Automorphic functions for a Kleinian group

MASAAKI YOSHIDA

In the paper 'Automorphic functions for a Whitehead-complement group' [5], Matsumoto, Nishi and Yoshida constructed automorphic functions on real 3–dimensional hyperbolic space for a Kleinian group called the Whitehead-link-complement group. For a Kleinian group (of the first kind), no automorphic function/form has been studied before. In this note, their motivation is presented with a historical background.

11F55, 14P05; 57M25

Dedicated to Professor Fred Cohen on his sixtieth birthday

1 Introduction

In [5] Matsumoto, Nishi and Yoshida constructed automorphic functions on real 3–dimensional hyperbolic space for a Kleinian group called the Whitehead-link-complement group. For a Kleinian group (of the first kind), no automorphic function/form has been studied before, except by Matsumoto and Yoshida [9].

In this note, we first recall (from a very elementary level) branched covers of the complex projective line, especially those with three branch points; these are prototypes of our branched cover story. We also define, as higher dimensional generalizations, several branched covers of complex projective spaces, when the universal ones are the complex ball (complex hyperbolic sapce) and we briefly mention the history.

The main topic of this note is another higher dimensional generalization: branched covers of the 3–sphere. One of the simplest links/knots whose complements admit hyperbolic structures is the Whitehead link L; there is a discrete subgroup W of the automorphism group of real hyperbolic 3–space $\mathbb{H}^3_{\mathbb{R}}$, such that the quotient space $\mathbb{H}^3_{\mathbb{R}}/W$ is homeomorphic to the complement $S^3 - L$. We construct automorphic fucntions on $\mathbb{H}^3_{\mathbb{R}}$ for W and make the projection $\mathbb{H}^3_{\mathbb{R}} \to S^3 - L$ explicit.

2 Covers of the complex projective line

Let $X = \mathbb{P}^1$ be the complex projective line, which is also called the Riemann sphere. We are interested in its coverings. Since X is simply connected, there is no non-trivial covering unless we admit branch points.

Published: 19 March 2008 DOI: 10.2140/gtm.2008.13.483

2.1 One branch point

If we admit only one branch point, which we can assume to be the point at infinity, then since $X - \{\infty\} \cong \mathbb{C}$ is still simply connected, there is no non-trivial cover.

2.2 Two branch points

If we admit two branch points, which we can assume to be the origin and the point at infinity, and assign $p, q \in \{2, 3, \cdots, \infty\}$ as the indices, respectively, then such a covering of X exists if and only if $p = q$. Let us denote such a *covering space* by Z, and the *projection* by $\pi\colon Z \to X$; its (multi-valaued) inverse is called the *developing map* $s\colon X \to Z$. The covering (deck) transformation group, which is a group of automorphisms of Z, is denoted by Γ; note that the projection π is Γ–automorphic, ie, invariant under the action of Γ.

2.2.1 Case $p = q < \infty$ The covering space and the projection are given by

$$\pi\colon \mathbb{P}^1 \cong Z \ni z \longmapsto x = z^p \in X,$$

and the developing map is the multi-valed map $x \mapsto z = x^{1/p}$. Note that the projection is invariant under the finite group

$$\Gamma = \{z \mapsto e^{\frac{2\pi i k}{p}} z \mid k = 1, \ldots, p\}.$$

2.2.2 Case $p = q = \infty$ The covering space and the projection are given by

$$\pi\colon \mathbb{C} \cong Z \ni z \longmapsto x = \exp z \in X$$

and the developing map is the multi-valed map $x \mapsto z = \log x$. Note that the projection is invariant under the infinite group

$$\Gamma = \{z \mapsto z + 2\pi i k \mid k \in \mathbb{Z}\}.$$

In this way, two points on a sphere – a simple geometric object – naturally leads to very important functions the exponential and the logarithm functions.

2.3 Three branch points

If we admit three branch points, which we can assume to be $x = 0, 1, \infty$, and assign $p, q, r \in \{2, 3, \cdots, \infty\}$ as the indices, respectively, then such a covering of X always exists; there are many such. Let Z be the biggest one, the universal branched covering with the pre-assigned indices; this is characterized as the simply connected one. There are only three simply connected 1–dimensional complex manifolds. The ramification indices determine the nature of Z. We tabulate the three cases with familiar names for the deck transformation groups.

type	$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}$	Z	deck transformation group Γ
elliptic	>1	$\mathbb{P}^1$	polyhedral groups
parabolic	$=1$	$\mathbb{C}$	1-dim$_{\mathbb{C}}$ crystallographic groups
hyperbolic	<1	$\mathbb{H}$	triangle (Fuchsian) groups

Complex crystallographic groups of dimension n are by definition, groups of affine transformations of $\mathbb{C}^n$ with compact quotients. A Fuchsian group (of the first kind) is by definition a discrete subgroup of $SL(2,\mathbb{R})$ acting on the upper half-space

$$\mathbb{H}=\{\tau\in\mathbb{C}\mid\Im\tau>0\},$$

such that a(ny) fundamental domain has finite volume.

