Roots of torsion polynomials and dominations

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We show that the nonzero roots of the torsion polynomials associated to the infinite cyclic covers of a given compact, connected, orientable 3–manifold M are contained in a compact part of $\mathbb{C}^*$ a priori determined by M. This result is applied to prove that when M is closed, it dominates at most finitely many Sol manifolds.

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Dedicated to the memory of Heiner Zieschang

1 Introduction

All manifolds are connected and orientable in this paper. All homology groups will have $\mathbb{Q}$–coefficients unless otherwise specified.

Suppose that M and N are compact 3–manifolds. We say that M *dominates* N if there is a nonzero degree map $f\colon (M, \partial M) \to (N, \partial N)$.

To each epimorphism $\psi\colon \pi_1(M) \to \mathbb{Z}$ of the fundamental group of a compact 3–manifold one can associate a torsion polynomial $\Delta_\psi^M(t)$. Our first result shows that the absolute values of the nonzero roots of such polynomials are pinched between two constants depending only on M, even though $\pi_1(M)$ has infinitely many epimorphisms to $\mathbb{Z}$ when its first Betti number is greater than one. We combine this result with a classical argument due to Wall for nonzero degree maps to show that the same conclusion holds for any 3–manifold dominated by M. As an application we prove that a closed 3–manifold M dominates at most finitely many Sol manifolds.

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2 Roots of torsion polynomials

Given a compact 3–manifold and an epimorphism $\psi\colon \pi_1(M) \to \mathbb{Z}$, let $\widetilde{M}_\psi \to M$ be the associated infinite cyclic cover. The action on $H_1(\widetilde{M}_\psi)$ of $t = (T_\psi)_*$ induced by the generator T_ψ of the deck transformation group corresponding to $1 \in \mathbb{Z}$ makes $H_1(\widetilde{M}_\psi)$ a finitely generated Γ–module, where $\Gamma = \mathbb{Q}[\pi_1(M)/\ker(\psi)] \cong \mathbb{Q}[t, t^{-1}]$. Since Γ is a principal domain, $H_1(\widetilde{M}_\psi) \cong \Gamma^k \oplus_{i=1}^n \Gamma/(p_i(t))$ where $0 \neq p_i(t) \in \Gamma$. The product $\Delta_\psi^M(t) = p_1(t)p_2(t)\ldots p_n(t)$, called the *torsion polynomial* of ψ, represents the order of the Γ–torsion submodule $\mathrm{Tor}(H_1(\widetilde{M}_\psi))$ of $H_1(\widetilde{M}_\psi)$ and is well-defined up to multiplication by some unit rt^i of Γ ($i \in \mathbb{Z}$, $0 \neq r \in \mathbb{Q}$). In particular, the set of nonzero roots $\{t_0 \in \mathbb{C}^* : \Delta_\psi^M(t_0) = 0\}$ is independent of the choice of $\Delta_\psi^M(t)$. A straightforward calculation shows that $\Delta_\psi^M(t)$ coincides, up to units, with the characteristic polynomial of the automorphism of the $\mathbb{Q}$–vector space $\oplus_{i=1}^n \Gamma/(p_i(t))$ corresponding to multiplication by t.

Theorem 2.1 *A compact, connected, orientable 3–manifold M determines a constant $c_M > 0$ with the following property: If $t_0 \in \mathbb{C}^*$ is a root of a torsion polynomial $\Delta_\psi^M(t)$ associated to an epimorphism $\psi\colon \pi_1(M) \to \mathbb{Z}$, then $1/c_M \leq |t_0| \leq c_M$.*

Proof Since $1/t_0$ is a root of $\Delta_{-\psi}^M(t)$, it suffices to prove the existence of a constant c_M such that $|t_0| \leq c_M$.

For a group G and $\alpha = \sum\limits_{g \in G} r_g g \in \mathbb{Q}[G]$, we set $\|\alpha\| = \sum\limits_{g \in G} |r_g|$.

Consider a finite presentation $\langle x_j : r_i \rangle$ of $\pi_1(M)$ and let $J = \left(\frac{\partial r_i}{\partial x_j}\right)$ be the associated Jacobian matrix. Define $k(\langle x_j : r_i \rangle) = \sum_{i,j} \left\| \frac{\partial r_i}{\partial x_j} \right\| \in \mathbb{N}$ and set

$$k_M = \min\{k(\langle x_j : r_i \rangle) : \langle x_j : r_i \rangle \text{ presents } \pi_1(M)\}.$$

We assume that $\langle x_j : r_i \rangle$ has been chosen to realize k_M and that the number m of generators is minimal among such presentations.

