

## A condition that prevents groups from acting nontrivially on trees

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We describe a simple criterion for showing that a group has Serre’s property FA. By exhibiting a certain pattern of finite subgroups, we show that this criterion is satisfied by  $\text{Aut}(F_n)$  and  $\text{SL}(n, \mathbb{Z})$  when  $n \geq 3$ .

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*Dedicated to the memory of Heiner Zieschang*

An  $\mathbb{R}$ -tree is a geodesic metric space in which there is a unique arc connecting each pair of points. A group  $\Gamma$  is said to have *property FR* if for every action of  $\Gamma$  by isometries on an  $\mathbb{R}$ -tree, the fixed point set  $\text{Fix}(\Gamma)$  is nonempty. Serre’s property FA is similar except that one considers only actions on simplicial trees. A group has FA if and only if it cannot be expressed as a nontrivial amalgamated free product or HNN extension.

**Lemma 1** *Let  $\Gamma$  be a group that is generated by the union of the subsets  $A_1, \dots, A_N$ . If  $H_{i,j} = \langle A_i \cup A_j \rangle$  has property FR for all  $i, j \in \{1, \dots, N\}$ , then  $\Gamma$  also has property FR.*

**Proof** Let  $C_1, \dots, C_N$  be connected subsets of an  $\mathbb{R}$ -tree. It is not difficult to see that if  $C_i \cap C_j$  is nonempty for  $i, j = 1, \dots, N$ , then  $C_1 \cap \dots \cap C_N$  is nonempty (cf Serre [7, p 65]). Setting  $C_i := \text{Fix}(A_i)$  proves the lemma, since  $C_i \cap C_j = \text{Fix}(H_{i,j})$  is assumed to be nonempty and  $C_1 \cap \dots \cap C_N = \text{Fix}(\Gamma)$ .  $\square$

Every finite group  $G$  has FR because the circumcentre of any  $G$ -orbit in an  $\mathbb{R}$ -tree will be a fixed point.

**Corollary 2** (The Triangle Criterion) *If  $\Gamma$  is generated by  $A_1 \cup A_2 \cup A_3$  and  $H_{i,j} = \langle A_i \cup A_j \rangle$  is finite for  $i, j = 1, 2, 3$ , then  $\Gamma$  has property FR.*

Let  $\text{Aut}(F_n)$  denote the automorphism group of the free group of rank  $n$ .

**Theorem 3** *If  $n \geq 3$  then  $\text{Aut}(F_n)$  and  $\text{SL}(n, \mathbb{Z})$  satisfy the Triangle Criterion and hence have property  $\text{FR}$ .*

J-P Serre [7] was the first to prove that  $\text{SL}(n, \mathbb{Z})$  has FA if  $n \geq 3$ , and his argument shows that these groups actually have  $\text{FR}$ . Our argument is very similar to his except that he exploited the pattern of nilpotent subgroups rather than finite ones. In the light of a theorem of J Tits [8], Serre’s argument shows that all subgroups of finite index in  $\text{SL}(n, \mathbb{Z})$  have FA. In contrast, there is a subgroup of finite index  $\Gamma \subset \text{Aut}(F_3)$  that does not have property FA (see McCool [6]), while it is unknown if  $\text{Aut}(F_n)$  has such subgroups when  $n \geq 4$ . O Bogopolski [1] was the first to prove that  $\text{Aut}(F_n)$  has FA. M Culler and K Vogtmann [5] gave a short proof based on their idea of “minipotent” elements.

The obvious appeal of Theorem 3 lies in the final phrase, but the stronger fact that these groups satisfy the Triangle Criterion is useful in my work on fixed point theorems for actions of automorphism groups of free groups on higher-dimensional  $\text{CAT}(0)$  spaces [2]. One can extend the theorem in various ways (cf Section 2) but I shall not present the details here as to do so would obscure the simple and transparent proof that  $\text{Aut}(F_n)$  has property FA, which is the main point of this note. I hope that it is a proof that Zieschang would have enjoyed.

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## 1 Generating $\text{Aut}(F_n)$ and $\text{SL}(n, \mathbb{Z})$ by finite subgroups

We assume that  $n \geq 3$  and fix a basis  $\mathcal{B} = \{a_1, \dots, a_n\}$  of  $F_n$ . For  $i = 1, \dots, n$ , let  $\varepsilon_i$  be the automorphism of  $F_n$  that sends  $a_i$  to  $a_i^{-1}$  and fixes the other basis elements. J Nielsen proved that  $\text{Aut}(F_n)$  is generated by the right Nielsen transformations  $\rho_{ij}: [a_i \mapsto a_i a_j, a_k \mapsto a_k \text{ if } k \neq i]$  and the involutions  $\varepsilon_i$ .

