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A characterisation of $\mathbb{S}^3$ among homology spheres

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We prove that an integral homology 3–sphere is $\mathbb{S}^3$ if and only if it admits four periodic diffeomorphisms of odd prime orders whose space of orbits is $\mathbb{S}^3$. As an application we show that an irreducible integral homology sphere which is not $\mathbb{S}^3$ is the cyclic branched cover of odd prime order of at most four knots in $\mathbb{S}^3$. A result on the structure of finite groups of odd order acting on integral homology spheres is also obtained.

57M40; 57S17, 57M60, 57M12, 57M50

To the memory of Heiner Zieschang

1 Introduction

A well-known property of the standard sphere $\mathbb{S}^3$ is to admit a periodic diffeomorphism ψ of any order and with trivial quotient $\mathbb{S}^3$. By definition we say that a periodic diffeomorphism ψ of an orientable 3–manifold M has *trivial quotient* if the underlying space of orbits of its action $|M/\psi|$ is homeomorphic to $\mathbb{S}^3$.

The goal of this article is to show that a much weaker version of the aforementioned property characterises the 3–sphere $\mathbb{S}^3$ among integral homology spheres. More precisely the main result of this article is:

Theorem 1 *An integral homology 3–sphere M is homeomorphic to the 3–sphere if and only if it admits four periodic diffeomorphisms with pairwise different odd prime orders and trivial quotients.*

Remark that this result is sharp, because the Brieskorn homology sphere with three exceptional fibres $\Sigma(p_1, p_2, p_3)$ is the p_i–fold cyclic cover of $\mathbb{S}^3$ branched along the $T(p_j, p_k)$ torus knot, where $\{i, j, k\} = \{1, 2, 3\}$ and the p_i's are three distinct odd prime numbers. These examples are Seifert manifolds. The existence of hyperbolic

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homology 3–spheres behaving in an analogous way can be obtained by applying the strongly almost identical (AID) imitation theory of Kawauchi [12].

Note moreover that the requirement that the diffeomorphisms have trivial quotient is essential. The Brieskorn homology sphere $\Sigma(p_1, \dots, p_n)$, $n \geq 4$, admits n periodic diffeomorphisms of pairwise distinct odd prime orders with nonempty fixed-point set but with nontrivial quotient.

In the following, we say that a nontrivial periodic diffeomorphism ψ of an orientable 3–manifold M is a *rotation* if it preserves the orientation of M and $\mathrm{Fix}(\psi)$ is nonempty and connected.

A basic observation is that a nontrivial periodic diffeomorphism ψ of an integral homology 3–sphere with odd prime order and trivial quotient is a rotation. Indeed, since the order is odd, the diffeomorphism must preserve the orientation of the manifold. Moreover, such diffeomorphism cannot act freely, for the quotient $\mathbb{S}^3 = |M/\psi|$ is simply connected. As the manifold is an integral homology sphere, standard Smith theory implies that the fixed-point set of the diffeomorphism is a circle, which projects to a knot in the quotient $\mathbb{S}^3$.

To prove Theorem 1 we need to understand the behaviour of rotations with trivial quotient acting on homology spheres. The key result is:

Theorem 2 *Let M be an irreducible integral homology 3–sphere which admits $n \geq 3$ rotations $\{\psi_i\}_{1 \leq i \leq n}$ with trivial quotient and of distinct odd prime orders. Then, up to conjugacy, the rotations $\{\psi_i\}_{1 \leq i \leq n}$ generate a cyclic subgroup of* $\mathrm{Diff}(M)$.

This theorem has the following consequence:

Corollary 1 *Let M be an irreducible integral homology 3–sphere which is not homeomorphic to $\mathbb{S}^3$. Then:*

(i) *There are at most four distinct knots in $\mathbb{S}^3$ having M as cyclic branched cover of odd prime order.*

(ii) *If M is hyperbolic or Seifert fibred then there are at most three distinct knots in $\mathbb{S}^3$ having M as cyclic branched cover of odd prime order; if there are three such knots then the three branching orders are distinct.*

(iii) *If M is the p_i–fold cyclic cover of $\mathbb{S}^3$ branched over a knot K_i for three distinct odd prime numbers p_i, then the three knots K_i are related by the standard abelian construction described in Section 6. Moreover, the knots K_i are pairwise non equivalent.*

Remark that the conclusions of Corollary 1 are no longer valid for covers of arbitrary prime order. Indeed, the Brieskorn sphere $\Sigma(p_1, \ldots, p_n)$, $n \geq 3$, is the double branched cover of $(n-1)!/2$ inequivalent Montesinos knots in $\mathbb{S}^3$. Moreover, for $n = 3$, the Montesinos knot and the torus knots $T(p_i, p_j)$ are not related by the standard abelian construction. On the other hand, part (i) of Corollary 1 is not the best possible, and one can prove that there are at most three distinct knots in $\mathbb{S}^3$ having a given irreducible integral homology 3–sphere as cyclic branched cover of odd prime order. This bound is clearly sharp because so is Theorem 1. The proof in the general case is however rather technical, and only a sketchy idea will be given at the end of Section 6.

Compare also Reni and Zimmermann [18] where the case of hyperbolic 3–manifolds is considered which are not necessarily homology 3–spheres.

If one is given n rotations of pairwise distinct odd prime orders acting on an integral homology sphere M and belonging to a finite subgroup $G \subset \mathrm{Diff}(M)$ of odd order, then Theorem 2 is a consequence of the following result on the structure of finite groups of odd order acting on integral homology spheres:

Theorem 3 *Let G be a finite group of odd order acting on an integral homology 3–sphere. Then G is cyclic or a direct product of two cyclic groups.*

In Section 2 we show how one can deduce Theorem 1 from Theorem 2. The proof of Theorem 2 consists of several steps: we start by establishing in Section 3 a preliminary result which states that Theorem 2 is true under the requirement that the rotations are contained in a finite group. Theorem 3 on the structure of finite groups of odd order acting on integral homology spheres will also be proved in Section 3. The actual proof of Theorem 2 will be subdivided into two parts according to the structure of the irreducible homology sphere under consideration, ie a sphere with trivial JSJ decomposition (Section 4) or not (Section 5) [10; 11]. Finally, in Section 6 we describe the standard abelian construction and prove Corollary 1.