Fix an epimorphism $\psi\colon \pi_1(M) \to \mathbb{Z}$ and let Ψ be the composition $\mathbb{Z}[\pi_1(M)] \to \mathbb{Q}[\pi_1(M)/\ker(\psi)] = \mathbb{Q}[t, t^{-1}]$. Recall that $J^\Psi = \left(\Psi\left(\frac{\partial r_i}{\partial x_j}\right)\right)$ presents the Γ–module $H_1(\widetilde{M}_\psi) \oplus \Gamma$ (see Burde and Zieschang [1, Section 9], for example). Set $q_{ij}(t) = \Psi\left(\frac{\partial r_i}{\partial x_j}\right) \in \mathbb{Q}[t, t^{-1}]$ and observe that $\|q_{ij}(t)\| \leq \left\| \frac{\partial r_i}{\partial x_j} \right\|$. Thus the following claim holds.

Claim 2.2 $\sum_{i,j} \|q_{ij}(t)\| \leq k_M$. □

If r denotes the Γ–rank of J^{Ψ}, then $\Delta_{\psi}^{M}(t)$ is, up to units, the g.c.d. of the r–rowed minors of J^{Ψ} (see Jacobson [2, Theorem 3.9], for example). Thus it suffices to show that the absolute values of the roots of some nonzero r–rowed minor of J^{Ψ} are bounded above by a constant depending only on M. To that end, fix such a minor $D(t) \in \mathbb{Z}[t,t^{-1}]$ which, without loss of generality, we can assume is polynomial in t, and let $D_0(t)$ be the monic polynomial with the same roots. Since $r \leq m$, the expansion of $D(t)$ in terms of the $q_{ij}(t)$ shows that $m!k_M^m$ is an upper bound for the sum of the absolute values of its coefficients (cf Claim 2.2). It is evident that the same inequality holds for $D_0(t) = t^s + b_{s-1}t^{s-1} + \ldots + b_0$. If $|t| > R = 1 + \sum_i |b_i|$, then $|D_0(t)| > R^n - (\sum_i |b_i| R^i) \geq R^{n-1}(R - \sum_i |b_i|) > 0$ so that the roots of $D_0(t)$ lie in the ball of radius $1 + \sum_i |b_i|$ centred at zero. Thus the theorem holds with $c_M = 1 + m!k_M^m$. □

We generalize this result with our applications in mind.

Theorem 2.3 *For a compact, connected, orientable 3–manifold M, there is a constant $c_M > 0$ with the following property: If N is a compact 3–manifold dominated by M and $t_0 \in \mathbb{C}^*$ is a root of a torsion polynomial $\Delta_{\psi}^{N}(t)$ of an epimorphism $\psi\colon \pi_1(N) \to \mathbb{Z}$, then $1/c_M \leq |t_0| \leq c_M$*

Proof Suppose that $f\colon M \to N$ is a nonzero degree map and fix an epimorphism $\psi\colon \pi_1(N) \to \mathbb{Z}$. Since $\deg(f) \neq 0$, there is an integer $n \geq 1$ such that the image $(\psi \circ f_\#)(\pi_1(M)) = n\mathbb{Z}$. Denote by $\theta\colon \pi_1(M) \to \mathbb{Z}$ the epimorphism $(1/n)(\psi \circ f_\#)$ and by $\Delta_{\theta}^{M}(t)$ the associated torsion polynomial. The theorem is a simple consequence of Theorem 2.1 and the following claim.

