The work of Fred Cohen

RAN LEVI

This paper gives an overview of Fred Cohen's work and is a summary of the talk which I gave during his 60^{th} birthday conference, held at the University of Tokyo in July 2005.

55R80, 55P99; 20F05, 20F36, 55P42

Summarizing Fred's contributions to mathematics in a 1–hour talk, or for that matter in a single paper, is a daunting task. With nearly a hundred papers in print, and collaborations with no less than 48 authors, the sheer volume of his publication record makes trying to choose the appropriate highlights a difficult task. What makes it even harder is the variety of topics the work touches on. To all active topologists Fred Cohen is a very familiar name, and readers are likely to have a pretty good idea of the main topics his name is mostly associated with. For instance the study of configuration spaces and their applications, or his work on homotopy exponents with Moore and Neisendorfer. I will of course mention those in this article. However, as all who know him are aware, Fred's interests are not limited to mainstream homotopy theory. Quite the contrary, he will never shy away from an opportunity to explore an unknown, possibly eccentric grounds. I will therefore attempt to explore some of the less known aspects of his work, which I find always interesting, and occasionally spectacular. This paper is not intended as a comprehensive summary of Fred's mathematical contributions, but rather as a sampler of some of his many achievements.

I wish to thank my co-editors in this volume for entrusting me with the task of delivering a talk on Fred Cohen's work during his 60^{th} birthday conference, and kindly allowing me to record it in print. But above all I wish to thank Fred himself for simply being there, and doing so much to enrich our mathematical lives.

1 Configuration spaces and applications

Fred's first contributions to the mathematical literature appear in two research announcements in the Bulletin of the AMS in 1973, as the outcomes of his PhD thesis [7; 8]. The real work however only appeared three years and five independent publications later in the collection by Cohen–Lada–May [18], which became more or less immediately

Published: 7 April 2009 DOI: 10.2140/gtm.2008.13.529

the definitive basic reference in the subject of its title – the homology of iterated loop spaces. To this colossal 490 page book, Fred contributed two articles with a total page length of 192 pages.

Very few papers in algebraic topology are more fundamental than these two. The following is the tip of an iceberg summary of what is done there.

"The homology of $\mathcal{C}_{n+1}$–spaces, $n \geq 0$" , [9], starts with a careful analysis of the Dyer–Lashof and Browder homology operations, the relationships between them and the Pontrijagin product in the homology of iterated loop space. The main idea is the utilization of May's little n–cubes operad and its action on the homology of an iterated loop space. This is then applied to the calculation of the homology of May's configuration space model $H_*(C_{n+1}X)$, and $H_*(\Omega^{n+1}\Sigma^{n+1}X, \mathbb{F}_p)$, as free objects, in the appropriate sense, on the homology of X. I can't possibly explain these results here in much detail, nonetheless how they are obtained. But to give the reader the flavour, here is a brief description.

Definition 1.1 A graded $\mathbb{F}_p$–vector space M is:

- an allowable R_n–module if there are homomorphisms

  $$Q^s\colon M_q \to M_{q+2s(p-1)}, \quad (Q^s\colon M_q \to M_{q+s} \quad \text{if p=2})$$

  for $0 \leq 2s < q+n$, $(s < q+n)$ such that $Q^s = 0$ for $2s < q$, $(s < q)$ and the composition of the Q^s satisfy the Adem relations.
- an allowable AR_n–module if, in addition M admits an action of the dual of the Steenrod algebra which satisfies the Nishida relations.
- an allowable AR_n–algebra, if in addition M is a commutative algebra satisfying
  (1) $Q^s x = x^p$, if $|x| = 2s$, $(|x| = s)$ for any $x \in M$,
  (2) $Q^s(1) = 0$ if $s > 0$ and
  (3) the Cartan formula for products.
- an allowable $AR_n\Lambda_n$–Hopf algebra (with conjugation), if M is a monoidal Hopf algebra satisfying further properties concerning the Browder operations λ_n and the "top" operation ξ_n, and their commutation relations with the Dyer Lashof operations and the Pontrijagin product.

Define

(1) a functor W_n to be the free functor left adjoint to the forgetful functor from allowable $AR_n\Lambda_n$–Hopf algebras to cocommutative coalgebras over the dual of the Steenrod algebra.

(2) A functor G from allowable AR–Hopf algebras to AR–Hopf algebras with conjuation to be
$$GW = W \otimes_{\pi W} G_0 W,$$
where W is an allowable AR–Hopf algebra, πW is the commutative monoid under the product in W, πGW is the commutative group generated by πW, and $G_0 W$ is its group ring. In other words GW is the localization of the ring W at the monoid πW.

With this terminology and notation, Fred proves the following:

Theorem 1.2 [9, Theorem 3.1] *For every space X there is an isomorphism of allowable $AR_n\Lambda_n$–Hopf algebras:*
$$\overline{\eta}_*\colon W_n H_*(X) \longrightarrow H_*(C_{n+1}X)$$

Theorem 1.3 [9, Theorem 3.2] *For every space X there is an isomorphism of allowable $AR_n\Lambda_n$–Hopf algebras with conjugation:*
$$\widetilde{\eta}_*\colon GW_n H_*(X) \longrightarrow H_*(\Omega^{n+1}\Sigma^{n+1}X)$$

The generality of the results implies of course that certain specific cases, particularly when one is not after full and complete answers, may be easier to compute directly than by referring to the theorems. But nevertheless, the importance of these results in the development of algebraic topology cannot be over estimated.

