On multiplicity of mappings between surfaces

SEMEON BOGATYI
JAN FRICKE
ELENA KUDRYAVTSEVA

Let M and N be two closed (not necessarily orientable) surfaces, and $f\colon M \to N$ a continuous map. By definition, the *minimal multiplicity* $\mathrm{MMR}[f]$ of the map f denotes the minimal integer k having the following property: f can be deformed into a map g such that the number $|g^{-1}(c)|$ of preimages of any point $c \in N$ under g is $\le k$. We calculate $\mathrm{MMR}[f]$ for any map f of positive absolute degree $A(f)$. The answer is formulated in terms of $A(f)$, $[\pi_1(N) : f_\#(\pi_1(M))]$, and the Euler characteristics of M and N. For a map f with $A(f)=0$, we prove the inequalities $2 \le \mathrm{MMR}[f] \le 4$.

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In grateful memory of Heiner, his wonderful collaboration and friendship

1 Introduction

For a continuous map $f\colon X \to Y$ between topological spaces, we define the *multiplicity* of f as $\max_{y\in Y} |f^{-1}(y)|$, and the *minimal multiplicity* of f as the minimal multiplicity of maps homotopic to f, that is

$$\mathrm{MMR}[f] := \min_{g\simeq f} \max_{y\in Y} |g^{-1}(y)|.$$

From now on, $\simeq$ means that the mappings are homotopic. The problem of determining $\mathrm{MMR}[f]$ arises. This problem is closely related to the *self-intersection problem* of determining the *minimal self-intersection number* (see Bogatyi, Kudryavtseva and Zieschang [4; 3])

$$\mathrm{MI}[f] := \min_{g\simeq f} |\mathrm{Int}(g)|, \qquad \mathrm{Int}(g) := \{(x,y)\in X\times X \mid x\ne y,\ g(x)=g(y)\}/\Sigma_2$$

(here Σ_2 is the symmetric group in two symbols, which acts on $X\times X$ by permutations of the coordinates), and to the problem of determining the *minimal (unordered) μ–tuple*

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self-intersection number

$$\mathrm{MI}_\mu[f] := \min_{g\simeq f} |\mathrm{Int}_\mu(g)|, \quad \mathrm{Int}_\mu(g) := \{I \subset X \mid |I| = \mu,\ |g(I)| = 1\}, \quad \mu \geq 2.$$

Clearly, $\mathrm{MI}[f] = \mathrm{MI}_2[f]$, and one easily shows[1] that $\mathrm{MI}_{\mu+1}[f] \leq (\mathrm{MI}_\mu[f])^2$, $\mu \geq 2$. The connection between $\mathrm{MMR}[f]$ and $\mathrm{MI}_\mu[f]$ is illustrated by the following properties:

$$\mathrm{MI}_\mu[f] = 0 \iff \mathrm{MMR}[f] < \mu \qquad \text{and} \qquad \mathrm{MI}_\mu[f] > 0 \iff \mathrm{MMR}[f] \geq \mu.$$

In particular, $\mathrm{MI}[f] = 0$ if and only if $\mathrm{MMR}[f] = 1$. The numbers $\mathrm{MMR}[f]$, $\mathrm{MI}[f]$, and $\mathrm{MI}_\mu[f]$, measure, in a sense, "complexity" of the self-intersection set $\mathrm{Int}(f)$.

It is natural to consider the above problem for maps $f\colon M^m \to N^n$ between closed connected (nonempty) smooth manifolds, where $m = \dim M$, $n = \dim N$. The problem is nontrivial for $0 < m \leq n \leq 2m$.

Hurewicz [14] proved that, if X is an m–dimensional compact metric space and $m+1 \leq n \leq 2m$, then any continuous map $f\colon X \to \mathbb{R}^n$ can be deformed, by means of an arbitrary small perturbation, to a map $g\colon X \to \mathbb{R}^n$ of multiplicity $\leq [\frac{n}{n-m}]$. A similar assertion is also valid if the Euclidean space $\mathbb{R}^n$ is replaced by an arbitrary smooth manifold N^n. Thus, for $m < n \leq 2m$, we have

$$\textbf{(1.1)} \qquad \mathrm{MMR}[f] \leq \left[\frac{n}{n-m}\right].$$

This inequality follows by observing that, for a "generic" map $g\colon M \to N$, the set $\mathrm{Int}_{\mu+1}(g) \subset M$ has dimension $(\mu+1)m - \mu n$, which is negative (and, thus, $\mathrm{MMR}[f] \leq \mu$) if $\mu > \frac{m}{n-m}$.

The special case $n = 2m$ is the classical self-intersection problem which gives rise to Whitney's work [21]. Here the estimation (**1.1**) gives $\mathrm{MMR}[f] \in \{1, 2\}$, and computing $\mathrm{MMR}[f]$ is equivalent to deciding whether $\mathrm{MI}[f] = 0$, ie whether the map f is homotopic to an embedding. Namely, we have $\mathrm{MMR}[f] = 1$ if $\mathrm{MI}[f] = 0$, and $\mathrm{MMR}[f] = 2$ if $\mathrm{MI}[f] > 0$. A useful tool for deciding whether $\mathrm{MI}[f] = 0$ is the *Nielsen self-intersection number* $\mathrm{NI}[f]$ of f [4; 3]. One can show by using the Whitney trick [21] that $\mathrm{MI}[f] = \mathrm{NI}[f]$ if $m \geq 3$. But, if $m \leq 2$, one has only the inequality $\mathrm{MI}[f] \geq \mathrm{NI}[f]$ (see our papers with Zieschang [4; 3] for $m = 1$). For $m = 1$, there are several combinatorial and geometric methods for deciding whether a closed curve