Claim 2.4 *If $t_0 \in \mathbb{C}^*$ is a root of $\Delta_{\psi}^{N}(t)$, then t_0^n is a root of $\Delta_{\theta}^{M}(t)$.*

Proof Let $\mathbb{Q}[t,t^{-1}]_f$ be the $\mathbb{Z}[\pi_1(M)]$–module whose underlying group is $\mathbb{Q}[t,t^{-1}]$ and whose $\pi_1(M)$ action is that determined by the homomorphism $f_\#\colon \pi_1(M) \to \pi_1(N)$. Thus for $x \in \pi_1(M)$ and $p(t) \in \mathbb{Q}[t,t^{-1}]$ we have $x \cdot p(t) = t^{(\psi \circ f_\#)(x)} p(t)$. When $n = 1$, this action coincides with that of $\mathbb{Z}[\pi_1(M)]$ on $\mathbb{Q}[\pi_1(M)/\ker(\psi \circ f_\#)]$ and so $H_1(M;\mathbb{Q}[t,t^{-1}]_f) \cong H_1(\widetilde{M}_\theta)$, where the latter has the Γ–action described above. In particular, since $\deg(f) \neq 0$, there is a Γ–module splitting

$$H_1(\widetilde{M}_\theta) = H_1(M;\mathbb{Q}[t,t^{-1}]_f)) \cong H_1(N;\mathbb{Q}[t,t^{-1}]) \oplus K = H_1(\tilde{N}_\psi) \oplus K$$

for some finitely generated Γ–submodule K of $H_1(\widetilde{M}_\theta)$ (see the proof of [7, Lemma 2.1]). Hence when $n = 1$, $\mathrm{Tor}(H_1(\widetilde{N}_\psi))$ is a Γ–submodule of $\mathrm{Tor}(H_1(\widetilde{M}_\theta))$, and so its order $\Delta_{\psi}^{N}(t)$ divides $\Delta_{\theta}^{M}(t)$, which implies the claim in this case.

Next suppose $n > 1$ and let $\widetilde{N}_{(\psi,n)} \to N$ be the n–fold cyclic cover with $\pi_1(\widetilde{N}_{(\psi,n)})$ the kernel of the (mod n) reduction of ψ. Then f lifts to a π_1–surjective, nonzero degree map $\tilde{f}\colon M \to \widetilde{N}_{(\psi,n)} = N'$. Let

$$\psi'\colon \pi_1(N') \to n\mathbb{Z} \xrightarrow{1/n} \mathbb{Z}$$

be the epimorphism induced by ψ. The case $n = 1$ shows that any nonzero root of $\Delta_{\psi'}^{N'}(t)$ is also a root of $\Delta_\theta^M(t)$. On the other hand, it is easy to see that $(T_{\psi'})_* = (T_\psi)_*^n$ on $H_1(\widetilde{N'}_{\psi'}) = H_1(\widetilde{N}_\psi)$ so that if $t_0 \in \mathbb{C}^*$ is a root of $\Delta_\psi^N(t)$, then $t_0^n \in \mathbb{C}^*$ is a root of $\Delta_{\psi'}^{N'}(t)$, and therefore of $\Delta_\theta^M(t)$. This completes the proof of the claim and therefore of Theorem 2.3. $\square$

3 $\mathbb{Q}$–Homology surface bundles

Let F be a compact surface and A an abelian group. An *A–homology $F \times I$* is a 3–manifold W with boundary containing two disjoint surfaces $F_1 \cong F_2 \cong F$ such that

(i) $\overline{\partial W \setminus (F_1 \cup F_2)} \cong \partial F \times I$ where $\partial F \times \{0\} = \partial F_1, \partial F \times \{1\} = \partial F_2$, and

(ii) the inclusion induced homomorphism $H_*(F_1; A) \to H_*(W; A)$ is an isomorphism.

(Duality and universal coefficients shows that (ii) is equivalent to each of the following three conditions: $H_*(W, F_1; A) = 0$; $H_*(W, F_2; A) = 0$; $H_*(F_2; A) \xrightarrow{\cong} H_*(W; A)$.) Note that W determines orientations on F_1 and F_2 well-defined up to simultaneous reversal. Thus the set $\mathrm{Homeo}(F_2, F_1)^-$ of orientation reversing homeomorphisms $F_2 \to F_1$ is well-defined. For each $\varphi \in \mathrm{Homeo}(F_2, F_1)^-$ we define W_φ to be the compact, orientable manifold obtained from W by identifying F_2 to F_1 via φ. The composition

$$H_1(F_1; A) \xrightarrow{\cong} H_1(W; A) \xrightarrow{\cong} H_1(F_2; A) \xrightarrow{\varphi_*} H_1(F_1; A)$$

determines an isomorphism

$$\varphi_*^W\colon H_1(F_1; A) \to H_1(F_1; A)$$

which we call the *algebraic monodromy* of W_φ. Set

$$\Delta_\varphi^W(t) = \det(\varphi_*^W - tI).$$

We call W_φ an *A–homology F bundle*.