In *"The Homology of $SF(n+1)$"*, [10], Fred studies the homology of the topological monoid of degree 1 self maps of the sphere S^{n+1}, where the monoid operation is given by composition. The main theorem is the statement that the Pontrijagin ring $H_*(SF(n+1), \mathbb{F}_p)$ is a commutative algebra which for odd primes is isomorphic as an algebra to $H_*(\Omega_0^{n+1}S^{n+1})$ (where the loop space structure is given as usual by juxtaposition of loops. The proof is by direct calculation, and utilizes similar techniques to those used in [9].

2 Braid groups

Artin's braid groups are fundamental objects which arise naturally in geometry, knot theory, ring theory and many other mathematical disciplines. They also have prominent roles to play in algebraic topology, to a large extent thanks to Fred's contributions. Fred's first published encounter with Braid groups appear as a 2 page appendix to [9], where he computes the homology of the braid group on r strands B_r as a module over the dual of the Steenrod algebra, as well as the rational and integral homology. The mod–2 results read as follows.

Theorem 2.1 [9, Theorem A.1] *There is an isomorphism of modules over the dual of the Steenrod algebra*

$$H_*(B_r, \mathbb{F}_2) \cong P[\xi_{2^j-1} \mid j \geq 1]/I,$$

where I is the ideal generated by the monomials

$$\xi_{j_1}^{k_1} \cdots \xi_{j_t}^{k_t} \quad \text{where} \quad \sum_{i=1}^{t} k_i 2^{j_i} > r.$$

Furthermore, the action of the dual Steenrod algebra is determined by the requirements that Sq_*^r acts trivially if $r > 1$, and that $Sq_*^1(\xi_{j+1}) = \xi_j^2$.

Mark Mahowald has shown in a 1977 Topology paper [29] that the Thom spectrum of the natural map $\overline{\eta}\colon \Omega^2 S^3 \longrightarrow BO$ is the Eilenberg–MacLane spectrum $K(\mathbb{Z}/2, 0)$. It is common in mathematics to look for different ways in which various bits of mathematical knowledge fit together. Fred has always been on the lookout for such 'non-accidents'. In [11] he shows the following:

Theorem 2.2 [11] *Let B_∞ denote the colimit of the braid groups B_r under the obvious inclusions. Then there is a homology isomorphism $\theta\colon K(B_\infty, 1) \longrightarrow \Omega^2 S^3$.*

From this he is able to deduce the following beautiful

Corollary 2.3 [11] *The Thom spectrum MB_∞ of the composite $\overline{\eta} \circ \theta$ (at any prime) is the $K(\mathbb{Z}/2, 0)$–spectrum. Thus every mod–2 homology class may be realized as a manifold whose stable normal bundle has a B_r–reduction.*

Fred's romance with the braid groups continued through his career and to the present day. One beautiful result was presented by him in this conference, based on the the paper [27] in this volume.

Another very recent work of a totally different flavour is a collaboration with Alejandro Adem and Dan Cohen [1]. A group Γ is said to be homologically toroidal if there is a homomorphism $\mathbb{Z}^{n_1} * \cdots * \mathbb{Z}^{n_k} \longrightarrow \Gamma$ inducing a split epimorphism on integral homology. An example of such a group is the pure braid group P_r.

The authors show the following:

Theorem 2.4 [1] *If Γ is a homologically toroidal group and $\Gamma \to U(n)$ is a representation then the composite*

$$B\Gamma \longrightarrow BU(n) \longrightarrow BU$$

is null homotopic, while if $\Gamma \longrightarrow O(n)$ is a representation the corresponding composite is null homotopic if and only if the first two Stiefel Whitney classes of the representation vanish.

In the process, the subgroup of elements in the K–theory of $B\Gamma$ which arise from orthogonal representations is also determined.

For the pure braid group, each quadratic relation in the cohomology ring $H^*(P_r)$ is shown to correspond to a spin representation of P_r. This representation is nontrivial, but it gives rise to a trivial bundle over the configuration space $F(\mathbb{C}, r) = K(P_r, 1)$.

3 Exponents in homotopy theory

The mid to late 70s saw a collaboration between Fred, Joe Neisendorfer and John Moore, the results of which mark some of the most beautiful results in unstable homotopy theory ever achieved. The subject of study is exponents in homotopy theory. Homotopy theorists, realizing that looking for explicit calculations in unstable homotopy groups couldn't possibly be feasible in great generality, started looking for qualitative, rather than quantitative results. The search for exponent results was one of the paths one could explore. Two famous conjectures, both open to this day, are worth mentioning, as they provided much of the motivation for the Cohen–Moore–Neisendorfer project.

The Barratt conjecture states that if a double suspension $X = \Sigma^2 Y$ has the property that the order of the class of the identity element in the abelian group $[X, X]$ is p^r for some prime p, then the p^{r+1} power map on $\Omega^2 X$ is null homotopic. The Moore conjecture is more general in its setup, but less specific in its conclusion. It states that if X is a finite p–local CW complex, then the torsion part of $\pi_*(X)$ has a global exponent if and only if the rational homotopy of X is globally finite dimensional, or using Moore's terminology, if and only if X is elliptic (as opposed to hyperbolic). Both conjectures were stated at a point were not a single example was known. A lucid discussion of these conjectures can be found in [30].