[1](Indeed, take a map $g \simeq f$ such that $\mathrm{MI}_\mu[f] = |\mathrm{Int}_\mu(g)| =: \ell$. We can assume that $\ell < \infty$. Then $\ell = \sum_{i\geq\mu} \sum_{y\in Y,\ |g^{-1}(y)|=i} \binom{i}{\mu}$. Hence, for every nonvanishing summand in this sum, one has $\binom{i}{\mu} \leq \ell$ and

$$\binom{i}{\mu+1} = \binom{i}{\mu}\frac{i-\mu}{\mu+1} < \binom{i}{\mu}\frac{i}{\mu} \leq \binom{i}{\mu}^2 \leq \ell\binom{i}{\mu}.$$

Therefore $\mathrm{MI}_{\mu+1}[f] \leq |\mathrm{Int}_{\mu+1}(g)| = \sum_{i>\mu} \sum_{y\in Y,\ |g^{-1}(y)|=i} \binom{i}{\mu+1}$, which is at most $\ell \sum_{i>\mu} \sum_{y\in Y,\ |g^{-1}(y)|=i} \binom{i}{\mu} \leq \ell^2$.)

on a surface is homotopic to a simple closed curve (see, for example, Gonçalves, Kudryavtseva and Zieschang [9] and references therein). An answer in terms of the Nielsen self-intersection number is given in Theorem 2.2. In the remaining case $m = 2$, we only know that $\mathrm{NI}[f] > 0$ implies $\mathrm{MI}[f] > 0$ (and thus $\mathrm{MMR}[f] = 2$), but the question whether $\mathrm{NI}[f] = 0$ implies $\mathrm{MI}[f] = 0$ is still open.

The present paper studies the number $\mathrm{MMR}[f]$ mainly in the case $m = n \leq 2$. Here $\mathrm{MMR}[f]$ is closely related to the *absolute degree* $A(f)$ (as defined in Hopf [13] or Epstein [7]; see also Kneser [16], Olum [18] and Skora [19]) of the map f. A definition of the absolute degree is also given in Definition 3.7 in the paper by Gonçalves, Kudryavtseva and Zieschang [10] of this volume. Theorem 2.1 computes the number $\mathrm{MMR}[f]$ for a self-mapping f of a circle ($m = n = 1$). In the case $m = n = 2$ (mappings between closed surfaces), the following results are obtained. We calculate $\mathrm{MMR}[f]$ in terms of $A(f)$, $\ell(f) := [\pi_1(N) : f_\# \pi_1(M)]$, and the Euler characteristics of the surfaces, for any map $f\colon M \to N$ with $A(f) > 0$ (Theorem 3.2 and Theorem 3.3). We also estimate $\mathrm{MMR}[f]$ for any map f with $A(f) = 0$ (Theorem 4.2). In particular, we prove that

$$\begin{aligned} &\mathrm{MMR}[f] \in \{A(f), A(f)+2\} && \text{if} \quad A(f) > 0, \\ &\mathrm{MMR}[f] \in \{2, 3, 4\} && \text{if} \quad A(f) = 0. \end{aligned}$$

The authors do not know whether $\mathrm{MMR}[f] \geq A(f)$ if $m = n \geq 3$.

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2 Computing MMR[f] for mappings of a circle

Any map $f\colon S^1 \to N$ with $\dim N \geq 3$ is homotopic to an embedding, thus $\mathrm{MMR}[f] = 1$. Consider the cases $\dim N = 1, 2$.

Theorem 2.1 *For any self-map $f\colon S^1 \to S^1$,*

$$\mathrm{MMR}[f] = \begin{cases} |\deg f|, & \deg f \neq 0, \\ 2, & \deg f = 0. \end{cases}$$

Proof We will identify the circle S^1 with the unit circle in the complex plane $\mathbb{C}$. Consider the projection $p\colon \mathbb{R} \to S^1$, $p(r) = e^{2\pi i r}$, $r \in \mathbb{R}$, of the universal covering $\mathbb{R}$ of S^1 to S^1.

Suppose that $\deg f \neq 0$. Then f is homotopic to the map sending $z \mapsto z^{\deg f}$, $z \in S^1$. Thus all points have exactly $|\deg f|$ preimages, hence $\mathrm{MMR}[f] \le |\deg f|$. Let us show that the number of preimages can not be reduced. Since $\deg f \neq 0$, for every point $s \in S^1$ there exists a point $t \in S^1$ such that $f(t) = s$. Let $r_0 \in \mathbb{R}$ be a point such that $p(r_0) = t$. Consider a lifting $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$ of $f\colon S^1 \to S^1$. Then $\tilde{f}(r_0 + 1) = \tilde{f}(r_0) + \deg f$, so by the Intermediate Value Theorem, there exist points $r_1, \dots, r_{|\deg f|-1} \in (r_0, r_0+1)$ such that $\tilde{f}(r_i) = \tilde{f}(r_0) + j\,\mathrm{sgn}(\deg f)$, $1 \le j \le |\deg f| - 1$. Thus $p(r_0), p(r_1), \dots, p(r_{|\deg f|-1})$ are different preimages of s under the mapping f. This shows $\mathrm{MMR}[f] \ge |\deg f|$.

