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A classification of special 2–fold coverings

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Starting with an $SO(2)$–principal fibration over a closed oriented surface F_g, $g \geq 1$, a 2–fold covering of the total space is said to be *special* when the monodromy sends the fiber $SO(2) \sim S^1$ to the nontrivial element of $\mathbb{Z}_2$. Adapting D Johnson's method [11], we define an action of $Sp(\mathbb{Z}_2, 2g)$, the group of symplectic isomorphisms of $(H_1(F_g; \mathbb{Z}_2), .)$, on the set of special 2–fold coverings which has two orbits, one with $2^{g-1}(2^g+1)$ elements and one with $2^{g-1}(2^g-1)$ elements. These two orbits are obtained by considering Arf-invariants and some congruence of the derived matrices coming from Fox Calculus. $Sp(\mathbb{Z}_2, 2g)$ is described as the union of conjugacy classes of two subgroups, each of them fixing a special 2–fold covering. Generators of these two subgroups are made explicit.

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1 Introduction

We consider an $SO(2)$–principal bundle over a closed oriented surface F_g of genus $g \geq 1$ as a S^1–principal bundle: $S^1 \hookrightarrow P \xrightarrow{p} F_g$. A 2–fold covering $\pi_\varphi\colon E_\varphi \longrightarrow P$ is said to be *special* if its monodromy $\varphi\colon \pi_1 P \longrightarrow \mathbb{Z}_2$ has the property that $\varphi(u_0) = 1$, where 1 is the nontrivial element of $\mathbb{Z}_2$, and u_0 is the image of the generator of $\pi_1 S^1$. The set $\mathcal{E}(q) = \{\varphi\colon \pi_1 P \longrightarrow \mathbb{Z}_2, \mid \varphi(u_0) = 1\}$ is not empty if and only if q, the Chern class of the principal bundle, is even. This condition coincides with the vanishing of the second Stiefel–Whitney class of the S^1–principal bundle $S^1 \hookrightarrow P \xrightarrow{p} F_g$. In the sequel, it will be a running hypothesis that q is even. In [8] we obtained a presentation of $\pi_1 E_\varphi$. For all $\varphi \in \mathcal{E}(q)$, these spaces E_φ are isomorphic to the total space of a S^1–bundle over F_g classified by $q/2$. The images $\pi_\varphi(\pi_1 E_\varphi)$ are not conjugate subgroups of $\pi_1 P$. Nevertheless, any $\varphi, \varphi' \in \mathcal{E}(q)$, $E_\varphi \longrightarrow P$ and $E_{\varphi'} \longrightarrow P$ are weakly equivalent in the sense that there exists an automorphism f of $\pi_1 P$ such that $\varphi = \varphi' \circ f$ (see Proposition 17).

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The purpose of this work is to introduce on $\mathcal{E}(q)$ a supplementary structure obtained by an action of the symplectic group $Sp(H_1(F_g\mathbb{Z}_2),.)$. The following theorem synthesizes the results obtained in Theorem 14 and Theorem 15.

Theorem 1 *Let ξ be a S^1–principal bundle over a closed surface F_g of genus $g \geq 1$ with even Chern class q. Choosing a system of generators for $\pi_1 F_g$ and $\pi_1 P$ gives rise to a quadratic section s: $H_1(F_g;\mathbb{Z}_2) \longrightarrow H_1(P;\mathbb{Z}_2)$ (see Proposition 7). The s–action of the symplectic group $Sp(H_1(F_g\mathbb{Z}_2),.)$ on the set $\mathcal{E}(q)$ of special 2–fold coverings associated to the principal bundle ξ produces two orbits, one with $2^{g-1}(2^g+1)$ elements and the other one with $2^{g-1}(2^g-1)$ elements. The number of orbits and the number of elements in each orbit do not depend on the quadratic section s.*

The quadratic section s generalizes the work done by D Johnson [11] to any S^1–principal bundle over F_g with even Chern class, when ξ is associated to the tangent bundle of F_g, $g \geq 1$. Note that in this case the Chern class is always even.

One motivation to study special 2–fold coverings is that they can be considered as Spin–structures associated to an oriented 2–vector bundle over F_g with even Chern class $q = 2c$; see Milnor [12] and the article by the last three authors [7]. When this oriented 2–vector bundle is the tangent bundle and F_g is orientable, Atiyah [2], Birman and Craggs [3] and Johnson [9; 10] studied the Torelli subgroup of the mapping class group of the surface F_g. In these works, the splitting of $\mathcal{E}(2c)$ into two classes is an important ingredient. Nevertheless the study of normal fibrations defined by embeddings of a surface in $\mathbb{C}^2$ (see Blanlœil and Saeki [4]) shows that it is also worthwhile to start with any oriented S^1–principal bundle over F_g.

The results of Theorem 1 are obtained in two different ways. The quadratic section s allows us to consider the set $\mathcal{E}(q)$ as the set of quadratic forms over $(H_1(F_g;\mathbb{Z}_2),.)$ where the symbol "." is the intersection product. The associated Arf-invariant gives the counts of orbits and elements in each orbit. Considering the elements of $\mathcal{E}(q)$ as 2–fold coverings leads us to use Crowell and Fox calculus and to define congruence of the associated derived matrices (Definition 32). This congruence gives a classification of the $\mathbb{Z}_2[\mathbb{Z}_2]$–module structure of $H_1(E_\varphi,(E_\varphi)_0;\mathbb{Z}), \varphi \in \mathcal{E}(q)$ (Theorem 37).

As shown by Atiyah [2], each symplectic automorphism fixes a quadratic form. In Corollary 22, $Sp(H_1(F_g\mathbb{Z}_2),.)$ is described as the union of conjugacy classes of two subgroups, each of them fixing a special 2–fold covering. Generators of these two subgroups are made explicit in Theorem 19 and in Theorem 21.

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2 First part

2.1 Notation for the generators; introduction to special 2–fold coverings

For $\pi_1(F_g, x)$ we take the usual presentation

$$\pi_1(F_g, x) = \Big\langle x_1, \cdots, x_{2g} \Big| \prod_{j=1}^{g} [x_{2j-1}, x_{2j}] \Big\rangle.$$

In $\pi_1(P, y)$ we choose elements $\{u_1, \cdots, u_{2g}\}$ such that $p_\sharp(u_i) = x_i$. Let us fix $\mathbf{U} := \{\{u_i\}_{1 \le i \le 2g}, u_0\}$ where u_0 is a fixed generator of the fiber of p. The presentation of $\pi_1(P, y)$ is:

$$\pi_1(P, y) = \Big\langle \mathbf{U} \Big| R_i = [u_i, u_0], 1 \le i \le 2g;\, R_0 = \prod_{\ell=1}^{g} [u_{2\ell-1}, u_{2\ell}] u_0^q \Big\rangle.$$

Definition 2 Let u_0 be the element of $\pi_1 P$ obtained from the fiber of p and consider the exact sequence associated to a 2–fold covering

$$1 \longrightarrow \pi_1 E_\varphi \xrightarrow{\pi_\varphi} \pi_1 P \xrightarrow{\varphi} \mathbb{Z}_2 \longrightarrow 0.$$

When $\varphi(u_0) = 1$, the nontrivial element of $\mathbb{Z}_2$, we will say that the 2–fold covering $\pi_\varphi\colon E_\varphi \longrightarrow P$ is *special*.

There exists a one-to-one correspondence between the set of all special 2–fold coverings $\pi_\varphi\colon E_\varphi \longrightarrow P$ and the set $\mathcal{E}(q) = \{\varphi\colon \pi_1 P \longrightarrow \mathbb{Z}_2 | \varphi(u_0) = 1\}$, which corresponds bijectively to the set of Spin–structures associated to the oriented 2–vector bundle over F_g with Chern class q. This set $\mathcal{E}(q)$ is not empty if and only if q is even (see the presentation of $\pi_1 P$ given above). This condition is valid throughout this work and coincides with the vanishing of the second Stiefel–Whitney class of the S^1–principal bundle: $S^1 \hookrightarrow P \xrightarrow{p} F_g$. The set $\mathcal{E}(q)$ has 2^{2g} elements.

One important property of these special 2–fold coverings is that they have isomorphic fundamental group [8]

$$\pi_1 E_\varphi = \Big\langle y_1, \cdots, y_{2g}, k \Big| [y_i, k], 1 \le i \le 2g; \prod_{\ell=1}^{g} [y_{2\ell-1}, y_{2\ell}] k^{\frac{q}{2}} \Big\rangle.$$

The injection $\pi_\varphi\colon \pi_1 E_\varphi \longrightarrow \pi_1 P$ is defined by $\pi_\varphi(y_i) = u_i$, if $\varphi(u_i) = 0$, or $\pi_\varphi(y_i) = u_i u_0^{-1}$, if $\varphi(u_i) = 1$, and $\pi_\varphi(k) = u_0^2$. There are 2^{2g} injections of this type defining 2^{2g} images $\pi_\varphi(\pi_1 E_\varphi)$ which are not conjugate subgroups in $\pi_1 P$. To be convinced of this fact, let us remark that $\pi_\varphi(\pi_1 E_\varphi) = \ker \varphi$, hence is a normal subgroup of $\pi_1 P$, and $\varphi = \varphi'$ if and only if $\ker \varphi = \ker \varphi'$.