The Cohen–Moore–Neisendorfer team set out in the mid 70's to fix the situation. The three of them together published five papers on the subject, two of which appeared in the Annals of Mathematics [21; 20; 19; 22; 23].

In the 1983 International Congress of Mathematicians in Warsaw, Fred delivered an invited address where he reported on the Cohen–Moore–Neisendorfer project. His report appeared as [12].

The Cohen–Moore–Neisendorfer papers contain enough ideas to keep a whole generation of topologists busy. They influenced the work of many topologists in the almost 30 years since the first paper was published. Among them Anick, Gray, Theriault, and Selick. Exponents in homotopy theory were studied before this project commenced, but arguably never before in such a systematic fashion. One of the most striking aspect of the project is the elegant and systematic use of techniques of differential graded Lie algebras. The authors apply these methods to the homotopy Bockstein spectral sequence, which is a differential graded Lie algebra with respect to the Samelson product. Using the homological information, they conclude the existence of a product splittings of certain loop spaces. The first theorem we quote is an example. For a prime p and a positive integer m, let $P^n(p^m)$ denote the homotopy cofibre of the degree p^m map on the sphere S^{n-1}, and let $S^{n-1}\{p^m\}$ denote the homotopy fibre of the same map In [21; 20] the authors restrict attention to primes $p>3$ and prove the following.

Theorem 3.1 [21] *Let p be an odd prime, and n a positive integer. Then*

$$\Omega P^{2n+2}(p^r) \simeq S^{2n+1}\{p^r\} \times \Omega\left(\bigvee_{m=0}^{\infty} P^{4n+2mn+3}(p^r)\right).$$

Another important ingredient in their analysis is the homotopy fibre $F_n\{p^r\}$ of the pinch map $P^n(p^r) \longrightarrow S^n$. In two consecutive theorems in [21] they provide a product splitting for $\Omega F_n\{p^r\}$.

Theorem 3.2 [21] *Let p be an odd prime, and n a positive integer. Then*

$$\Omega F^{2n+1}\{p^r\} \simeq S^{2n-1} \times \prod_{k=1}^{\infty} S^{2p^k n-1}\{p^{r+1}\} \times P,$$

and

$$\Omega F^{2n}\{p^r\} \simeq \Omega S^{2n-1} \times S^{4n-3} \times \prod_{k=1}^{\infty} S^{4p^k n-2p^k-1}\{p^r\} \times P'.$$

Here P and P' denote (different) infinite products of loop spaces on mod p^r Moore spaces.

These product splittings allows the authors to construct a map

$$\pi\colon \Omega^2 S^{2n+1} \longrightarrow S^{2n-1},$$

whose composition with the double Freudenthal suspension map

$$S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1} \xrightarrow{\pi} S^{2n-1}$$

is homotopic to the degree p map on S^{2n-1}. Combined with work of Toda this allows them to show that the p–torsion in $\pi_*(S^{2n+1})$ has exponent p^{n+1}.

In the same paper they also show that the $\pi_*(P^n(p^r))$ contains infinitely many elements of order p^{r+1}.

A refinement of the their methods in [20] allows them to chose a map π as above, such that the composition the other way

$$\Omega^2 S^{2n+1} \xrightarrow{\pi} S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1}.$$

is the double loops of the p^{th} power map on $\Omega^2 S^{2n+1}$.

All that is needed now is to iterate this composite n times, and the p^n power map on $\Omega^2 S^{2n+1}$ factors through S^1. This implies that p^n annihilates the homotopy of S^{2n+1}.

In the next two papers [19] and [22], they show the existence of exponents for the Moore spaces $P^n(p^r)$. In [19] they prove that p^{2r+1} annihilates the homotopy of an even dimensional mod p^r Moore space at odd primes. To improve on this, as to fit with the Barratt conjecture which predicts an exponent p^{r+1}, the missing ingredient is a product splitting for $\Omega P^{2n+1}(p^r)$. This is done in [22], and in a subsequent paper by Neisendorfer, the predicted exponent is obtained.

Examples of the Moore and Barratt conjectures have been constructed by numerous authors following Cohen–Moore–Neisendorfer, but as general statements these conjectures remain as intact today as they were when they were originally stated.

4 An early curiosity

In the 70's between his thesis and the work he did with Moore and Neisendorfer, Fred wrote a number of papers where he applied his configuration spaces techniques to various problems – most notably a powerful generalization of the Borsuk–Ulam theorem. The contributions already mentioned above are among Fred's greatest achievements. In this short section however, I have chosen to mention a much less familiar paper he collaborated on early in his career.

Every mathematician who has been active long enough knows what Erdös numbers are. Paul Erdös wrote according to MathSciNet more than 1500 papers, most of them in collaboration with other mathematicians. This uncommon prolificacy yielded the concept. A mathematician has Erdös number 1 if they wrote a paper with Erdös himself, and Erdös number $\leq n$ if they collaborated with a mathematician whose Erdös number is $n-1$. It is conjectured that any mathematician who ever collaborated on a paper has a finite Erdös number. However, since Erdös was a number theorist, it appears unlikely for a homotopy theorist to have a small Erdös number. Fred, never in the habit of putting a title on what interested him, earned an Erdös number 2 due to a collboration with number theorist Selfridge in [24]. The main theorem in this paper is of the kind Erdös himself would probably approve of.

Theorem 4.1 [24] *There exist infinitely many odd numbers M, such that neither $M+2^n$ not $|M-2^n|$ is a prime power for any n.*

The authors also construct an explicit 94 digit example of the theorem.