Suppose that $\deg f = 0$. Let us show that there exists $g \simeq f$ with $|g^{-1}(s)| \le 2$ for any $s \in S^1$. Indeed, take g to be the map given by the following rule: $g(z) = z$ if $\operatorname{Im} z \ge 0$, $g(z) = \bar{z}$ if $\operatorname{Im} z \le 0$. It remains to show that for any $f\colon S^1 \to S^1$, $\deg f = 0$, there exists a point $s \in S^1$ with $|f^{-1}(s)| \ge 2$. Such a map f lifts to a map $\bar{f}\colon S^1 \to \mathbb{R}$, thus it is enough to show that $\bar{f}$ is not an embedding. This can be easily deduced by taking two points $s_0, s_1 \in S^1$ with $\bar{f}(s_0) = \min_{s\in S^1} \bar{f}(s)$, $\bar{f}(s_1) = \max_{s \in S^1} \bar{f}(s)$, and applying the Intermediate Value Theorem to the restriction of $\bar{f}$ to two segments in S^1 having endpoints at s_0, s_1. □

Consider a closed curve $f\colon S^1 \to N^2$ on a closed surface N^2. Then computing $\mathrm{MMR}[f]$ is equivalent to deciding whether the homotopy class $[f]$ of the curve f contains a simple closed curve. Namely, $\mathrm{MMR}[f] = 1$ if $[f]$ contains a simple curve, and $\mathrm{MMR}[f] = 2$ otherwise.

Theorem 2.2 [4; 3] *A closed curve $f\colon S^1 \to N^2$ on a closed surface N^2 is homotopic to a simple closed curve if and only if $\mathrm{NI}[f] = 0$ and one of the following conditions is fulfilled: the curve f is not homotopic to a proper power of any closed curve on N, or $f \simeq g^2$ for some orientation-reversing closed curve $g\colon S^1 \to N$.* □

An analogue of Theorem 2.2 was proved by Turaev and Viro [20, Corollary II], in terms of the intersection index introduced therein.

3 MMR(f) for maps of positive degree between surfaces

In the following, $M = M^2$ and $N = N^2$ are arbitrary connected closed surfaces, ie 2–dimensional manifolds. By $\chi(M)$, we denote the Euler characteristic of M. For a continuous mapping $f\colon M \to N$, $A(f)$ denotes its *absolute degree* (see Hopf [13], Epstein [7], Kneser [16], Olum [18], Skora [19] or Gonçalves, Kudryavtseva and Zieschang [8]). Denote the index of the image of the fundamental group of M in the fundamental group of N by $\ell(f) := [\pi_1(N, f(x_0)) : f_\#(\pi_1(M, x_0))]$ for some $x_0 \in M$. Actually the number $\ell(f)$ does not depend on the choice of the point x_0.

The following consequence of Kneser's inequality will be central in the proof of our main result.

Proposition 3.1 *If $f\colon M \to N$ has absolute degree $d = A(f) > 0$ then there are at most $d \cdot \chi(N) - \chi(M)$ points in N whose preimages have cardinality $\le d-1$. Moreover, if pairwise different points $y_1, \dots, y_r$ of N have $\mu_1, \dots, \mu_r$ preimages, respectively, then*

$$d \cdot \chi(N) \ge \chi(M) + \sum_{i=1}^{r} (d - \mu_i).$$

Proof In the case when $r = 1$ and f is orientation-true, the latter inequality was proved in Theorem 2.5 (a) of [8]. In the general case, the inequality can be proved using the techniques in [1; 8; 11; 2], as follows.

If f is not orientation-true and $d = A(f) > 0$ then $d = \ell(f)$, due to the result of Kneser [15; 16]. On the other hand, one has $\mu_i \ge \ell(f)$, $1 \le i \le r$, since the map f admits a lifting $\hat{f}\colon M \to \hat{N}$ such that $f = p \circ \hat{f}$, where $p\colon \hat{N} \to N$ is an $\ell(f)$–fold covering corresponding to the subgroup $f_\#(\pi_1(M, x_0))$ of $\pi_1(N, f(x_0))$, and $A(\hat{f}) = 1$, hence $\hat{f}$ is surjective. Therefore $\sum_{i=1}^{r} (d - \mu_i) \le 0$. This, together with the Kneser inequality [16], $d \cdot \chi(N) \ge \chi(M)$, implies the desired inequality.

If f is orientation-true, one proceeds as in the proof of Proposition 2.5 (a) of [8], where one replaces the single point $y_0 \in N$ by the set of r points $y_1, \dots, y_r$. More specifically, by applying a suitable deformation, one can assume that there are small pairwise disjoint disks D_i, D_{ij}, $1 \le i \le r$, $1 \le j \le \mu_i$, around the points y_i of N and the points of $f^{-1}(y_i)$ such that $f^{-1}(\mathring{D}_i) = \bigcup_{j=1}^{\mu_i} \mathring{D}_{ij}$, and $f|_{D_{ij}}$ is a branched covering of type $z \mapsto z^{d_{ij}}$ for some positive integer d_{ij}. Therefore the complement of these open disks are two compact surfaces $F \subset M$, $G \subset N$ such that the restriction of f induces a proper map carrying the boundary into the boundary, $f|_F\colon (F, \partial F) \to (G, \partial G)$. By Proposition 1.6 of [8] (or by a more general Theorem 4.1 of [19]), $\chi(F) \le A(f) \cdot \chi(G)$. This, together with $\chi(F) = \chi(M) - \sum_{i=1}^{r} \mu_i$, $\chi(G) = \chi(N) - r$, gives the desired inequality. $\square$

Theorem 3.2 *Suppose that $f\colon M \to N$ has absolute degree $d = A(f) > 0$. If $\ell(f) \ne d$, or $\ell(f) = d$ and $d \cdot \chi(N) = \chi(M)$, then $\mathrm{MMR}[f] = d$.*

Proof The inequality $\mathrm{MMR}[f] \geq A(f)$ follows from the first part of Proposition 3.1.