Theorem 3.1 *For a compact, connected, orientable 3–manifold M, there is a constant $c_M > 0$ with the following property: If W_φ is a $\mathbb{Q}$–homology surface bundle which is dominated by M, then the absolute values of the roots of the characteristic polynomial $\Delta_\varphi^W(t)$ of φ_*^W are pinched between $1/c_M$ and c_M.*

Proof Let $F \subset W_\varphi$ be the nonseparating surface corresponding to $F_1 = \varphi(F_2)$. It determines a nonzero class $[F] \in H_2(W_\varphi)$, well-defined up to sign, and an epimorphism

$$\psi\colon \pi_1(W_\varphi;\mathbb{Z}) \to \mathbb{Z}, \alpha \mapsto \alpha \cdot [F].$$

Let $\widetilde{W}_\varphi \to W_\varphi$ be the infinite cyclic cover associated to this epimorphism ψ. Note that $H_1(\widetilde{W}_\varphi) = H_1(W_\varphi;\Gamma)$ where Γ is the $\mathbb{Z}[\pi_1(W_\varphi)]$–module $\mathbb{Q}[\pi_1(W_\varphi)/\ker(\psi)] \cong \mathbb{Q}[\mathbb{Z}] \cong \mathbb{Q}[t,t^{-1}]$. The $\mathbb{Z}[\pi_1(W_\varphi)]$ action on $H_1(\widetilde{W}_\varphi)$ factors through one of Γ in such a way that $t = (T_\varphi)_*$ where $T_\varphi\colon \widetilde{W}_\varphi \to \widetilde{W}_\varphi$ is a generator of the group of deck transformations of $\widetilde{W}_\varphi \to W_\varphi$.

Claim 3.2 *$H_1(\widetilde{W}_\varphi)$ is a torsion module over Γ whose order is represented by $\Delta_\varphi^W(t)$.*

Proof The quotient map $W \to W_\varphi$ lifts to an inclusion of W into $\widetilde{W}_\varphi$ with image $\widetilde{W}_0$ say. Let $\widetilde{F}_0 \subset \partial\widetilde{W}_0$ correspond to F_1 and set $\widetilde{W}_j = T_\varphi^j(\widetilde{W}_0)$, $\widetilde{F}_j = T_\varphi^j(\widetilde{F}_0)$. Then $\widetilde{W}_\varphi = \cup_j \widetilde{W}_j$ where $\widetilde{W}_j \cap \widetilde{W}_k = \varnothing$ if $|j-k| > 1$ and $\widetilde{W}_j \cap \widetilde{W}_{j-1} = \widetilde{F}_j$. Since W is a $\mathbb{Q}$–homology $F_1 \times I$, the composition $H_1(F_1) = H_1(\widetilde{F}_0) \to H_1(\widetilde{W}_\varphi)$ is an isomorphism under which the algebraic monodromy $\varphi_*^W\colon H_1(F_1) \to H_1(F_1)$ corresponds to $(T_\varphi)_*\colon H_1(\widetilde{W}_\varphi) \to H_1(\widetilde{W}_\varphi)$.

It is now clear that $H_1(\widetilde{W}_\varphi)$ is a torsion module over Γ since $H_1(\widetilde{W}_\varphi) \cong H_1(F_1)$ is finite dimensional over $\mathbb{Q}$. Hence the order of $H_1(\widetilde{W}_\varphi)$ as a Γ–module corresponds to the characteristic polynomial of the automorphism $(T_\varphi)_*$ of the $\mathbb{Q}$–vector space $H_1(\widetilde{W}_\varphi)$, at least up to multiplication by some unit Γ. Since $(T_\varphi)_*$ corresponds to φ_*^W under $H_1(F_1) \xrightarrow{\cong} H_1(\widetilde{W}_\varphi)$, $\Delta_\varphi^W(t)$ also represents the order of $H_1(\widetilde{W}_\varphi)$. □

Claim 3.2 shows that, up to multiplication by a unit, $\Delta_\varphi^W(t)$ is the torsion polynomial of the epimorphism ψ. Theorem 3.1 now follows from Theorem 2.3. □

Corollary 3.3 *Let W be a $\mathbb{Q}$–homology $F \times I$. A compact, connected, orientable 3–manifold M determines a finite subset $\mathcal{P}_{(M,W)}$ of $\mathbb{Q}[t]$ such that if M dominates W_φ, then the characteristic polynomial of φ_*^W is contained in $\mathcal{P}_{(M,W)}$.*