Remark 3 Let us denote by E^m the total space of a S^1–fibration over F_g classified by the integer m. Each E^c, with c an odd integer, is the start of an infinite graph with vertices $E^{2^n c}$, and 2^{2g} arrows $E^{2^n c} \longrightarrow E^{2^{n+1} c}$ the projections of nonisomorphic special 2–fold coverings.

Proposition 4 *Two special 2–fold coverings $E_\varphi \longrightarrow P$ and $E_{\varphi'} \longrightarrow P$, are always weakly equivalent in the sense that there exists an automorphism f of $\pi_1 P$ such that $\varphi = \varphi' \circ f$.*

Instead of proving this proposition, we will prove a stronger one, Proposition 17 in Section 2.5, where we impose f to be a lift of an automorphism of $\pi_1 F_g$. Let us recall some facts about these different lifts.

Lemma 5 (1) *Let $\mathrm{Homeo}^+(F_g)$ be the group of homeomorphisms of F_g preserving the orientation. The projection*

$$\mathrm{Homeo}^+(F_g) \longrightarrow Sp(H_1(F_g; \mathbb{Z}_2), .)$$

is an epimorphism.

(2) *An orientable homeomorphism of F_g admits a lift as an orientable fiber homeomorphism of P.*

Proof **(1)** The group of symplectic isomorphisms of $(H_1(F_g; \mathbb{Z}_2), .)$ is generated by the transvections, which are transformations of the form $A(x) = x + (x.a)a$ for some vector a [13]. These transvections define Dehn twists, which are orientable homeomorphisms of the surface F_g [14].

(2) Cutting the surface F_g along a cut system produces a $4g$–polygon Y. Let D be a disk in the interior of the polygon Y. The restriction to $Y - \mathring{D}$ of the S^1–fibration P is homeomorphic to $(Y - \mathring{D}) \times S^1$. To the boundary of the hole, has to be attached a torus $D \times S^1$ after q turns, where q is the Chern class of P. Let f be an orientable homeomorphism of F_g, we define $\hat{f}$ to be $f|_{Y - \mathring{D}} \times \mathrm{id}$. The curve $f(\partial D)$ is a simple closed curve. After turning q times, the gluing of $\hat{f}((Y - \mathring{D}) \times S^1)$ with $f(D) \times S^1$ is homeomorphic to P. □

The above results and considerations suggest that there exists an action of the group of symplectic isomorphism of $(H_1(F_g; \mathbb{Z}_2), .)$ on $\mathcal{E}(q)$.

2.2 Action of $Sp(H_1(F_g;\mathbb{Z}_2),.)$ on $\mathcal{E}(q)$

When P is the S^1–principal bundle associated to the tangent bundle of F_g, Johnson defines an action of $Sp(H_1(F_g;\mathbb{Z}_2),.)$ which has two orbits [11]. The definition of this action is given by means of a choice of a section of the projection $H_1(P,\mathbb{Z}_2)\longrightarrow H_1(F_g,\mathbb{Z}_2)$. In [11], the section reflects the geometry of the tangent bundle. We adapt this construction to make it work for any oriented S^1–principal bundle over F_g.

2.3 Johnson's lift of $p_\star\colon H_1(P;\mathbb{Z}_2)\longrightarrow H_1(F_g;\mathbb{Z}_2)$

Notation 6 Let us denote by h_M the composition $\pi_1 M\to H_1(M;\mathbb{Z})\to H_1(M;\mathbb{Z}_2)$, where the first morphism is the Hurewicz epimorphism. An element $\varphi\in\mathcal{E}(q)$ determines a unique $\widetilde{\varphi}\colon H_1(P;\mathbb{Z}_2)\longrightarrow\mathbb{Z}_2$ such that $\varphi=\widetilde{\varphi}\circ h_P$ and $\varphi(u_0)=\widetilde{\varphi}\circ h_P(u_0)=1$, the nontrivial element of $\mathbb{Z}_2$. This allows us to identify $\mathcal{E}(q)$ with $\{\widetilde{\varphi}\colon H_1(P;\mathbb{Z}_2)\longrightarrow\mathbb{Z}_2\mid\widetilde{\varphi}(u_0)=1\}$.

The family $\boldsymbol{\sigma}=\{\sigma_i\}_{1\leq i\leq 2g}$, $\sigma_i:=h_{F_g}(x_i)$ where $\{x_i\}$ are the fixed generators of $\pi_1 F_g$, is a symplectic basis in $(H_1(F_g;\mathbb{Z}_2),.)$ where $.$ is the intersection product. The family $\boldsymbol{\nu}:=\{\nu_i\}_{0\leq i\leq 2g}$; $\nu_i:=h_P(u_i)$ is a basis of $H_1(P;\mathbb{Z}_2)$.

Proposition 7 *Choose a family $\{s_i\}_{1\leq i\leq 2g}$ in $\oplus_{0\leq i\leq 2g}\nu_i\mathbb{Z}_2=H_1(P;\mathbb{Z}_2)$ such that $p_\star(s_i)=\sigma_i$ from the 2^{2g} possible choices. Then the following holds:*

(1) *$\{\{s_i\}_{1\leq i\leq 2g},\nu_0\}$ is a basis of $H_1(P;\mathbb{Z}_2)$.*

(2) *For all i, $s_i=\nu_i+r_i\nu_0, r_i\in\mathbb{Z}_2$, so 2^{2g} possible choices for $\{s_i, 1\leq i\leq 2g\}$.*

(3) *There exists a unique map*

$$s\colon \oplus_{1\leq i\leq 2g}\sigma_i\mathbb{Z}_2=H_1(F_g;\mathbb{Z}_2)\longrightarrow\oplus_{0\leq i\leq 2g}\nu_i\mathbb{Z}_2=H_1(P;\mathbb{Z}_2),$$

defined by $s(\sigma_i)=s_i, 1\leq i\leq 2g$ such that for all $a,b\in H_1(F_g;\mathbb{Z}_2)$

$$s(a+b)=s(a)+s(b)+(a.b)\nu_0. \tag{2–1}$$

Notation 8 The map s obtained in Proposition 7 will be called a *quadratic section*.

Proof of Proposition 7 **(1)** If $\Sigma\alpha_i s(\sigma_i)+\gamma\nu_0=0$, then $p_\star(\Sigma\alpha_i s(\sigma_i)+\gamma\nu_0)=\Sigma\alpha_i\sigma_i=0$; so $\alpha_i=0$ and $\gamma=0$. This implies that $\{\{s(\sigma_i)\}_{1\leq i\leq 2g},\nu_0\}$ is a basis of $H_1(P;\mathbb{Z}_2)$.

(2) This is true because $\ker p_\star=\langle\nu_0\rangle$.

(3) Take $a = \Sigma a_i \sigma_i, a_i \in \mathbb{Z}_2$. Because of condition (2–1), we must define $s(a)$ by:

$$s(a) = \Sigma a_i s(\sigma_i) + (\Sigma a_{2i-1} a_{2i})\nu_0.$$

Now, if the map s is defined by this equation, then

$$\begin{aligned} s(\Sigma a_i\sigma_i + \Sigma b_i\sigma_i) &= s(\Sigma(a_i + b_i)\sigma_i) \\ &= \Sigma(a_i + b_i)s(\sigma_i) + [(\Sigma(a_{2i-1} + b_{2i-1})(a_{2i} + b_{2i})]\nu_0 \\ &= s(\Sigma a_i\sigma_i) + s(\Sigma b_i\sigma_i) + [\Sigma(a_{2i}b_{2i-1} + a_{2i-1}b_{2i})]\nu_0 \\ &= s(\Sigma a_i\sigma_i) + s(\Sigma b_i\sigma_i) + (\Sigma a_i\sigma_i).(\Sigma b_i\sigma_i)\nu_0, \end{aligned}$$

because the coefficients are in $\mathbb{Z}_2$. □

Remark 9 In the case where the fiber bundle P is the S^1–principal bundle associated to the tangent bundle of the surface F_g, the geometry imposes the choice of $r_i = 1, 1 \leq i \leq 2g$ [11]. Hence, if necessary, it is possible to normalize the choice of the maps s imposing this condition on the family r_i as in Arf [2].

Let U be a set of generators of $H_1(P; Z_2)$. We also impose that for each chosen generator of $\pi_1 F_g$ there will be one element in U which is a lift of it. Hence two such systems of generators $\mathbf{U}$ and $\mathbf{U}'$ of $\pi_1 P$ are related by $u_i' = u_0^{-\alpha_i} u_i$ (or equivalently $u_i = u_0^{\alpha_i} u_i'$), $\alpha_i \in \{0, 1\}$. An element $\varphi \in \mathcal{E}(q)$ is then changed into $\varphi'(u_i) = \varphi(u_i) + \alpha_i$ and $\varphi'(u_0) = \varphi(u_0)$. Note that such a change of generators is equivalent to a change of the quadratic section s.