Let us show the converse inequality, $\mathrm{MMR}[f] \leq A(f)$. It follows from [6; 19; 16], respectively, that the mapping f is homotopic to a d–fold covering which is branched in the first case and unbranched in the second case. Thus, we found a mapping which is homotopic to f, and the preimage of any point of N has cardinality $\leq d$. □

Theorem 3.3 *Suppose that* $f\colon M \to N$ *has absolute degree* $d = A(f) > 0$. *If* $\ell(f) = d$ *and* $d \cdot \chi(N) \neq \chi(M)$, *then* $\mathrm{MMR}[f] = d + 2$.

Proof **Case 1** Suppose that $d = A(f) = 1$. It follows from [6; 19] that the mapping f is homotopic to a pinching map where the pinched subsurface $M' \subset M$, $\partial M' \simeq S^1$, is different from the 2–disk D^2 (here the natural projection $M \to M/M'$ is called a pinching map).

Let us show that such a pinching map is homotopic to a map g of multiplicity ≤ 3. For this, we construct a proper continuous map $g'\colon (M', \partial M') \to (D^2, \partial D^2)$ whose restriction to $\partial M'$ is a homeomorphism, and whose multiplicity equals 3. Such a map g' is shown in Figure 1. We may identify N with the surface which is obtained by gluing of $M \setminus \mathring{M}'$ and D^2 by means of the aforementioned homeomorphism of the boundary circles, where $\mathring{M}'$ denotes the interior of M'. Define $g\colon M \to N$ as $g|_{M\setminus M'} = \mathrm{id}_{M\setminus M'}$ and $g|_{M'} = g'$. Clearly, $f \simeq g$, since g' is homotopic relative boundary to a pinching map. In Case 2 below, we will use the following property of the constructed map g: its restriction to the preimage of the complement $N \setminus D^2$ of a disk is injective.

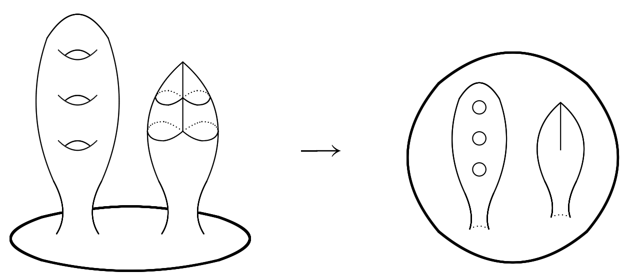


Figure 1: A proper map $g'\colon M' \to D^2$ of multiplicity 3

It follows from the inequality of Euler characteristics of M and N that f is not homotopic to an embedding. (Indeed, otherwise such an embedding g is a homeomorphism onto $g(M)$; it follows from Brouwer's Theorem on Invariance of Domain [5] that g

is surjective and, therefore, it is a homeomorphism.) Suppose that f is homotopic to a map $g\colon M \to N$ of multiplicity 2, we will show that this leads to a contradiction. Let $y \in N$ be a point with $g^{-1}(y) = \{x_1, x_2\}$. Then the local degree of g at each of the points x_1 and x_2 is defined modulo 2, and

$$\deg(g, x_1) + \deg(g, x_2) \equiv A(g) \equiv A(f) \equiv 1 \mod 2.$$

Without loss of generality, we may assume that $\deg(g, x_1) \neq 0$. This implies that the image of any neighbourhood of x_1 contains a neighbourhood of $y = g(x_1)$, since otherwise one could construct a map $F\colon D^2 \to S^1$ with $\deg(F|_{\partial D^2}) = \deg(g, x_1) \neq 0$. Therefore the restriction of g to an appropriate neighbourhood of x_2 is injective and, thus (by Brouwer's Theorem on Invariance of Domain [5]), is a homeomorphism onto a neighbourhood of y. This implies that $\deg(g, x_2) = \pm 1$. Similar arguments show that $\deg(g, x_1) = \pm 1$, a contradiction.

Case 2 Suppose that $d = A(f) = \ell(f) \geq 2$. Let us construct a map g which is homotopic to f and has multiplicity $A(f)+2$. Consider a covering $p\colon \widetilde{N} \to N$ which corresponds to the subgroup $f_\#(\pi_1(M, x_0))$ of $\pi_1(N, f(x_0))$. So, this is an $\ell(f)$–fold covering. Let $y \in N$ be an arbitrary point and D a small closed neighbourhood which is homeomorphic to the disk D^2. Let $D_1, \dots, D_d$ be the connected components of $p^{-1}(D)$.