2.3.1 Developing maps In each case the developing map is given by the Schwarz map

$$s\colon X\ni x\mapsto z=u_0(x)/u_1(x)\in Z,$$

defined by the ratio of two (linealy independent) solutions of the hypergeometric differential equation $E(a,b,c)$:

$$x(1-x)u''+\{c-(a+b+1)x\}u'-abu=0,$$

where the parameters a,b,c are determined by the condition

$$|1-c|=\frac{1}{p},\quad |c-a-b|=\frac{1}{q},\quad |a-b|=\frac{1}{r}.$$

See Yoshida [15] for hypergeometric functions, thir fundamental properties and the Schwarz map.

2.3.2 Projections In each elliptic case, the projection is given by an invariant of a polyhedral group Γ, which is classically known.

In each parabolic case, the projection is given by an elliptic function.

However in hyperbolic cases, only for a few examples, are the projections explicitly constructed. Here we explain a typical example when $(p,q,r)=(\infty,\infty,\infty)$.

The developing map is the Schwarz map of the equation $E(1/2,1/2,1)$, the deck transformation group is (conjugate to) the congruence subgroup

$$\Gamma(2)=\{g\in SL(2,\mathbb{Z})\mid g\equiv \text{identity mod } 2\}$$

of the elliptic full modular group $SL(2,\mathbb{Z})$. The projection is given by the lambda function

$$\lambda(\tau)=\left(\frac{\vartheta_{01}(\tau)}{\vartheta_{00}(\tau)}\right)^4,\quad \tau\in\mathbb{H},$$

where

$$\vartheta_{00}(\tau)=\sum_{n\in\mathbb{Z}}q^{2n^2},\quad \vartheta_{01}(\tau)=\sum_{n\in\mathbb{Z}}(-1)^n q^{2n^2},\qquad q=\exp\pi i\tau/2.$$

2.3.3 Note Three points on a sphere – a simple geometric object – naturally leads to interesting kind of mathematics such as polyhedral groups and their invariants, Fuchsian automorphic forms/functions and the hypergeometric differential equation/functions.

2.4 Four or more branch points

If we admit four branch points, which we can assume $x=0,1,\infty,t$, and assign any four indices $\mathbf{p}=(p_0,\cdots,p_3)$, then such a covering of X always exists. Let Z be the biggest one. If the indices are $\mathbf{p}=(2,\cdots,2)$ then $Z\cong\mathbb{C}$, otherwise $Z\cong\mathbb{H}$. The developing map is the Schwarz map of a second-order Fuchsian equation $E(t,\mathbf{p})$. The t–dependence of the coefficients of this equation is difficult to analyse, and when t is in a generic position, the coefficients have no explicit expressions. The projection is known only for sporadic values of t and $\mathbf{p}$. Though there have been many attempts for tackling this difficulty the goal is still far away. I am afraid that 'four points on $\mathbb{P}^1$' is an object too difficult for human beings. There is no hope for more than four points.

3 Covers of the complex projective spaces

When one encounters a serious difficulty, one of the ways to proceed is to turn to high dimensional analogues.

3.1 Covers of $\mathbb{P}^2$

The first attempt was made by E Picard [10]. He considers the six lines

$$x_0x_1x_2(x_0-x_1)(x_1-x_2)(x_2-x_0)=0$$

on the projective plane $X(\cong \mathbb{P}^2)$ with homogeneous coordinates $x_0 : x_1 : x_2$, and studies the universal branched covering Z of X ramifying along the six lines with indices 3. It turns out that Z is isomorphic to the complex 2–dimentional ball

$$\mathbb{B}^2 = \{z_0 : z_1 : z_2 \in \mathbb{P}^2 \mid |z_0|^2 - |z_1|^2 - |z_2|^2 > 0\},$$

and that the developing map is the Schwarz map

$$s\colon X \ni x \longmapsto z = u_0(x) : u_1(x) : u_2(x) \in Z,$$

defined by linearly independent solutions u_0, u_1 and u_2 of Applell's hypergeometric differential equation F_1 (with special parameters), a system of linear partial differential equations defined on X of rank three with singularities along the six lines. The deck transformation group is an arithmetic subgoup of the group of automorphisms of $\mathbb{B}^2$ defined over $\mathbb{Z}(1^{1/3})$, and the projection can be expicitly expressed in terms of Riemann theta functions (this part is completed by H Shiga [11]).

