Minimizing the number of Nielsen preimage classes

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We find conditions on topological spaces X, Y and nonempty subset B of Y which guarantee that for each continuous map $f\colon X \to Y$ there exists a map $g \sim f$ such that Nielsen preimage classes of $g^{-1}(B)$ are all topologically essential.

54H99, 55M99; 55S35

I feel very honoured to have the possibility to contribute a paper to this volume dedicated to the memory of an outstanding mathematician and a pleasant good-humoured person: Heiner Zieschang. In 2002–2003 in M V Lomonosov Moscow State University Heiner gave a series of lectures on fixed points and coincidence theory, which I was lucky to attend. In the same period I learned the German language at his seminars. During a nice voyage in summer 2003 from Moscow to Saint Petersburg, in which I was invited to take part, I made the acquaintance with his wife Ute and daughter Kim; two years later I met his other daughter Tanja. Heiner guided my study of coincidences, intersections and preimages during my visit in November–December 2004 in Ruhr-Universität Bochum. It was planned, to continue the project in 2005. But that hope was doomed to disappointment. . . .

1 Introduction

Let $f\colon X \to Y$ be a continuous map of topological spaces and B be a nonempty subsets of Y. The so-called preimage problem considers the preimage set $f^{-1}(B)$, that is, $\{x \in X \mid f(x) \in B\}$. The minimization problem in its "classical" setting is to compute or at least find a good (lower) estimate for the number

$$\mathrm{MP}(f, B) = \min_{g \sim f} |g^{-1}(B)|,$$

where $|g^{-1}(B)|$ is the cardinality of the set $g^{-1}(B)$ and minimum is taken over all maps g homotopic to f.

This problem was considered in detail by Dobreńko and Kucharski [8] (see also Schirmer [31] and Jezierski [20]); if B is a point, the problem is called the root problem and dates back to Hopf [16; 17]. Other problems of this type are, for example, fixed

Published: 29 April 2008 DOI: 10.2140/gtm.2008.14.193

point, coincidence and intersection problems; see Jiang [21], Bogatyĭ, Gonçalves and Zieschang [3] and McCord [26] ([27] for their interrelations). All these problems can be attacked by a Nielsen-type technique. So, in order to estimate the number $\mathrm{MP}(f,B)$, the preimage set $f^{-1}(B)$ is divided into equivalence classes bearing the name of Nielsen. The Nielsen number $\mathrm{N_t}(f,B)$ is the number of so-called (topologically; therefore we put a straight letter "t" in the notation, which is not to be mixed up with a homotopy or isotopy parameter) essential classes. It is a homotopy invariant, and $\mathrm{N_t}(f,B)\leqslant \mathrm{MP}(f,B)$ [8, Theorem (1.9)]. If $\mathrm{N_t}(f,B)=\mathrm{MP}(f,B)$, it is said that the setting $f\colon X\to Y\supset B$ has the Wecken property, or a Wecken type theorem holds true. Assuming $\mathrm{N_t}(f,B)<\infty$ (see eg Corollary 3 and Remark 4), the Wecken property holds iff there exists a map $g\sim f$ which has exactly $\mathrm{N_t}(f,B)$ Nielsen classes, moreover, each of them contains only one point.

For X, Y, B manifolds with $\dim X=\dim Y-\dim B\geqslant 3$ Wecken type theorems hold true; see Dobreńko and Kucharski [8, Theorem (3.4)], Jezierski [20, Theorem (3.2)] and Frolkina [10; 11] for maps of pairs of smooth manifolds. The case of a surface X is more complicated. If Y is also a surface and B is a finite set, this problem is solved; see Bogatyĭ, Gonçalves, Kudryavtseva and Zieschang [2, p 17, Remark (e)]. But, as it is noted by McCord [27, p 175], for each surface S of negative Euler characteristic reasoning of Jiang [22; 23] provides examples of preimage problems of the form $f\Delta\,\mathrm{id}_S\colon S\to S\times S\supset\Delta S$ (ΔS is the diagonal) such that $1=\mathrm{N_t}(f\Delta\,\mathrm{id}_S,\Delta S)<\mathrm{MP}(f\Delta\,\mathrm{id}_S,\Delta S)=2$. If $\dim X>\dim Y-\dim B$, the number $\mathrm{MP}(f,B)$ is "usually" infinite; we could consider instead the minimal number of path components of the preimage set, as is done by Koschorke [24] for coincidences. We will not deal with such a problem here.