Let $\widetilde{f}\colon M \to \widetilde{N}$ be a lifting of f. Then $A(\widetilde{f}) = \ell(\widetilde{f}) = 1$. By Case 1, there exists a map $\widetilde{g}\colon M \to \widetilde{N}$ which is homotopic to $\widetilde{f}$ and has multiplicity ≤ 3. Then the map $g := p \circ \widetilde{g}$ is homotopic to $f = p \circ \widetilde{f}$. By Case 1, we may also assume that $\widetilde{g}$ is injective on $\widetilde{g}^{-1}(\widetilde{N} \setminus D_1)$. Therefore the map g has multiplicity $\ell(f)+2 = A(f)+2$.

Let us show that the multiplicity of f is $\geq \ell(f)+2$. Let $\widetilde{f}\colon M \to \widetilde{N}$ be a lifting of f to this $\ell(f)$–fold covering, thus $A(\widetilde{f}) = \ell(\widetilde{f}) = 1$. By Case 1, there exists a point $\widetilde{y} \in \widetilde{N}$ whose preimage under $\widetilde{f}$ has cardinality ≥ 3. Since $A(\widetilde{f}) > 0$, every point of $p^{-1}(p(\widetilde{y}))$ has a nonempty preimage under $\widetilde{f}$. Therefore $f^{-1}(p(\widetilde{y}))$ has cardinality at least $\ell(f) + 2 = A(f) + 2$. □

4 Estimates for MMR(f) if $A(f) = 0$

Suppose that M is a connected orientable closed surface of genus $g \geq 0$. Consider the standard presentation of the closed surface M as the boundary of a solid surface in $\mathbb{R}^3$ which is obtained from a closed 3-ball by attaching g solid handles; see Figure 2 (a). Choose a base point $x_0 \in M$ and consider a system of simple closed curves $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ on M based at x_0 which form a *canonical system of cuts*; see

Figure 2 (a). Then the fundamental group $\pi_1(M, x_0)$ admits a canonical presentation

$$\pi_1(M, x_0) = \left\langle a_1, b_1, \dots, a_g, b_g \;\middle|\; \prod_{j=1}^{g} [a_j, b_j] \right\rangle,$$

where a_j, b_j are the homotopy classes of the based loops α_j, β_j, respectively. Denote by V_g the bouquet of g circles $\alpha_1 \cup \ldots \cup \alpha_g$ if $g \geq 1$, $V_0 := \{x_0\}$ if $g = 0$, and by ϱ a retraction $\varrho\colon M \to V_g$ which maps all loops β_j to the point x_0. We can assume that the curves $\alpha_1, \dots, \alpha_g$ are contained in the plane $\Pi \subset \mathbb{R}^3$ which is tangent to M at x_0. (In Figure 2, the plane Π is parallel to the plane of the picture.)

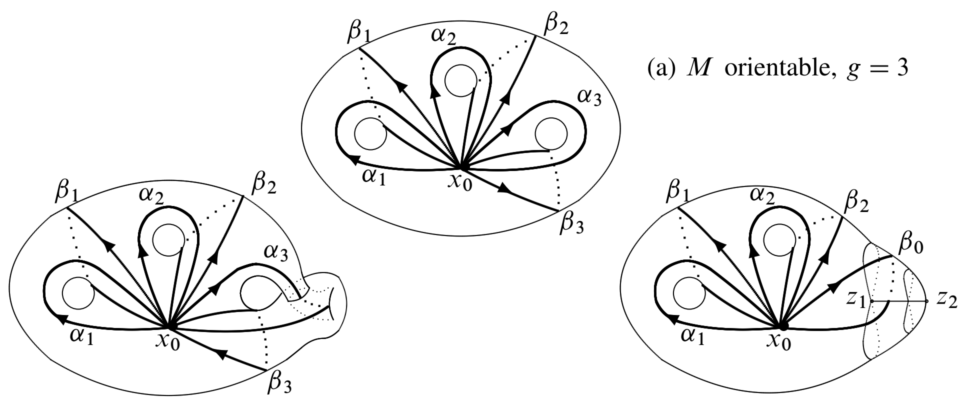
Let $i\colon M \to \mathbb{R}^3$ denote the inclusion, and $p_\Pi\colon \mathbb{R}^3 \to \Pi$ the orthogonal projection. The following properties of the map $p = p_\Pi \circ i\colon M \to \Pi$ can be assumed without loss of generality, and will be used later:

(p1) The restriction of p to a neighbourhood U of the base point $x_0 \in M$ is a homeomorphism onto a neighbourhood of the point $p(x_0)$ in Π. Moreover, $p|_{V_r}\colon V_r \to \Pi$ is an embedding, and all curves $p|_{\alpha_j}\colon \alpha_j \to \Pi$ are regular;

(p2) All curves $p|_{\beta_j}$ are contractible in $p(M)$;

(p3) $p(M)$ is a regular neighbourhood of the graph $p(V_r)$ in Π;

(p4) The map p has multiplicity 2.