3.2 Covers of $\mathbb{P}^n$

T Terada [12; 13] considers the hyperplanes

$$\prod_{i=1}^{n} x_i \cdot \prod_{i,j=1, i\neq j}^{n} (x_i - x_j) = 0$$

on the projective n–space $X(\cong \mathbb{P}^n)$ with homogeneous coordinates $x_0 : \cdots : x_n$, and studies the universal branched covering Z of X ramifying along these hyperplanes with different indices. It turns out, for some choices (finite possibilities) of indices, that Z is isomorphic to the complex n–dimentional ball

$$\mathbb{B}^n = \{z_0 : \cdots : z_n \in \mathbb{P}^n \mid |z_0|^2 - \sum_{i=1}^{n} |z_n|^2 > 0\},$$

and that the developing map is the Schwarz map

$$s\colon X \ni x \longmapsto z = u_0(x) : \cdots : u_n(x) \in Z,$$

defined by linearly independent solutions $u_0, \cdots, u_n$ of Applell's hypergeometric differential equation F_D (with parameters corresponding to the indices chosen), a system of linear partial differential equations defined on X of rank $n+1$ with singularities along those hyperplanes. The deck transformation group is a discrete subgoup (not necessarily arithmetic, see Deligne and Mostow [3]) of the group of automorphisms of $\mathbb{B}^n$. For several cases, the projection can be explicitly expressed in terms of Riemann theta functions (Shiga, [11], Matsumoto and Terasoma [7; 8]).

3.3 Note

Recently further studies have been made; the objects and techniques require considerably more algebraic geometry and representation theory. For more detail see Allcock, Carlson and Toledo [1], or Couwenberg, Heckman and Looijenga [2].

4 Covers of the 3–sphere

If one recalls that the complex projective line is also called the (Riemann) sphere and that the Poincaré upper half-plane $\mathbb{H}$ is just the real hyperbolic 2–space, another high-dimensional generalization would be made by the 3–dimensional sphere S^3 and the real hyperbolic 3–space

$$\mathbb{H}^3_{\mathbb{R}} = \{2 \times 2 \text{ positive Hermitian matrices}\}/\mathbb{R}_{>0}.$$

Since the branch locus must be a submanifold of codimension 2, we consider links/knots as branch loci.

If our branch locus is a trivial knot, nothing interesting happens.

If our branch locus is the trefoil knot, several interesting things happen, but the universal cover (branched cover with branch index $= \infty$) is not $\mathbb{H}^3_{\mathbb{R}}$.

If our branch locus is the Whitehead link or the figure eight knot, it is known that its universal cover is $\mathbb{H}^3_{\mathbb{R}}$. I believe that these are the simplest ones, and that these are the ones we should/could study. For more complicated links/knots, their complements would permit hyperbolic structures, but one can not expect nice mathematics from the view point of function theory. In this note, we just treat the Whitehead link.

4.1 Whitehead-link-complement group

It is known that the complement of the Whitehead link L admits a hyperbolic structure; that is, the universal cover of $S^3 - L$ is $\mathbb{H}^3_{\mathbb{R}}$, or there is a discrete subgroup W of the automorphisms of $\mathbb{H}^3_{\mathbb{R}}$, such that $S^3 - L$ is *homeomorphic* to the quotient space $\mathbb{H}^3_{\mathbb{R}}/W$.

The group of automorphisms of $\mathbb{H}^3_{\mathbb{R}}$ is generated by $PGL(2,\mathbb{C})$ acting as

$$GL(2,\mathbb{C}) \ni g\colon \mathbb{H}^3_{\mathbb{R}} \ni z \longmapsto gz\,{}^t\overline{g} \in \mathbb{H}^3_{\mathbb{R}},$$

and the (orientation reversing) transpose operation

$$T\colon \mathbb{H}^3_{\mathbb{R}} \ni z \longmapsto {}^t z \in \mathbb{H}^3_{\mathbb{R}}.$$

The Whitehead-link-complement group W is generated by the two elemnts

$$\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 1+i & 1 \end{pmatrix}.$$

This group is a subgroup of the principal congruence subgroup $\Gamma(1+i)$ of the full modular group

$$\Gamma = GL(2, \mathbb{Z}[i])$$

of finite index. It is not normal; the conjugate one is $\bar{W}$ (complex conjugate). We have the homeomorphism

$$\text{home: } \mathbb{H}^3_{\mathbb{R}}/W \ \sim \ S^3 - L.$$

4.2 What should be studied?

We stated in subsection 2.3.2 that we have an isomorphism (of complex analytic varieties)

$$\lambda\colon \mathbb{H}/\,\Gamma(2) \ \xrightarrow{\cong} \ \mathbb{P}^1 - \{0, 1, \infty\}.$$

If it is just a homeomorphism

$$\mathbb{H}^2_{\mathbb{R}}/\,\Gamma(2) \sim S^2 - \{0, 1, \infty\},$$

the picture is easy to see: a schoolchild could glue the corresponding sides of the fundamental domain of $\Gamma(2)$ (see Figure 1) and obtain a balloon with three holes. The explicit expression (in terms of the theta functions) of λ makes this isomorphism into interesting mathematics.