For general topological spaces, obtaining Wecken type theorems seems to be a very complicated problem whose solution depends on concrete spaces and maps; see Brooks [5, p 102; 6, Example (3.15)] for the root problem. In this paper we will consider (for preimage case) the following question: does there exist a map $g\sim f$ which has exactly $\mathrm{N_t}(f,B)$ Nielsen classes (not necessary consisting of one point each). That is, defining

$$\mathrm{MP_{cl}}(f,B)=\min_{g\sim f}|\{\text{Nielsen classes of } g^{-1}(B)\}|,$$

we have $\mathrm{N_t}(f,B)\leqslant\mathrm{MP_{cl}}(f,B)\leqslant\mathrm{MP}(f,B)$, and the question concerns exactness of the left inequality. But we will not restrict ourselves to the case $\mathrm{N_t}(f,B)<\infty$. Therefore we extend the posed question as follows: does there exist a map $g\sim f$, whose Nielsen classes are all topologically essential. This is not always possible; a counterexample can be obtained converting (see McCord [27] or the end of Section 1) the coincidence problem of [14, Example 2.4] into the preimage problem. For coincidences and roots, this problem was introduced by Brooks [5], who stated sufficient conditions for a positive answer. (Refer also to the paper Gonçalves and Aniz [13] devoted to the question of simultaneous minimization of a number of points in all root classes.) Our main theorem (Theorem 1) unites and generalizes the results [5, Theorems 1,2]. We also discuss in detail Nielsen classes and Nielsen number.

Before starting, we would like to underline the following. The classical Nielsen fixed point number (the idea of its definition belongs to J Nielsen [28]) is the number of *algebraically* essential fixed point classes, that is, classes of nonzero index. Local fixed point index is defined for maps of compact metric ANRs; but there does not seem to be a well-developed index theory for the root problem, except in, for example, the manifold case (see Brooks [6, pp 376, 381–382, section 4] and Gonçalves [12, p 24]). Therefore Brooks prefers to use the notion of *topological* essentiality which does not depend on the existence of local root index. Since the root problem is a particular case of the preimage problem, by the same reason we prefer to consider topological essentiality rather than algebraic; the corresponding "topological" Nielsen number was defined in Dobreńko and Kucharski [8]. (Note that for X, Y, B manifolds with $\dim X = \dim Y - \dim B \geqslant 3$ the two notions of essentiality coincide [8; 20].) It is defined for arbitrary spaces; in particular, this allows us to omit compactness assumptions.

Conventions and notation

Throughout this paper spaces X, Y are Hausdorff, connected, locally path connected; moreover, Y is semilocally simply connected; B is a nonempty subset of Y; the same is suggested for X', Y', B'. In our main statements we will additionally repeat this and, if necessary, require something else.

For topology of infinite polyhedra, the reader should refer to Spanier [33, Chapter 3]; when we make use of a concrete theorem, we will give a more detailed reference. For a polyhedron X we denote by $X^{(n)}$ its n–skeleton; I is the unit segment $[0,1]$.

As usual, all covering spaces are (assumed to be) connected.

All maps are assumed to be continuous. By id_X we denote the identity map of a space X; fg is the composition of maps f and g; $\Delta\{f_i\}$ is the diagonal product of the family of maps $\{f_i\}$; for $A \subset X$ and an integer r the symbol $\Delta A \subset X^r$ is used for the image of A under the diagonal product of r embeddings $A \hookrightarrow X$. For a subset $A \subset X \times I$ by its t–section, where $t \in I$, we mean the set $A \cap X \times \{t\}$. Speaking about homeomorphisms and homotopy equivalences of triples of spaces, we mean of course morphisms of the appropriate category.