Definition 10 Let A be the symplectic matrix $(a_{ij})_{i,j\leq 2g}$ written in the basis $\boldsymbol{\sigma}$, of a symplectic isomorphism $f\colon H_1(F_g; \mathbb{Z}_2) \longrightarrow H_1(F_g; \mathbb{Z}_2)$. We define

$$f_s\colon \oplus_{0\leq i\leq 2g}\nu_i\mathbb{Z}_2 = H_1(P; \mathbb{Z}_2) \longrightarrow \oplus_{0\leq i\leq 2g}\nu_i\mathbb{Z}_2 = H_1(P; \mathbb{Z}_2)$$

by linearity from

$$f_s(s(\sigma_i)) := s(f(\sigma_i)), f_s(\nu_0) := \nu_0.$$

The matrix of f_s in the basis $\boldsymbol{\nu}$ is $\left(\begin{smallmatrix} A & 0 \\ W & 1 \end{smallmatrix}\right)$ where W is a line with $2g$ terms $w_j = \Sigma a_{ij} r_i + S_j + r_j$, $S_j = \Sigma a_{2i,j} a_{2i-1,j}$, $r_j \nu_0 = s(\sigma_j) + \nu_j$.

Notice that $f_s \circ s = s \circ f$. We have $(f_1 f_2)_s = (f_1)_s (f_2)_s$ and $(\mathrm{id}_{Sp(\mathbb{Z}_2,2g)})_s = \mathrm{id}_{Sl(\mathbb{Z}_2,2g+1)}$. This proves the following proposition, where $Sp(\mathbb{Z}_2, 2g)$ denotes the group of the symplectic $2g \times 2g$ matrices with coefficients in $\mathbb{Z}_2$:

Proposition 11 *The injective map*

$$\begin{array}{rcl} J\colon Sp(\mathbb{Z}_2, 2g) & \longrightarrow & Sl(\mathbb{Z}_2, 2g+1) \\ A & \mapsto & \widetilde{A} = \begin{pmatrix} A & 0 \\ W & 1 \end{pmatrix} \end{array}$$

with $A = (a_{ij})_{i,j \le 2g}$ *and* $W = (w_1 \cdots w_{2g})$ *where* $w_j = \Sigma a_{ij} r_i + S_j + r_j$ *and* $S_j = \Sigma a_{2i,j} a_{2i-1,j}$ *is a monomorphism.*

2.4 s–Relation between two special 2–fold coverings

Definition 12 Two special 2–fold coverings φ and φ' are s–related if there exists a symplectic isomorphism $f\colon H_1(F_g;\mathbb{Z}_2) \longrightarrow H_1(F_g;\mathbb{Z}_2)$ such that

$$\widetilde{\varphi} = \widetilde{\varphi}' \circ f_s,$$

with f_s given in Definition 10, or equivalently: $\widetilde{\varphi} \circ s = \widetilde{\varphi}' \circ s \circ f$.

Proposition 13 *Two special* 2*–fold coverings* φ *and* φ' *are* s*–related if and only if*

$$\forall j, \varphi(u_j) = \Sigma_{i=1,\dots,2g} a_{ij} \varphi'(u_i) + w_j, \tag{2–2}$$

where $A = (a_{ij})$ *is the matrix of a symplectic isomorphism in the basis* σ, *and* $w_j = \Sigma a_{ij} r_i + \Sigma a_{2i,j} a_{2i-1,j} + r_j$, r_j *determined by the choice of* s. □

Let us recall or introduce some terminology needed for Theorem 14.

- For any automorphism K of $H_1(F_g;\mathbb{Z}_2)$, a *lift* of K is an automorphism k of $\pi_1 P$ such that $h_P \circ k = K \circ h_P$, where h_P is defined in Notation 6.
- Two special 2–fold coverings $\pi_\varphi\colon E_\varphi \longrightarrow P$ and $\pi_{\varphi'}\colon E_{\varphi'} \longrightarrow P$ are *weakly equivalent* by $k \in \mathrm{Aut}(\pi_1(P))$ if and only if $\varphi = \varphi' \circ k$.

Theorem 14 (1) *For any symplectic automorphism* f *of* $H_1(F_g;\mathbb{Z}_2)$, *there exists a lift* $f_\sharp\colon \pi_1 P \longrightarrow \pi_1 P$ *of* f_s *(Definition 10).*

(2) *For any such* f *and* $f_\sharp$, $\varphi, \varphi' \in \mathcal{E}(q)$ *are* s*–related by* f *if and only if* $\pi_\varphi, \pi_{\varphi'}$ *are weakly equivalent by* $f_\sharp$.

Proof **(1)** Using geometric arguments we proved in Lemma 5 that there exists an automorphism g of $\pi_1 P$ such that $g(u_0) = u_0$ and $f \circ h_{F_g} \circ p_\sharp = h_{F_g} \circ p_\sharp \circ g$. This implies that the morphism $f_s \circ h_P - h_P \circ g\colon \pi_1(P) \longrightarrow H_1(P;\mathbb{Z}_2)$ takes its values in the subgroup $\ker(p_*) = \mathbb{Z}_2 \nu_0$, ie, it is of the form $x \mapsto \overline{\rho}(x)\nu_0$ for some morphism $\overline{\rho}\colon \pi_1(P) \longrightarrow \mathbb{Z}_2$. If we are able to construct a lift $\rho\colon \pi_1(P) \longrightarrow \mathbb{Z}$ of $\overline{\rho}$, then we just have to define $f_\sharp\colon \pi_1(P) \longrightarrow \pi_1(P)$ by $f_\sharp(x) = g(xu_0^{\rho(x)})$ to get an automorphism $f_\sharp$ of $\pi_1(P)$ satisfying $h_P \circ f_\sharp = f_s \circ h_P$. In order to construct such a lift ρ, notice that $\overline{\rho}$ factorizes through $H_1(P;\mathbb{Z})$: denoting by $hz\colon \pi_1(P) \longrightarrow H_1(P;\mathbb{Z})$ the Hurewicz morphism, we have $\overline{\rho} = \overline{r} \circ hz$ for some morphism $\overline{r}\colon H_1(P;\mathbb{Z}) \longrightarrow \mathbb{Z}_2$, for which we want a lift $r\colon H_1(P;\mathbb{Z}) \longrightarrow \mathbb{Z}$. There are many such r's, since $\overline{r}(hz(u_0)) = \overline{\rho}(u_0) = 0$ and $H_1(P;\mathbb{Z})/\langle hz(u_0)\rangle \simeq H_1(F_g;\mathbb{Z})$ is a free $\mathbb{Z}$–module.

(2) Recall that h_P is an epimorphism, hence we have the following equivalences:

$$\widetilde{\varphi}' \circ f_s = \widetilde{\varphi} \Leftrightarrow \widetilde{\varphi}' \circ f_s \circ h_P = \widetilde{\varphi} \circ h_P \Leftrightarrow \widetilde{\varphi}' \circ h_P \circ f_\sharp = \widetilde{\varphi} \circ h_P \Leftrightarrow \varphi' \circ f_\sharp = \varphi. \quad \square$$

2.5 Arf type invariant

The purpose of this section is to prove that there are two orbits under the s–action, one with $2^{g-1}(2^g+1)$ elements and one with $2^{g-1}(2^g-1)$ elements. We use the quadratic section s defined and fixed in the above subsection to associate bijectively a special 2–fold covering φ and a quadratic form $\omega_\varphi = \widetilde{\varphi} \circ s$.