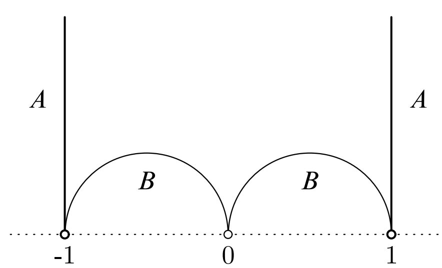

Figure 1: A fundamental domain for $\Gamma(2)$

A fundamenatal domain (consisting of two pyramids) of the Whitehead-link-complement group W is shown in Figure 2 (cf Wielenberg [14]); here hyperbolic 3–space is realized as the upper half-space model

$$\{(z, t) \in \mathbb{C} \times \mathbb{R} \mid t > 0\}.$$

The faces of the two pyramids are patched as follows: the eight walls with the same

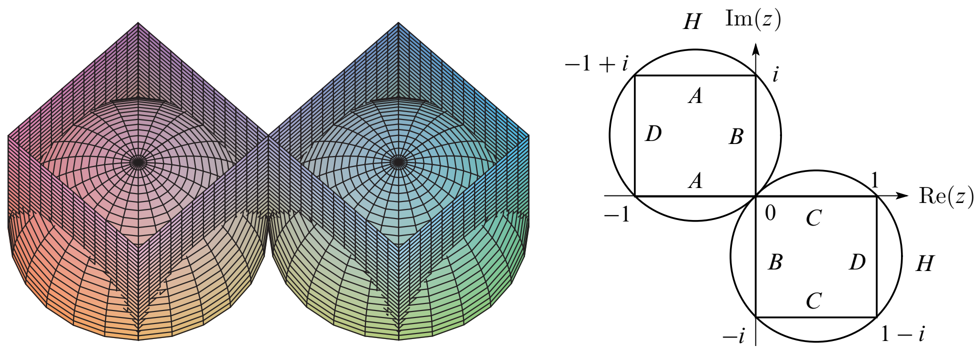

Figure 2: Fundamental domain FD of W in $\mathbb{H}^3_{\mathbb{R}}$

labels (A, B, C, D) are patched together, and the two hemi-spheres labelled H are also patched. The group W has two cusps. They are represented by the vertices of the pyramids:

$$(z,t) = (*, +\infty), \quad (0,0) \sim (\pm i, 0) \sim (\pm 1, 0) \sim (\mp 1 \pm i, 0).$$

Though it is not so easy as in the 2–dimensional case above to see that the fundamental domain modulo this patching is homeomorphic to the complement of the Whitehead link $L = L_0 \cup L_\infty$ (in $S^3 = \mathbb{R}^3 \cup \{\square\}$) shown in Figure 3, it is still not advanced mathematics.

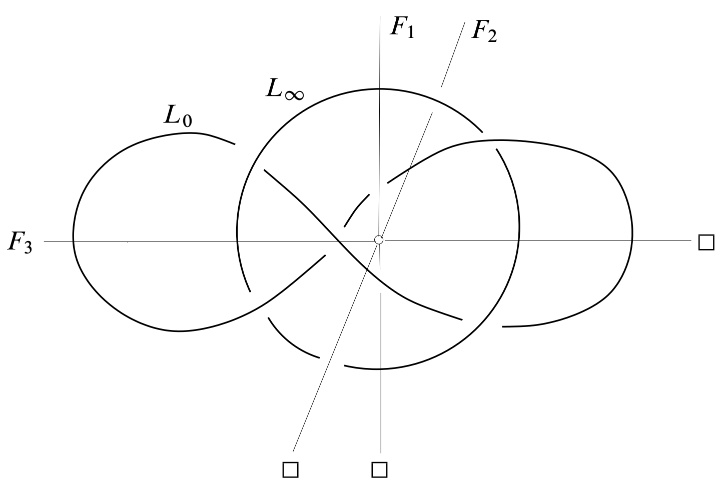

Figure 3: Whitehead link with its symmetry axes

Our goal is to find functions on $\mathbb{H}^3_{\mathbb{R}}$ invariant under the action of W, and make the projection

$$\pi\colon\quad \mathbb{H}^3_{\mathbb{R}} \longrightarrow S^3 - L$$

(which induces home: $\mathbb{H}^3_{\mathbb{R}}/W \xrightarrow{\sim} S^3 - L$) explicit.

Though the automorphic forms/functions on the Poincaré upper half-plane $\mathbb{H}$ and those on their high complex-dimensional generalizations (such as Siegel upper half-spaces, complex balls, and other Hermitian symmetric spaces) have been studied in detail, those on $\mathbb{H}^3_{\mathbb{R}}$ have never been studied before; this is really surprising.

As 'function theory' usually means 'theory of complex analytic functions', most (explicit) functions are defined naturally in complex domains. So we had to rely first on a Hermitian (complex anlytic) setting, and then to restrict to a real subvariety isomorphic to $\mathbb{H}^3_{\mathbb{R}}$.