5 "You'd think it's easy to decide whether something is divisible by 2"

The title of this section is actually a quote of Fred in his 1990 algebraic topology class, which I was a part of. This is of course all about the so called *strong form of the Kervaire invariant conjecture*, a statement which mystified and deceived topologists for more than 50 years. The statement is simple:

The Whitehead square $\omega_{2n+1}=[\iota_{2n+1},\iota_{2n+1}]\in\pi_{4n+1}(S^{2n+1})$ is divisible by 2 if $n=2^k-1$.

It is easy to see $\omega_{2n+1}=0$ for $n=0,1,3$ (since the spheres in question are $H-$ spaces). In [13], which is a textbook treatment of this and many other aspects of classical homotopy theory, Fred gives no less than 5 equivalent formulations of this question. In the next statement cohomology is taken with coefficients in $\mathbb{F}_2$.

Theorem 5.1 [13, Proposition 11.4] *Let $n\neq 0,1,3$. Then the following statements are equivalent.*

(1) *The Whitehead product ω_{2n+1} is divisible by 2.*

(2) *The short exact sequence*

$$0\longrightarrow\mathbb{Z}/2\longrightarrow\pi_{4n+l}(S^{2n+1})\longrightarrow\pi_{4n+2}(S^{2n+2})\longrightarrow 0$$

is not split .

(3) *There is a map* $P^{4n+2}(2) \longrightarrow \Omega S^{2n+2}$ *which is non-zero in homology*

(4) *There exists a space X with* $\bar{H}^i(X) = \mathbb{Z}/2$ *for* $i = 2n+2, 4n+3,$ *and* $4n+4$, *and zero otherwise, with* $Sq^{2n+2}\colon H^{2n+2}(X) \longrightarrow H^{4n+4}(X)$ *and* $sq^l\colon H^{4n+3}(X) \longrightarrow H^{4n+4}(X)$ *isomorphisms.*

(5) $\Omega^2[-1]$ *is homotopic to* -1 *on* $\Omega^2 S^{2n+l}$.

For graduate students in Rochester Fred's special "homework problems" are a familiar concept. I remember vividly how after proving the theorem above in class he suggested, "well, here is a homework problem for you. Prove one of these statements". Anybody who ever heard a lecture by Fred is likely to have been assigned a homework problem in the context of the Kervaire invariant conjecture, and many other subjects. Those who know him are aware of three basic facts: (1) He is genuinely interested to know the answer, (2) he has tried it himself, and (3) you may spend your lifetime trying to solve this homework problem, and there is no partial credit.

But Fred tends to be very serious about his homework assignments, and when the students are struggling he always tries to help. So if five formulations are not enough, then in [14] he gives yet another formulation. This one is quite special in that it relates the question to the real Cayley–Dickson algebras. He constructs a certain subspace $K(n,\epsilon)$ of the topological vector space given by the polynomial ring $\mathbb{R}[x,y]$. He shows the following:

Lemma 5.2 [14] *If* $\epsilon > 0$ *and* $n \geq 2$, *the space* $K(n,\epsilon)$ *is homotopy equivalent to the* $(4n-1)$*–skeleton of* $\Omega^2 S^{2n+1}$.

He then uses the multiplication induced from the Cayley–Dickson algebra to construct a model Sq for the degree 2 map on S^k. He then proceeds to show

Lemma 5.3 [14] *If* $k = 2^n - 1$ *and* $n \geq 3$, *then* ω_k *is divisible by 2 if the loop squaring map and the map* $\Omega^2(Sq)$ *are homotopic when restricted to the* $(4n-3)$*–skeleton.*

Using these two lemmas, Fred concludes the following

Proposition 5.4 [14] *If* $k = 2^n - 1$, *then* ω_k *is divisible by 2 if and only if the composites*

$$K(\frac{k-1}{2},\epsilon) \longrightarrow \Omega^2 S^k \xrightarrow{\Omega^2(Sq)} \Omega^2 S^k, \quad \text{and}$$
$$K(\frac{k-1}{2},\epsilon) \longrightarrow \Omega^2 S^k \xrightarrow{2} \Omega^2 S^k$$

are homotopic.

This gives a way (which unfortunately fails) to attempt an explicit homotopy that does the job.

6 Some general homology calculations

Fred's work includes many computational results, some specific, and others very general. It is the second kind this section deals with. One example of such a calculation is of course Fred's thesis, where he gives a complete description of the homology of $\Omega^n\Sigma^n X$, but there are many others, some of which I will touch on below.

Many problems in algebraic topology involve understanding mapping spaces. The most obvious example is that of iterated loop spaces. The n–fold loop space $\Omega^n X$ can be identified with the pointed mapping space $\mathrm{Map}_*(S^n, X)$. Along totally different lines, the Sullivan conjecture, and subsequent work by Miller and Lannes involve studying mapping spaces of the form $\mathrm{Map}(BV, X)$, where V is an elementary abelian p–group.

When X and Y are arbitrary spaces, identification of $\mathrm{Map}_*(X, Y)$ is practically impossible. In a joint work with Larry Taylor [25], the authors study these spaces under certain hypotheses on X and Y, for which they obtain a rather explicit result.