2 Proof of Theorem 1

In this section we prove Theorem 1, assuming Theorem 2.

Assume that $M = \mathbb{S}^3$. Then it is trivial to see that for each integer $n \geq 2$, M admits a rotation of order n about a standard circle (ie the trivial knot) with quotient again $\mathbb{S}^3$. In particular, $\mathbb{S}^3$ admits four rotations with pairwise distinct odd prime orders and trivial quotients.

We now prove the converse. Let us assume that M is an integral homology 3–sphere admitting four rotations with trivial quotients and pairwise distinct odd prime orders.

Claim 1 *We can assume M to be irreducible.*

Proof Since $\mathbb{S}^3$ is irreducible, the equivariant sphere theorem shows that each rotation must leave invariant and induce a rotation on each prime summand of a decomposition for M. Moreover the induced rotation must have trivial quotient, for the only possible decompositions of $\mathbb{S}^3$ as connected sum, contain only $\mathbb{S}^3$ summands. Each summand of the prime decomposition of M is again an integral homology sphere, and thus must be irreducible. □

Since M is irreducible, according to Theorem 2 it admits four commuting rotations with trivial quotient and pairwise different odd prime orders. Fix one of these rotations ψ. The projection $M \longrightarrow |M/\psi|$ is a cyclic cover of the 3–sphere $\mathbb{S}^3 = |M/\psi|$ branched along a knot K. The three remaining rotations, which commute with and thus normalise ψ, induce rotations of the pair $(\mathbb{S}^3, K)$. Moreover, since these rotations commute, they generate a cyclic group of diffeomorphisms of the pair $(\mathbb{S}^3, K)$.

Claim 2 *Let $M \neq \mathbb{S}^3$ be an irreducible manifold admitting two commuting rotations ψ and φ with trivial quotients and distinct orders. Let K be the knot* $\mathrm{Fix}(\psi)/\psi \subset \mathbb{S}^3$ *and let ϕ the rotation of the pair $(\mathbb{S}^3, K)$ induced by φ. The rotation ϕ has* trivial quotient knot, *ie the quotient of K by the action of ϕ is the trivial knot.*

Proof The proof of this claim will be given in Section 6. □

The above claim implies that the knot K admits three rotations with pairwise distinct odd prime orders and trivial quotient knots. The proof is now a consequence of the following result, which is a special case of [2, Theorem 3]. For completeness we give the proof in this special case where the symmetries commute.

Lemma 1 *Let K be a knot in $\mathbb{S}^3$ admitting three commuting rotational symmetries φ_i, $i = 1, 2, 3$ with trivial quotient knots and whose orders are three pairwise coprime numbers p_i, $i = 1, 2, 3$. Then K is the trivial knot.*

Proof Assume first that two of the symmetries – say φ_1, φ_2 – have the same axis. Since the three symmetries commute, φ_2 induces a rotation of the trivial knot K/φ_1 which is non trivial for the order of φ_2 and that of φ_1 are coprime. The axis of this induced symmetry is the image of $\mathrm{Fix}(\varphi_2) = \mathrm{Fix}(\varphi_1)$ in the quotient by the action of φ_1. In particular K/φ_1 and $\mathrm{Fix}(\varphi_1)/\varphi_1$ form a Hopf link and K is the trivial knot: this follows from the equivariant Dehn lemma; see Hillman [9].

We can thus assume that the axes are pairwise disjoint. In this case we would have that the axis of φ_1, which is a trivial knot, admits two commuting rotations, φ_2 and φ_3, with distinct axes, which is impossible: this follows, for instance, from the fact (see Edmonds and Livingston [8, Theorem 5.2]) that one can find a fibration of the complement of the trivial knot which is equivariant with respect to the two symmetries. □

3 Finite groups acting on homology 3–spheres

In this section we prove Theorem 2 in the case where the $n \geq 3$ rotations belong to a finite subgroup of diffeomorphisms of M.

Proposition 1 *Let M be an integral homology 3–sphere and $G \subset \mathrm{Diff}(M)$ be a finite subgroup. If G contains $n \geq 3$ rotations $\{\psi_i\}_{1\leq i\leq n}$ of distinct odd prime orders, then, up to conjugacy in G, the rotations $\{\psi_i\}_{1\leq i\leq n}$ generate a cyclic subgroup of $\mathrm{Diff}(M)$.*

Proof The first step in the proof is a consequence of the classification of finite groups which can admit actions on integral homology 3–spheres given in [15, Theorem 2, page 677].

Lemma 2 *Let M be an integral homology 3–sphere and $G \subset \mathrm{Diff}(M)$ be a finite subgroup. If G contains $n \geq 3$ rotations $\{\psi_i\}_{1\leq i\leq n}$ of distinct odd prime orders, then, up to conjugacy in G, the rotations belong to a subgroup of odd order of G.*

Proof First we show that G must be solvable:

Claim 3 *Let M be an integral homology 3–sphere and $G \subset \mathrm{Diff}(M)$ be a finite subgroup. If G contains a rotation of prime order $p \geq 7$, then G is solvable. In particular G is solvable if it contains at least $n \geq 3$ rotations of distinct odd prime orders.*

Proof In [15, Theorem 2, page 677] a list of the finite nonsolvable groups which can admit actions on integral homology spheres is given.

According to [15, Theorem 2, page 677] a finite group G acting on an integral homology 3–sphere is solvable or isomorphic to a group of the following list: $\mathcal{A}_5$, $\mathcal{A}_5 \times \mathbb{Z}/2$, $\mathcal{A}_5^* \times_{\mathbb{Z}/2} \mathcal{A}_5^*$ or $\mathcal{A}_5^* \times_{\mathbb{Z}/2} C$, where $\mathcal{A}_5$ is the dodecahedral group (alternating group on 5 elements), $\mathcal{A}_5^*$ is the binary dodecahedral group (isomorphic to $SL_2(5)$), C is a solvable group with a unique involution and $\times_{\mathbb{Z}/2}$ denotes a central product, ie the quotient of the two factors in which the two central involutions are identified.