Proof Let $\beta_1(F)$ be the first Betti number of F. The reader will verify that since W is a $\mathbb{Q}$–homology $F \times I$, we can choose bases of for $H_1(F_1;\mathbb{Z})$ and $H_1(F_2;\mathbb{Z})$ with respect to which the matrix X of $H_1(F_1) \to H_1(W) \to H_1(F_2)$ lies in $SL_{\beta_1(F)}(\mathbb{Q})$

and the matrix Y of $H_1(F_2) \xrightarrow{\varphi_*} H_1(F_1)$ lies in $SL_{\beta_1(F)}(\mathbb{Z})$. Now φ_*^W is represented by YX so the denominators of its entries are bounded above by some constant N. Thus the coefficients of $\Delta_\varphi^W(t) = \det(YX - tI)$ have denominators bounded above by $N^{\beta_1(F)}$ and since its degree is $\beta_1(F)$, the corollary follows from Theorem 3.1. □

Remark 3.4 (1) The finite set $\mathcal{P}_{(M,W)}$ described in the corollary depends on both M and W. In the case when $W \cong F \times I$, the matrix X of the proof of Corollary 3.3 lies in $SL_{\beta_1(F)}(\mathbb{Z})$, so it is easy to see that $\mathcal{P}_{(M,W)}$ depends only on M and the Euler characteristic of the fibre.

(2) A given compact 3–manifold M can be the total space of infinitely many distinct surface bundles over the circle. Moreover, there are cases where the Euler characteristic of the fibres are unbounded. However, Theorem 3.1 provides the following constraint on the monodromy of any such bundle structure on M.

Corollary 3.5 *Given a compact* 3*–manifold M, there is a constant $c_M > 0$ such that the absolute values of the roots of the characteristic polynomial of the algebraic monodromy of any surface bundle structure on M are pinched between $1/c_M$ and c_M.* □

Recall that an element $\varphi \in SL_2(\mathbb{Z})$ is called *hyperbolic* if $|\mathrm{trace}(\varphi)| > 2$.

Corollary 3.6 *A closed, connected, orientable* 3*–manifold M dominates only finitely many Sol manifolds.*

Proof First suppose that M dominates a torus bundle over the circle with hyperbolic monodromy $\varphi \in SL_2(\mathbb{Z})$. Corollary 3.3 shows that there are only finitely many possibilities for $\mathrm{trace}(\varphi)$, which is the negative of the coefficient of t in $\Delta_\varphi^{T^2 \times I}(t)$. On the other hand, there are only finitely many $SL_2(\mathbb{Z})$ conjugacy classes of hyperbolic elements of $SL_2(\mathbb{Z})$ with a given trace (eg see Wang and Zhou [9, Lemma 8]). Since the homeomorphism type of a torus bundle over the circle depends only on the conjugacy class of its monodromy $\varphi \in SL_2(\mathbb{Z})$, it follows that a closed, connected, orientable 3–manifold can dominate at most finitely many torus bundles over the circle with hyperbolic monodromy. But a closed, connected Sol manifold N is double covered by such a bundle $\widetilde{N}$ and so if M dominates N, some double cover of $\widetilde{M}$ dominates $\widetilde{N}$. Since M has only finitely many double covers, there are only finitely many possibilities for $\widetilde{N}$, and therefore for N [4; 3]. □

It is known that if a closed, orientable 3–manifold dominates a manifold which admits a geometric structure based on the geometries $\mathbb{S}^3$, Nil, or $\widetilde{SL}_2$, then it dominates infinitely many distinct such manifolds [8]. This is false for the remaining geometries.

Corollary 3.7 *A closed, orientable 3–manifold dominates at most finitely many manifolds admitting an $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{E}^3$, $\mathbb{H}^3$, $\mathbb{H}^2 \times \mathbb{R}$, or Sol structure.*

Proof The corollary holds for dominations of $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{E}^3$ manifolds since there are only finitely many such spaces (see eg Scott [5]). It holds for dominations of hyperbolic manifolds by Soma [6], for $\mathbb{H}^2 \times \mathbb{R}$ manifolds by Wang and Zhou [9], and for Sol manifolds by Corollary 3.6. □

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