9). Then we clearly have $(\tau(\boldsymbol{y}) \cdot \lambda)^A = \tau^A(\boldsymbol{y}^A) \cdot \lambda^A$. By Theorem 4.5, the map τ^A is injective.

Therefore τ is injective and hence bijective. □

Theorem 4.8 *Conjecture 3.6 holds in the case of simple Lie algebras of type* B_N*.*

Proof Similarly to the case $N = 2$, the N–tuple $(y_1, \dots, y_N)$ represents a critical point of type B_N if and only if the $(2N-1)$–tuple $(y_1, \dots, y_{N-1}, y_N, y_{N-1}, \dots, y_1)$ represents a critical point of type A_{2N-1}, see also [9]. The tuple $(y_1, \dots, y_N)$ is super-fertile in B_N sense because the $(2N-1)$–tuple $(y_1, \dots, y_{N-1}, y_N, y_{N-1}, \dots, y_1)$ is super-fertile in A_{2N-1} sense. □

4.3 The case of the root systems of types B, C, D, E and F

From the sl_2, sl_3 and B_2 cases, we obtain the general case (except for G_2).

Theorem 4.9 *Conjecture 3.9 holds in the case of the root systems of types* B, C, D, E *and* F*.*

Proof Let $\mathfrak{g}$ be a rank r simple Lie algebra of type B, C, D, E or F.

The Weyl group of $\mathfrak{g}$ is a finite group described by the generators and relations:

$$\mathcal{W} = < s_1, \dots, s_r > / (s_i^2 = (s_i s_j)^{-(\alpha_i, \alpha_j)+2} = 1, \ i, j = 1, \dots, r, \ i \neq j).$$

Here $< s_1, \dots, s_r >$ denotes the free group with generators $s_1, \dots, s_r$.

Note that in our case (α_i, α_j) takes values $2, 0, -1$ or -2.

Let λ be strongly non-integral, $\boldsymbol{y}$ represent a $\mathfrak{g}$–critical point associated to $(\Lambda_\infty, \lambda)$ and let $P = P(\boldsymbol{y}, \lambda)$ be the corresponding population.

We have the corresponding relations in the populations among descendents:

- $\boldsymbol{y}^{(i,i)} = \boldsymbol{y}$,
- $\boldsymbol{y}^{(i,j)} = \boldsymbol{y}^{(j,i)}$ if $(\alpha_i, \alpha_j) = 0$,
- $\boldsymbol{y}^{(i,j,i)} = \boldsymbol{y}^{(j,i,j)}$ if $(\alpha_i, \alpha_j) = -1$,
- $\boldsymbol{y}^{(i,j,i,j)} = \boldsymbol{y}^{(j,i,j,i)}$ if $(\alpha_i, \alpha_j) = -2$.

The first relation follows from the case of sl_2, Lemma 3.4, the second relation is obvious, the third relation follows from the case of sl_3, Theorem 4.5, and the fourth relation follows from the case of B_2, Theorem 4.7.

Therefore, P has at most $|\mathcal{W}|$ elements. □

4.4 The case of G_2

In the case of G_2 we have two roots α_1, α_2 such that $(\alpha_1, \alpha_1) = 2$, $(\alpha_2, \alpha_2) = 6$, $(\alpha_1, \alpha_2) = -3$.

The key observation is that G_2 populations can be embedded in C_3 populations.

Given a G_2 weight Λ, define the C_3 weight Λ^C by

$$\langle \Lambda^C, (\alpha_1^C)^\vee \rangle = \langle \Lambda^C, (\alpha_3^C)^\vee \rangle = \langle \Lambda, \alpha_1^\vee \rangle, \qquad \langle \Lambda^C, (\alpha_2^C)^\vee \rangle = \langle \Lambda, \alpha_2^\vee \rangle,$$

where α_i^C are roots of C_3.

Note that if Λ is strongly non-integral then Λ^C is strongly non-integral.

Lemma 4.10 *A pair $\boldsymbol{y} = (y_1, y_2)$ represents a G_2 critical point associated to $(\boldsymbol{\Lambda}, \boldsymbol{z}; \Lambda_\infty, \lambda)$ if and only if the triple $\boldsymbol{y}^C = (y_1, y_2, y_1)$ represents a C_3 critical point associated to $(\boldsymbol{\Lambda}_i^C, \boldsymbol{z}; \Lambda_\infty^C, \lambda^C)$. Moreover there is an embedding $P(\boldsymbol{y}, \lambda) \to P^C(\boldsymbol{y}^C, \lambda^C)$ which sends $((\tilde{y}_1, \tilde{y}_2), \tilde{\lambda}) \in P(\boldsymbol{y}, \lambda)$ to $((\tilde{y}_1, \tilde{y}_2, \tilde{y}_1), \tilde{\lambda}^C) \in P^C(\boldsymbol{y}^C, \lambda^C)$.*

Proof Follows immediately from definitions. □

Theorem 4.11 *Conjecture 3.9 holds in the case of the root system G_2.*

Proof Recall the surjective maps $\tau : P \to \mathcal{W}$ and $\tau^C : P^C \to \mathcal{W}^C$, where $\mathcal{W}$ and $\mathcal{W}^C$ are the G_2 and C_3 Weyl groups, see (9). Then we clearly have $(\tau(\boldsymbol{y}) \cdot \lambda)^C = \tau^C(\boldsymbol{y}^C) \cdot \lambda^C$. By Theorem 4.9, the map τ^C is injective. □

To summarize, we have

Corollary 4.12 *Conjecture 3.9 holds in the case of all simple Lie algebras.*

5 Weyl group actions on Bethe vectors

5.1 The trigonometric Gaudin operators and Bethe vectors

Let $\mathfrak{g}$ be a simple complex Lie algebra of rank r with the Killing form $(\ ,\ )$. We choose a Cartan subalgebra $\mathfrak{h}$, simple roots α_i, $i=1,\dots,r$. We identify $\mathfrak{h}$ with $\mathfrak{h}^*$ using the Killling form on $\mathfrak{g}$. Let F_i, E_i, $i=1,\dots,r$, be the Chevalley generators of $\mathfrak{g}$.

Let Δ be the root system, let $\Delta_\pm$ be the sets of positive and negative roots and let $\mathfrak{g}=\mathfrak{h}\oplus\sum_{\alpha\in\Delta} g_\alpha$ be the root decomposition. Let e_α, $\alpha\in\Delta$, be generators of g_α such that $(e_\alpha,e_{-\alpha})=1$. Let $\{h_j\}_{j=1,\dots,r}$ be an orthonormal basis of the Cartan algebra $\mathfrak{h}\subset\mathfrak{g}$.

Set

$$\Omega^0=\frac{1}{2}\sum_{j=1}^{r} h_j\otimes h_j,\qquad \Omega^+=\Omega^0+\sum_{\alpha\in\Delta_+} e_\alpha\otimes e_{-\alpha},\qquad \Omega^-=\Omega^0+\sum_{\alpha\in\Delta_+} e_{-\alpha}\otimes e_\alpha.$$

The *trigonometric* R*–matrix* is defined by

$$r(z)=\frac{\Omega^+z+\Omega^-}{z-1}.$$

We fix $\boldsymbol{z},\boldsymbol{\Lambda},\Lambda_\infty,\boldsymbol{l},\lambda$ as in Section 2.2. Let $L_1,\dots,L_n$ be irreducible $\mathfrak{g}$–modules with highest weights $\Lambda_1,\dots,\Lambda_n$ and let $V=L_1\otimes\cdots\otimes L_n$. Let $V[\mu]\subset V$ be the subspace of V of all vectors of weight μ.