Let $\{\sigma_1, \sigma_2, \cdots, \sigma_{2g-1}, \sigma_{2g}\}$ be a symplectic basis of $(\mathbb{Z}_2^{2g}, .)$. This means that $\sigma_{2i-1} \,.\, \sigma_{2i} = \sigma_{2i} \,.\, \sigma_{2i-1} = 1, 1 \le i \le g$, and all the others $\sigma_i.\sigma_j = 0$. The Arf-invariant of a quadratic form $\omega\colon (\mathbb{Z}_2^{2g}, .) \longrightarrow \mathbb{Z}_2$ is defined by

$$\alpha(\omega) = \Sigma\omega(\sigma_{2j-1})\omega(\sigma_{2j}).$$

Theorem 15 *Two special 2–fold coverings φ and φ' are s–related if and only if the Arf-invariants of ω_φ and $\omega_{\varphi'}$ are equal [1], explicitly:*

$$\Sigma\widetilde{\varphi}(s(\sigma_{2j-1}))\widetilde{\varphi}(s(\sigma_{2j})) = \Sigma\widetilde{\varphi}'(s(\sigma_{2j-1}))\widetilde{\varphi}'(s(\sigma_{2j})).$$

Proof Proposition 7 proved that $s(a+b) = s(a) + s(b) + (a.b)\nu_0$, so φ determines a quadratic form

$$\begin{array}{rcl} \omega_\varphi\colon \mathbb{Z}_2^{2g} & \longrightarrow & \mathbb{Z}_2 \\ a & \mapsto & \omega_\varphi(a) = \widetilde{\varphi}(s(a)). \end{array}$$

A quadratic form ω determines $\varphi \in \mathcal{E}(q)$ by $\widetilde{\varphi}(s(\sigma_i)) = \omega(\sigma_i)$ and $\widetilde{\varphi}(\nu_0) = \nu_0$. Two special 2–fold coverings φ and φ' are s–related (Definition 12) if and only if there exists a symplectic map $f\colon H_1(F_g, \mathbb{Z}_2) \longrightarrow H_1(F_g, \mathbb{Z}_2)$ such that $\omega_\varphi = \omega_{\varphi'} \circ f$, which is equivalent to the equality of the Arf-invariants of ω_φ and $\omega_{\varphi'}$ [1]. We give below a short proof of this classical property. $\square$

Proposition 16 *There exists a symplectic map $f\colon (\mathbb{Z}_2^{2g}, .) \longrightarrow (\mathbb{Z}_2^{2g}, .)$ such that $\omega = \omega' \circ f$ if and only if $\alpha(\omega) = \alpha(\omega')$. We will denote this by $\omega \sim \omega'$.*

Proof Let $\omega, \omega'\colon (\mathbb{Z}_2^{2g}, .) \longrightarrow \mathbb{Z}_2$ be any two quadratic forms. Their difference is a linear form

$$\omega'(x) - \omega(x) = V.x.$$

By an elementary computation we have:

$$\alpha(\omega') - \alpha(\omega) = \omega(V).$$

For any vector Y, let us denote by T_Y the symplectic transvection defined by $T_Y(x) = x + (Y.x)Y$. We then obtain $\omega(T_Y(x)) = \omega(x) + \omega((Y.x)Y) + Y.x$, hence

$$(\omega \circ T_Y)(x) - \omega(x) = (1 + \omega(Y))Y.x.$$

Using these two equations we deduce:

- $\alpha(\omega') = \alpha(\omega) \Rightarrow \omega(V) = 0 \Rightarrow \omega \circ T_V - \omega = V. = \omega' - \omega \Rightarrow \omega \circ T_V = \omega' \Rightarrow \omega' \sim \omega$.
- Conversely, $\omega' = \omega \circ T_Y \Rightarrow V = (1 + \omega(Y))Y \Rightarrow \alpha(\omega') - \alpha(\omega) = (1 + \omega(Y))\omega(Y) = 0$. Hence (since transvections generate the group of symplectic isomorphisms [13]) $\omega' \sim \omega \Rightarrow \alpha(\omega') = \alpha(\omega)$. □

The following proposition will prove a stronger property than weak equivalence for any pair of special 2–fold coverings:

Proposition 17 *Given two special 2–fold coverings $E_\varphi \longrightarrow P$, $E_{\varphi'} \longrightarrow P$, it is possible to choose a quadratic section $s(\varphi, \varphi')$ such that these 2–fold coverings are $s(\varphi, \varphi')$–related (Definition 12).*

Proof First, it is possible to choose a quadratic section $s = s(\varphi, \varphi')$ such that $\alpha(\widetilde{\varphi} \circ s) = 0 = \alpha(\widetilde{\varphi}' \circ s)$. In fact, because $\alpha(\widetilde{\varphi} \circ s) = \sum_{i=1}^{g} (\widetilde{\varphi}(\nu_{2i-1}) + r_{2i-1})(\widetilde{\varphi}(\nu_{2i}) + r_{2i})$ (the same for φ'), it is enough to choose for example $r_i = \widetilde{\varphi}(\nu_i)$ for i odd and $r_i = \widetilde{\varphi}'(\nu_i)$ for i even. By Proposition 16 or [1] there exists $f \in Sp(H_1(F_g; \mathbb{Z}_2), .)$ such that $\widetilde{\varphi} \circ s = \widetilde{\varphi}' \circ s \circ f$. □

2.6 Subgroups $Sp_\omega(\mathbb{Z}_2, 2g)$ of the symplectic automorphisms which fix a quadratic form ω

2.6.1 Generators As shown by Atiyah [2], each symplectic automorphism fixes a quadratic form ω. Let us study the subgroup $Sp_\omega(\mathbb{Z}_2, 2g)$ of symplectic automorphisms which fix ω (this ω may be of the form $\omega_\varphi := \widetilde{\varphi} \circ s$).

It suffices to study the two subgroups Sp_i ($i = 0$ or 1) corresponding to ω_i, with $\omega_0(x) := \sum x_{2k-1}x_{2k}$ and $\omega_1(x) := \omega_0(x) + x_1 + x_2$. Then, if $\alpha(\omega) = i$, Sp_ω is a conjugate of Sp_i (by any $f \in Sp(\mathbb{Z}_2, 2g)$ such that $\omega = \omega_i \circ f$).

Lemma 18 *The actions of Sp_0 on $H_0 := \{x \neq 0, \omega_0(x) = 0\}$ and on $H_1 := \{x, \omega_0(x) = 1\}$ are transitive.*

Proof We assume that $g > 1$ ($g = 1$ is obvious). Note that Sp_0 contains all symplectic permutations, and all transvections T_u such that $\omega_0(u) = 1$.

If $x \in H_0$, since $x \neq 0$, up to some symplectic permutation, we may assume that $x.e_1 = 1$. Let $u := x + e_1$. Then $\omega_0(u) = 1$ and $T_u(e_1) = x$.

If $x \in H_1$, we have:

Case 1: If $x.(e_{2k-1} + e_{2k}) = 1$ for some k, up to some symplectic permutation, we may assume that $k = 1$. Let $u := x + e_1 + e_2$. Then $\omega_0(u) = 1$ and $T_u(e_1 + e_2) = x$.

Case 2: If $x.(e_{2k-1} + e_{2k}) = 0$ for all k's. Since $x \neq 0$, up to some symplectic permutation, we may assume that $x.e_1 = 1$. Let $u' := e_1 + e_3 + e_4$ (hence $\omega_0(u') = 1$) and $x' := T_{u'}(x) = x + u'$. Then $x'.(e_1 + e_2) = u'.(e_1 + e_2) = 1$ hence we are led to the first case. □

Theorem 19 *Any element of Sp_0 is a product of:*

(1) *symplectic permutations,*

(2) *(if $g \geq 2$) the matrix $B_1 := \begin{pmatrix} A_1 & 0 \\ 0 & I_{2g-4} \end{pmatrix}$ with $A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.*

Proof Take $g > 1$ ($g = 1$ is obvious) and assume the property true for $g - 1$. Call "type R" all matrices of the form $\left(\begin{smallmatrix} I_2 & 0 \\ 0 & A \end{smallmatrix}\right)$ (which, by induction hypothesis, are products of these generators). Let $\gamma \in Sp_0$ and $V := \mathrm{Vect}(e_1, e_2)$.

Case 0: $\gamma(V) = V$. Then γ fixes or exchanges e_1 and e_2; hence (up to some product by a symplectic transposition) γ is of type R.

Case 1: $\gamma(V) \neq V$ but there exists a (nonzero) $x \in V$ such that $\gamma(x) \in V$.

1.1: $x = e_1$ or e_2. Up to symplectic permutation(s), $\gamma(e_2) = e_2$. Then $\gamma(e_1) = y + z$ with $y \in V, z \in V^{\perp}$, $z \neq 0$, $y.e_2 = 1$ (hence $y = e_1$ or $e_1 + e_2$), and $\omega_0(z) = \omega_0(y)$.

1.1.1: $y = e_1$, $\omega_0(z) = 0$. Hence by the lemma we may assume $z = e_3$ (up to some product by a type R matrix). In this case, $B_1^{-1}\gamma$ is of type R.

1.1.2: $y = e_1 + e_2$, $\omega_0(z) = 1$. Hence (by the lemma again) we may assume $z = e_3 + e_4$. In this case, $B_1^{-1}\gamma$ fixes e_2 and sends e_1 to $e_1 + e_4$, hence it falls into the subcase 1.1.1.

1.2: $x=e_1+e_2$. For $i=1,2$, $\gamma(e_i)=y_i+z$ with $y_i\in V, z\in V^\perp, z\neq 0, y_1+y_2=e_1+e_2, y_1.y_2=1$. Hence (up to symplectic transposition) $y_i=e_i$, so that $\omega_0(z)=0$, hence, by the lemma again, we may assume that $z=e_3$. In that case, $B_1^{-1}\gamma$ fixes e_1; hence it belongs to the subcase 1.1 (or to case 0).