Theorem 6.1 [25] *Let Y be an m–fold suspension, and let X be a finite complex of dimension less than $m/2$, which is itself the suspension of a connected space. Then there is a mod–p homology isomorphism of graded vector spaces*

$$H_*(\mathrm{Map}(X, Y)) \approx \bigotimes H_*((\Omega^i Y)^{\beta_i(X)}),$$

where $\beta_i(X)$ is the i^{th} Betti number of X, and the tensor product runs over all i such that $\beta_i(X) \neq 0$. Furthermore, if X is also a double suspension, then the isomorphism is as Hopf algebras.

Another family of spaces which features frequently in Fred's work are configuration spaces $C^k(M)$, already mentioned in the context of his thesis. Fred's interest in configuration spaces never withered, and they keep coming up in his work in a variety of contexts. Here are a few examples.

In [5] the authors Bödigheimer, Taylor and Fred study the configuration space $F(M, k)$ of k distinct points in a smooth compact m–manifold M, possibly with boundary. The paper determines the additive structure of the homology $H_*(F(M, k); \mathbb{F})$ where $\mathbb{F}$ is any field if m is odd, and $\mathbb{F}_2$ otherwise. This is a well cited paper, but like many important results its significance was not discovered immediately. This paper appeared in 1989, was first cited in 2000, and since then twelve more times, none of which by any of the authors themsleves.

In [26] the authors study the cohomology of the configuration space $F(\mathbb{R}^m, r)$ as a module over the symmetric group Σ_r. Although most of the work is done with integer coefficients, the most specific results, including identification of specific characters, are obtained with rational coefficients. The authors identify a class $A_{2,1} \in H^{m-1}(F(\mathbb{R}^m, 2); \mathbb{Z})$. Letting $\pi_{i,j}\colon F(\mathbb{R}^m, r) \longrightarrow F(\mathbb{R}^m, 2)$ be defined by the formula

$$\pi_{i,j}(x_1, \ldots, x_r) = (x_i, x_j),$$

they define $A_{i,j} = \pi_{i,j}(A_{2,1})$. These classes have many good properties and they play a key role in of $H^*(F(\mathbb{R}^m, r); \mathbb{Z})$. This paper is quoted by a number of authors in various applications, notably in a recent paper by Arone, Lambrechts and Volic [2].

Finally, in a collaboration with Sam Gitler [17], the loop space homology of $F(M, k)$ for certain manifolds M is studied. Most of the work is concerned with the case where M is obtained by removing a single point from a closed manifold. For instance, if $M = \mathbb{R}^m$, $m \geq 3$, the authors prove that the primitive elements of the integral homology ring $S = H_*(\Omega F(\mathbb{R}^m, k))$ form a Lie algebra generated by elements $B_{i,j}$ subject to the infinitesmal braid relations and that S itself is the universal enveloping algebra of this Lie algebra. The classes $B_{i,j}$ are related to the generators used by Fred in the calculation of $H^*(F(\mathbb{R}^m, k))$ in his thesis. More generally, if M is a simply connected punctured manifold of dimension $m \geq 3$, then the authors show that $\Omega F(M, k)$ is homotopy equivalent to a direct product, one factor of which is $\Omega F(\mathbb{R}^m, k)$. The remaining factors, which are identified explicitly and involve M, are also loop spaces, but the equivalence in the product decomposition is not multiplicative. Thus, to describe the ring structure in homology, one must determine the twisting among the factors, for instance, between the various $B_{i,j}$ and classes coming from the homology of M. Under appropriate hypotheses on M and with coefficients in certain fields $\mathbb{F}$, the authors find a complete set of relations describing this twisting, thereby determining $H^*(\Omega F(M, k); \mathbb{F})$ completely. They also show that the hypotheses are necessary by giving examples where some of the relations do not hold.

7 Mapping class groups

The homotopy theory associated to the mapping class groups is another subject Fred contributed very substantially to, with no less than nine papers to his name with the phrase "mapping class group" in the title. In these papers Fred and his coauthors explore various connections of the mapping class groups to homotopy theory, or perform various cohomology calculations (a good example of the latter is [4]).

A very beautiful example of the way Fred explores connections among mathematical objects is in his paper [16]. Let M^g be a closed orientable surface of genus g and let Γ_g denote its mapping class group. The hyperelliptic mapping class group Δ_g is defined to be the centralizer in Γ_g of the hyperelliptic involution which acts on M^g and fixes $2g+2$ points. In this paper Fred studies the groups Δ_g. For $g=2$, $\Delta_d = \Gamma_g$, but for $g > 2$ these subgroups are neither normal nor of finite index in Γ_g. Let Γ^n denote the mapping class group of S^2, with n fixed points. The group Γ^n was studied from the group theoretic point of view by Magnus. The relevance to Δ_g comes from the existence of a central extension

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \Delta_g \longrightarrow \Gamma^{2g+2} \longrightarrow 1.$$

In this paper Fred uses techniques of classical homotopy theory to study topological and homological properties of Δ_g. In particular he constructs spaces of type $K(\pi,1)$, where $\pi\Delta_g$. The constructions involves properties of the Lie groups $SO(3)$ and $Spin^c(3)$, and particularly a work of S. Smale, who showed that the natural inclusion $SO(3) \longrightarrow \mathrm{Diff}^+(S^2)$ is a homotopy equivalence.

8 Combinatorial group theory in homotopy theory

Fred's work on combinatorial group theory in homotopy theory is, in my mind, one of his most beautiful and original contributions. The core paper, for reasons he must know better than I do, remains unpublished [6]. When I say, the work remains unpublished, I'm lying a bit.