An easy check shows that, if G is not solvable, either it cannot contain a rotation of prime order $p \geq 7$, or we are in the last case and the rotation of prime order $p \geq 7$ is contained in the solvable factor C. However, according to [15, Theorem 2, page 677] the elements of C must act freely, so that they cannot be rotations. One can also see this directly by observing that the normaliser of the element contained in C cannot be of the form described in the following Remark 1, for it contains $\mathcal{A}_5^*$. □

Remark 1 Let G be a finite group of diffeomorphisms acting on a 3–manifold M. It is straightforward to see that one can choose a Riemannian metric on M with respect to which G acts by isometries. Let now $g \in G$ be such that $\mathrm{Fix}(g)$ is a circle. Since the normaliser $\mathcal{N}_G(g)$ of g in G must leave such circle invariant, we deduce that $\mathcal{N}_G(g)$ is a finite subgroup of $\mathbb{Z}/2 \ltimes (\mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z})$, where the element of order 2 acts by sending each element of the direct sum to its inverse.

Now the proof of Lemma 2 is a consequence of the theory of Sylow subgroups in solvable groups. Applying [25, Theorem 5.6, page 104], up to conjugacy, we can assume that all the rotations belong to a Hall subgroup of maximal odd order of G. □

By Lemma 2 we can assume that G itself has odd order. Then Proposition 1 is a consequence of Theorem 3. □

To prove Theorem 3, which is interesting in it own right, we shall need the following Lemmas; a proof of the first can be found in Mecchia and Zimmermann [15, Proposition 4].

Lemma 3 *For an odd prime p, let $G = \mathbb{Z}_p \times \mathbb{Z}_p$ be a finite group of diffeomorphisms of a mod p homology 3–sphere M. There are exactly two subgroups $\mathbb{Z}_p$ of G with nonempty fixed-point set, and each fixed-point set is connected (a simple closed curve).* □

Lemma 4 *Let G be a finite group acting on a mod p homology 3–sphere M. If p is an odd prime, then a Sylow p–subgroup S_p of G is cyclic or a direct product of two cyclic groups.*

Proof If the finite p–group S_p acts freely on the mod p homology sphere M then, by [5, Theorem 8.1, page 148], S_p has no subgroup $\mathbb{Z}_p \times \mathbb{Z}_p$; since the center of a finite p–group is nontrivial, S_p has a unique subgroup of order p, and by [6, Theorem VI.9.7], S_p is cyclic (because p is odd).

Suppose that some nontrivial element h of S_p has nonempty fixed-point set $\mathrm{Fix}(h)$; by general Smith fixed-point theory (see Bredon [5]), $\mathrm{Fix}(h)$ is connected and hence a simple closed curve. We denote by $N := \mathcal{N}_{S_p} H$ the normaliser in S_p of the subgroup

$H = \langle h \rangle$ generated by h. Then N maps the fixed-point set $\mathrm{Fix}(h)$ of H to itself, and it follows easily that N is cyclic or the direct product of two cyclic groups (acting as standard rotations along and about $\mathrm{Fix}(h)$ in a regular neighbourhood of $\mathrm{Fix}(h)$); see Remark 1.

Now Lemma 3 implies that the union of the fixed-point sets of nontrivial elements of N consists of one or two simple closed curves; one of them is the fixed-point set $\mathrm{Fix}(h)$ of H. The normaliser $\widetilde{N}$ of N in S_p maps this union to itself. Since p is odd, $\widetilde{N}$ maps $\mathrm{Fix}(h)$ to itself and hence normalises H, therefore $\widetilde{N} = N$. By [24, Chapter 2, Theorem 1.6] the normaliser of a proper subgroup of a p–group is strictly larger than the subgroup, hence $N = S_p$ and S_p is cyclic or a product of two cyclic groups. □

Proof of Theorem 3 Suppose that G has odd order. If G acts freely then, by [6, VI.9.3], each Sylow p–subgroup of G is cyclic. By a theorem of Burnside (cf [28, 5.4]), G is a metacyclic group. The cohomological period of G divides four; the period of a metacyclic group is determined in [26], and the only metacyclic groups of odd order and of period dividing four are cyclic.

We can therefore assume that some element $g \in G$ of prime order p has nonempty connected fixed-point set $\mathrm{Fix}(g)$. By Lemma 4, a Sylow p–subgroup of G is cyclic or a product of two cyclic groups. It follows as in the proof of Lemma 4 that the normaliser of S_p in G maps $\mathrm{Fix}(g)$ to itself and hence is abelian (because G has odd order). We apply the Burnside transfer theorem; this states that if a Sylow p–subgroup S_p of a group G is contained in the center of its normaliser, then G has a characteristic subgroup U_1 such that $G = U_1 S_p$ and $U_1 \cap S_p = 1$ (see Suzuki [25, Chapter 5,Theorem 2.10]).

If U_1 acts freely, then it is cyclic. Assume that some element in U_1, of prime order q different from p, has nonempty connected fixed-point set. The group S_p acts by conjugation on the set of q–Sylow subgroups of U_1; by a Sylow theorem, the number of elements of this set divides the order of U_1. The number of elements of each orbit of the action of S_p is a power of p. Since p does not divide the order of U_1, some orbit must have one element. Hence S_p normalizes a Sylow q–subgroup S_q of U_1; since some element of S_q has nonempty connected fixed-point set invariant under both S_q and S_p, these two groups commute element-wise and generate a subgroup $S_q \times S_p$; note that this subgroup is cyclic or a product of two cyclic groups. Also, by the Burnside transfer theorem there is a characteristic subgroup U_2 of U_1 such that $U_1 = U_2 S_q$, $U_2 \cap S_q = 1$.

Iterating the construction, we find a decomposition $G = US$, $U \cap S = 1$ such that U is a cyclic (maybe trivial) characteristic subgroup of G acting freely on M, and S is cyclic or a direct product of two cyclic groups (a direct product of Sylow subgroups of G corresponding to different prime numbers).