Case 2: $\gamma(x)\in V^\perp$ for some nonzero $x\in V$. By the lemma we may assume (up to some product by a type R matrix) that $\gamma(x)=e_3$ or $\gamma(x)=e_3+e_4$ (depending whether $\omega_0(x)$ equals 0 or 1). By symplectic permutation the situation is reduced to case 0 or 1.

Case 3: None of the three nonzero elements of V is sent by γ to $V\cup V^\perp$. Let $\gamma(e_1)=y+z, \gamma(e_2)=y'+t$ with $y,y'\in V, z,t\in V^\perp$. Then y,y' are nonzero and distinct, hence at least one of them equals some e_i (with $i=1$ or 2). We may assume that $\gamma(e_1)=e_1+z$, hence $\omega_0(z)=0$. Since $z\neq 0$, we may assume $z=e_3$. Then $B_1^{-1}\gamma$ fixes e_1, hence it belongs to case 0 or 1. □

Remark 20 A classical set of generators for the whole group $Sp(\mathbb{Z}_2,2g)$ consists of these generators of the subgroup Sp_0, and the matrix B_0 corresponding to $A_0:=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ (cf O'Meara [13]).

Theorem 21 *Any element of Sp_1 is a product of:*

(1) *elements of the subgroup $Sp(\mathbb{Z}_2,2)\times Sp_0(\mathbb{Z}_2,2g-2)$,*

(2) *(if $g\geq 2$) the matrix* $B_2:=\begin{pmatrix}A_2&0\\0&I_{2g-4}\end{pmatrix}$ *with* $A_2=\begin{pmatrix}1&0&0&0\\0&1&0&1\\1&0&1&1\\0&0&0&1\end{pmatrix}$.

Proof If $g=1$, $Sp_1=Sp(\mathbb{Z}_2,2)$.

Let $\gamma\in Sp_1$ and $V=\mathrm{Vect}(e_1,e_2)$.

Case 0: $\gamma(V)=V$. Then $\gamma\in Sp(\mathbb{Z}_2,2)\times Sp_0$.

Case 1: $\gamma(V)\neq V$ but there exists a (nonzero) $x\in V$ such that $\gamma(x)\in V$. Assume (up to products by elements of $Sp(\mathbb{Z}_2,2)=GL(2,\mathbb{Z}_2)$) $\gamma(e_2)=e_2$ and $\gamma(e_1)=e_1+z$, with $z\in V^\perp$, nonzero, and such that $\omega_0(z)=0$. Assume moreover (up to some product by an element of Sp_0, by Lemma 18) $z=e_3$. Then $B_2^{-1}\gamma$ belongs to the subgroup $Sp_0(\mathbb{Z}_2,2g-2)$.

Case 2: For some $x\in V$, $\gamma(x)\notin V\cup V^\perp$. Using the same arguments as above, we may assume that $\gamma(e_1)=e_1+e_3=B_2(e_1)$, hence $B_2^{-1}\gamma$ satisfies the condition in case 0 or 1.

Case 3: For all $x \in V, \gamma(x) \in V^{\perp}$. We may assume that $\gamma(e_1) = e_3 + e_4 = B_2(e_2 + e_4)$, hence $B_2^{-1}\gamma$ satisfies the condition in case 2. □

For each ω such that $\alpha(\omega) = \alpha(\omega_i), i = 0, 1$, let us choose the transvection T_{Y_ω} where Y_ω is the vector such that for all x, $\omega(x) - \omega_i(x) = Y_\omega.x$. Recall that we have shown in the proof of Proposition 16 that $\alpha(\omega) - \alpha(\omega_i) = \omega_i(Y_\omega)$. Now let us define the two subsets $\alpha_i := \{Y \mid \omega_i(Y) = 0\}$. The family α_0 has $2^{g-1}(2^g+1)$ elements and α_1 has $2^{g-1}(2^g-1)$ elements. We get the corollary:

Corollary 22

$$Sp(\mathbb{Z}_2, 2g) = \bigcup_{Y \in \alpha_0} [T_Y^{-1} Sp_0 T_Y] \cup \bigcup_{Y \in \alpha_1} [T_Y^{-1} Sp_1 T_Y].$$

The generators of Sp_0 and Sp_1 (Theorems 19, 21) admit lifts, described for example in Zieschang, Vogt and Coldewey [14], as homeomorphisms of the surface F_g. When a quadratic section s is chosen, we may view these homeomorphisms as homeomorphisms fixing a Spin–structure associated to an oriented 2–vector bundle over F_g with Chern class equal to q.

Corollary 23 (1) *Under the action defined in Definition 12 the set $\mathcal{E}(q)$ of special 2–fold coverings is divided into two orbits: $\mathcal{E}(q)^0$ with $2^{g-1}(2^g+1)$ elements and $\mathcal{E}(q)^1$ with $2^{g-1}(2^g-1)$.*

(2) *The stabilizers of an element of $\mathcal{E}(q)^i$ is a conjugate of $Sp_i, i = 0, 1$.*

Remark 24 Let us emphasize that after a change of the generators of $\pi_1 P$, which are lifts of the fixed generators of $\pi_1 F_g$, or after a change in the choice of the quadratic section s (see Proposition 7 (2)), only the number of orbits of $\mathcal{E}(q)$ and the number of elements in each orbit do not change.

3 Second part

3.1 Derived matrix

In this section we apply to the special 2–fold coverings the classical tools of Fox derivatives. We will give a description of the $\mathbb{Z}[\mathbb{Z}_2]$–module structure of $H_1(E_\varphi, (E_\varphi)_0; \mathbb{Z})$, using the Reidemeister method as referred in [5, Chapter 9] (also [6]), where $(E_\varphi)_0$ is the fiber with two elements above the base point of P. The exact sequence of the pair $(E_\varphi, (E_\varphi)_0$ is:

$$0 \longrightarrow H_1(E_\varphi; \mathbb{Z}) \longrightarrow H_1(E_\varphi, (E_\varphi)_0; \mathbb{Z}) \longrightarrow \mathbb{Z}[\mathbb{Z}_2] \longrightarrow \mathbb{Z} \longrightarrow 0.$$

A notion of congruence is defined on the matrices. It leads to the same relation between the data as the necessary relations to be s–related (2–2) or Arf related Theorem 15. The last step is to add a $*$–product on $H_1(E_\varphi, (E_\varphi)_0; \mathbb{Z}_2)$ and to find a relation between the $\mathbb{Z}[\mathbb{Z}_2]$–module structures of $H_1(E_\varphi, (E_\varphi)_0; \mathbb{Z}_2)$ and $H_1(E_{\varphi'}, (E_{\varphi'})_0; \mathbb{Z}_2)$ when φ and φ' are s–related.

3.1.1 Summary of Crowell and Fox calculus Let $\varphi \in \mathcal{E}(q)$. In the exact sequence of the homotopy groups of the special 2–fold covering $\pi\colon E_\varphi \longrightarrow P$:

$$0 \longrightarrow \pi_1(E_\varphi, x) \xrightarrow{\pi_\sharp} \pi_1(P, y) \xrightarrow{\varphi} \mathbb{Z}_2 \longrightarrow 0$$

the group $\mathbb{Z}_2$ is the multiplicative group of deck transformation of the covering. Writing the ring $\mathbb{Z}[\mathbb{Z}_2] = \mathbb{Z}[t]/(1-t^2)$, the homomorphism $\varphi\colon \pi_1 P \longrightarrow \mathbb{Z}_2$ extends to a group ring morphism $\mathbb{Z}[\pi_1 P] \longrightarrow \mathbb{Z}[\mathbb{Z}_2]$, also denoted by φ. This morphism verifies in particular $\varphi(u_0) = t, \varphi(1) = 1$, ($u_0$ coming from the fiber S^1) and $\varphi(0) = 0$.

3.1.2 Explicit computations of the derived matrix Recall that we make choices such that the presentation of $\pi_1(P, y)$ is:

$$\pi_1(P, y) = \left\langle \mathbf{U} \middle| R_i = [u_i, u_0], 1 \le i \le 2g;\ R_0 = \prod_1^g [u_{2\ell-1}, u_{2\ell}] u_0^{2c} \right\rangle.$$

Let M_2 be the free $\mathbb{Z}[\mathbb{Z}_2]$–module generated by the set $\mathbf{R} := \{R_i, 0 \le i \le 2g\}$ and M_1 the free $\mathbb{Z}[\mathbb{Z}_2]$–module generated by $\mathbf{U} = \{u_i, 0 \le i \le 2g\}$. Using the Fox derivation $\partial R_i/\partial u_j$, a $\mathbb{Z}[\mathbb{Z}_2]$–morphism $d_\varphi\colon (M_2, \mathbf{R}) \longrightarrow (M_1, \mathbf{U})$ is defined by $d_\varphi R_i = \sum_j m_{ji} u_j$, where $m_{ji} = \varphi q(\partial R_i/\partial u_j)$ and q is the ring morphism obtained from the group projection from the free group generated by the set $\mathbf{U}$ to $\pi_1(P, y)$. So there is an exact sequence of $\mathbb{Z}[\mathbb{Z}_2]$–modules:

$$(M_2, \mathbf{R}) \xrightarrow{d_\varphi} (M_1, \mathbf{U}) \longrightarrow (M_1/\mathrm{Im}\ d_\varphi, \bar{\mathbf{U}}) \longrightarrow 0,$$

where $\bar{\mathbf{U}} := \{\bar{u_i}\}_{0 \le i \le 2g}$, $\bar{u_i}$ class of u_i modulo $\mathrm{Im} d_\varphi$.