The object of study in this paper is the group $[\Omega\Sigma X, \Omega\Sigma X]$ of pointed homotopy classes of maps, where X is any reasonable space. Within this group one can single out two types of elements:

(1) For each natural number k the class of the k^{th} power map

$$\Omega\Sigma X \xrightarrow{\Delta} (\Omega\Sigma X)^{\times k} \xrightarrow{\mu} \Omega\Sigma X,$$

where Δ is the k–fold diagonal map, and μ is the loop space multiplication map.

(2) For each natural number k the class of the composite

$$\Omega\Sigma X \xrightarrow{h_k} \Omega\Sigma X^{(k)} \xrightarrow{\Omega\omega_k} \Omega\Sigma X,$$

where h_k is the k^{th} James–Hopf invariant, $X^{(k)}$ is the k–fold smash power of X, and ω_k is the k–fold iterated Whitehead product.

Naturality of these maps implies that they can be considered as endomorphisms (ie, self natural transformations) of the functor $\Omega\Sigma$ on the homotopy category of spaces. The set of all such endomorphisms forms a group under loop multiplication at the target (rather than under composition, which gives a monoid, but not generally a group, structure). In this group, which we denote $[\Omega\Sigma(-), \Omega\Sigma(-)]$, one can consider the subgroup H_∞ generated by the elements above.

Let $J_n(X)$ denote the n^{th} stage of the James construction on X. Thus, $J_\infty(X) = \cup_n J_n(X)$ is homotopy equivalent to $\Omega\Sigma X$. Then, as above, one can consider the group (under loop multiplication) of homotopy classes of natural transformations from $J_n(-)$ to $\Omega\Sigma(-)$, which we denote by $[J_n(-), \Omega\Sigma(-)]$. The James construction is naturally filtered by subfunctors $J_n(-)$, and this filtration induces a filtration on the group H_∞ by subgroups $H_n \le H_{n+1} \le \cdots \le H_\infty$. As Whitehead products, Hopf invariants, and compositions of such are among the most important maps in classical homotopy theory, studying the group H_∞ may give a breakthrough in our understanding of these maps and the relationships between them.

A similar exercise can be done in the group of natural transformations in the homotopy category $[(-)^n, \Omega\Sigma(-)]$ (again, under loop multiplication). In this group consider the subgroup generated by the elements p_i, given as the classes of the composites

$$(-)^n \xrightarrow{\mathrm{proj}_i} (-) \xrightarrow{E} \Omega\Sigma(-).$$

Let $K_n \le [(-)^n, \Omega\Sigma(-)]$ be the subgroup generated by these.

Here is one of the fundamental theorems proven in [6].

Theorem 8.1 [6] *The group K_n is a finitely presented, torsion free nilpotent group of class n. A specific presentation is given by the group generated by elements $x_1, \dots x_n$, subject to the relations:*

(i) $[x_{i_1}, \dots, x_{i_k}] = 1$ *if* $x_{i_j} = x_{i_k}$ *for* $j < k$.

(ii) $[x_{i_1}^{n_1}, \dots, x_{i_k}^{n_k}] = [x_{i_1}, \dots, x_{i_k}]^{n_1 \cdots n_k}$.

He then proceeds by identifying the groups H_n as subgroups given by a given set of generators inside the groups K_n.

Although [6] is not published, it has inspired further works by J Wu and others, some in collaboration with Fred.

9 Classifying spaces – a personal note

Classifying spaces are not a main theme in Fred's work, but like so many other subjects it is one on which he touched and inspired others – in this case myself. A mathematician is measured by his work and contributions to mathematics, but the hard work of teaching and inspiring students is often neglected. To do so in Fred Cohen's case will be to miss out on what I think is a major aspect of his mathematical persona. Of course, I only have my own experiences as Fred's student to share, but I dare guess that my story is not atypical.

In 1989 I went to graduate school in Rochester. My advisor at the time, Emmanuel Farjoun, recommended it to me very highly as one of the best places in the world to do a PhD in Topology. He told me who was there, and gave me a brief description of each person and his work. On Fred he said that he's been through some hard times health-wise, but in spite of that he was a wonderful mathematician.

I got to Rochester, and I remember very vividly the very first class with Fred, whom until then I never met. We were all seated waiting for him to arrive. Based on Emmanuel's description, my own prejudice made me expect a weak and tortured figure. Instead, in came a man, anything but weak and tortured, almost running with the aid of his cane, with this huge smile on his face, and a strong confident voice. Said hello, and started one of the most illuminating lectures I've ever heard until then. It was at this point, I think, that I decided this man will be my thesis advisor.

During my first year in Rochester I had ample opportunity to talk mathematics to Fred. I loved his lectures. A condition of participation in Fred's classes was that each one of us had to solve at least one homework problem in public, and he gave us plenty to think about. His manner was deceptively very casual. I remember thinking – oh what a wonderful new way of proving statements by saying "well, what could it be?" in a convincing tone. It didn't quite occur to me at that stage that one has to be prepared to explain why it couldn't be anything else. The first time I tried this technique on Fred in a private session, he replied "Ran, it could be many things. Go back and work out exactly what it is.". And, of course, he had a good reason for that. I had confidently "proved" a very wrong statement. This was very educational. Fred is always very friendly and informal with students, but they have to get it right or meet his "steel".

My own work with Fred was an example of how wide his interests range. In 1990 he participated in the Barcelona conference, which took place in the beautiful beach town of San Feliu on the Catalan Costa Brava. Trying to study a question of Dave Benson which occurred in a discussion between them in the Hotel's bar, Fred "gave birth" to my thesis subject.