4.3 A Hermitian setting

More than fifteen years ago, we made in [6] a generalization of the story (explained above) of the isomorphism

$$\lambda\colon \mathbb{H}/\Gamma(2) \quad \xrightarrow{\cong} \quad \mathbb{P}^1 - \{0, 1, \infty\}.$$

The developing map $\mathbb{P}^1 - \{0, 1, \infty\} \to \mathbb{H}^1$ can be considered as the period map of the family of elliptic curves

$$t^2 = (s - x_1) \cdots (s - x_4),$$

where $(x_1, \dots, x_4)$ is a quadruple of distict points in $\mathbb{P}^1$, which can be, modulo $PGL(2, \mathbb{C})$ action, normalized as

$$(0, 1, \infty, x), \quad x \in \mathbb{P}^1 - \{0, 1, \infty\}.$$

We consider the family of K3 surfaces

$$t^2 = \ell_1(s_1, s_2) \cdots \ell_6(s_1, s_2),$$

where ℓ_j $(j = 1, \dots, 6)$ are linear forms in $(s_1, s_2) \in \mathbb{P}^2$ in general position; modulo $PGL(3, \mathbb{C})$ action, one can check that this family is 4–dimensional. Let us call this parameter space X, that is,

$$X = PGL(3, \mathbb{C}) \backslash \left\{ (\ell_1, \cdots, \ell_6) \mid \text{six ordered lines in } \mathbb{P}^2 \text{ in general position} \right\}.$$

The periods of these surfaces – these are also solutions of the generalized hypergeometric differential equation of type $(3, 6)$ defined in X – defines a map from X to the 2×2

upper half-space

$$\mathbb{H}_{2\times 2} = \left\{ \tau :\ 2\times 2 \text{ complex matrix } \mid \frac{\tau - {}^t\overline{\tau}}{2i} > 0 \right\};$$

the deck transormation group (the monodromy group) is the principal congruence subgroup

$$\Delta(1+i) = \{ g \in \Delta \mid g \equiv I_4 \mod (1+i) \}$$

of the full modular group

$$\Delta = GL(4, \mathbb{Z}[i]) \cap \{ g \in GL(4, \mathbb{C}) \mid g J_4\, {}^t\overline{g} = J_4 \}, \quad \text{where} \quad J_4 = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}.$$

The group Δ acts on $\mathbb{H}_{2\times 2}$ as

$$g\cdot\tau = (A\tau + B)(C\tau + D), \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Delta,\ \tau \in \mathbb{H}_{2\times 2}.$$

The projection $\pi\colon \mathbb{H}_{2\times 2} \to X$ can be expressed in terms of the theta functions

$$\theta\begin{pmatrix} a \\ b \end{pmatrix}(\tau) = \sum_{n\in\mathbb{Z}[i]^2} \exp \pi i \{ (n+a)\tau\, {}^t\overline{(n+b)} + 2\Re(n \cdot {}^t\overline{b}) \}$$

with characteristics

$$a, b \in \left(\frac{\mathbb{Z}[i]}{1+i} \right)^2.$$

The projection π induces the isomorphism $\mathbb{H}_{2\times 2}/\Delta(1+i) \cong X$.

4.4 Restriction to $\mathbb{H}^3_{\mathbb{R}}$ in $\mathbb{H}_{2\times 2}$

The real hyperbolic 3–space $\mathbb{H}^3_{\mathbb{R}}$ lives in $\mathbb{H}_{2\times 2}$ as

$$\mathbb{H}_{2\times 2} = \mathrm{Her}_2 + i\,\mathrm{Her}_2^+ \supset 0 + i\,\mathrm{Her}_2^+ \supset 0 + i\{ z \in \mathrm{Her}_2^+ \mid \det z = 1 \} \cong \mathbb{H}^3_{\mathbb{R}},$$

where Her_2 stands for the space of 2×2–Hermitian matrices, and Her_2^+ that of positive definite ones. We restrict the groups Δ and $\Delta(1+i)$ onto the subvariety $\mathbb{H}^3_{\mathbb{R}} \subset \mathbb{H}_{2\times 2}$; we have

$$\Delta|_{\mathbb{H}^3_{\mathbb{R}}} = \Gamma, \quad \Delta(1+i)|_{\mathbb{H}^3_{\mathbb{R}}} = \Gamma(1+i).$$

Since the Whitehead-link-complement group W is a subgroup of Γ of finite index, and so W is commensurable with $\Gamma(1+i)$, one can expect that the restrictions of the theta functions in the Hermitian setup would give the desired projection $\pi\colon \mathbb{H}^3_{\mathbb{R}} \to S^3 - L$.