The structure of $\mathbb{Z}[\mathbb{Z}_2]$–module of $M_1/\mathrm{Im}\ d_\varphi$ is denoted by H_φ.

Let u be an element of $\pi_1(P, y)$ and select a loop $\alpha \in u$. By the path-lifting property of covering spaces, there exists a unique path $\alpha'\colon I \longrightarrow E$ such that the projection of α' is α and $\alpha'(0) = y$. Its relative homology class is denoted by $\widetilde{u}$. From [5, Chapter 9; 6], we know that there exists a $\mathbb{Z}[\mathbb{Z}_2]$–isomorphism

$$H_\varphi \longrightarrow H_1(E_\varphi, (E_\varphi)_0; \mathbb{Z}), \bar{u}_i \mapsto \widetilde{u}_i.$$

Up to this isomorphism, we have to study the $\mathbb{Z}[\mathbb{Z}_2]$–module H_φ.

We introduce the notation $n=(n_1,\cdots,n_{2g})$ where $n_i=0$ if $\varphi(u_i)=1$ and $n_i=-1$ if $\varphi(u_i)=t$. For convenience, we also denote $\varepsilon(2s)=n_{2s-1}$ and $\varepsilon(2s-1)=-n_{2s}$.

Proposition 25 *The Fox derivatives associated to $\varphi\in\mathcal{E}(q)$ define a $\mathbb{Z}[\mathbb{Z}_2]$–linear map denoted by*

$$d_\varphi\colon M_2=\Sigma_{1\le i\le 2g}\mathbb{Z}[\mathbb{Z}_2]R_i+\mathbb{Z}[\mathbb{Z}_2]R_0\longrightarrow M_1=\Sigma_{1\le i\le 2g}\mathbb{Z}[\mathbb{Z}_2]u_i+\mathbb{Z}[\mathbb{Z}_2]u_0.$$

Its matrix, with coefficients in $\mathbb{Z}[\mathbb{Z}_2]$, has the following form:

$$\begin{pmatrix} 1-t & 0 & \cdots & 0 & 0 & \varepsilon(1)(1-t) \\ 0 & 1-t & \cdots & 0 & 0 & \varepsilon(2)(1-t) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1-t & 0 & \varepsilon(2g-1)(1-t) \\ 0 & 0 & \cdots & 0 & 1-t & \varepsilon(2g)(1-t) \\ n_1(1-t) & n_2(1-t) & \cdots & n_{2g-1}(1-t) & n_{2g}(1-t) & c(1+t) \end{pmatrix}$$

Proof The coefficients m_{ji} are:

$$\begin{aligned} m_{ii}&=\varphi q(1-u_iu_0u_i^{-1})=\varphi(1-u_0)=1-t, i\neq 0;\\ m_{ji}&=0, i\neq j, i\neq 0, j\neq 0;\\ m_{0i}&=\varphi q(u_i-[u_i,u_0])=\varphi(u_i)-1, i\neq 0;\\ m_{(2j-1),0}&=1-\varphi(u_{2j});\\ m_{2j,0}&=\varphi(u_{2j-1})-1;\\ m_{00}&=c(1+t). \end{aligned}$$

□

The relation $\Sigma_{i=1}^{2g}n_i\varepsilon(i)=0$ implies that the $\mathbb{Z}[\mathbb{Z}_2]$–module Im d_φ is generated by $\{(1-t)v_i, 1\le i\le 2g\}$ and $c(1+t)u_0$ with $v_i=u_i+n_iu_0$.

Notation 26 $\mathbf{V}:=\{v_i, 1\le i\le 2g, v_0=u_0\}$ and $\mathbf{Q}:=\{R_1,\cdots,R_{2g},Q\}, Q=R_0-\Sigma\varepsilon(i)R_i$. Also $\bar{\mathbf{V}}$ is the notation for $\mathbf{V}$ modulo Im d_φ.

The structure of $\mathbb{Z}[\mathbb{Z}_2]$–module of H_φ is:

$$(H_\varphi,\bar{\mathbf{V}})=\bigoplus_{1\le i\le 2g}\frac{\mathbb{Z}[t]}{((1-t^2),(1-t))}\overline{v}_i\oplus\frac{\mathbb{Z}[t]}{((1-t^2),c(1+t))}\overline{u}_0;$$

$$\frac{\mathbb{Z}[t]}{((1-t^2),(1-t))}\simeq\frac{\mathbb{Z}[t]}{(1-t)}\simeq\mathbb{Z};\quad\frac{\mathbb{Z}[t]}{((1-t^2),c(1-t))}\simeq\frac{\mathbb{Z}}{c\mathbb{Z}}\times\mathbb{Z}.$$

Definition 27 The matrix of $d_\varphi \otimes \mathrm{id}_{\mathbb{Z}_2}\colon (M_2 \otimes \mathbb{Z}_2, \mathbf{R}) \longrightarrow (M_1 \otimes \mathbb{Z}_2, \mathbf{U})$ is called the derived matrix associated to $\varphi \in \mathcal{E}(q)$.

This matrix is

$$*(1+t)\begin{pmatrix} 1 & 0 & \cdots & \cdots & n_2 \\ 0 & 1 & \cdots & \cdots & n_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & n_{2g-1} \\ n_1 & n_2 & \cdots & n_{2g} & c \bmod 2 \end{pmatrix}$$

with $n_i = \varphi(u_i) \in \{0,1\}$.

Proposition 28 (1) *The following sequence is exact:*

$$0 \longrightarrow M_2 \otimes \mathbb{Z}_2 \overset{d_\varphi \otimes \mathrm{id}_{\mathbb{Z}_2}}{\longrightarrow} M_1 \otimes \mathbb{Z}_2 \longrightarrow H_\varphi \otimes \mathbb{Z}_2 \longrightarrow 0.$$

(2) *When c is odd, the matrix of $d_\varphi \otimes \mathrm{id}_{\mathbb{Z}_2}\colon (M_2 \otimes \mathbb{Z}_2, \mathbf{Q}) \longrightarrow (M_1 \otimes \mathbb{Z}_2, \mathbf{V})$ is $(1-t)\mathrm{Id}_{2g+1}$, and*

$$(H_\varphi \otimes Z_2, \bar{\mathbf{V}}) = \oplus_{1 \le i \le 2g} \mathbb{Z}_2 \bar{v}_i \oplus \mathbb{Z}_2 \bar{u}_0.$$

(3) *When c is even, the matrix of $d_\varphi \otimes \mathrm{id}_{\mathbb{Z}_2}\colon (M_2 \otimes \mathbb{Z}_2, \mathbf{Q}) \longrightarrow (M_1 \otimes \mathbb{Z}_2, \mathbf{V})$ is*

$$(1+t)\begin{pmatrix} I_{2g} & 0 \\ 0 & 0 \end{pmatrix},$$

then
$$H_\varphi \otimes Z_2 \simeq \oplus_{1 \le i \le 2g} \mathbb{Z}_2 \bar{v}_i \oplus \mathbb{Z}_2[\mathbb{Z}_2] \bar{u}_0.$$ □

3.2 Congruence of derived matrices

Let φ and φ' be elements of $\mathcal{E}(q)$, and consider the following diagram:

$$\begin{array}{ccc} (M_2 \otimes \mathbb{Z}_2, \mathbf{R}) & \xrightarrow{d_\varphi \otimes \mathrm{Id}_{\mathbb{Z}_2}} & (M_1 \otimes \mathbb{Z}_2, \mathbf{U}) \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\psi} \\ (M_2 \otimes \mathbb{Z}_2, \mathbf{R}) & \xrightarrow{d_{\varphi'} \otimes \mathrm{Id}_{\mathbb{Z}_2}} & (M_1 \otimes \mathbb{Z}_2, \mathbf{U}) \end{array} \tag{3–1}$$

where the matrix of ψ in the basis $\mathbf{U}$ is $J(A)$, $A \in Sp(\mathbb{Z}_2, 2g)$ (see Proposition 11 for the definition of J).