We write $X^{(k)}$ for an operator $X\in\mathfrak{g}$ acting on the k-th factor. Similarly we write $X^{(k,l)}$ for an operator $X\in\mathfrak{g}\otimes\mathfrak{g}$ acting on the k-th and l-th factors.

The *trigonometric Gaudin operators* $H_i(\lambda)$, $i=1,\dots,n$, are defined by

$$H_i(\lambda)=\lambda^{(i)}+\sum_{j=1,\dots,n,\ j\neq i} r^{(i,j)}(z_i/z_j).$$

The trigonometric Gaudin operators depend on $\lambda\in\mathfrak{h}$ and act in V. The trigonometric Gaudin operators all commute, $[H_i(\lambda),H_j(\lambda)]=0$, $i,j=1,\dots,n$. The trigonometric Gaudin operators commute with the action of $\mathfrak{h}$ on V and therefore preserve every weight subspace of V.

For a given λ, common eigenvectors of the trigonometric Gaudin operators $H_i(\lambda)$ in the weight subspace $V[\Lambda_\infty]$ can be constructed by the Bethe Ansatz method as follows.

Let $l = l_1 + \cdots + l_n$. Let c be the unique non-decreasing function from $\{1,\ldots,l\}$ to $\{1,\ldots,r\}$ such that $\sharp\, c^{-1}(i) = l_i$, $i = 1,\ldots,r$.

Let $P(\boldsymbol{l},n)$ be the set of sequences $I = (i_1^1,\ldots,i_{j_1}^1;\ \ldots;\ i_1^n,\ldots,i_{j_n}^n)$ of integers in $\{1,\ldots,r\}$ such that for all $i = 1,\ldots,r$, the integer i appears in I precisely l_i times. For $I \in P(\boldsymbol{l},n)$, and a permutation $\sigma \in S_l$ set $\sigma_1(i) = \sigma(i)$ for $i = 1,\ldots,j_1$, and $\sigma_s(i) = \sigma(j_1 + \cdots + j_{s-1} + i)$ for $s = 2,\ldots,n$ and $i = 1,\ldots,j_s$. Define

$$S(I) \;=\; \{\, \sigma \in S_l \mid c(\sigma_s(j)) = i_s^j \text{ for } s = 1,\ldots,n \text{ and } j = 1,\ldots,j_s \,\}\,.$$

For $I \in P(\boldsymbol{l},n)$ we define a vector in $V[\Lambda_\infty]$ by the formula

$$F_I v \;=\; F_{i_1^1}\ldots F_{i_{j_1}^1} v_1 \otimes \cdots \otimes F_{i_1^n}\ldots F_{i_{j_n}^n} v_n.$$

For $I \in P(\boldsymbol{l},n)$, $\sigma \in S(I)$, we define a rational function of $\boldsymbol{t} = (t_i^{(j)})_{j=1,\ldots,r}^{i=1,\ldots,l_j}$ by the formula

$$\omega_{I,\sigma}(\boldsymbol{t}) \;=\; \omega_{\sigma_1(1),\ldots,\sigma_1(j_1)}(z_1;\boldsymbol{t}) \,\cdots\, \omega_{\sigma_n(1),\ldots,\sigma_n(j_n)}(z_n;\boldsymbol{t}),$$

where

$$\omega_{i_1,\ldots,i_j}(z;\boldsymbol{t}) \;=\; \frac{1}{(t_{i_1}-t_{i_2})\cdots(t_{i_{j-1}}-t_{i_j})(t_{i_j}-z)}$$

and $(t_1,\ldots,t_l) = (t_1^{(1)},\ldots,t_{l_1}^{(1)},t_1^{(2)},\ldots,t_{l_2}^{(2)},\ldots,t_1^{(r)},\ldots,t_{l_r}^{(r)})$.

We define *the weight function* by

$$\omega(\boldsymbol{t}) \;=\; \sum_{I\in P(\boldsymbol{l},n)}\ \sum_{\sigma\in S(I)} \omega_{I,\sigma}(\boldsymbol{t})\; F_I v\,. \tag{15}$$

The weight function $\omega(\boldsymbol{t})$ is a rational function of $\boldsymbol{t}$ with values in $V[\Lambda_\infty]$.

Proposition 5.1 *If $\boldsymbol{t}$ is a critical point of the master function* (2) *associated to* (Λ_∞,λ), *then $\omega(\boldsymbol{t})$ is a well defined vector of weight Λ_∞ in V which is an eigenvector of the operators $H_i(\lambda+\rho+\Lambda_\infty/2)$, $i = 1,\ldots,n$.*

Proof Proposition 5.1 follows from the corresponding fact for the rational Gaudin operators, see [15] and the relation between rational and trigonometric Gaudin operators, see [5, Appendix B]. □

The vector $\omega(\boldsymbol{t})$ is called the *Bethe vector associated to the critical point* $\boldsymbol{t}$. It is expected that for generic values of parameters, all critical points are non-degenerate and the Bethe vectors form a basis in V. In particular, the number of orbits of critical points and thus the number of populations should match the dimension of the subspace of all vectors of weight Λ_∞ in V.

5.2 Counting sl_{N+1} critical points

Let L_Λ denote the irreducible sl_{N+1} module of highest weight Λ.

Proposition 5.2 *For almost all λ the number of orbits of sl_{N+1} critical points associated to $(\Lambda_\infty, \lambda)$ and counted with multiplicity does not exceed the dimension of the subspace of the weight Λ_∞ in the tensor product $L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n}$.*

Proof By Lemma 2.1, for almost all λ, the number of critical points associated to $(\Lambda_\infty, \lambda)$ is finite. Therefore, there is a Zariski open set $O \subset \bar{\mathfrak{h}}^*$, such that the number of orbits of sl_{N+1} critical points associated to $(\Lambda_\infty, \lambda)$ and counted with multiplicities is the same for all $\lambda \in O$.

If λ is a dominant integral weight, then the number of orbits of critical points associated to $(\Lambda_\infty, \lambda)$ and counted with multiplicities is bounded from above by the multiplicity of L_{Λ_∞} in the tensor product $L_\lambda \otimes L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n}$, see [9; 1].

For any integer $M > 0$, let $\mathcal{C}_M$ be the set of all weights $\lambda \in \mathfrak{h}^*$ such that the scalar products (λ, α_i) are integers greater than M.

If $\lambda \in \mathcal{C}_M$ and M is large enough, then any singular vector of weight Λ_∞ in the tensor product $L_\lambda \otimes L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n}$ is uniquely determined by its projection to $v_\lambda \otimes L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n}$, where v_λ is the highest weight vector of L_λ. Therefore, the multiplicity of L_{Λ_∞} in the tensor product $L_\lambda \otimes L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n}$ equals the dimension of the subspace of weight Λ_∞ in the tensor product $L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n}$.

For any $M \in \mathbb{Z}_{\geq 0}$, the set $\mathcal{C}_M$ is not contained in any proper algebraic subset in $\bar{\mathfrak{h}}^*$ and thus the proposition follows. □

Proposition 5.3 *For almost all λ and almost all $(z_1, \dots, z_n) \in \mathbb{C}^n$, all of the critical points are non-degenerate and the number of orbits of sl_2 critical points associated to $(\Lambda_\infty, \lambda)$ equals the dimension of the subspace of weight Λ_∞ in the tensor product $L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n}$.*

Proof For almost all λ the number of orbits of sl_2 critical points associated to $(\Lambda_\infty, \lambda)$ is the same.

If λ is dominant integral, then the number of orbits of critical points associated to $(\Lambda_\infty, \lambda)$ for generic $\boldsymbol{z}$ equals the multiplicity of L_{Λ_∞} in the tensor product $L_\lambda \otimes L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n}$ and all these orbits are non-degenerate, see [16, Theorem 1].