Suppose that $U \cong \mathbb{Z}_n$ is a nontrivial cyclic group of order n. We will show that S acts trivially on U by conjugation. Since M is a homology 3–sphere, the quotient $\overline{M} := M/U$ has first homology $\mathbb{Z}_n$ and is a homology lens space. Any element s of S normalises U and projects to a diffeomorphism $f = f_s$ of $\overline{M}$, and the induced action f_* of f on the first homology $H_1(\overline{M}) = \mathbb{Z}_n$ coincides with the action, by conjugation, of s on $U = \mathbb{Z}_n$. Suppose that $f_*\colon \mathbb{Z}_n \to \mathbb{Z}_n$ is multiplication by an integer x. It is a consequence of Poincaré duality that linking numbers (in the following denoted by $\circledcirc$) induce a nonsingular bilinear form on $H_1(\overline{M})$, with values in $\mathbb{Q}/\mathbb{Z}$; in particular, denoting by α a generator of $H_1(\overline{M})$, there exists α^* in $H_1(\overline{M})$ such that $\alpha \circledcirc \alpha^* = [1/n] \in \mathbb{Q}/\ Z$ (see eg [23, Satz 14.7.11]). By some properties of linking numbers [23, Satz 14.7.12],

$$[1/n] = \alpha \circledcirc \alpha^* = f_*(\alpha) \circledcirc f_*(\alpha^*) = x\alpha \circledcirc x\alpha^* = x^2(\alpha \circledcirc \alpha^*) = x^2[1/n] = [x^2/n],$$

and hence $(x^2-1)/n \in \mathbb{Z}$, $x^2 \equiv 1 \mod n$. It follows that the automorphism of $\mathbb{Z}_n$ induced by f and s has order one or two; since G has odd order, it has order one and s acts trivially on $U = \mathbb{Z}_n$.

It follows that G is the direct product of U and S and hence is cyclic or a direct product of two cyclic groups (because the orders of U and S are coprime). □

4 Geometric homology spheres

We are now ready to prove Theorem 2 when M has trivial JSJ decomposition. Note that according to the orbifold theorem (see Boileau and Porti [3], Boileau, Maillot and Porti [1] and Cooper, Hodgson and Kerckhoff [7]), an irreducible manifold admitting a rotation has a geometric decomposition. In particular, if its JSJ decomposition is trivial it admits either a hyperbolic or a Seifert fibred structure. We shall consider two cases according to the structure of M.

Proposition 2 *Let M be a hyperbolic integral homology sphere. If M admits three rotations $\{\psi_i\}_{i=1,2,3}$ with pairwise distinct odd prime orders, then* $\mathrm{Isom}^+(M)$ *is solvable and, up to conjugacy, the three rotations generate a cyclic subgroup of* $\mathrm{Isom}^+(M)$.

Proof We shall exploit the fact that, by the orbifold theorem [3], a rotation acting on a hyperbolic manifold M can be assumed, up to conjugacy, to act as an isometry for the unique hyperbolic structure on M. Note, moreover, that $\mathrm{Isom}^+(M)$ is a finite group. Assuming that M admits $n \geq 3$ rotations, Claim 3 shows that the group of isometries of M is solvable. Moreover Proposition 1 implies that, up to conjugacy, the given rotations generate a cyclic group. □

The proof of the Smith conjecture implies that Theorem 2 is true for the 3–sphere S^3 since any rotation can be conjugated to an orthogonal rotation about a given unknotted great circle. For Seifert fibred integral homology spheres, not homeomorphic to $\mathbb{S}^3$, Theorem 2 follows from:

Proposition 3 *Let M be a Seifert fibred integral homology sphere which is not homeomorphic to $\mathbb{S}^3$. Then any rotation of M of order > 2 is conjugated into the circle action $S^1 \subset \mathrm{Diff}^+(M)$ inducing the Seifert fibration.*

Proof A homological computation [21] shows that a Seifert fibred integral homology sphere has singular fibres of coprime orders and base $\mathbb{S}^2$: they are Brieskorn spheres. Since M is not homeomorphic to $\mathbb{S}^3$, there are at least 3 singular fibres and in particular M admits a unique Seifert fibration, up to homeomorphism by [17; 22; 27]. By the orbifold theorem [3], up to conjugacy, the rotations can be chosen in such a way as to preserve the Seifert fibration of M. Since the base of the fibration is a 2–sphere with at least three cone points which cannot be permuted (because they have different orders), the action on the base induced by each rotation is trivial. Indeed, since the order of the rotation is > 2, the action on the base cannot be a reflection in a great circle containing the cone points. A rotation cannot be a product of vertical Dehn twists along incompressible saturated tori (see Johannson [11] or McCullough [14]) because its fixed-point set has empty interior. Hence the rotations belong to the circle action $S^1 \subset \mathrm{Diff}^+(M)$ inducing the Seifert fibration. □

5 Integral homology spheres with nontrivial JSJ decomposition

In this section we deal with the case where the JSJ decomposition of the homology sphere is not empty. We shall use the fact that the rotations preserve the JSJ decomposition and act *geometrically* (see below) on each piece to prove the following proposition:

Proposition 4 *Let M be an irreducible integral homology sphere with a nontrivial JSJ decomposition. If M admits $n \geq 3$ rotations $\{\psi_i\}_{i=1,\dots,n}$ with trivial quotient and pairwise distinct odd prime orders, then, up to conjugacy, they generate a cyclic subgroup of $\mathrm{Diff}^+(M)$.*

Proof Consider the JSJ decomposition for M. Since it is non trivial, M decomposes into geometric pieces which admit either a complete hyperbolic structure with finite volume or a product structure $\mathbb{H}^2 \times \mathbb{R}$. Since M is a homology sphere, the base orbifolds of the Seifert pieces of the decomposition are orientable and planar. In particular, all Seifert pieces admit a unique Seifert fibration (see also Corollary 2). By the orbifold theorem [3], we can assume, after conjugacy, that each rotation is *geometric*, ie it preserves the JSJ decomposition of M, acts isometrically on the hyperbolic pieces and respects the product structure on the Seifert pieces.

Let Γ be the dual graph of the JSJ decomposition which is in fact a tree, for M is a homology sphere. Let G denote the group of diffeomorphisms of M generated by the geometric rotations ψ_i, $i = 1, \dots, n$. Let G_Γ denote the finite group which is the image of the natural representation of G in $Aut(\Gamma)$. Since rotations of finite odd order cannot induce an inversion, a standard result in the theory of group actions on trees implies that G_Γ fixes point-wise a nonempty subtree Γ_f of Γ.

The idea of the proof is now as follows: We shall start by showing that, up to conjugacy, the rotations can be chosen to generate a cyclic group on the submanifold $M_f \subset M$ corresponding to the subtree Γ_f. We shall then consider the maximal subtree Γ_c corresponding to a submanifold $M_c \subset M$ on which the rotations commute up to conjugacy and prove that such subtree is in fact Γ.