(a) M orientable, $g = 3$

(b) M nonorientable, $g = 6$

(c) M nonorientable, $g = 5$

Figure 2: A canonical system of cuts on a closed surface M

Suppose that M is a connected nonorientable closed surface of genus $g \geq 1$. Choose a base point $x_0 \in M$. Then the fundamental group of M admits the following canonical presentation:

$$\pi_1(M, x_0) = \left\langle a_1, b_1, \dots, a_{g/2}, b_{g/2} \;\middle|\; \left(\prod_{j=1}^{g/2-1} [a_j, b_j] \right) \cdot [a_{g/2}, b_{g/2}]_- \right\rangle \quad \text{if } g \text{ is even,}$$

$$\pi_1(M, x_0) = \left\langle a_1, b_1, \dots, a_{[g/2]}, b_{[g/2]}, b_0 \;\middle|\; \left(\prod_{j=1}^{(g-1)/2} [a_j, b_j] \right) \cdot b_0^2 \right\rangle \qquad \text{if } g \text{ is odd,}$$

where we use the notation

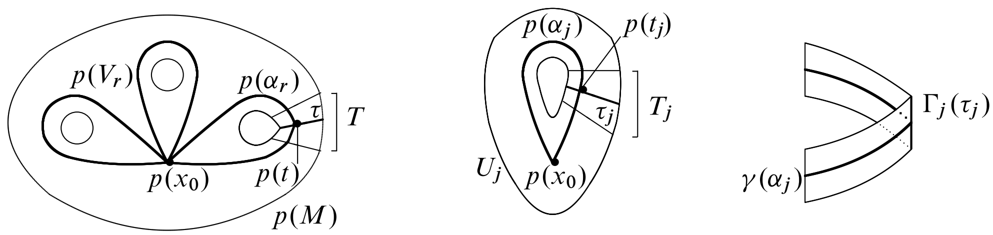
$$[x, y] = xyx^{-1}y^{-1}, \qquad [x, y]_- = xyx^{-1}y.$$

This presentation of the group $\pi_1(M, x_0)$ corresponds to a system of simple closed curves $\alpha_1, \beta_1, \dots, \alpha_{[g/2]}, \beta_{[g/2]}, \beta_0$ on M based at x_0, which form a *canonical system of cuts*; see Figure 2 (b), (c). Here the curve β_0 appears only if g is odd. Denote by V_r the bouquet of $r = [g/2]$ circles $\alpha_1 \cup \ldots \cup \alpha_{[g/2]}$ for $g \geq 2$, $V_0 = \{x_0\}$ for $g = 1$, and by ϱ a retraction $\varrho\colon M \to V_r$ which maps all loops β_j to the point x_0. We consider a realization of M in $\mathbb{R}^3$ via a map $i\colon M \to \mathbb{R}^3$ which is an immersion if g is even (see Figure 2 (b)), while, for g odd, the restriction $i|_{M \setminus \{z_1, z_2\}}$ to the complement of the set of two points $z_1, z_2 \in M \setminus \{x_0\}$ is an immersion; see Figure 2 (c). We can assume that $i|_{V_r}$ is an embedding with $i(V_r) \subset \Pi$, moreover Π coincides with the tangent plane to $i(M)$ at $i(x_0)$.

Let $p_\Pi\colon \mathbb{R}^3 \to \Pi$ denote the orthogonal projection. Without loss of generality, we may assume that the map $p = p_\Pi \circ i\colon M \to \Pi$ has the properties (p1), (p2), (p3) from above. Moreover, (p4) holds if g is odd, while the following property holds if g is even:

(p4$'$) The map p has multiplicity 4. Moreover, the set of all points of $p(M)$, whose preimage under p contains more than 2 points, lies in a regular neighbourhood T in $p(M)$ of a simple arc $\tau \subset p(M)$, where the endpoints of τ lie on the boundary of $p(M)$, τ intersects the graph $p(V_r)$ at the unique point $p(t)$, for some $t \in \alpha_r \setminus \{x_0\}$, and the intersection of τ and $p(\alpha_r)$ at the point $p(t)$ is transverse; see Figure 3 (a).

Proposition 4.1 *Suppose that M is an (orientable or nonorientable) closed surface of genus g, and $f\colon M \to N$ has absolute degree $A(f) = 0$. Then there exists a self-homeomorphism φ of M and a map $\gamma\colon V_r \to N$ such that $f \simeq \gamma \circ \varrho \circ \varphi$. Here $r = 2g$ if M is orientable, $r = [g/2]$ if M is nonorientable, and $\varrho\colon M \to V_r$ is the retraction defined above.*



(a) $T \subset p(M)$ (b) $T_j \subset U_j$ (c) $\Gamma_j(T_j) \subset N$

Figure 3: The strips T, T_j and "folding" of T_j via Γ_j

Proof Since $A(f) = 0$, it follows from [16] or [7] that f is homotopic to a map h which is not surjective; thus $h(M) \subset N^* = N \setminus \mathring{D}^2$ for an appropriate disk $D^2 \subset N$. Since the fundamental group of N^* is a free group, we obtain a homomorphism $h_\#\colon \pi_1(M) \to \pi_1(N^*)$ to the free group $\pi_1(N^*)$.

Suppose that M is orientable. It has been proved in Satz 2 of Zieschang [22] using the Nielsen method (see also Zieschang, Vogt and Coldewey [23], or Proposition 1.2 of Grigorchuk, Kurchanov and Zieschang [12]) that there is a sequence of "elementary moves" of the system of generators $a_1, b_1, \dots, a_g, b_g$ and the corresponding sequence of "elementary moves" of the system of cuts $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ on M (see above), such that the resulting system of cuts $\widetilde{\alpha}_1, \widetilde{\beta}_1, \dots, \widetilde{\alpha}_g, \widetilde{\beta}_g$ is also canonical (this means there exists a self-homeomorphism φ of M such that $\alpha_j = \varphi(\widetilde{\alpha}_j)$, $\beta_j = \varphi(\widetilde{\beta}_j)$), and the loops $h|_{\widetilde{\beta}_j}\colon \widetilde{\beta}_j \to N^*$ are contractible in N^*. From this, using the fact that $\pi_2(N^*) = 0$, one can prove that $h \simeq \gamma \circ \varrho \circ \varphi$ where $\gamma := h|_{V_g}$.