For dominant integral values of λ which are large enough, the multiplicity of L_{Λ_∞} in the tensor product $L_\lambda \otimes L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n}$ equals the dimension of the subspace of weight Λ_∞ in the tensor product $L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n}$.

In this case, almost all λ means all but finitely many and therefore the proposition follows. □

5.3 Actions of the Weyl group on Bethe vectors

Let $\mathfrak{g}$ be a simple Lie algebra, G the corresponding connected and simply connected Lie group. The group G acts on any finite-dimensional irreducible representation of $\mathfrak{g}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, and $T \subset G$ the corresponding torus. The Weyl group of $\mathfrak{g}$ can be described as N/T where $N = \{g \in G \mid gTg^{-1} = T\}$. In particular, this defines a projective action of the Weyl group on any tensor product of finite-dimensional irreducible representations of $\mathfrak{g}$. The projective action becomes an action in the zero weight subspace.

We fix our $\boldsymbol{z}, \boldsymbol{\Lambda}, \Lambda_\infty, \boldsymbol{l}, \lambda$ as in Section 2.2. Let $L_1, \dots, L_n$ be irreducible $\mathfrak{g}$–modules with highest weights $\Lambda_1, \dots, \Lambda_n$ and let $V = L_1 \otimes \cdots \otimes L_n$. Let $V[\mu]$ be the subspace of all vectors in V of weight μ. Let $P(V) = \{\mu,\ V[\mu] \neq 0\}$ be the set of all nontrivial weights in V.

We define the dynamical Weyl group acting on V following [18].

Let M_μ denote the Verma module with highest weight μ, v_μ a highest weight vector in M_μ.

Let M_μ, M_λ be Verma modules. Two cases are possible:

a) $\mathrm{Hom}_{\mathfrak{g}}(M_\mu, M_\lambda) = 0$ or
b) $\mathrm{Hom}_{\mathfrak{g}}(M_\mu, M_\lambda) = \mathbb{C}$ and every nontrivial homomorphism $M_\mu \to M_\lambda$ is an embedding.

Let M_λ be a Verma module with dominant weight $\lambda \in P^+$. Then $\mathrm{Hom}_{\mathfrak{g}}(M_\mu, M_\lambda) = \mathbb{C}$ if and only if there is $w \in \mathcal{W}$ such that $\mu = w \cdot \lambda$.

Let $w = s_{i_k} \dots s_{i_1}$ be a reduced presentation of an element of the Weyl group $\mathcal{W}$. Set $\alpha^1 = \alpha_{i_1}$ and $\alpha^j = (s_{i_1} \dots s_{i_{j-1}})(\alpha_{i_j})$ for $j = 2, \dots, k$. Let $n_j = (\lambda + \rho, (\alpha^j)^\vee)$. For a dominant $\lambda \in P^+$, the numbers n_j are positive integers. Define a singular vector $v^\lambda_{w\cdot\lambda} \in M_\lambda$ by

$$v^\lambda_{w\cdot\lambda} = \frac{(E_{-\alpha_{i_k}})^{n_k}}{n_1!} \cdots \frac{(E_{-\alpha_{i_1}})^{n_1}}{n_k!} v_\lambda . \tag{16}$$

This vector does not depend on the reduced presentation, see [18].

For all $\lambda \in P^+$, $w \in \mathcal{W}$, fix an embedding $M_{w\cdot\lambda} \hookrightarrow M_\lambda$ sending $v_{w\cdot\lambda}$ to $v^\lambda_{w\cdot\lambda}$.

We say that $\lambda \in P^+$ is generic with respect to V if

- For any $\nu \in P(V)$ and any $v \in V[\nu]$, there exist a unique intertwining operator $\Phi_\lambda^v : M_\lambda \to M_{\lambda-\nu} \otimes V$ such that $\Phi_\lambda^v(v_\lambda) = v_{\lambda-\nu} \otimes v +$ terms of lower weight in the first factor.
- For any $w, w' \in \mathcal{W}$, $w \neq w'$, and any $\nu \in P(V)$, the vector $w \cdot \lambda - w' \cdot (\lambda - \nu)$ does not belong to $P(V)$.

If $\lambda = \sum_i \lambda_i \omega_i$, where ω_i are fundamental weights and λ_i are large enough positive numbers then λ is generic with respect to V.

Lemma 5.4 [18] *Let $\lambda \in P^+$ be generic with respect to V. Let $v \in V[\nu]$. Consider the intertwining operator $\Phi_\lambda^v : M_\lambda \to M_{\lambda-\nu} \otimes V$. For $w \in \mathcal{W}$, consider the singular vector $v_{w\cdot\lambda}^\lambda \in M_\lambda$. Then there exists a unique vector $A_w(\lambda)(v) \in V[w(\nu)]$ such that*

$$\Phi_\lambda^v(v_{w\cdot\lambda}^\lambda) = v_{w\cdot(\lambda-\nu)}^{\lambda-\nu} \otimes A_w(\lambda)(v) \, + \, \textit{terms of lower weight in the first factor}.$$

For generic $\lambda \in P^+$, Lemma 5.4 defines a linear operator $A_w(\lambda) : V \to V$ such that $A_w(\lambda)(V[\nu]) \subset V[w(\nu)]$ for all $\nu \in P(V)$. This operator is extended to other values of λ as a rational function of λ.

The collection of rational functions $A_w(\lambda)$, $w \in \mathcal{W}$, is called *the dynamical Weyl group acting on V*.

Introduce new linear operators $\mathcal{A}_w(\lambda) : V \to V$ for $w \in \mathcal{W}$. Namely, for any $w \in \mathcal{W}$, $\nu \in P(V)$, $v \in V[\nu]$, set

$$\mathcal{A}_w(\lambda)\, v \; = A_w(\lambda + \nu)\, v \; .$$

We still have $\mathcal{A}_w(\lambda)(V[\nu]) \subset V[w(\nu)]$ for all $\nu \in P(V)$.

Lemma 5.5 [18]

- *For any $w_1, w_2 \in \mathcal{W}$ and $\nu \in P(V)$, we have*

  $$\big(\mathcal{A}_{w_1}(w_2 \cdot \lambda)\, \mathcal{A}_{w_2}(\lambda)\big)\, |_{V[\nu]} = c_{w_1,w_2,\lambda,\nu}\; \mathcal{A}_{w_1 w_2}(\lambda)\, |_{V[\nu]} \, ,$$

  where $c_{w_1,w_2,\lambda,\nu}$ is a constant depending on w_1, w_2, λ, ν.
- *For any $w, w_1, w_2 \in \mathcal{W}$, $\nu \in P(V)$, the limits*

  $$\mathcal{A}_w(\infty) = \lim_{\lambda\to\infty} \mathcal{A}_w(\lambda), \qquad c_{w_1,w_2,\nu} = \lim_{\lambda\to\infty} c_{w_1,w_2,\lambda,\nu}$$

  do exist. Therefore, we have

  $$(\mathcal{A}_{w_1}(\infty)\, \mathcal{A}_{w_2}(\infty))\, |_{V[\nu]} \; = \; c_{w_1,w_2,\nu}\; \mathcal{A}_{w_1 w_2}(\infty)\, |_{V[\nu]}.$$

  Moreover, the collection of operators $\mathcal{A}_w(\infty)$, $w \in \mathcal{W}$, gives the canonical projective action of $\mathcal{W}$ on V.