We shall need the following result which describes the Seifert fibred pieces of a manifold admitting a geometric rotation of odd prime order with trivial quotient, as well as the action of the rotation on the pieces. The proof is standard and can be found in Boileau and Paoluzzi [2] (see also Kojima [13, Lemma 2]).

Lemma 5 *Let M be an irreducible 3–manifold with a nontrivial JSJ decomposition. Let p be an odd prime integer. Assume that M admits a geometric rotation ψ of order p with trivial quotient. Let V be a Seifert piece of the JSJ decomposition for M. According to its base B, the action of ψ on a Seifert piece V of the JSJ decomposition of M can be described as follows:*

(1) *A disc with 2 cone points corresponding to singular fibres. In this case either ψ freely permutes p copies of V or leaves V invariant and belongs to the circle action $S^1 \subset \mathrm{Diff}(V, \partial V)$ inducing the Seifert fibration.*

(2) *A disc with p cone points corresponding to singular fibres. In this case ψ leaves V invariant and cyclically permutes the singular fibres while fixing set-wise a regular one.*

(3) *A disc with $p+1$ cone points corresponding to singular fibres. In this case ψ leaves V invariant and cyclically permutes p singular fibres while fixing set-wise the remaining one.*

(4) *An annulus with* 1 *cone point corresponding to a singular fibre. In this case either* ψ *freely permutes* p *copies of* V *or leaves* V *invariant and belongs to the circle action* $S^1 \subset \mathrm{Diff}(V, \partial V)$ *inducing the Seifert fibration.*

(5) *An annulus with* p *cone points corresponding to singular fibres. In this case* ψ *leaves* V *invariant and cyclically permutes the* p *singular fibres.*

(6) *A disc with* $p-1$ *holes and* 1 *cone point corresponding to a singular fibre. In this case* ψ *leaves* V *invariant and cyclically permutes all its boundary components while fixing set-wise the singular fibre and a regular one.*

(7) *A disc with* p *holes and* 1 *cone point corresponding to a singular fibre. In this case* ψ *leaves* V *invariant and cyclically permutes* p *boundary components while fixing set-wise the singular fibre and the remaining boundary component.*

(8) *A disc with* k *holes,* $k \geq 2$*. In this case either* ψ *freely permutes* p *copies of* V *or leaves* V *invariant. In this latter case either* ψ *belongs to the circle action* $S^1 \subset \mathrm{Diff}(V, \partial V)$ *inducing the Seifert fibration, or* $k = p - 1$ *and* ψ *permutes all the boundary components while fixing set-wise two regular fibres, or* $k = p$ *and* ψ *permutes* p *boundary components, while fixing set-wise the remaining one and a regular fibre.* □

In the case where M is a homology sphere, the Seifert fibration of V embeds in a Seifert homology sphere M' in such a way that a fibration of M' induces that of V. Hence the Seifert fibred piece V is obtained from some Brieskorn sphere by removing the tubular neighborhoods of a finite numbers of fibres. In particular, the singular fibres of V have coprime orders and cannot be exchanged by a rotation. So we have the following corollary:

Corollary 2 *Let* M *be an irreducible integral homology sphere with a nontrivial JSJ decomposition which admits a geometric rotation* ψ *of odd prime order* p *and with trivial quotient. Under this hypothesis only cases 1, 4, 6, 7 and 8 of Lemma 5 can occur.* □

The following consequence will be useful:

Corollary 3 *Let* M *be an irreducible integral homology sphere. Assume that* M *admits two geometric rotations* ϕ *and* ψ *with trivial quotients and distinct odd prime orders* p *and* q*. If* ϕ *and* ψ *leave invariant a Seifert piece* V *of the JSJ decomposition for* M*, then their restrictions* ϕ_V *and* ψ_V *to* V *generate a finite cyclic group of isometries of order* pq*.*

Proof If the JSJ decomposition is trivial, Proposition 3 applies and the result follows. Else, by Corollary 2, at least one of the rotation, say ϕ, induces the identity on the base of V. Hence its restriction ϕ_V belongs to the circle action $S^1 \subset \mathrm{Diff}(V, \partial V)$, inducing the Seifert fibration of V, and commutes with ψ_V. □

Consider now Γ_f. Since the rotations have odd orders, either Γ_f contains an edge, or it consists of a single vertex. We shall analyse these two cases.

Claim 4 *Assume that Γ_f contains an edge and let T denote the corresponding torus. Then the geometric rotations commute on the geometric pieces of M adjacent to T.*

Proof First of all notice that the geometric pieces adjacent to T are left invariant by the rotations. Let V denote one of the two adjacent geometric pieces. Two possible cases can arise according to the geometry of V.

V **is hyperbolic.** In this case all rotations act as isometries and leave a cusp invariant. Since their order is odd, the rotations must act as translations along horospheres, and thus commute. Note that, even in the case of rotations of order 3, their fixed-point set cannot meet a JSJ torus, for each such torus is separating and the fixed-point set is connected.

V **is Seifert fibred.** This case is covered by Corollary 3. □

Claim 5 *Assume that Γ_f consists of just one vertex and let V denote the corresponding geometric piece. Up to conjugacy by geometric diffeomorphisms of M, the geometric rotations commute on V.*

Proof Again we need to consider two cases according to the geometry of V.

V **is hyperbolic.** Each component W of $M \setminus \mathrm{int}(V)$ is an integral homology solid torus. On its boundary torus $T_W = \partial W$ there is a unique simple closed curve, up to isotopy, μ_W that bounds a properly embedded surface F_W in W. The surfaces F_W can be chosen to be incompressible and ∂–incompressible in W.

By pinching the surface F_W onto a disc D^2, for each component W of $M \setminus \mathrm{int}(V)$, we can define a degree-one map $p\colon M \to M'$, where M' is the integral homology sphere obtained by Dehn filling each torus T_W along the curve μ_W.