Suppose that M is nonorientable. The method to prove Satz 2 of [22] can be successfully applied to construct a canonical system of cuts $\widetilde{\alpha}_1, \widetilde{\beta}_1, \dots, \widetilde{\alpha}_{[g/2]}, \widetilde{\beta}_{[g/2]}, \widetilde{\beta}_0$ on M (this means there exists a homeomorphism φ of M with $\alpha_j = \varphi(\widetilde{\alpha}_j)$, $\beta_j = \varphi(\widetilde{\beta}_j)$) such that the loops $h|_{\widetilde{\beta}_j}\colon \widetilde{\beta}_j \to N^*$ are contractible in N^*; see Ol'shanskiĭ [17] or Proposition 1.5 of [12]. (Again the curve β_0 is considered only if g is odd.) Similarly to the orientable case, this implies that $h \simeq \gamma \circ \varrho \circ \varphi$ where $\gamma := h|_{V_g}$. □

Theorem 4.2 *Suppose that $f\colon M \to N$ has absolute degree $A(f) = 0$. Then* $2 \le \mathrm{MMR}[f] \le 4$.

Proof Suppose that h is homotopic to f and has multiplicity 1. Then h is a homeomorphism onto $h(M)$. It follows from Brouwer's Theorem on Invariance of

Domain [5] that h is surjective and, therefore, it is a homeomorphism. Then $A(h)=1$, a contradiction. Therefore $\mathrm{MMR}[f]\geq 2$.

Let us prove the second inequality. Since $A(f)=0$, by Proposition 4.1, $f\simeq\gamma\circ\varrho\circ\varphi$ for a self-homeomorphism φ of M, the retraction $\varrho\colon M\to V_r$, and a map $\gamma\colon V_r\to N$, where $r=g$ if M is an orientable surface of genus g, $r=[g/2]$ if M is a nonorientable surface of genus g. Without loss of generality, we may assume that γ has the following properties:

(γ1) There exists a homeomorphism ψ of the neighbourhood U of x_0 in M onto a neighbourhood of $\gamma(x_0)$ in N such that $\gamma|_{V_r\cap U}=\psi|_{V_r\cap U}$. In other words, $\gamma|_{V_r\cap U}$ extends to an embedding $\psi\colon U\to N$;

(γ2) The restriction of γ onto each curve $\alpha_1,\dots,\alpha_r$ is an immersion $S^1\to N$. Moreover, γ has multiplicity ≤ 2, and it has only finitely many double points (ie pairs of distinct points of V_r having the same image).

Case 1 Suppose that the surface M is either orientable (thus $r=g$), or nonorientable with g odd (thus $r=(g-1)/2$). In both cases, the map $p=p_\Pi\circ i\colon M\to\Pi=\mathbb{R}^2$ of M to the plane Π has the properties (p1), (p2), (p3), (p4); see above.

Subcase 1 Suppose that N is orientable. Since every closed curve $\gamma|_{\alpha_j}$ is orientation-preserving, it follows from the properties (γ1), (γ2), (p1), (p3) that the map $\hat\gamma=\gamma\circ p^{-1}\colon p(V_r)\to N$ can be extended to an immersion $\Gamma\colon p(M)\to N$ of the regular neighbourhood $p(M)$ of $p(V_r)$ in the plane Π to N, such that Γ has multiplicity ≤ 2.

Consider the composition $\hat\varrho=p\circ\varrho\colon M\to\Pi$. Observe that the maps $\hat\varrho$ and p are homotopic as maps $M\to p(M)\subset\Pi$ with the target $p(M)$, due to $\hat\varrho|_{V_r}=p|_{V_r}$, (p2), and $\pi_2(p(M))=0$. From this and $\gamma=\Gamma\circ p|_{V_r}$, we have

$$f\simeq\gamma\circ\varrho\circ\varphi=\Gamma\circ p\circ\varrho\circ\varphi\simeq\Gamma\circ p\circ\varphi. \tag{4.3}$$

Since φ is bijective and each of Γ and p has multiplicity ≤ 2 (see (p4)), the multiplicity of the composition $\Gamma\circ p\circ\varphi$ is $\leq 2\cdot 2\cdot 1=4$.

Subcase 2 Suppose that N is nonorientable. So in general, the immersion $\hat\gamma\colon p(V_r)\to N$ can not be extended to an immersion of the regular neighbourhood $p(M)$ of $p(V_r)$ in $\Pi=\mathbb{R}^2$. However, due to (γ1), (γ2), and (p1), we can extend $\hat\gamma$ to an immersion $\tilde\Gamma\colon p(D\cup V_r)\to N$, where $D\subset U$ is a small disk centred at x_0.