- *For any vector $v \in V[\nu]$ and $w \in \mathcal{W}$, we have*

$$\mathcal{A}_w(\lambda)\ H_i\left(\lambda+\rho+\frac{\nu}{2}\right)\ v\ =\ H_i\left(w\cdot\lambda+\rho+\frac{w(\nu)}{2}\right)\ \mathcal{A}_w(\lambda)\, v\ .$$

Proof The first statement follows from [18, Theorems 8 and 10]. The second statement is [18, Corollary 14]. The statement of [18, Lemma 18], which holds for any root system, gives the last statement of our lemma. □

According to this lemma, if ω is an eigenvector of the operators $H_i(\lambda+\rho+\frac{\nu}{2})$, then $\mathcal{A}_w(\lambda)\,\omega$ is an eigenvector of the operators $H_i(w\cdot\lambda+\rho+\frac{w(\nu)}{2})$.

Let λ be generic. Let $\boldsymbol{t}$ be a solution of the Bethe Ansatz equation associated to (Λ_∞,λ) and let $\boldsymbol{y}$ be the corresponding r–tuple of polynomials. Then $\omega(\boldsymbol{t})$ is an eigenvector of the operators $H_i(\lambda+\rho+\Lambda_\infty/2)$.

By Corollary 4.12, for each element w of the Weyl group, we have the descendent $w\boldsymbol{y}$ of $\boldsymbol{y}$ obtained via the reproduction procedure. Let $w\boldsymbol{y}$ represent the tuple $\boldsymbol{t}_w$. Moreover, if $w\boldsymbol{y}$ is off-diagonal, then $\boldsymbol{t}_w$ is a critical point associated to $(w\Lambda_\infty, w\cdot\lambda)$ and $\omega(\boldsymbol{t}_w)$ is an eigenvector of the operators $H_i(w\cdot\lambda+\rho+w\Lambda_\infty/2)$.

We conjecture that the action of the operator $\mathcal{A}_w(\lambda)$ coincides with the action of the Weyl group, induced by the reproduction procedure (when the latter action is well-defined). More precisely, we have

Conjecture 5.6 *Let λ be generic. Let $\boldsymbol{t}$ be a critical point of the master function* (2) *associated to (Λ_∞,λ) and let $\boldsymbol{y}$ be the corresponding r–tuple of polynomials. Let $\omega(\boldsymbol{t})\in V[\Lambda_\infty]$ be the corresponding Bethe vector. Let $w\in\mathcal{W}$. Assume that $w\boldsymbol{y}$ is off-diagonal. Let $w\boldsymbol{y}$ represent the tuple $\boldsymbol{t}_w$.*

Then the vector $\mathcal{A}_w(\lambda)\,\omega(\boldsymbol{t})$ is a scalar multiple of the Bethe vector $\omega(\boldsymbol{t}_w)$.

Below we prove this conjecture for sl_2, see Theorem 5.7.

5.4 The case of sl_2

Let $L_1,\dots,L_n$ be irreducible finite-dimensional sl_2 modules of highest weights $\Lambda_1,\dots,\Lambda_n\in\mathbb{Z}_{\ge 0}$. Let $v_1,\dots,v_n$ be the corresponding highest weight vectors. Let $V=L_1\otimes\cdots\otimes L_n$. We also fix an n–tuple of non-zero distinct complex numbers $\boldsymbol{z}=(z_1,\dots,z_n)$ and $l\in\mathbb{Z}_{\ge 0}$. We set $\Lambda_\infty=\sum_{s=1}^n\Lambda_s-2l$.

In the case of sl_2 the weight function $\omega(\boldsymbol{t})$ can be rewritten in the following form. We say $\boldsymbol{m} = (m_1, \dots, m_n) \in \mathcal{C}(\boldsymbol{\Lambda}, \Lambda_\infty)$ if $m_s \in \{0, \dots, \Lambda_i\}$, $s = 1, \dots, n$, and $\sum_{s=1}^n m_s = l$. Set

$$\omega_{\boldsymbol{m}}(\boldsymbol{t}) = (\prod_{j=1}^n (m_j!)^{-1}) Sym \prod_{s=1}^n \prod_{i=m_1+\dots+m_{s-1}+1}^{m_1+\dots+m_s} \frac{1}{t_i - z_s},$$

where Sym denotes the symmetrization with respect to $t_1, \dots, t_l$. Let

$$F^{\boldsymbol{m}} \boldsymbol{v} := F^{m_1} v_1 \otimes \dots \otimes F^{m_n} v_n.$$

Then we explicitly have

$$\omega(\boldsymbol{t}) = \sum_{\boldsymbol{m} \in \mathcal{C}(\boldsymbol{\Lambda}, \Lambda_\infty)} \omega_{\boldsymbol{m}}(\boldsymbol{t}) F^{\boldsymbol{m}} \boldsymbol{v}.$$

Recall that if $\boldsymbol{t}$ is a critical point of the master function (2) then the vector $\omega(\boldsymbol{t})$ is called the Bethe vector.

It follows from [16], that there exists a Zariski open set $U_1 = U(\boldsymbol{\Lambda})$ in $\mathbb{C}^n$ such that for any $\boldsymbol{z} \in U_1$ there exists a Zariski open set $U_2 = U_2(\boldsymbol{\Lambda}, \boldsymbol{z})$ in $\mathbb{C}$ such for all $\lambda \in U_2(\boldsymbol{z})$, the number of orbits of critical points of the sl_2 master function (2) associated to $(\Lambda_\infty, \lambda)$ equals to the dimension of the subspace of V of vectors of weight $\Lambda_\infty = \sum_{s=1}^n \Lambda_s - 2l$. Moreover all critical points are non-degenerate and the corresponding Bethe vectors form a basis in this subspace.

Theorem 5.7 *Let w be the generator of the sl_2 Weyl group. For $\boldsymbol{z} \in U_1$, there exists a Zariski open set $U_3(\boldsymbol{z}) \subset \mathbb{C}$ with the following property. Let $\lambda \in U_3(\boldsymbol{z})$, let $\boldsymbol{t}$ be a critical point associated to $(\Lambda_\infty, \lambda)$ and let y be the corresponding polynomial. Let $\boldsymbol{t}_w$ be the tuple represented by the polynomial wy.*

Then all roots of the polynomial wy are simple and the vector $\mathcal{A}_w(\lambda)\,\omega(\boldsymbol{t})$ is a non-zero scalar multiple of the Bethe vector $\omega(\boldsymbol{t}_w)$.

Proof In the sl_2 case $\nu, \mu \in \mathbb{C}$, and the Casimir operator is given by $C = h \otimes h/2 + e \otimes f + f \otimes e$.

We claim that the joint spectrum of $H_k(\lambda + \rho + \Lambda_\infty/2)$, $k = 1, \dots, n$, acting in $V[\Lambda_\infty]$, is generically simple. Indeed, in the limit $\lambda \to \infty$ the main term is given by the operators $\lambda h^{(k)}$. The joint spectrum of commuting operators $h^{(k)}$, $k = 1, \dots, n$, is simple. Therefore $H_k(\lambda + \rho + \Lambda_\infty/2)$, $k = 1, \dots, n$, for generic λ have a simple joint spectrum as well.

It follows that the dynamical Weyl group maps the Bethe vectors to the Bethe vectors.

Now we compare the two actions. We do it in the same limit $\lambda \to \infty$.

The common eigenvectors of operators $h^{(i)}$ are monomial vectors $F^{\boldsymbol{m}}\boldsymbol{v}$. The Weyl group of sl_2 is generated by the element w, $w^2 = id$ which acts on the weight vectors by

$$w(F^{m_1}v_1 \otimes \cdots \otimes F^{m_n}v_n) = cF^{\Lambda_1 - m_1}v_1 \otimes \cdots \otimes F^{\Lambda_n - m_n}v_n, \tag{17}$$

where c is some non-zero constant depending on m_i, Λ_i.