Let G be the group of isometries of V generated by the rotations. Each rotation acts equivariantly on the set of isotopy classes of curves $\mu_W \subset \partial W$. Therefore the action of the finite group G on V extends to M'. Each rotation ψ_i extends to a rotation ψ_i' of M' because either the fixed-point set of the rotation is contained in V or there exists a unique component W which contains its axis. In the latter case, by [8, Corollary 2.2], the rotation ψ preserves a representative of μ_W and hence ψ_i' has nonempty fixed-point set in the solid torus glued to T_W to obtain M', giving rise again to a rotation.

We can now apply Proposition 1 to conclude that the rotations ψ_i' commute, up to conjugacy in G. Hence the restrictions of the rotations ψ_i commute on V, up to conjugacy by geometric diffeomorphisms of M.

V **is Seifert fibred.** Once more this case is covered by Corollary 3. □

To conclude that the rotations can be chosen to commute on the submanifold of M corresponding to Γ_f we need the following gluing lemma:

Lemma 6 *If the rotations preserve a JSJ torus T then they commute on the union of the two geometric pieces adjacent to T.*

Proof The lemma follows from two claims.

Claim 6 *Let ψ be a periodic diffeomorphism of the product $T^2 \times [0,1]$ which is isotopic to the identity and whose restriction to each boundary torus $T \times \{i\}$, $i = 0, 1$, is a translation with rational slopes α_0 and α_1 in $H_1(T^2;\mathbb{Z})$. Then $\alpha_0 = \alpha_1$.*

Proof By Meeks and Scott [16, Theorem 8.1] (see also Bonahon and Seibenmann [4, Proposition 12]), there is a Euclidean product structure on $T^2 \times [0,1]$ preserved by ψ such that ψ acts by translation on each fiber $T \times \{t\}$ with rational slope α_t. By continuity the rational slopes α_t are constant. □

Let V and W be the two geometric pieces adjacent to T. By Claim 4 the rotations commute on V and W, hence their restrictions on V and W generate two cyclic groups of the same finite order. Let g_V and g_W be generators of these two cyclic groups. They both act by translation on T. The fact that these two actions can be glued follows from the following claim:

Claim 7 *The translations $g_V|_T$ and $g_W|_T$ have the same slope in $H_1(T^2;\mathbb{Z})$.*

Proof Let p_i the order of ψ_i and $q_i = \Pi_{j\neq i} p_j$. Then the slopes α_V and α_W of $g_V|_T$ and $g_W|_T$ verify: $q_i\alpha_V = q_i\alpha_W$ for $i = 1, ..., n$, by applying Claim 6 to each ψ_i. Since the GCD of the q_i is 1, it follows that $\alpha_V = \alpha_W$. □

This finishes the proof of Lemma 6. □

Together with Claim 5, Lemma 6 implies that the rotations commute on the submanifold of M corresponding to Γ_f, up to conjugacy by geometric diffeomorphisms of M.

Let Γ_c be the largest subtree of Γ containing Γ_f, such that, up to conjugacy by geometric diffeomorphisms of M, the rotations commute on the corresponding invariant submanifold M_c of M. We need to show that $\Gamma_c = \Gamma$. If this is not the case, we can choose an edge contained in Γ corresponding to a boundary torus T of M_c. Denote by U the submanifold of M adjacent to T but not contained in M_c and by $V \subset U$ the geometric piece adjacent to T.

Let G be the subgroup of geometric diffeomorphisms of M generated by the n rotations ψ_i. The restriction of G to M_c is cyclic. Since $\Gamma_f \subset \Gamma_c$, the G–orbit of T cannot be reduced to only one element.

If no rotation leaves T invariant, the G–orbit of T contains as many elements as the product of the orders of the rotations, for they commute on M_c. In particular, only the identity (which extends to U) stabilises a torus in the orbit of T. Note now that all components of ∂M_c in the G–orbit of T bound a manifold homeomorphic to U.

Since the rotation ψ_i acts freely on the G–orbit of U, U is a knot exterior in the quotient $M/\psi_i = \mathbb{S}^3$. Hence there is a well defined meridian-longitude system on $T = \partial U$ and also on each torus of the G–orbit of T. This set of meridian-longitude systems is cyclically permuted by each ψ_i and thus equivariant under the action of G.

Let M_c/G be the quotient of M_c by the induced cyclic action of G on M_c. Then there is a unique boundary component T' which is the image of the G–orbit of T. We can glue a copy of U to M_c/G along T' by identifying the image of the meridian-longitude system on ∂U with the projection on T' of the equivariant meridian-longitude system on the G–orbit of T. Denote by N the resulting manifold. For all $i = 1, \dots, n$, consider the cyclic (possibly branched) cover of N of order $q_i = \prod_{j \neq i} p_j$ which is induced by the cover $\pi_i \colon M_c/\psi_i \longrightarrow M_c/G$. Observe that this makes sense because $T' \subset N$ is such that $\pi_1(T') \subset \pi_{i*}(\pi_1(M_c/\psi_i))$. Call $\widetilde{N}_i$ the total space of such covering. By construction it follows that $\widetilde{N}_i$ is the quotient $(M_c \cup G \cdot U)/\psi_i$. This clearly implies that the ψ_i's commute on $M_c \cup G \cdot U$ contradicting the maximality of Γ_c.

We can thus assume that some rotations fix T and some do not. Since all rotations commute on M_c, we see that the orbit of T consists of as many elements as the products of the orders of the rotations which do not fix T and each element of the orbit is fixed by the rotations which leave T invariant. The rotations which fix T commute on the orbit of V according to Claim 4 and Lemma 6, and form a cyclic group generated by, say, γ. Reasoning as in the previous situation we see that the rotations which act freely on the orbit of T also commute on the orbit of U and thus on the orbit of V, and form again a cyclic group generated by, say, η. To reach a contradiction to the maximality of M_c, we only need to show that γ, after perhaps some conjugation, commutes with η on the G–orbit of V (ie γ and $\eta\gamma\eta^{-1}$ coincide on $G \cdot V$). Note now that η acts freely and transitively on the G–orbit of V so that there is a natural and well-defined way to identify each element of the orbit $G \cdot V$ to V itself.