Let  $\Sigma_n \subset \text{Aut}(F_n)$  be the group generated by permutations<sup>1</sup> of  $\mathcal{B}$ . Conjugation by a permutation  $\sigma$  sends  $\rho_{ij}$  to  $\rho_{\sigma(i)\sigma(j)}$  and  $\varepsilon_i$  to  $\varepsilon_{\sigma(i)}$ . Therefore  $\text{Aut}(F_n)$  is generated by  $\rho_{12}$ ,  $\Sigma_n$  and  $\varepsilon_n$ . In particular  $\text{Aut}(F_n)$  is generated by  $\rho_{12} \circ \varepsilon_2$  and the subgroup  $W_n \cong (\mathbb{Z}_2)^n \rtimes \Sigma_n$  generated by  $\Sigma_n$  and the  $\varepsilon_i$ . (The action of  $\text{Aut}(F_n)$  on the abelianisation of  $F_n$  gives a epimorphism  $\text{Aut}(F_n) \rightarrow \text{GL}(n, \mathbb{Z})$ , and the image of  $W_n$  under this map is the group of monomial matrices.)

<sup>1</sup>We shall write  $(a_i a_j)$  to denote the transposition of  $a_i$  and  $a_j$ .

We write  $\Sigma_{n-2} \subset \Sigma_n$  and  $W_{n-2} \subset W_n$  for the subgroups corresponding to the sub-basis  $\{a_3, \dots, a_n\}$ . Let  $\theta := \rho_{12} \circ \varepsilon_2$ , let  $\tau := (a_2 a_3) \circ \varepsilon_1$  and  $\eta := (a_1 a_2) \circ \varepsilon_1 \circ \varepsilon_2$ , and note that each is an involution. Define

$$A_1 = \{\varepsilon_n, \eta\} \cup \Sigma_{n-2}, \quad A_2 = \{\theta\}, \quad A_3 = \{\tau\}.$$

**Lemma 1.1** *Aut( $F_n$ ) is generated by  $A_1 \cup A_2 \cup A_3$ .*

**Proof** Conjugating  $(a_n a_3) \in \Sigma_{n-2}$  by  $\tau$  we get  $(a_n a_2)$ , which conjugates  $\varepsilon_n$  to  $\varepsilon_2$  and  $(a_n a_3)$  to  $(a_2 a_3)$ . Thus  $\varepsilon_1 = (a_2 a_3) \circ \tau$  and  $(a_1 a_2) = \eta \circ \varepsilon_1 \circ \varepsilon_2$  are in the subgroup generated by the  $A_i$ ; hence  $\Sigma_n$  and  $W_n$  are too. We already noted that  $\text{Aut}(F_n)$  is generated by  $W_n$  and  $\theta$ .  $\square$

**Lemma 1.2** *The groups  $H_{ij} = \langle A_i \cup A_j \rangle$  are finite.*

**Proof**  $\Sigma_{n-2}$  and  $\varepsilon_n$  commute with the involutions  $\theta$  and  $\eta$ , and  $\theta \circ \eta$  has order 3, so  $H_{12} \cong W_{n-2} \times D_6$ . As  $(\theta \circ \tau)^4 = 1$ , we have  $H_{23} \cong D_8$ . And  $H_{13} \subset W_n$ .  $\square$

These lemmas prove that  $\text{Aut}(F_n)$  satisfies the Triangle Criterion if  $n \geq 3$ , and by taking the images of the  $A_i$  under the natural map  $\text{Aut}(F_n) \rightarrow \text{GL}(n, \mathbb{Z})$  we see that  $\text{GL}(n, \mathbb{Z})$  does too. When  $n$  is odd, we obtain the corresponding result for  $\text{SL}(n, \mathbb{Z})$  by replacing the image  $\bar{\gamma}$  of each  $\gamma \in A_i$  by  $\det(\bar{\gamma}) \cdot \bar{\gamma}$ ; let  $A_i^+(n)$  denote the image of  $A_i$  modified in this manner.

When  $n \geq 4$  is even we need to adjust the  $A_i$  a little more. Let  $\alpha = \varepsilon_n \circ (a_n a_{n-1})$  and note that  $\text{SL}(n, \mathbb{Z})$  is generated by the image  $\bar{\alpha}$  of  $\alpha$  and the subgroup  $\text{SL}(n-1, \mathbb{Z}) \subset \text{SL}(n, \mathbb{Z})$  corresponding to the sub-basis  $\{a_1, \dots, a_{n-1}\}$ . If  $n \geq 4$  then the groups  $H_{12}$  and  $H_{13}$  remain finite if we add  $\alpha$  to  $A_1$ . Thus the sets  $A_1^+(n-1) \cup \{\alpha\}$ ,  $A_2^+(n-1)$ ,  $A_3^+(n-1)$  demonstrate that  $\text{SL}(n, \mathbb{Z})$  satisfies the Triangle Criterion.