4.5 Toward an embedding of $\mathbb{H}^3_{\mathbb{R}}/W$

The expectation stated at the end of the previous section becomes true; we can construct automorphic functions (out of the theta functions) f_j defined in $\mathbb{H}^3_{\mathbb{R}}$ invariant under W, and get a map

$$\mathbb{H}^3_{\mathbb{R}} \ni x \longmapsto (f_j(x)) \in \mathbb{R}^N, \qquad N = 13,$$

which gives an embedding of $\mathbb{H}^3_{\mathbb{R}}/W$ onto a semi-algebraic subset of $\mathbb{R}^N$ homeomorphic to $S^3 - L$; refer to our paper [5] for precise formulation and proofs. Here we describe what is going on.

4.5.1 Base camp The miraculously lucky fact is that the theta functions $\theta\binom{a}{b}(\tau)$ for $a, b \in (\mathbb{Z}[i]/(1+i))^2$ are real valued on $\mathbb{H}^3_{\mathbb{R}}$. (We do not know yet a basic reason for this.)

Let $\Gamma(2)$ be the principal congruence subgroup of Γ of level 2. The group $\Gamma^T(2) := \langle \Gamma(2), T\rangle$ is a (hyperbolic) Coxeter group with (ideal) octahedral Weyl chamber (see Figure 4). For suitable four such thetas $x_0, \dots, x_3$, this Weyl chamber ($\sim \mathbb{H}^3_{\mathbb{R}}/\,\Gamma^T(2)$) is mapped diffeomorphically by the map

$$\mathbb{H}^3_{\mathbb{R}} \ni (t, z) \longmapsto \frac{1}{x_0}(x_1, x_2, x_3) \in \mathbb{R}^3,$$

onto the euclidean octahedron

$$\{(t_1, t_2, t_3) \in \mathbb{R}^3 \mid |t_1| + |t_2| + |t_3| \le 1\}$$

minus the six vertices. Note that this embedding of $\mathbb{H}^3_{\mathbb{R}}/\,\Gamma^T(2)$ is of codimention 0. This embedding plays a role of a base camp of our orbifold-embedding-tour.

4.5.2 Symmetries of L Though the codimension of the embedding announced in the beginning of this subsection (4.5) is high ($N - 3 = 10$), we can nevertheless *see* what is happening as follows. We consider a chain of supgroups of W:

$$W \overset{2}{\subset} W_1 \overset{2}{\subset} W_2 \overset{2}{\subset} \Lambda := \langle \Gamma^T(2), W\rangle,$$

each inclusion being of index 2, such that the double coverings

$$(S^3 - L \sim)\ \mathbb{H}^3_{\mathbb{R}}/W \xrightarrow{p_1} \mathbb{H}^3_{\mathbb{R}}/W_1 \xrightarrow{p_2} \mathbb{H}^3_{\mathbb{R}}/W_2 \xrightarrow{p_3} \mathbb{H}^3_{\mathbb{R}}/\Lambda$$

of the orbifolds have clear geometric interpretations.

- p_1 is the projection modulo the order-2–rotation with axis $F_1 \subset \mathbb{H}^3_{\mathbb{R}}/W \sim S^3 - L$ (cf Figures 3, 5),

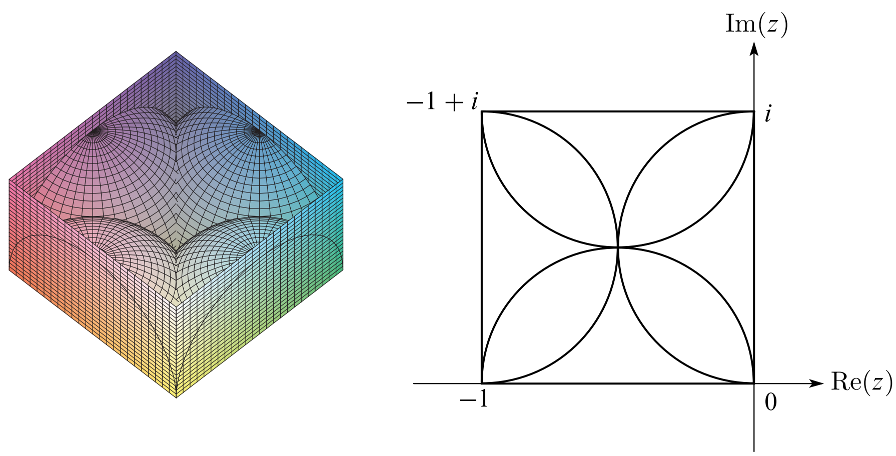

Figure 4: Weyl chamber of $\Gamma^T(2)$

.