Claim 8 *Assume that V is Seifert fibred and that the restriction of γ induces a nontrivial action on the base of V. Then γ induces a nontrivial action on the base of each component of the G–orbit of V. Moreover, up to conjugacy on $G \cdot V \setminus V$ by diffeomorphisms which extend to M, we can assume that the restrictions of γ to these components induce the same permutation of their boundary components and the same action on their bases.*

Proof By hypothesis γ and $\eta\gamma\eta^{-1}$ coincide on ∂M_c. The action of γ on the base of V is nontrivial if and only if its restriction to the boundary circle corresponding to the torus T is nontrivial. Therefore the action of γ is nontrivial on the base of each component of $G \cdot V$.

By Corollary 2 the base of V consists of a disc with p holes, where p is the order of one of the rotations which generate γ, and at most one singular fibre. Moreover, the restriction of γ on the elements of $G \cdot V$ cyclically permutes their boundary components which are not adjacent to M_c. Up to performing Dehn twists, along vertical tori, which permute the boundary components, we can assume that the restriction of γ induces the same cyclic permutations on the boundary components of each element of $G \cdot V$. We only need to check that Dehn twists permuting two boundary components extend to the whole manifold M. This follows from the fact that the manifolds adjacent to these components are all homeomorphic and that Dehn twist act trivially on the homology of the boundary.

Since the actions of the restrictions of γ on the bases of the elements of $G \cdot V$ are combinatorially equivalent, after perhaps a further conjugacy by an isotopy, the different restrictions can be chosen to coincide on the bases. $\square$

We can now deduce that the restrictions of γ and $\eta\gamma\eta^{-1}$ to the orbit of V commute, up to conjugacy of γ. This follows from Claim 4 in the hyperbolic case, and from Corollary 3 and Claim 8 for the Seifert fibred one. Since γ and $\eta\gamma\eta^{-1}$ coincide on the G–orbit of T, we can conclude that they coincide on the G–orbit of V. This finishes the proof of Proposition 4 and of Theorem 2. $\square$

6 Branched covers of $\mathbb{S}^3$

The aim of this section is to prove Corollary 1. We start by describing how one can build different knots with the same cyclic branched cover.

We recall that a rotation ψ with trivial quotient on M induces a cyclic cover $M \longrightarrow \mathbb{S}^3 = |M/\psi|$ of $\mathbb{S}^3$, branched along a knot K which is the image of $\mathrm{Fix}(\psi)$ in the quotient $|M/\psi|$. Let $L = L_1 \cup L_2$ be a link with two trivial components. One can construct two knots in the following way: take the cyclic p_i–fold cover of $\mathbb{S}^3$ branched along L_i, where $p_1, p_2 \geq 2$ are two coprime integers. The resulting manifold is $\mathbb{S}^3$ and the lift of L_j, $j \neq i$, is a knot K_j provided that p_i and the linking number of L_1 and L_2 are coprime. The p_1–fold cyclic cover of $\mathbb{S}^3$ branched along K_1 coincides with the p_2–fold cyclic cover of $\mathbb{S}^3$ branched along K_2 and is the $\mathbb{Z}_{p_1} \oplus \mathbb{Z}_{p_2}$ cover of $\mathbb{S}^3$ branched along $L = L_1 \cup L_2$.

Conversely, assume now that $M \neq \mathbb{S}^3$ admits two commuting rotations ψ_i, $i = 1, 2$, of coprime orders p_i, with trivial quotients. Denote by K_i the knot $\mathrm{Fix}(\psi_i)/\psi_i$. Because $M \neq \mathbb{S}^3$, the knots K_i are not trivial. Observe that, since the two rotations commute, ψ_j, $j \neq i$, induces a rotational symmetry φ_j of K_i of order p_j, ie a rotation of $\mathbb{S}^3$ such that $\varphi_j(K_i) = K_i$. The axis of φ_j is the image of the axis of ψ_j and is the trivial knot because of Smith's conjecture, in particular K_i and $\mathrm{Fix}(\varphi_j)$ are distinct. Moreover, since the rotations ψ_i, $i = 1, 2$, commute, K_i and $\mathrm{Fix}(\varphi_j)$ are in fact disjoint. By taking the quotient $(\mathbb{S}^3, K_i \cup \mathrm{Fix}(\varphi_j))/\varphi_j$ one gets a link with two components L_i and L_j which are the images of K_i and $\mathrm{Fix}(\varphi_j)$ respectively, where L_j is trivial. It is easy to convince oneself that M is the $\mathbb{Z}_{p_i} \oplus \mathbb{Z}_{p_j}$ cover of $\mathbb{S}^3$ branched along the components of L. By exchanging the roles of i and j it is now clear that both components of L are trivial, so that φ_j is a rotational symmetry with trivial quotient knot. This implies that K_i is a prime knot and that M is irreducible [2, Lemma 3; 19, Theorem 4].

We remark that the above discussion proves also the following claim which was originally stated in Section 2.

Claim 2 *Let $M \neq \mathbb{S}^3$ be an irreducible manifold admitting two commuting rotations ψ and φ with trivial quotients and distinct orders. Let K be the knot $\mathrm{Fix}(\psi)/\psi \subset \mathbb{S}^3$ and let ϕ the rotation of the pair $(\mathbb{S}^3, K)$ induced by φ. The rotation ϕ has trivial quotient knot.*

If we now start with three commuting rotations ψ_i, $i = 1, 2, 3$ with trivial quotient, and pairwise coprime orders p_i, we get three knots admitting each two rotational symmetries with trivial quotient knot. Observe that the above discussion implies that the fixed-point sets of the rotations ψ_i, $i = 1, 2, 3$, are pairwise disjoint, thus M is a cover of $\mathbb{S}^3$ branched along a link L with three components. According to the proof of Lemma 1, the axes of the two rotational symmetries of each knot form a Hopf link so that each two-component sublink of L is again a Hopf link.

We shall now describe the converse of the above description, ie how one can recover three knots starting with an appropriate three component link. We shall call this method a *standard abelian construction*. Let p_1, p_2 and p_3 be three different integers which are pairwise coprime. Let $L = \bar{K}_1 \cup \bar{K}_2 \cup \bar{K}_3 \subset \mathbb{S}^3$ be a link of three trivial components such that any two components of L form a Hopf link. The p_3–fold cyclic branched cover of $\bar{K}_3$ is the 3–sphere, and the preimages K_1' of $\bar{K}_1$ and K_2' of $\bar{K}_2$ form a link of two trivial components of linking number p_3. The preimage of K_1' in the p_2–fold cyclic branched covering of K_2' (which is again the 3–sphere) is a knot K_1 in S^3. Finally, the p_1–fold cyclic branched covering of K_1 is a 3–manifold M which, by construction, is also the regular branched $(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3})$–cover of the link L.