The $\mathbb{Z}_2[\mathbb{Z}_2]$–map θ is supposed invertible. Its matrix is denoted by

$$B=\begin{pmatrix} B_1 & B_2 \\ B_3 & b \end{pmatrix}, \text{ with } B_1=(b_{ij}; i,j\le 2g),\ B_2=\begin{pmatrix} c_1 \\ \vdots \\ c_{2g} \end{pmatrix} \text{ and } B_3=(b_1,\cdots,b_{2g}).$$

We write $n_i:=\varphi(u_i), n'_i:=\varphi'(u_i)$, $\varepsilon(2s):=n_{2s-1}, \varepsilon'(2s):=n'_{2s-1}$ and $\varepsilon(2s-1):=n_{2s}, \varepsilon'(2s-1):=n'_{2s}$.

Remark 29 The condition "the matrix of ψ in the basis $\mathbf{U}$ is $J(A), A\in Sp(\mathbb{Z}_2,2g)$" implies that the inverse of ψ in the basis $\mathbf{U}$ is also in $J(Sp(\mathbb{Z}_2,2g))$.

Proposition 30 *Let ψ and θ be as above. The diagram* (3–1) *is commutative if and only if the parameters verify the following conditions* mod $(1+t)$

$$(\alpha)\qquad b_{ij}=a_{ij}+b_j\varepsilon'(i)$$

$$(\beta)\qquad c_i=\Sigma a_{ij}\varepsilon(j)+b\varepsilon'(i)$$

$$(\gamma)\qquad w_j+n_j=\Sigma n'_i a_{ij}+b_j c$$

$$(\delta)\qquad 0=(1+b+\Sigma b_j\varepsilon(j)),$$

with $w_j=\Sigma a_{i,j}r_i+S_j+r_j$, $S_j=\Sigma a_{2i,j}a_{2i-1,j}$.

Proof Mod $(1+t)$, the commutativity of the diagram (3–1) gives the following equations:

$$(\alpha)\qquad a_{ij}=b_{ij}+b_j\varepsilon'(i)$$

$$(\beta)\qquad \Sigma a_{ij}\varepsilon(j)=c_i+b\varepsilon'(i)$$

$$(\gamma')\qquad w_j+n_j=\Sigma n'_i b_{ij}+b_j c$$

$$(\delta')\qquad \Sigma w_i\varepsilon(i)+c=\Sigma n'_j c_j+bc.$$

Using the fact that $\Sigma n_i\varepsilon(i)=0, \Sigma n'_i\varepsilon'(i)=0$, the equation (α) implies that $\Sigma n'_i b_{ij}=\Sigma n'_i a_{ij}$. Hence, the equation (γ') is now (γ) : $w_j+n_j=\Sigma n'_i a_{ij}+b_j c$. The equation (γ) implies that $\Sigma_j w_j\varepsilon(j)=\Sigma_{i,j}n'_i a_{ij}\varepsilon(j)+(\Sigma b_j\varepsilon(j))c$ and (β) implies that $\Sigma_{i,j}n'_i a_{ij}\varepsilon(j)=\Sigma n'_i c_i$. Now (δ') becomes $c(1+b+\Sigma b_j\varepsilon(j))=0$.

If c is odd, the relation (δ) is true.

If c is even, we have to add the relation $(\delta): 0=(1+b+\Sigma b_j\varepsilon(j))$, mod $(1+t)$ which is the condition to get the invertibility of the matrix B. This is obtained from the following computations:

Write $B = B_0 + (1+t)K$ with $B_0 \in GL(\mathbb{Z}_2, 2g)$. An element $(x_1, \cdots, x_{2g}, x_0) \in \ker B_0$ verifies, $\mod (1+t)$

$$\begin{aligned} \forall i, \Sigma_j (a_{ij} + b_j \varepsilon'(i)) x_j + (\Sigma_j a_{ij} \varepsilon(j) + b\varepsilon'(i)) x_0 &= 0 \\ \Sigma b_j x_j + b x_0 &= 0. \end{aligned}$$

The matrix (a_{ij}) is invertible, so for all j, $x_j = \varepsilon(j) x_0$ and $x_0(\Sigma b_j \varepsilon(j) + b) = 0$. This proves that B_0 is bijective if and only if $b = 1 + \Sigma b_j \varepsilon(j) \mod (1+t)$.

Moreover, we have that B bijective if and only if B_0 is bijective. One implication is evident. To prove the converse, let us write $B = B_0 + (1+t)K$ with $B_0 \in GL(\mathbb{Z}_2, 2g)$, then $(B_0^{-1}B)^2 = (\mathrm{Id} + (1+t)B_0^{-1}K)^2 = \mathrm{Id}$, as a matrix with entries in $\mathbb{Z}_2[t]/(1-t^2)$. So $B_0^{-1}BB_0^{-1}$ is the inverse of B. □

Remark 31 Once chosen the basis $\mathbf{U}$, a symplectic matrix $A = (a_{i,j})$ and any pair φ, φ', we have:

(1) If c is even, (γ) becomes $w_j + n_j = \Sigma n'_i a_{ij}$, which involves a relation between φ and φ', which is the necessary and sufficient condition for the existence of θ.

(2) If c is odd, we choose b_j such that (γ) is fulfilled, then b_{ij} and b such that (α) and (δ) are true and then c_i. This means that it is possible to find an isomorphism θ such that the diagram (3–1) commutes, hence we have the following definition:

Definition 32 Let φ and φ' be elements of $\mathcal{E}(q)$, the derived matrices $d_\varphi \otimes \mathrm{Id}_{\mathbb{Z}_2}$ and $d_{\varphi'} \otimes \mathrm{Id}_{\mathbb{Z}_2}$ are said congruent via (ψ, θ) if there exist $\mathbb{Z}_2[\mathbb{Z}_2]$–isomorphisms ψ and θ such that the following diagram commutes:

$$\begin{tikzcd} (M_2 \otimes \mathbb{Z}_2, \mathbf{R}) \arrow[r, "d_\varphi \otimes \mathrm{Id}_{\mathbb{Z}_2}"] \arrow[d, "\theta"'] & (M_1 \otimes \mathbb{Z}_2, \mathbf{U}) \arrow[d, "\psi"'] \\ (M_2 \otimes \mathbb{Z}_2, \mathbf{R}) \arrow[r, "d_{\varphi'} \otimes \mathrm{Id}_{\mathbb{Z}_2}"] & (M_1 \otimes \mathbb{Z}_2, \mathbf{U}) \end{tikzcd}$$

with the constraints that the matrix of ψ in the basis $\mathbf{U}$ is an element of $J(Sp(\mathbb{Z}_2, 2g))$ and the matrix of θ in the basis $\mathbf{R}$ is of the following type:

$$\begin{pmatrix} B_1 & B_2 \\ 0 & 1 \end{pmatrix}$$

With this definition, independently of the parity of c, the only condition remaining to get the congruence of the derived matrices is the condition (γ) of Proposition 30. So we get the main theorem:

Theorem 33 *Two special 2–fold coverings φ and φ' are s–related (see Equation (2–2)) if and only if the derived matrices associated to φ and φ' are congruent.*

3.2.1 The $*$–product We need to lift the intersection product from $H_1(F_g;\mathbb{Z}_2)$ to $(H_\varphi \otimes Z_2, \bar{\mathbf{V}})$.

Replacing t by 1 gives the description of the projection $H_1(E_\varphi,(E_\varphi)_0;\mathbb{Z}_2) \longrightarrow H_1(P;\mathbb{Z}_2)$. Considering a new basis $\tau := \{\tau_i = \nu_i + \varphi(u_i)\nu_0, 1 \le i \le 2g; \tau_0 = \nu_0\}$ of $H_1(P;\mathbb{Z}_2)$, we define successively

$$\begin{aligned}\pi_\varphi\colon (H_\varphi \otimes \mathbb{Z}_2, \bar{\mathbf{V}}) &\longrightarrow (H_1(P;\mathbb{Z}_2), \tau);\\ \Sigma d_i \bar{v}_i + y(t)\bar{u}_0 &\mapsto \Sigma d_i \tau_i + y(1)\tau_0,\end{aligned}$$

where $y(t)$ is in fact a constant in $\mathbb{Z}_2$ if c is odd and an element of $\mathbb{Z}_2[\mathbb{Z}_2]$ if c is even, and $p_\varphi = p_\star \circ \pi_\varphi$ the composition of the projections

$$H_\varphi \otimes \mathbb{Z}_2 \xrightarrow{\pi_\varphi} H_1(P;\mathbb{Z}_2) \xrightarrow{p_\star} H_1(F_g;\mathbb{Z}_2).$$

Definition 34 A product, denoted by $*$, is defined in $H_\varphi \otimes \mathbb{Z}_2$ by lifting the intersection product in $H_1(F_g;\mathbb{Z}_2)$:

$$x, y \in H_\varphi \otimes \mathbb{Z}_2 \mapsto x * y = p_\varphi(x).p_\varphi(y) \in \mathbb{Z}_2,$$

where $a.b$ is the intersection product of two elements of $H_1(F_g;\mathbb{Z}_2)$.