Now, for each curve α_j, we will extend the immersion $\tilde\Gamma_j=\tilde\Gamma|_{p(D\cup\alpha_j)}\colon p(D\cup\alpha_j)\to N$ to a regular neighbourhood $U_j\supset p(D)$ of $p(\alpha_j)$ in Π as follows. If the curve $\gamma|_{\alpha_j}$ is orientation-preserving then, similarly to Case 1, the immersion $\tilde\Gamma_j\colon p(D\cup\alpha_j)\to N$

can be extended to an immersion $\Gamma_j\colon U_j \to N$. If the curve $\gamma|_{\alpha_j}$ is orientation-reversing, let us choose a point $t_j \in \alpha_j \setminus D$ such that t_j is the only preimage of the point $\gamma(t_j)$ under γ. Consider a simple arc $\tau_j \subset U_j \setminus p(D)$, which transversally intersects $p(\alpha_j)$ at the only point $p(t_j)$, and whose endpoints lie on the boundary of U_j. Let T_j be a regular neighbourhood of the arc τ_j in $U_j \setminus p(D)$, thus T_j is a "strip" in the annulus U_j; see Figure 3 (b). Outside the interior of the strip T_j, we extend $\widetilde{\Gamma}_j$ to an immersion $\overline{\Gamma}_j\colon (U_j \setminus T_j) \cup p(\alpha_j) \to N$ similarly to above. Now we extend the obtained immersion $\overline{\Gamma}_j$ to the whole annulus U_j, giving a map $\Gamma_j\colon U_j \to N$ which coincides with $\overline{\Gamma}_j$ outside $T_j \setminus p(\alpha_j)$ and has a "folding" along the arc $\tau_j \subset T_j$, as shown in Figure 3 (c).

Without loss of generality, we may assume that $U_j \subset p(M)$, and any two annuli U_j, U_k have only the disk $p(D)$ in common. Since the constructed mappings $\Gamma_j\colon U_j \to N$ agree on the common part $p(D)$, they determine an extension $\overline{\Gamma}\colon U \to N$ of the map $\widetilde{\Gamma}$, where $U = U_1 \cup \ldots \cup U_r$ is a regular neighbourhood of $p(V_r)$ in $\Pi = \mathbb{R}^2$. The above construction can be performed in such a way that the map $\overline{\Gamma}$ has multiplicity ≤ 2, due to $(\gamma 2)$ and the choice of the points $t_j \in \alpha_j$. Obviously, the map $\overline{\Gamma}$ can be extended to the regular neighbourhood $p(M)$ of $p(D \cup V_r)$ (see (p3)) and the extended map $\Gamma\colon p(M) \to N$ also has multiplicity ≤ 2.

Similarly to Subcase 1, the composition $\Gamma \circ p \circ \varphi$ has multiplicity $\leq 2 \cdot 2 \cdot 1 = 4$, and (4.3) holds. This completes the proof in Case 1.

Case 2 Suppose that M is a nonorientable closed surface of even genus g, thus $r = g/2$, and the map $p = p_\Pi \circ i\colon M \to \Pi = \mathbb{R}^2$ of M to the plane Π has the properties (p1), (p2), (p3), (p4$'$); see above. We may assume, without loss of generality, that the map $\gamma\colon V_r \to N$ has the following additional property:

$(\gamma 3)$ The point $t \in \alpha_r$ considered in (p4$'$) is the only preimage of $\gamma(t)$ under γ, and the analogous property holds for any point $\tilde{t} \in \alpha_r \cap p^{-1}(T)$.

Subcase 1 Suppose that N is orientable. Similarly to Subcase 1 of Case 1, one shows using $(\gamma 1)$, $(\gamma 2)$, (p1), (p3) that the immersion $\hat{\gamma} = \gamma \circ p^{-1}\colon p(V_r) \to N$ extends to an immersion $\Gamma\colon p(M) \to N$ of multiplicity 2, and using (p2) that **(4.3)** holds. Taking into account (p4$'$) and $(\gamma 3)$, one can show that the multiplicity of $\Gamma \circ p \circ \varphi$ is ≤ 4.

Subcase 2 Suppose that N is nonorientable. We proceed as in Subcase 2 of Case 1. Namely, for those curves α_j whose image under γ is orientation-preserving, we extend the immersion $\widetilde{\Gamma}_j\colon p(D \cup \alpha_j) \to N$ to U_j, as in Case 1. For each of the remaining curves α_j, we choose a point $t_j \in \alpha_j \setminus D$ which is the only preimage of $\gamma(t_j)$ under γ, and we extend the corresponding immersion $\widetilde{\Gamma}_j$ to a map $\overline{\Gamma}_j\colon U_i \to N$ having a "folding" along an arc $\tau_j \subset T_j \subset U_j$, which transversally intersects $p(V_r)$ at the unique point $p(t_j)$; see Case 1. As above, this allows one to construct a map $\Gamma\colon\ p(M) \to N$ of multiplicity ≤ 2 which is an extension of $\widehat{\gamma}$, and to show that **(4.3)** holds. Observe now that, if the curve $\gamma|_{\alpha_r}$ is orientation-reversing, we can choose the point $t_r \in \alpha_r$ in such a way that it is "far enough" from the point $t \in \alpha_r$ considered in (p4′). This, together with $(\gamma 3)$, shows that the above construction can be performed in such a way that the composition $\Gamma \circ p \circ \varphi$ has multiplicity ≤ 4. This completes the proof of Theorem 4.2. □

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SB, EK: *Department of Mathematics and Mechanics, Moscow State University*
Moscow 119992, Russia

JF: *Fachbereich 6 – Mathematik, Universität Siegen*
57068 Siegen, Germany

bogatyi@inbox.ru, fricke@math.uni-siegen.de, eakudr@mech.math.msu.su

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