By cyclically permuting the roles of the components $\bar{K}_1$, $\bar{K}_2$ and $\bar{K}_3$ of L, we get three knots K_1, K_2 and K_3 in S^3 such that M is the p_1–fold cyclic branched cover of K_1, the p_2–fold cyclic branched cover of K_2 and the p_3–fold cyclic branched cover of K_3. Then we say that the knots K_i, $i = 1, 2, 3$ are *related by a standard abelian construction*.

Proof of Corollary 1 Part (i) It was shown in [2, Theorem 1] that for any fixed odd prime p, an irreducible manifold can be the p–fold cyclic branched cover of at most two inequivalent knots. Theorem 1 states that an integral homology sphere not homeomorphic to $\mathbb{S}^3$ can be the cyclic cover of the 3–sphere branched along some knot for at most three odd primes. If an irreducible integral homology sphere M is the branched cover of $\mathbb{S}^3$ for at most two odd primes orders, then the assertion is clearly verified. We can thus assume that M admits three rotations ψ_i with trivial quotient and pairwise distinct odd prime orders p_i. We want to prove that for each prime p_i, M is the p_i–fold cyclic branched cover of precisely one knot. Assume now by contradiction that for a prime, say p_1, M is the p_1–fold cyclic branched cover of two non equivalent knots with non conjugate cyclic groups of covering transformations generated by ψ and ψ'. We can now apply Theorem 2 twice to the rotations ψ, ψ_2 and ψ_3 and to ψ', ψ_2 and ψ_3, to conclude that both ψ and ψ' commute up to conjugacy with ψ_2. The desired contradiction follows now from the following assertion, keeping in mind that ψ and ψ' cannot be conjugate into the same cyclic group:

Claim 9 *Let $n \geq 3$ be a fixed odd integer. Let ρ be a rotation with trivial quotient of an irreducible manifold M. All the rotations of M of order n which commute with ρ are conjugate in* $\mathrm{Diff}(M)$ *into the same cyclic group of order n.*

Proof Each rotation of order n induces a rotational symmetry of order n of the prime knot $K = \mathrm{Fix}(\rho)/\rho$. According to [19, Theorem 3], a prime knot admits a unique symmetry of a given odd order up to conjugacy, and the conclusion follows. $\square$

Part (ii) Suppose that M is hyperbolic. If the isometry group of M is solvable, then by the generalisation of the Sylow theorems for solvable groups we can assume that all rotations of odd order belong to a maximal subgroup U of odd order which, by Theorem 3, is cyclic or a product of two cyclic groups. Suppose that, for a prime p, U contains a subgroup $\mathbb{Z}_p$ generated by a rotation with trivial quotient. Then, for any different prime q, U does not contain a subgroup $\mathbb{Z}_q \times \mathbb{Z}_q$ (otherwise its projection to $M/\mathbb{Z}_p$ would contradict the Smith conjecture), so M is a q–fold cyclic branched cover of at most one knot in $\mathbb{S}^3$. Also, by Theorem 1, there are at most three rotations with trivial quotient and pairwise different odd prime orders.

On the other hand, suppose that the isometry group G of M is nonsolvable. The list of possible groups G is given in the proof of Claim 3, and the only possible odd orders of rotations are 3 and 5. The solvable groups C act freely and hence have cyclic Sylow 3– and 5–subgroups. Suppose that the Sylow 5–subgroup of G has a subgroup $U \cong \mathbb{Z}_5 \times \mathbb{Z}_5$. By Lemma 3, exactly two of the six subgroups $\mathbb{Z}_5$ of U have nonempty connected fixed-point set, and it follows easily that these two subgroups have to be conjugate in G (noting that $\mathcal{A}_5^*$ has two conjugacy classes of elements of order 5). Hence M cannot be a 5–fold cyclic branched cover of two different knots in $\mathbb{S}^3$, and similarly for 3–fold covers.

If $M \neq \mathbb{S}^3$ is Seifert fibred, then it is a cyclic branched cover of some torus knot and has precisely three exceptional fibres (one can reason as in Lemma 5 and Corollary 2). More precisely, the preimage of each torus knot corresponds to a singular fibre whose order of singularity coincides with the order of the cyclic branched cover (see also Proposition 3). This finishes the proof of (ii).

Part (iii) The fact that the three knots are related by a standard abelian construction is a straightforward consequence of the above discussion. Since the odd prime branching indices p_i, $i = 1, 2, 3$ are distinct, volume considerations show that the knots K_i must be inequivalent; see Salgueiro [20]. This finishes the proof of Corollary 1. $\square$

Remark 2 One can improve part (i) of Corollary 1 by showing that any irreducible integral homology sphere M not homeomorphic to $\mathbb{S}^3$ is the cyclic branched cover of odd prime order of at most three prime knots; see [2, Section 5].

Here is a brief idea of how one can handle the general case. According to part (ii) of Corollary 1, we can assume that M has a non trivial JSJ decomposition. According to the proof of part (i), we can assume that M is the cyclic branched cover of $\mathbb{S}^3$ for precisely two distinct odd primes, say p and q. We can moreover assume that, for each prime, M is the branched covering of two distinct knots with covering transformations ψ, ψ' of order p and φ, φ' of order q. If each rotation of order p commutes with each rotation of order q up to conjugacy, then we reach a contradiction as in the proof of part (i). Else, consider the subgroup $G = \langle \psi, \psi', \varphi, \varphi' \rangle$ of diffeomorphisms of M. According to the proof of Proposition 4, each rotation of order p commutes with each rotation of order q up to conjugacy, unless the induced action of G on the dual tree of the JSJ decomposition for M fixes precisely one vertex corresponding to a hyperbolic piece V of the decomposition and $\{p, q\} = \{3, 5\}$. In this case, one deduces as in the proof of part (ii) that the restrictions of ψ and ψ' (respectively φ and φ') coincide up to conjugacy on V. Using the same techniques seen in the last part of Section 5 we see that ψ and ψ' (respectively φ and φ') coincide up to conjugacy on M and the conclusion follows.

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