By Lemma 5.5, the limit $\lambda \to \infty$, the dynamical Weyl group action on the Bethe vectors coincides (up to a scalar) with the action of the Weyl group (17).

Let us consider the limit of the action defined in terms of the reproduction procedure. It is shown in the proof of Theorem 3.2 that if y represents an sl_2 critical point and $\tilde{y}$ is the immediate descendent, then for almost all λ, $y, \tilde{y}$ can be included in a family of critical points y_a, and their descendents $\tilde{y}_a$ and in the limit $\lambda_a \to \infty$ the product $y_a \tilde{y}_a$ tends to $T = \prod_{i=1}^n (t - z_i)^{\Lambda_i}$.

Finally we claim that if the polynomials y_a of degree $l = \sum_{i=1}^n m_i$, represent critical points associated to λ_a and the limit of y_a as λ_a tend to ∞ is $\prod_i (x - z_i)^{m_i}$, then the corresponding Bethe vectors tend to a scalar multiple of the monomial vector $F^{\boldsymbol{m}}\boldsymbol{v}$.

For $i = 1, \ldots, l$, let $s(i) \in \{1, \ldots, n\}$ be such that the i-th root of y, t_i, tends to $z_{s(i)}$. Then we write $t_i(\lambda) = z_{s(i)} + c_i/\lambda + o(1/\lambda)$. The Bethe Ansatz equation for t_i implies that for any $j = 1, \ldots, n$, the set of $\{c_i \mid s(i) = j\}$ satisfy the Bethe Ansatz equation with $n = 1$:

$$-\frac{\Lambda_j}{c_i} + \sum_{k,\ k \neq i,\ s(k) = s(i)} \frac{2}{c_i - c_k} = 1.$$

These equations are solved explicitly. The solutions are limits of [19, formulas (1.3.2)] as $\beta \to \infty$. It follows that all c_i with $s(i) = j$ are different from zero and from each other.

Now consider the limit of the corresponding Bethe vector. The dominant term is

$$\lambda^l \prod_{j=1}^n (m_j!)^{-1} \prod_{i=1}^l c_i^{-1} F^{\boldsymbol{m}}\boldsymbol{v}.$$

This finishes the proof of the claim and the theorem. □

6 Exponential populations

We considered in detail the trigonometric Gaudin model, where the Bethe Ansatz equation takes the form (3). There are other models, where the reproduction procedure for the solutions of the Bethe Ansatz equation works in the same way and one obtains a transitive and free Weyl group action on each population. One such model, the quasi-periodic Gaudin model, is considered in this section, another one, the quasi-periodic XXX model, is considered in Section 7.

We fix our $\mathfrak{g}, \boldsymbol{\Lambda}, \Lambda_\infty, \boldsymbol{l}, \lambda$ as in Section 2.2. Let $z_1, \dots, z_n$ be any distinct complex numbers. Consider the *master function with exponents*

$$\Phi^{exp}(\boldsymbol{t}; \Lambda_\infty; \lambda) = \prod_{i=1}^{r} \prod_{j=1}^{l_i} e^{-(\lambda,\alpha_i) t_j^{(i)}} \prod_{i=1}^{r} \prod_{j=1}^{l_i} \prod_{s=1}^{n} (t_j^{(i)} - z_s)^{-(\Lambda_s,\alpha_i)} \times \tag{18}$$
$$\prod_{i=1}^{r} \prod_{1\le j<s\le l_i} (t_j^{(i)} - t_s^{(i)})^{(\alpha_i,\alpha_i)} \prod_{1\le i<j\le r} \prod_{s=1}^{l_i} \prod_{k=1}^{l_j} (t_s^{(i)} - t_k^{(j)})^{(\alpha_i,\alpha_j)}.$$

We call $\boldsymbol{t} = (t_j^{(i)})_{i=1,\dots,r}^{j=1\dots l_i}$ *a critical point of the master function with exponents associated to* $(\Lambda_\infty, \lambda)$ if

$$-(\lambda,\alpha_i) - \sum_{s=1}^{n} \frac{(\Lambda_s,\alpha_i)}{t_j^{(i)} - z_s} + \sum_{s,\ s\neq i} \sum_{k=1}^{l_s} \frac{(\alpha_s,\alpha_i)}{t_j^{(i)} - t_k^{(s)}} + \sum_{s,\ s\neq j} \frac{(\alpha_i,\alpha_i)}{t_j^{(i)} - t_s^{(i)}} = 0, \tag{19}$$

for $i = 1, \dots, r,\ j = 1, \dots, l_i$.

We have analogs of Propositions 5.2 and 5.3.

Proposition 6.1 *Let $\mathfrak{g} = sl_{N+1}$. For almost all λ the number of orbits of critical points of the master function with exponents associated to $(\Lambda_\infty, \lambda)$ and counted with multiplicities does not exceed the dimension of the subspace of the weight Λ_∞ in the tensor product $L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n}$.*

Proof The number of critical points of the master function with exponents (18) is finite for almost all λ, see [6].

Replacing the factors $e^{-(\lambda,\alpha_i)t_j^{(i)}}$ in the master function (18) with $(1 + t_j^{(i)}/m)^{-(\lambda,\alpha_i)m}$ we obtain a master function of type (2). Therefore the function (18) is the limit of master functions of type (2) as $m \to \infty$. The proposition now follows from Proposition 5.2 and the fact that the number of orbits of isolated critical points of a function counted with multiplicity does not change under small deformations of the function. □

A different proof of Proposition 6.1 which uses Schubert Calculus is given in [6].

Proposition 6.2 *Let $\mathfrak{g} = sl_2$. For almost all $\lambda \in \mathbb{C}$ and almost all $(z_1, \dots, z_n) \in \mathbb{C}^n$, the number of orbits of critical points of the master function with exponents associated to $(\Lambda_\infty, \lambda)$ equals the dimension of the subspace of weight Λ_∞ in the tensor product $L_{\Lambda_1} \otimes \dots \otimes L_{\Lambda_n}$. Moreover all these points are non-degenerate.*

Proof If λ is a large positive integer then the proposition is proved by using methods of [12]. The rest is similar to the proof of Proposition 5.3 $\square$

Let $\mathfrak{g}$ be a Kac–Moody algebra. As in Section 3.2 we represent a tuple $\boldsymbol{t} = (t_j^{(i)})_{i=1,\dots,r}^{j=1,\dots,l_i}$ by the r–tuple of polynomials $\boldsymbol{y} = (y_1, \dots, y_r)$, where $y_i = \prod_{j=1}^{l_i}(x - t_j^{(i)})$, $i = 1, \dots, r$. We make no distinction between $(y_1, \dots, y_r)$ and $(c_1 y_1, \dots, c_r y_r)$ where $c_1, \dots, c_r$ are non-zero complex numbers. We introduce polynomials T_i, $i = 1, \dots, r$, by formula (7).

We call an r–tuple of polynomials $\boldsymbol{y}$ *exponentially off-diagonal* if its roots do not belong to the union of singular hyperplanes in (19). Namely $\boldsymbol{y}$ is exponentially off-diagonal if for $i = 1, \dots, r$, all roots of the polynomial y_i are simple, different from the roots of the polynomials y_j for all j such that $(\alpha_j, \alpha_i) \neq 0$ and different from the roots of the polynomial T_i.