3.2.2 Preserving the $*$–product Suppose that φ and φ' are two special 2–fold coverings and $\Psi\colon (H_\varphi \otimes \mathbb{Z}_2, \bar{\mathbf{V}}) \longrightarrow (H_{\varphi'} \otimes \mathbb{Z}_2, \bar{\mathbf{V}}')$ is a $\mathbb{Z}_2[\mathbb{Z}_2]$–isomorphism. Let us denote by $\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$ the matrix of Ψ. Here A is a $(2g \times 2g)$–matrix, B is a column with coefficients in $\mathbb{Z}_2[t]/(1-t) \simeq \mathbb{Z}_2$, C is a line and D is an element in $\mathbb{Z}_2[t]/((1-t), c(1+t))$. This ring $\mathbb{Z}_2[t]/((1-t), c(1+t))$ is isomorphic to $\mathbb{Z}_2$ if c is odd, and to $\mathbb{Z}_2[t]/(1-t^2)$ if c is even.

A generator of $\ker p_\varphi$ is $\tau_0 = \nu_0$ and for all V, $\nu_0 * V = 0$ and $\bar{v}_i * \bar{v}_j = p_\star(\nu_i).p_\star(\nu_j) = \sigma_i.\sigma_j$; hence we have the following proposition:

Proposition 35 *A $\mathbb{Z}_2[\mathbb{Z}_2]$–isomorphism $\Psi\colon (H_\varphi \otimes \mathbb{Z}_2, \bar{\mathbf{V}}) \longrightarrow (H_{\varphi'} \otimes \mathbb{Z}_2, \bar{\mathbf{V}}')$ respects the product, ie, $\Psi(x) * \Psi(y) = x * y \in \mathbb{Z}_2$, if and only if there exists a symplectic isomorphism $f\colon H_1(F_g;\mathbb{Z}_2) \longrightarrow H_1(F_g;\mathbb{Z}_2)$ such that*

$$f \circ p_\varphi = p_{\varphi'} \circ \Psi.$$

Proof $\Psi(x) * \Psi(y) = x * y$ if and only if $A \in Sp(\mathbb{Z}_2, 2g)$ and $B = 0$. These conditions are equivalent to $\Psi(\ker p_\varphi) = \ker p_{\varphi'}$ and the existence of such a symplectic map $f\colon H_1(F_g;\mathbb{Z}_2) \longrightarrow H_1(F_g;\mathbb{Z}_2)$ such that

$$f \circ p_\varphi = p_{\varphi'} \circ \Psi. \qquad \square$$

Let $f\colon (H_1(F_g;\mathbb{Z}_2),\sigma) \longrightarrow (H_1(F_g;\mathbb{Z}_2),\sigma)$ be a symplectic isomorphism with $A = (a_{ij})$ as symplectic matrix in the basis σ.

Let us denote by:

- Ψ_f the isomorphism from $(H_\varphi \otimes \mathbb{Z}_2, \bar{\mathbf{V}})$ to $(H_{\varphi'} \otimes \mathbb{Z}_2, \bar{\mathbf{V}}')$, with matrix $\left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We have $f \circ p_\varphi = p_{\varphi'} \circ \Psi_f$;
- ψ_f the automorphism of $(M_1 \otimes \mathbb{Z}_2, U)$ with matrix $J(A)$ (see Definition 10 and Proposition 11). Its matrix in the basis (V, V') is $\left(\begin{smallmatrix} A & 0 \\ M & 1 \end{smallmatrix}\right)$, with $M = (w_j + n_j + \Sigma a_{i,j} n'_i)$.

If c is even, then

$$(**) \qquad \psi_f(\mathrm{Im}\ d_\varphi) = \mathrm{Im}\ d_{\varphi'}$$

if and only if $M = 0$. If so, the quotient isomorphism is equal to Ψ_f and there exists an isomorphism θ as in Definition 32.

If c is odd, then the relation $(**)$ is always true for any M and the quotient isomorphism, in the basis (V, V'), is also $\left(\begin{smallmatrix} A & 0 \\ M & 1 \end{smallmatrix}\right)$. Nevertheless $M = 0$ is the condition to be added for getting an isomorphism θ as in Definition 32.

It is possible to synthesize this study into a definition:

Definition 36 Let Ψ be an isomorphism of $H_\varphi \otimes \mathbb{Z}_2$ to $H_{\varphi'} \otimes \mathbb{Z}_2$ respecting the product, and f the symplectic isomorphism of $H_1(F_g, \mathbb{Z}_2)$ associated by Proposition 35. We will say that Ψ is a quotient if the following conditions are fulfilled: Ψ is equal to Ψ_f and is a quotient isomorphism of ψ_f. (When c is even, these two conditions are equivalent).

Theorem 37 *There exists a $\mathbb{Z}_2[\mathbb{Z}_2]$–isomorphism*

$$\Psi\colon H_1(E_\varphi, (E_\varphi)_0; \mathbb{Z}_2) \longrightarrow H_1(E_{\varphi'}, (E_{\varphi'})_0; \mathbb{Z}_2)$$

which is a quotient if and only if φ and φ' are s–related.

3.2.3 Effect of a change of generators of $\pi_1 P$ on the derived matrices associated to some $\varphi \in \mathcal{E}(q)$ The derived matrix associated to $\varphi \in \mathcal{E}(q)$ is the matrix of the linear map $d_\varphi \otimes \mathrm{id}_{\mathbb{Z}_2}\colon (M_2 \otimes \mathbb{Z}_2, \mathbf{R}) \longrightarrow (M_1 \otimes \mathbb{Z}_2, \mathbf{U})$ defined by

$$d(R_j) = \Sigma\varphi\Big(\frac{\partial R_j}{\partial u_i}\Big) u_i.$$

Comparing with Section 3.1.2, we forget the map $q\colon \mathbb{Z}_2[F] \longrightarrow \mathbb{Z}_2[\pi_1 P]$, where F is the free group with $2g+1$ generators. Suppose that $(u'_i)_{0\le i\le n}$ is another choice of generators of $\pi_1 P$ such that for each i, $u_i = w_i(u'_0, \cdots, u'_n)$ is a word. We are in the situation where, if $R_j = W_j(u_1, \cdots, u_n, u_0)$, the new relations are $R'_j = W_j(w_1, \cdots, w_n, w_0)$ and $H_1(E_\varphi, (E_\varphi)_0; \mathbb{Z}_2) = \oplus \mathbb{Z}_2[\mathbb{Z}_2]u'_i / \mathrm{Im}\, d'_\varphi \otimes \mathrm{id}_{\mathbb{Z}_2}$ with

$$d'_\varphi(R'_j) = \Sigma \varphi\Big(\frac{\partial R'_j}{\partial u'_i}\Big) u'_i.$$

By induction on the length of the word W_j, it is possible to prove that

$$\frac{\partial R'_j}{\partial u'_i} = \Sigma_k \frac{\partial R_j}{\partial u_k} \frac{\partial u_k}{\partial u'_i}.$$

Let us denote by C the matrix with entries $(\partial u_j / \partial u'_i)$, M and M' the matrices of $d_\varphi \otimes \mathrm{id}_{\mathbb{Z}_2}\colon (M_2 \otimes \mathbb{Z}_2, \mathbf{R}) \longrightarrow (M_1 \otimes \mathbb{Z}_2, \mathbf{U})$ and $d'_\varphi \otimes \mathrm{id}_{\mathbb{Z}_2}\colon (M_2 \otimes \mathbb{Z}_2, \mathbf{R}') \longrightarrow (M_1 \otimes \mathbb{Z}_2, \mathbf{U}')$. We have the relation:

$$M' = \varphi(C) M.$$

Let us also remark that the matrix C' with entries $\partial u'_j / \partial u_i$ verifies $\varphi(C)\varphi(C') = \mathrm{Id}$ so $M = \varphi(C')M'$. Here two systems of generators must be the lifts of a fixed choice of generators of $\pi_1 F_g$, hence they are related by $u'_i = u_0^{-\alpha_i} u_i$ (or equivalently $u_i = u_0^{\alpha_i} u'_i$), $\alpha_i \in \{0, 1\}$. The matrix M' of $d'_\varphi \otimes \mathrm{id}_{\mathbb{Z}_2}\colon (M_2 \otimes \mathbb{Z}_2, \mathbf{R}') \longrightarrow (M_1 \otimes \mathbb{Z}_2, \mathbf{U}')$ may be considered as the matrix of $d_{\varphi'} \otimes \mathrm{id}_{\mathbb{Z}_2}\colon (M_2 \otimes \mathbb{Z}_2, \mathbf{R}) \longrightarrow (M_1 \otimes \mathbb{Z}_2, \mathbf{U})$ with $\varphi'(u_i) = \varphi(u_i) + \alpha_i, \varphi'(u_0) = \varphi(u_0)$. The effect is like changing the quadratic section s (see Remark 9).

The conclusion is that the only invariants, (independent of the choice of the generators of $\pi_1 P$, lifting some fixed canonical system of generators of $\pi_1 F_g$), are the number of classes under the s–relation and the number of elements in each class.

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