**Remark** If  $n \geq 4$  then by modifying the sets  $A_i$  slightly one can also show that  $\text{SAut}(F_n)$ , the inverse image in  $\text{Aut}(F_n)$  of  $\text{SL}(n, \mathbb{Z})$ , satisfies the Triangle Criterion.

## 1.1 The geometry of the $H_{ij}$

It would be unfair of me to leave the reader to guess the origin of the finite subgroups used in the above proof, so let me explain the geometry behind the construction.

Any finite subgroup of  $\text{Aut}(F_n)$  can be realised as a group of basepoint-preserving isometries of a graph of Euler characteristic  $1 - n$ . [Figure 1<sup>2</sup>](#) below gives such

<sup>2</sup>I am grateful to Karen Vogtmann for producing this figure.

realisations  $Y_{ij}$  for the groups  $H_{ij}$ . An important point is that if  $i \notin \{j, k\}$  then  $A_i$  cannot be realised as a group of symmetries of  $Y_{jk}$ . I wanted to obtain the generating set  $W_n \cup \{\theta\}$  that proved useful in my work with Karen Vogtmann [4]. Thus, starting with the rose and the graph  $Y_{12}$  for  $\theta$ , I looked for a third graph where  $\theta$  could be realised together with a symmetry intertwining  $\{a_1, a_2\}$  and  $\{a_3, \dots, a_n\}$ .

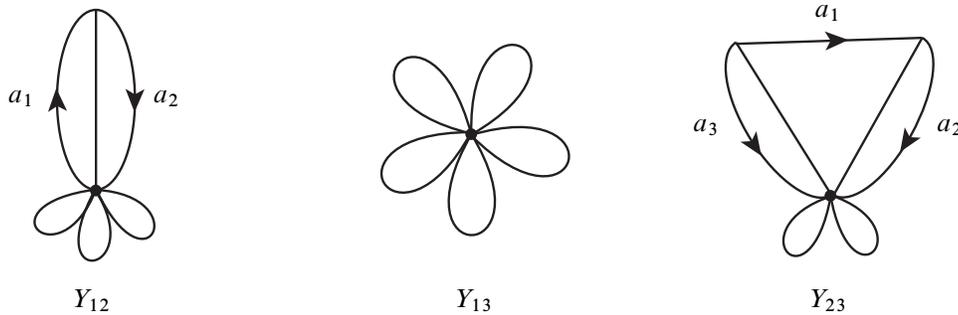


Figure 1: The graphs  $Y_{ij}$  exhibiting the finiteness of  $H_{ij}$

## 2 Variations on the theme

I have concentrated on configurations of finite subgroups in this note but [Lemma 1](#) can also be applied to situations where the subgroups  $\langle A_i \rangle$  are infinite. For example, if  $\gamma \in \Gamma$  lies in the commutator subgroup of its centralizer, then  $\text{Fix}(\gamma)$  will be nonempty whenever  $\Gamma$  acts by isometries on an  $\mathbb{R}$ -tree. (This is a special instance of a general fact about semisimple actions on CAT(0) spaces [3].) By exploiting such facts in conjunction with [Lemma 1](#) one can prove, for example, that the mapping class group of a surface of genus at least 3 has property  $\text{FR}$ , a result first proved in [5].

One can also strengthen [Lemma 1](#) using an argument due to J-P Serre [7, p 64]: it suffices to require that the  $A_i$  have  $\text{FR}$  and, in any action of  $\Gamma$  on an  $\mathbb{R}$ -tree, that  $a_i a_j$  have a fixed point, for every  $a_i \in A_i$  and  $a_j \in A_j$ . To see this, one reduces to the case  $n = 2$  and argues that if the fixed point sets of  $A_1$  and  $A_2$  did not intersect then the point of  $\text{Fix}(A_1)$  closest to  $\text{Fix}(A_2)$  would be fixed by all  $a_1 a_2$  with  $a_1 \in A_1$  and  $a_2 \in A_2$ , which is a contradiction.

A quite different strengthening begins with the observation that the behaviour of convex sets described in the proof of [Lemma 1](#) is a manifestation of the fact that trees are 1-dimensional objects. A suitable version of Helly's Theorem provides constraints on

the way in which convex sets can intersect in higher-dimensional  $\text{CAT}(0)$  spaces, and by applying these constraints to the fixed point sets of finite subgroups one can prove far-reaching generalisations of [Theorem 3](#); this is the theme of [\[2\]](#).

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