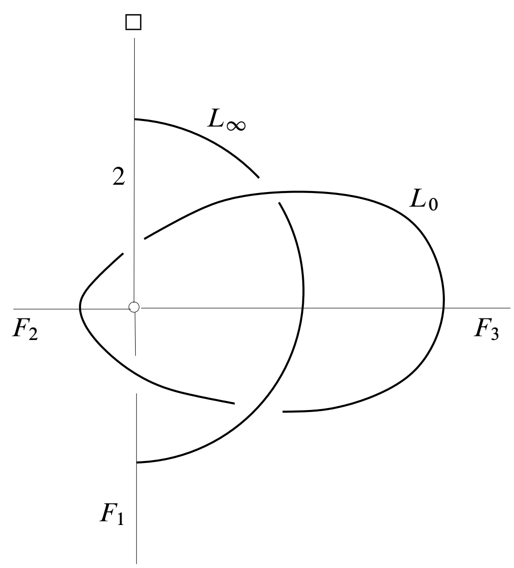

Figure 5: The orbifold $\mathbb{H}^3_{\mathbb{R}}/W_1$

- p_2 is the projection modulo the order-2–rotation with axis $p_1(F_2) \cup p_1(F_3) \subset \mathbb{H}^3_{\mathbb{R}}/W_1$ (cf Figures 5, 6),

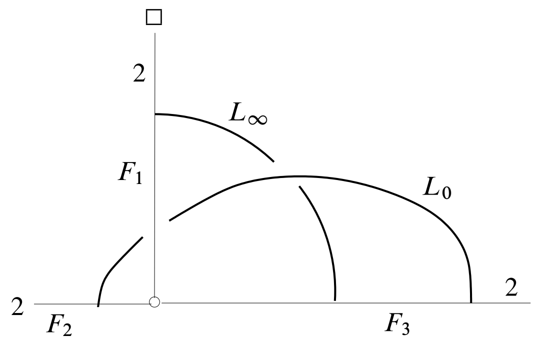

Figure 6: The orbifold $\mathbb{H}^3_{\mathbb{R}}/W_2$

- p_3 is the projection modulo an order-2–reflection with a mirror ($\sim S^2$) in $\mathbb{H}^3_{\mathbb{R}}/W_2$ (cf Figures 6, 7). The mirror is shown in Figure 8 as the union of four *triangles* labeled as a, b, c and d.

The orbifold $\mathbb{H}^3_{\mathbb{R}}/\Lambda$ is homeomorphic to a 3–ball with two holes, corresponding to the two strings L_0 and L_∞ forming L. We have a very simple embedding of $\mathbb{H}^3_{\mathbb{R}}/\Lambda$ of codimension 1, as we see in subsection 4.5.3.

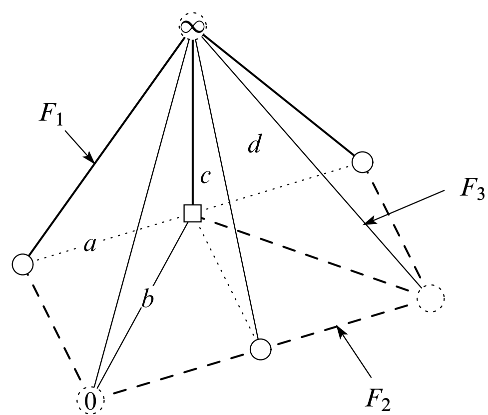

Figure 7: A fundamental domain for W_2

4.5.3 Climb up and down The group $\Gamma^T(2)$ is a normal subgroup of Λ, and $\Lambda/\Gamma^T(2)$ is isomorphic to the dihedral group of order eight. To climb up from $\Gamma^T(2)$ to Λ is easy. We have only to find rational functions in $x_0, \dots, x_3$ invariant under the dihedral group. Actuallly, by the map

$$\mathbb{H}^3_{\mathbb{R}} \ni (z, t) \longmapsto (\lambda_1, \dots, \lambda_4) = (\xi_1^2 + \xi_2^2, \xi_1^2\xi_2^2, \xi_3^2, \xi_1\xi_2\xi_3) \in \mathbb{R}^3,$$

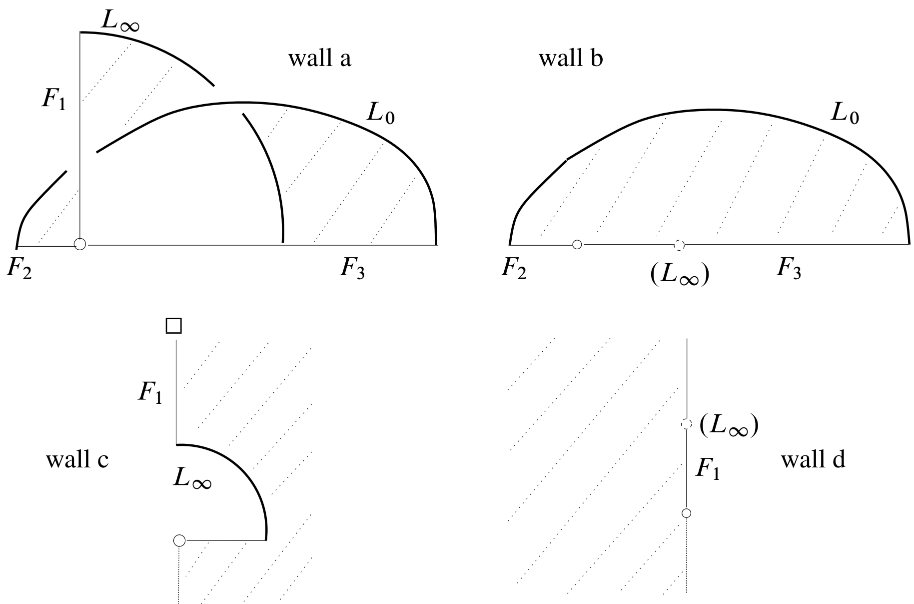

Figure 8: The mirror of the reflection in the orbifold $\mathbb{H}^3/W_2$ is shown as the union of four parts.