We have the corresponding exponential reproduction procedure. Namely, an r–tuple of polynomials $\boldsymbol{y}$ is called *exponentially fertile in the i-th direction with respect to λ*, $i \in \{1, \dots, r\}$, if there exists a polynomial $\tilde{y}_i$ such that

$$W(y_i, e^{\langle \lambda, \alpha_i^\vee \rangle x} \tilde{y}_i) = e^{\langle \lambda, \alpha_i^\vee \rangle x} T_i \prod_{j=1, j \neq i}^{r} y_j^{-a_{ij}}.$$

Then the r–tuple of polynomials $y^{(i)} = (y_1, \dots, \tilde{y}_i, \dots, y_r)$ is called *an exponential immediate descendent of $\boldsymbol{y}$ with respect to λ in the direction i.*

An r–tuple of polynomials is called *exponentially fertile with respect to λ* if it is exponentially fertile with respect to λ in all directions $i = 1, \dots, r$.

Theorem 6.3 *Let $\mathfrak{g}$ be a Kac–Moody algebra. An r–tuple of polynomials $\boldsymbol{y}$ represents a $\mathfrak{g}$ critical point of the master function with exponents associated to $(\Lambda_\infty, \lambda)$ if and only if $\deg y_i = l_i$, $\boldsymbol{y}$ is exponentially off-diagonal and exponentially fertile with respect to λ. Moreover, if $\boldsymbol{y}$ represents a critical point of the master function with exponents associated to $(\Lambda_\infty, \lambda)$ and if the immediate descendent of $\boldsymbol{y}$ with respect λ in the i-th direction, $y^{(i)} = (y_1, \dots, \tilde{y}_i, \dots, y_r)$, is exponentially off-diagonal then*

$y^{(i)}$ represents a critical point of the master function with exponents associated to $(s_i\Lambda_\infty, s_i\lambda)$.

Proof The proof is similar to the proof of Theorem 3.5. □

An r–tuple of polynomials $\boldsymbol{y}^{(i_1,i_2,\dots,i_m)}$, where $i_k \in \{1,\dots,r\}$, $k=1,\dots,m$, is called *an exponential descendent of $\boldsymbol{y}$ with respect to λ in the directions $(i_1,\dots,i_m)$* if there exist r–tuples of polynomials $\boldsymbol{y}^{(i_1,i_2,\dots,i_k)}$, $k=1,\dots,m-1$, such that for $k=1,\dots,m$, the r–tuple $\boldsymbol{y}^{(i_1,i_2,\dots,i_k)}$ is an exponential immediate descendent of $\boldsymbol{y}^{(i_1,i_2,\dots,i_{k-1})}$ with respect to $s_{i_{k-1}}\dots s_{i_2}s_{i_1}\lambda$ in the i_k-th direction.

An r–tuple of polynomials $\boldsymbol{y}$ is called *exponentially super-fertile with respect to λ* if it is exponentially fertile with respect to λ and all exponential descendents $\boldsymbol{y}^{(i_1,i_2,\dots,i_m)}$ of $\boldsymbol{y}$ with respect to λ in the directions $(i_1,\dots,i_m))$ are exponentially fertile with respect to $s_{i_m}\dots s_{i_1}\lambda$.

For any N–tuple of functions $\boldsymbol{y}$ and an sl_{N+1} weight λ, we set $y_{N+1}=1$ and define the linear differential operator of order $N+1$:

$$D^{exp}(\boldsymbol{y},\lambda) = \prod_{i}^{N\to 0}\left(\partial \;-\; \ln'\left(\frac{y_{i+1}\prod_{s=1}^{i} e^{(\lambda,\alpha_s)x}T_s}{y_i}\right)\right).$$

Proposition 6.4 *Let $\boldsymbol{y}$ be an N–tuple of polynomials and $l_i = \deg y_i$, $i=1,\dots,N$. Let $\boldsymbol{\Lambda}_\infty$ be given by* (1)*. Let $\boldsymbol{y}$ represent an sl_{N+1} critical point of the master function with exponents associated to (Λ_∞,λ) or let $\boldsymbol{y}$ be exponentially super-fertile with respect to λ. Then the kernel of the operator $D^{exp}(\boldsymbol{y},\lambda)$ is spanned by functions of the form*

$$p_0,\, p_1e^{(\lambda,\alpha_1)x},\dots,\, p_Ne^{(\lambda,\alpha_1+\dots+\alpha_N)x}, \tag{20}$$

where p_i is a polynomial of degree $\deg y_1 + (\Lambda_\infty, \alpha_1+\dots+\alpha_i)$. The only singular points of the operator $D^{exp}(\boldsymbol{y},\lambda)$ in $\mathbb{C}$ are regular singular points located at $z_1,\dots,z_n$, and the exponents at z_i, $i=1,\dots,n$, are

$$z_i : 0, (\Lambda_i+\rho,\alpha_1), (\Lambda_i+\rho,\alpha_1+\alpha_2),\dots,(\Lambda_i+\rho,\alpha_1+\dots+\alpha_N), \tag{21}$$

Conversely, if a linear differential operator D of order $N+1$ has the kernel spanned by functions of the form (20) *and the only singular points of D in $\mathbb{C}$ are regular singular points at $z_1,\dots,z_n$ with the exponents given by* (21)*, then the N–tuple $\boldsymbol{y}$ given by the divided Wronskians*

$$y_i = \frac{W(p_0, p_1e^{(\lambda,\alpha_1)x},\dots,p_{i-1}e^{(\lambda,\sum_{j=1}^{i-1}\alpha_j)x})}{e^{(\lambda,\sum_{j=1}^{i-1}(i-j)\alpha_j)x}\prod_{j=1}^{i-1}T_j^{i-j}},$$

$i = 1, \dots, N$, is an N–tuple of polynomials which is exponentially super-fertile with respect to λ and satisfies $\deg y_i = l_i$, *$i = 1, \dots, N$.*

Proof The proof is similar to the proof of Corollary 4.4. □

Conjecture 6.5 *If an r–tuple of polynomials $\boldsymbol{y}$ represents a critical point of the master function with exponents associated to $(\Lambda_\infty, \lambda)$ then $\boldsymbol{y}$ is exponentially super-fertile with respect to λ.*

Theorem 6.6 *Conjecture 6.5 holds for the case of simple Lie algebras of types A_N and B_N.*

Proof The proof is similar to the proof of Theorems 4.5, 4.8. □

For an r–tuple of polynomials $\boldsymbol{y}$ and a $\mathfrak{g}$ weight λ, we denote $P^{exp}(\boldsymbol{y}, \lambda)$ the set of all pairs of the form $(\boldsymbol{y}^{(i_1, i_2, \dots, i_m)}, s_{i_m} \dots s_{i_2} s_{i_1} \lambda)$, where $m \in \mathbb{Z}_{\geq 0}$, $i_k \in \{1, \dots, r\}$, $k = 1, \dots, m$, and $\boldsymbol{y}^{(i_1, i_2, \dots, i_m)}$ is an exponential descendent of $\boldsymbol{y}$ with respect to λ in the directions $(i_1, \dots, i_m)$.

We call the set $P^{exp}(\boldsymbol{y}, \lambda)$ *the exponential prepopulation originated at* $(\boldsymbol{y}, \lambda)$. Let an r–tuple of polynomials $\boldsymbol{y}$ be exponentially super-fertile with respect to λ. We call the set $P^{exp}(\boldsymbol{y}, \lambda)$ *the exponential population originated at* $(\boldsymbol{y}, \lambda)$.

Theorem 6.7 *Let $\mathfrak{g}$ be any simple Lie algebra and let λ be a strongly non-integral $\mathfrak{g}$–weight. Let an r–tuple of polynomials $\boldsymbol{y}$ be exponentially super-fertile with respect to λ. Then the map $P^{exp}(\boldsymbol{y}, \lambda) \to \mathcal{W}\lambda$ such that $(\widetilde{\boldsymbol{y}}, \widetilde{\lambda}) \mapsto \widetilde{\lambda}$ is a bijection of the exponential population originated at $(\boldsymbol{y}, \lambda)$ and of the orbit of the Weyl group.*

Proof The proof is similar to the proof of Corollary 4.12. □

7 Difference reproduction

In this section we describe the Bethe Ansatz equation corresponding to the quasi-periodic XXX model. In this case the reproduction procedure works in a similar way and one obtains a free and transitive Weyl group action on a population.