For a homotopy $\{f_t\}\colon X \to Y$ and a path $\alpha\colon I \to X$, by $\{f_t\alpha(t)\}$ we denote clearly a path $F(\alpha\Delta\,\mathrm{id}_I)$ in Y, where $F\colon X \times I \to Y$ is given by $(x,t) \mapsto f_t(x)$.

The symbol $\sim$ means homotopy of maps and homotopy of paths relative to end points; $[\alpha]$ is the homotopy class of a path α (again relative to end points); by $\alpha\cdot\beta$ we denote the product of paths α and β with $\alpha(1)=\beta(0)$ and by $[\alpha]\cdot[\beta]$ the product of their homotopy classes.

For a map $f\colon (X,x_0)\to(Y,y_0)$ we denote by $f_\#\colon \pi_1(X,x_0)\to\pi_1(Y,y_0)$ the induced homomorphism. If in the notation of (relative) homotopy groups $\pi_m(X,A,x_0)$ (also for the set $\pi_1(X,A,x_0)$) we omit the base point, we have in mind that A is (suggested to be) path connected.

We use singular (co)homology with coefficients in local systems of groups; if no coefficients are designated, they are usual ("constant") integers.

Other notation is either standard or is introduced in the text.

Statement of the main theorem

Theorem 1 *Suppose that the spaces X, Y are connected and locally path connected; moreover, Y is semilocally simply connected, and B is a nonempty locally path connected closed subspace. Suppose that for some integer $n \geqslant 3$ the space X is dominated by a polyhedron of dimension less or equal to n and $\pi_m(Y,Y-B)=0$ for all $1\leqslant m\leqslant n-1$. Then for each map $f\colon X\to Y$ there exists a map $g\sim f$ such that each Nielsen preimage class of $g\colon X\to Y\supset B$ is topologically essential; in particular,* $\mathrm{N_t}(f,B)=\mathrm{MP_{cl}}(f,B)$.

This means that under the above conditions we can delete all inessential preimage classes of f *at once* (recall that each single inessential class can be deleted by definition; see Definition 5 below).

Remark 1 For a path connected space Y and a subspace B, $\pi_1(Y,Y-B)=0$ if and only if B can be bypassed in Y; see Schirmer [32, Theorem 5.2]. Recall from [32, Definition 5.1] that a subspace $B\subset Y$ *can be bypassed in* Y if every path in Y with end points in $Y-B$ is homotopic to a path in $Y-B$. Recall also that for a triangulated manifold Y, its boundary ∂Y or a subpolyhedron B with $\dim Y-\dim B\geqslant 2$ provide examples of subspaces which can be bypassed [32, p 468].

Remark 2 As in Brooks [6, p 382–383], recall the following well-known results. If an n–dimensional paracompact space is dominated by a polyhedron, then it is dominated by a polyhedron of dimension n or less; see Granas and Dugundji [15, Theorem 17.7.16(c), p 483]. Each ANR is dominated by a polyhedron by Hu [19, Chapter I, Exercise R, p 32] (see also Borsuk [4, Corollary (V.4.5)]).

Remark 3 For arbitrary compact space X, other partial results of Wecken type (for roots) can be found in Gonçalves and Wong [14].

It is clear that the root problem is a particular case of the preimage problem. But the coincidence problem of $f, g\colon X \to Y$ is also equivalent to the preimage problem $f\Delta g\colon X \to Y \times Y \supset \Delta Y$ (for details, see Section 7 of the present paper and references there). Therefore the results [5, Theorems 1, 2] can be derived from our Theorem 1.

In order to prove Theorem 1, we firstly define and investigate Nielsen classes (see Theorem 2). We will need special properties of Nielsen number (see Lemma 2) which will be used also to prove homotopy invariance (on spaces) of Nielsen number (see Theorem 3).

2 Nielsen classes

Preimage points $f^{-1}(B)$ are divided into so-called Nielsen preimage classes (we also say simply "preimage classes" or "Nielsen classes"; sometimes we speak about classes of the problem $f\colon X \to Y \supset B$ or of the map f).