where $\xi_i = x_i/x_0$, the orbifold $\mathbb{H}^3_{\mathbb{R}}/\Lambda$ is embedded into a subdomain of the quadratic hypersurface $\lambda_2\lambda_3 = \lambda_4^2$.

To climb down from Λ to W_2, we need functions which separate the two sheets of the double cover $p_3\colon \mathbb{H}^3_{\mathbb{R}}/W_2 \to \mathbb{H}^3_{\mathbb{R}}/\Lambda$. These functions are made from the theta functions $\theta\binom{a}{b}(\tau)$ with deeper characteristics $a, b \in (\mathbb{Z}[i]/2)^2$. To determine the image of the embedding, we make use of the transformation formulas and algebraic relations among these thetas obtained by Matsumoto [4].

When we climb down from W_2 to W_1, and finally from W_1 to W, we need several functions made by such thetas. Anyway, by functions made by thetas, we can thus embed $\mathbb{H}^3_{\mathbb{R}}/W$ into $\mathbb{R}^{13}$.

References

[1] **D Allcock**, **J A Carlson**, **D Toledo**, *The complex hyperbolic geometry of the moduli space of cubic surfaces*, J. Algebraic Geom. 11 (2002) 659–724 MR1910264

[2] **W Couwenberg**, **G Heckman**, **E Looijenga**, *Geometric structures on the complement of a projective arrangement*, Publ. Math. Inst. Hautes Études Sci. (2005) 69–161 MR2217047

[3] **P Deligne**, **G D Mostow**, *Monodromy of hypergeometric functions and nonlattice integral monodromy*, Inst. Hautes Études Sci. Publ. Math. (1986) 5–89 MR849651

[4] **K Matsumoto**, *Algebraic relations among some theta functions on the bounded symmetric domain of type* $\mathrm{I}_{r,r}$, Kyushu J. Math. 60 (2006) 63–77 MR2216949

[5] **K Matsumoto**, **H Nishi**, **M Yoshida**, *Automorphic functions for the Whitehead-link-complement group*, Osaka J. Math. 43 (2006) 839–876 MR2303553

[6] **K Matsumoto**, **T Sasaki**, **M Yoshida**, *The monodromy of the period map of a* 4*-parameter family of* $K3$ *surfaces and the hypergeometric function of type* $(3,6)$, Internat. J. Math. 3 (1992) 164 MR1136204

[7] **K Matsumoto**, **T Terasoma**, *Theta constants associated to cubic threefolds*, J. Algebraic Geom. 12 (2003) 741–775 MR1993763

[8] **K Matsumoto**, **T Terasoma**, *Theta constants associated to coverings of* $\mathbf{P}^1$ *branching at eight points*, Compos. Math. 140 (2004) 1277–1301 MR2081157

[9] **K Matsumoto**, **M Yoshida**, *Invariants for some real hyperbolic groups*, Internat. J. Math. 13 (2002) 415–443 MR1912188

[10] **E Picard**, *Sur des fonctions de deux variables indépendantes analogues aux fonctions modulaires*, Acta Math. 2 (1883) 114–135 MR1554595

[11] **H Shiga**, *On the representation of the Picard modular function by* θ *constants. I, II*, Publ. Res. Inst. Math. Sci. 24 (1988) 311–360 MR966178

[12] **T Terada**, *Problème de Riemann et fonctions automorphes provenant des fonctions hypergéométriques de plusieurs variables*, J. Math. Kyoto Univ. 13 (1973) 557–578 MR0481156

[13] **T Terada**, *Fonctions hypergéométriques* F_1 *et fonctions automorphes. II. Groupes discontinus arithmétiquement définis*, J. Math. Soc. Japan 37 (1985) 173–185 MR780657

[14] **N Wielenberg**, *The structure of certain subgroups of the Picard group*, Math. Proc. Cambridge Philos. Soc. 84 (1978) 427–436 MR503003

[15] **M Yoshida**, *Hypergeometric functions, my love*, Modular interpretations of configuration spaces, Aspects of Mathematics, E32, Friedr. Vieweg & Sohn, Braunschweig (1997) MR1453580

Department of Mathematics, Kyushu University
Fukuoka 810-8560, Japan

`myoshida@math.kyushu-u.ac.jp`

Received: 24 November 2005 Revised: 12 December 2006