Let $h \in \mathbb{C}$ be a complex non-zero number. We fix $\mathfrak{g}, \boldsymbol{\Lambda}, \Lambda_\infty, \boldsymbol{l}, \lambda$ as in Section 2.2. Let $z_1, \dots, z_n$ be any distinct complex numbers, subject to the conditions $z_i - z_j \notin h\mathbb{Z}$ for all $i, j \in \{1, \dots, n\}$, $i \neq j$.

Consider the *exponential* XXX *Bethe equation* on variables $\boldsymbol{t}=(t_j^{(i)})_{i=1,\dots,r}^{j=1,\dots,l_i}$:

$$e^{\langle\lambda,\alpha_i^\vee\rangle h}=\prod_{s=1}^{n}\frac{t_j^{(i)}-z_s+(\Lambda_s,\alpha_i)h/2}{t_j^{(i)}-z_s-(\Lambda_s,\alpha_i)h/2}\times \prod_{\substack{m=1,\dots,r,\\ m\neq i}}\left(\prod_{k=1}^{l_m}\frac{t_j^{(i)}-t_k^{(m)}+h/2}{t_j^{(i)}-t_k^{(m)}-h/2}\right)^{-a_{im}}\prod_{\substack{k=1,\dots,l_i,\\ k\neq j}}\frac{t_j^{(i)}-t_k^{(i)}-h}{t_j^{(i)}-t_k^{(i)}+h}, \tag{22}$$

where $i=1,\dots,r,\ j=1,\dots,l_i$.

As in Section 3.2 we represent a tuple $\boldsymbol{t}=(t_j^{(i)})_{i=1,\dots,r}^{j=1,\dots,l_i}$ by the r–tuple of polynomials $\boldsymbol{y}=(y_1,\dots,y_r)$, where $y_i=\prod_{j=1}^{l_i}(x-t_j^{(i)})$, $i=1,\dots,r$. We make no distinction between $(y_1,\dots,y_r)$ and $(c_1y_1,\dots,c_ry_r)$ where $c_1,\dots,c_r$ are non-zero complex numbers.

For $i=1,\dots,r$, set

$$T_i^{(h)}(x)=\prod_{s=1}^{n}\prod_{j=1}^{(\Lambda_s,\alpha_i)}(x-z_s-(\Lambda_s,\alpha_i)h/2+jh).$$

An r–tuple of polynomials $\boldsymbol{y}$ is called *exponentially difference off-diagonal with respect to* $(\boldsymbol{\Lambda},\boldsymbol{z};h)$ if for $i=1,\dots,r$ the polynomial $y_i(x)$ has only simple roots, different from the roots of polynomials $y_m(x+h/2)$, whenever $(\alpha_i,\alpha_m)\neq 0$, and different from the roots of polynomials $T_i^{(h)}$, $y_i(x+h)$.

A solution $\boldsymbol{t}$ of (22) is called *off-diagonal* if the corresponding r–tuple of polynomials is exponentially difference off-diagonal.

Lemma 7.1 *A polynomial y of degree l represents an sl_2 off-diagonal solution of exponential* XXX *Bethe equation associated to (Λ_∞,λ) if and only if y is exponentially difference off-diagonal and there exists a polynomial $B(x)$ such that*

$$y(x+h)e^{\langle\lambda,\alpha^\vee\rangle h}\prod_{s=1}^{n}(x-z_s-\frac{(\Lambda_s,\alpha)h}{2}) + B(x)y(x)+y(x-h)\prod_{s=1}^{n}(x-z_s+\frac{(\Lambda_s,\alpha)h}{2})=0.$$

Proof The lemma is proved similarly to Lemma 3.1. □

Proposition 7.2 *Let $\mathfrak{g} = sl_{N+1}$. For almost all λ the number of orbits of off-diagonal solutions of the exponential XXX Bethe Ansatz equations associated to $(\Lambda_\infty, \lambda)$ does not exceed the dimension of the subspace of the weight Λ_∞ in the tensor product $L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n}$.*

Proof The proof is similar to the proof of the Proposition 6.1 with the help of [8, Corollary 4.15]. □

Proposition 7.3 [18] *Let $\mathfrak{g} = sl_2$. For almost all λ and almost all $(z_1, \dots, z_n) \in \mathbb{C}^n$, the number of orbits of solutions $\boldsymbol{t}$ of the exponential XXX Bethe Ansatz equation associated to $(\Lambda_\infty, \lambda)$ such that $t_i \neq t_j$ equals the dimension of the subspace of weight Λ_∞ in the tensor product $L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n}$. Moreover all such solutions are non-degenerate.*

We now describe the corresponding exponential difference reproduction procedure.

Denote W_h the discrete Wronskian:

$$W_h(f_1, \dots, f_N) := \det(f_i(x + (j-1)h))_{i,j=1,\dots,N}.$$

An r–tuple of polynomials $\boldsymbol{y}$ is called *exponentially difference fertile with respect to λ in the i-th direction*, $i \in \{1, \dots, r\}$, if there exists a polynomial $\tilde{y}_i$ such that

$$W_h(y_i, e^{\langle \lambda, \alpha_i^\vee \rangle x} \tilde{y}_i) = e^{\langle \lambda, \alpha_i^\vee \rangle x} \; T_i^{(h)}(x) \prod_{m=1,\, m \neq i}^{r} (y_m(x + h/2))^{-a_{im}}.$$

Then the r–tuple of polynomials $y^{(i)} = (y_1, \dots, \tilde{y}_i, \dots, y_r)$ is called *an exponential difference immediate descendent of $\boldsymbol{y}$ with respect to λ in the i-th direction*.

An r–tuple is called *exponentially difference fertile with respect to λ* if it is exponentially difference fertile with respect to λ in all directions $i = 1, \dots, r$.

Theorem 7.4 *An r–tuple of polynomials $\boldsymbol{y}$ represents an off-diagonal solution of the exponential XXX Bethe Ansatz equation associated to $(\Lambda_\infty, \lambda)$ if and only if $\boldsymbol{y}$ is exponentially difference off-diagonal,* $\deg y_i = l_i$, $i = 1, \dots, r$, *and $\boldsymbol{y}$ is exponentially difference fertile with respect to λ. Moreover, if $\boldsymbol{y}$ represents an off-diagonal solution of the exponential XXX Bethe Ansatz equation associated to $(\Lambda_\infty, \lambda)$ and if the exponential difference immediate descendent of $\boldsymbol{y}$ with respect λ in the i-th direction, $y^{(i)} = (y_1, \dots, \tilde{y}_i, \dots, y_r)$, is exponentially difference off-diagonal then $y^{(i)}$ represents an off-diagonal solution of the exponential XXX Bethe Ansatz equation associated to $(s_i \Lambda_\infty, s_i \lambda)$.*

Proof The proof is similar to the proof of Theorem 3.5, cf [10] and also [13; 8]. □

An r–tuple of polynomials $\boldsymbol{y}^{(i_1,i_2,\dots,i_m)}$, where $m \in \mathbb{Z}_{\geq 0}$, $i_k \in \{1,\dots,r\}$, $k = 1,\dots,m$, is called *an exponential difference descendent of* $\boldsymbol{y}$ *with respect to* λ *in the directions* $(i_1,\dots,i_m)$ if there exist r–tuples of polynomials $\boldsymbol{y}^{(i_1,i_2,\dots,i_k)}$, $k = 1,\dots,m-1$, such that for $k = 1,\dots,m$, the r–tuple $\boldsymbol{y}^{(i_1,i_2,\dots,i_k)}$ is an exponential difference immediate descendent of $\boldsymbol{y}^{(i_1,i_2,\dots,i_{k-1})}$ with respect to $s_{i_{k-1}}\dots s_{i_2}s_{i_1}\lambda$ in the i_k-th direction.