Two background concepts before we explain the question. A group is said to be "perfect" if its first integral homology vanishes. If G is a perfect group and X is a space with $\pi_1(X) = G$, then the Quillen's "plus" construction associates with X a simply connected space X^+ with the same homology as X.

Benson's question was: what can be said on the homotopy type of BG^+, where G is a finite perfect group? Fred reacted by doing one of the things he does best. He looked at a few examples of finite perfect groups to which he applied $B(-)^+$, took the corresponding loop spaces, and calculated the living daylight out of them. Within the course of the evening, he managed to calculate a few examples which exhibited quite a curious behavior. They were all "finitely resolvable by fibrations over spheres and loop spaces on spheres". In other words he produced a finite sequence of fibrations, the total space in the first of which is ΩBG^+, where the fibre in the n^{th} fibration is the total space in the $(n+1)^{\text{st}}$, and where all base spaces where either spheres or loop spaces on spheres. He recorded his thoughts in a little paper, which he published in the Conference Proceedings [15].

Here is an easy example. Let p be an odd prime, and let $n \geq 2$ be an integer dividing $p-1$. Then the cyclic group $\mathbb{Z}/n$ act on $\mathbb{Z}/p$ by automorphisms and one can form the semidirect product $G(p,n) = \mathbb{Z}/p \rtimes \mathbb{Z}/n$. This group is not perfect, but it is p–perfect, and one can replace the "plus" construction by p–completion for a p–local version. The observation is that

$$\Omega BG(p,n)^\wedge_p \simeq S^{2n-1}\{p\},$$

where the right hand side is the fibre of the degree p map on the sphere. Thus one has a length 2 resolution

$$\Omega S^{2n-1} \longrightarrow \Omega BG^\wedge_p \longrightarrow S^{2n-1}.$$

This is of course a very easy example. The article [15] contained quite a few more, some of them far from obvious. This was convincing enough for Fred to make what he called a "Rush Conjecture", that in general ΩBG^+ is finitely spherically resolvable.

When he came back to Rochester and told me about this amusing discovery, I was totally fascinated – in fact, more than fascinated, I was hooked. I always loved group theory, especially finite, and the chance to work on a combination of group theory and homotopy theory seemed too good to let pass. So I almost immediately asked him to become my advisor and to let me work on this question.

Since I spent many hours before this moment in time with Fred discussing mathematics, I was very surprised, not to mention disappointed, when he wasn't fast to agree. Not only he wasn't going to let me work on this wonderfully eccentric subject; he didn't want to be my advisor, or so I thought. I was crushed! A day later he left a note in my mailbox telling me that he would in fact be happy to talk to me about whatever I wanted, including becoming my thesis advisor, if I insisted. Later I learnt the reason for his initial reluctance. Fred always regarded being an advisor as a great responsibility, almost a type of fatherhood. He was genuinely concerned, and for a good reason, about things like getting a job after graduation, and about certain problems being too bizarre for a PhD project, and this one was certainly an example of such a problem. So, in a sense he maybe wanted to make sure that his own students know exactly what it is they are getting themselves into. It seems to me that in most if not all cases, they knew. I certainly did.

Fred's "Rush Conjecture" became my thesis subject, and I kept at it for a number of years after graduating. It turned out to be a subject much richer and more interesting than could have been predicted during that pub chat with Benson, and the following Proceedings article. The conjecture itself turned actually to be wrong, as I proved about a year after graduation. There are examples, in fact rather easy examples, of finite p–perfect groups which are not spherically resolvable [28]. However other aspects of these spaces remain very interesting, and inspired a number of other mathematicians. One remarkable example is a recent work of Benson [3], where he gives a purely algebraic interpretation of the mod p loop space homology of $BG^{\wedge}_{p}$.

Fred and I wrote a few more papers together, on classifying spaces and other subjects. Several other of his students have shared the same pleasure with me. These were and still are illuminating and fruitful interactions, for which I am thoroughly grateful. At the time of writing this summary it is too late to wish Fred a happy birthday, but I will conclude by saying:

All the best to you Fred,
for many years of mathematics to come.

References

[1] **A Adem**, **D Cohen**, **F R Cohen**, *On representations and K-theory of the braid groups*, Math. Ann. 326 (2003) 515–542 MR1992276

[2] **G Arone**, **P Lambrechts**, **I Volić**, *Calculus of functors, operad formality, and rational homology of embedding spaces*, Acta Math. 199 (2007) 153–198 MR2358051

[3] **D Benson**, *An algebraic model for chains on $\Omega BG^{\wedge}_{p}$*, Trans. Amer. Math. Soc. 361 (2009) 2225–2242 MR2465835

[4] **D J Benson**, **F R Cohen**, *Mapping class groups of low genus and their cohomology*, Mem. Amer. Math. Soc. 90, 443 (1991) MR1052554

[5] **C-F Bödigheimer**, **F R Cohen**, **L Taylor**, *On the homology of configuration spaces*, Topology 28 (1989) 111–123 MR991102

[6] **F R Cohen**, *Combinatorial group theory in homotopy thoery, I*, preprint Available at `http://www.math.rochester.edu/people/faculty/cohf/`

[7] **F R Cohen**, *Cohomology of braid spaces*, Bull. Amer. Math. Soc. 79 (1973) 763–766 MR0321074