An r–tuple of polynomials $\boldsymbol{y}$ is called *exponentially difference super-fertile with respect to* λ if it is exponentially difference fertile with respect to λ and all exponential descendents $\boldsymbol{y}^{(i_1,i_2,\dots,i_m)}$ of $\boldsymbol{y}$ with respect to λ in the directions $(i_1,i_2,\dots,i_m)$ are exponentially difference fertile with respect to $s_{i_m}\dots s_{i_1}\lambda$.

For any N–tuple of functions $\boldsymbol{y}$ and an sl_{N+1} weight λ, we set $y_{N+1}=1$ and define the linear difference operator:

$$D_h^{exp}(\boldsymbol{y},\lambda) = \prod_i^{N\to 0}\left(\partial_h - \frac{y_{i+1}(x+(i+2)h/2)}{y_{i+1}(x+ih/2)}\frac{y_i(x+(i-1)h/2)}{y_i(x+(i+1)h/2)} \times \prod_{s=1}^{i}\frac{e^{h(\lambda,\alpha_s)}T_s(x+(2i-s+1)h/2)}{T_s(x+(2i-s-1)h/2)}\right),$$

where ∂_h is the operator acting on functions of x by the formula $\partial_h(f(x)) = f(x+h)$.

Let V be a space spanned by functions of the type $p_0e^{\lambda_0 x}, p_1e^{\lambda_1 x},\dots,p_Ne^{\lambda_N x}$ where p_i, $i=0,\dots,N$, are polynomials and $\lambda_i \in \mathbb{C}$, $i=0,\dots,N$. We say the space V *has no base points* if for any $z\in\mathbb{C}$ there exists $f\in V$, such that $f(z)\neq 0$.

Assume V has no base points. For $i=2,\dots,N$, let U_i be the monic polynomial of the greatest possible degree such that $W_h(f_1,\dots,f_i)/U_i$ is a holomorphic function for all $f_1,\dots,f_i\in V$. Following [8], we call an N–tuple of monic polynomials $(T_1,\dots,T_N)$ *a frame of space* V if for $i=2,\dots,N$ we have $U_i = \prod_{j=1}^{i-1}\prod_{s=1}^{i-j} T_j(x+(s-1)h)$.

Lemma 7.5 *Let V be a space spanned by functions of the type*

$$p_0e^{\lambda_0 x}, p_1e^{\lambda_1 x},\dots,p_Ne^{\lambda_N x}.$$

Let V have no base points. Then there exists a unique frame of V.

Proof The proof is similar to the proof of [8, Lemma 4.9]. □

Proposition 7.6 *Let $\boldsymbol{y}$ be an N–tuple of polynomials and $l_i = \deg y_i$, $i = 1, \dots, N$. Let $\boldsymbol{\Lambda}_\infty$ be given by* (1)*. Let $\boldsymbol{y}$ represent an off-diagonal solution of sl_{N+1} exponential* XXX *Bethe Ansatz equation associated to $(\Lambda_\infty, \lambda)$ or let $\boldsymbol{y}$ be exponentially difference super-fertile with respect to λ. Then the kernel of the operator $D_h^{exp}(\boldsymbol{y}, \lambda)$ is spanned by functions of the form*

$$p_0,\, p_1 e^{(\lambda,\alpha_1)x}, \dots, p_N e^{(\lambda,\alpha_1+\dots+\alpha_N)x}, \tag{23}$$

where p_i is a polynomial of degree $\deg y_1 + (\Lambda_\infty, \alpha_1 + \dots + \alpha_i)$. Moreover, the N–tuple

$$(T_1^{(h)}(x), T_2^{(h)}(x+h/2), \dots, T_N^{(h)}(x+(N-1)h/2)) \tag{24}$$

is the frame of the kernel of the operator $D_h^{exp}(\boldsymbol{y}, \lambda)$.

Conversely, if a linear difference operator D of order $N+1$ has the kernel spanned by functions of the form (23) *with the frame* (24) *then the N–tuple $\boldsymbol{y}$ given by*

$$y_i = \frac{W_h(p_0, p_1 e^{(\lambda,\alpha_1)x}, \dots, p_{i-1} e^{(\lambda,\sum_{j=1}^{i-1}\alpha_j)x})}{e^{(\lambda,\sum_{j=1}^{i-1}(i-j)\alpha_j)x}\prod_{s=1}^{i-j} T_j^{(h)}(x+(s+j/2-3/2)h)},$$

$i = 1, \dots, N$, is an N–tuple of polynomials which is exponentially difference super-fertile with respect to λ and satisfies $\deg y_i = l_i$, $i = 1, \dots, N$.

Proof The proof is similar to the proof of Corollary 4.4. □

Conjecture 7.7 *Let $\mathfrak{g}$ be any simple Lie algebra. If an r–tuple of polynomials $\boldsymbol{y}$ represents an off-diagonal solution of the exponential* XXX *Bethe Ansatz equation associated to $(\Lambda_\infty, \lambda)$ then $\boldsymbol{y}$ is exponentially difference super-fertile with respect to λ.*

Theorem 7.8 *Conjecture 7.7 holds for the case of simple Lie algebras of types A_N and B_N.*

Proof The proof is similar to the proof of Theorems 4.5, 4.8. □

For an r–tuple of polynomials $\boldsymbol{y}$ we denote $P_h^{exp}(\boldsymbol{y}, \lambda)$ the set of all pairs of the form $(\boldsymbol{y}^{(i_1,i_2,\dots,i_m)}, s_{i_m} \dots s_{i_2} s_{i_1} \lambda)$, where $m \in \mathbb{Z}_{\geq 0}$, $i_k \in \{1, \dots, r\}$, $k = 1, \dots, m$, and $\boldsymbol{y}^{(i_1,i_2,\dots,i_m)}$ is an exponential difference descendent of $\boldsymbol{y}$ with respect to λ in directions $(i_1, \dots, i_m)$.

We call the set $P_h^{exp}(\boldsymbol{y}, \lambda)$ *the exponential difference prepopulation originated at* $(\boldsymbol{y}, \lambda)$. If an r–tuple of polynomials $\boldsymbol{y}$ is exponentially difference super-fertile with respect to λ, then we call the set $P_h^{exp}(\boldsymbol{y},\lambda)$ *the exponential difference population originated at* $(\boldsymbol{y},\lambda)$.

Theorem 7.9 *Let $\mathfrak{g}$ be any simple Lie algebra and let λ be a strongly non-integral $\mathfrak{g}$–weight. Let an r–tuple of polynomials $\boldsymbol{y}$ be exponentially difference super-fertile with respect to λ. Then the map $P_h^{exp}(\boldsymbol{y},\lambda)\to\mathcal{W}\lambda$ such that $(\tilde{\boldsymbol{y}},\tilde{\lambda})\mapsto\tilde{\lambda}$ is a bijection of the exponential difference population originated at $(\boldsymbol{y},\lambda)$ and of the orbit of the Weyl group.*

Proof The proof is similar to the proof of Corollary 4.12. □

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Department of Mathematical Sciences, Indiana University – Purdue University Indianapolis
402 North Blackford St, Indianapolis, IN 46202-3216, USA

Department of Mathematics, University of North Carolina at Chapel Hill
Chapel Hill, NC 27599-3250, USA

`mukhin@math.iupui.edu, anv@email.unc.edu`

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