[8] **F R Cohen**, *Homology of $\Omega^{(n+1)}\Sigma^{(n+1)}X$ and $C_{(n+1)}X$, $n \geq 0$*, Bull. Amer. Math. Soc. 79 (1973) 1236–1241 (1974) MR0339176

[9] **F R Cohen**, *The homology of $\mathcal{C}_{n+1}$-spaces, $n \geq 0$*, from: "The homology of iterated loop spaces", Lecture Notes in Mathematics 533, Springer, Berlin (1976) 207–351 MR0436146

[10] **F R Cohen**, *The Homology of $SF(n+1)$*, from: "The homology of iterated loop spaces", Lecture Notes in Mathematics 533, Springer, Berlin (1976) 352–398 MR0436146

[11] **F R Cohen**, *Braid orientations and bundles with flat connections*, Invent. Math. 46 (1978) 99–110 MR493954

[12] **F R Cohen**, *Applications of loop spaces to classical homotopy theory*, from: "Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983)", PWN, Warsaw (1984) 621–626 MR804718

[13] **F R Cohen**, *A course in some aspects of classical homotopy theory*, from: "Algebraic topology (Seattle, 1985)", Lecture Notes in Math. 1286, Springer, Berlin (1987) 1–92 MR922923

[14] **F R Cohen**, *On the Whitehead square, Cayley-Dickson algebras, and rational functions*, Bol. Soc. Mat. Mexicana (2) 37 (1992) 55–62 MR1317562 Papers in honor of José Adem (Spanish)

[15] **F R Cohen**, *Remarks on the homotopy theory associated to perfect groups*, from: "Algebraic topology (San Feliu de Guíxols, 1990)", Lecture Notes in Math. 1509, Springer, Berlin (1992) 95–103 MR1185963

[16] **F R Cohen**, *On the mapping class groups for punctured spheres, the hyperelliptic mapping class groups, $\mathrm{SO}(3)$, and $\mathrm{Spin}^c(3)$*, Amer. J. Math. 115 (1993) 389–434 MR1216436

[17] **F R Cohen**, **S Gitler**, *On loop spaces of configuration spaces*, Trans. Amer. Math. Soc. 354 (2002) 1705–1748 MR1881013

[18] **F R Cohen**, **T J Lada**, **J P May**, *The homology of iterated loop spaces*, Lecture Notes in Mathematics 533, Springer, Berlin (1976) MR0436146

[19] **F R Cohen**, **J C Moore**, **J A Neisendorfer**, *Decompositions of loop spaces and applications to exponents*, from: "Algebraic topology (Aarhus 1978)", Lecture Notes in Math. 763, Springer, Berlin (1979) 1–12 MR561210

[20] **F R Cohen**, **J C Moore**, **J A Neisendorfer**, *The double suspension and exponents of the homotopy groups of spheres*, Ann. of Math. (2) 110 (1979) 549–565 MR554384

[21] **F R Cohen**, **J C Moore**, **J A Neisendorfer**, *Torsion in homotopy groups*, Ann. of Math. (2) 109 (1979) 121–168 MR519355

[22] **F R Cohen**, **J C Moore**, **J A Neisendorfer**, *Note on higher torsion in the homotopy groups of single suspensions*, Illinois J. Math. 24 (1980) 619–623 MR586800

[23] **F R Cohen**, **J C Moore**, **J A Neisendorfer**, *Exponents in homotopy theory*, from: "Algebraic topology and algebraic K-theory (Princeton, 1983)", Ann. of Math. Stud. 113, Princeton Univ. Press, Princeton, NJ (1987) 3–34 MR921471

[24] **F R Cohen**, **J L Selfridge**, *Not every number is the sum or difference of two prime powers*, Math. Comp. 29 (1975) 79–81 MR0376583 Collection of articles dedicated to Derrick Henry Lehmer on the occasion of his seventieth birthday

[25] **F R Cohen**, **L R Taylor**, *Homology of function spaces*, Math. Z. 198 (1988) 299–316 MR946606

[26] **F R Cohen**, **L R Taylor**, *On the representation theory associated to the cohomology of configuration spaces*, from: "Algebraic topology (Oaxtepec, 1991)", Contemp. Math. 146, Amer. Math. Soc., Providence, RI (1993) 91–109 MR1224909

[27] **F R Cohen**, **J Wu**, *On braid groups and homotopy groups*, from: "Groups, homotopy and configuration spaces (Tokyo 2005)", (N Iwase, T Kohno, R Levi, D Tamaki, J Wu, editors), Geom. Topol. Monogr. 13 (2008) 169–193 Conference in honour of Fred Cohen's 60$^{\text{th}}$ birthday

[28] **R Levi**, *A counter-example to a conjecture of Cohen*, from: "Algebraic topology: new trends in localization and periodicity (Sant Feliu de Guíxols, 1994)", Progr. Math. 136, Birkhäuser, Basel (1996) 261–269 MR1397737

[29] **M Mahowald**, *A new infinite family in* ${}_2\pi_*{}^s$, Topology 16 (1977) 249–256 MR0445498

[30] **P Selick**, *Moore conjectures*, from: "Algebraic topology—rational homotopy (Louvain-la-Neuve, 1986)", Lecture Notes in Math. 1318, Springer, Berlin (1988) 219–227 MR952582

Department of Mathematical Sciences, University of Aberdeen
Meston Building 339, Aberdeen AB24 3UE, UK

ran@maths.abdn.ac.uk

Received: 30 November 2008