Definition 1 [8, Definition (1.2)] Points $x_0, x_1 \in f^{-1}(B)$ are said to be (Nielsen) equivalent if there are paths

$$\alpha\colon (I, 0, 1) \to (X, x_0, x_1) \quad \text{and} \quad \beta\colon (I, 0, 1) \to (Y, f(x_0), f(x_1))$$

such that $\beta(I) \subset B$ and $f\alpha \sim \beta$ (homotopy in Y).

Let $\{f_t\}\colon X \to Y$ be a homotopy.

Definition 2 A point $x_0 \in f_0^{-1}(B)$ is said to be $\{f_t\}$–related to a point $x_1 \in f_1^{-1}(B)$ if there exist paths $\alpha\colon (I, 0, 1) \to (X, x_0, x_1)$ and $\beta\colon (I, 0, 1) \to (Y, f_0(x_0), f_1(x_1))$ such that $\beta(I) \subset B$ and $\{f_t\alpha(t)\} \sim \beta$ (homotopy in Y).

(Another equivalent definition of the $\{f_t\}$–relation is [8, Definition (1.6)]; see also Item (5) of Theorem 2 below.)

Note that two preimage points of $f\colon X \to Y \supset B$ are Nielsen equivalent iff they are related by a constant homotopy $\{f_t = f\}$.

It is clear that the following definition makes sense:

Definition 3 A preimage class $A_0 \subset f_0^{-1}(B)$ is $\{f_t\}$–related to a preimage class $A_1 \subset f_1^{-1}(B)$ if at least one (and hence every) preimage point $x_0 \in A_0$ is $\{f_t\}$–related to at least one (and therefore every) preimage point $x_1 \in A_1$.

Note that each preimage class of f_0 is $\{f_t\}$–related to at most one preimage class of f_1 but may not be $\{f_t\}$–related to any class of f_1; see Definition 5 below.

A useful tool in connection with Nielsen classes is the following:

Definition 4 A Hopf covering and a Hopf lift for a map $f\colon X \to Y$ are, respectively, a covering $p\colon \widehat{Y} \to Y$ and a lift $\widehat{f}\colon X \to \widehat{Y}$ of f such that $p_\#(\pi_1(\widehat{Y}, \widehat{f}(x))) = f_\#(\pi_1(X, x))$ for each $x \in X$.

We use here the name of Hopf following Brooks [6], since Hopf was the first who used such covering and lifts in Nielsen root theory [17].

If it is desirable to underline that the covering $(\widehat{Y}, p)$ depends on a given map f, we will write $(\widehat{Y}_f, p_f)$ or $(\widehat{Y}, p_f)$.

The following remarks (see Hopf [17, Section 2] and Brooks [6, p 379]) will be used below. Recall that we assume the conventions of Section 1.

Proposition 1 (1) *Hopf coverings and lifts always exist.*

(2) *A Hopf covering is unique up to covering space isomorphism.*

(3) *For a Hopf covering $p_f\colon \widehat{Y}_f \to Y$ and two Hopf lifts $\widehat{f}^{(1)}$, $\widehat{f}^{(2)}$ of f, there exists a covering transformation $\rho\colon \widehat{Y}_f \to \widehat{Y}_f$ such that $\widehat{f}^{(1)} = \rho \widehat{f}^{(2)}$.*

(4) *If $(\widehat{Y}_f, p_f)$ is a Hopf covering for f, then it is a Hopf covering for each map $f_1 \sim f$.*

(5) *If $\widehat{f}$ is a Hopf lift for f and $\{\widehat{f}_t\}$ is a lift of a homotopy $\{f_t\}\colon f_0 = f \sim f_1$ such that $\widehat{f}_0 = \widehat{f}$, then $\widehat{f}_1$ is a Hopf lift for f_1.*

Proof (1) Fix an arbitrary point $x_0 \in X$. Take a covering $p\colon \widehat{Y} \to Y$ that corresponds to the subgroup $f_\#(\pi_1(X, x_0)) \subset \pi_1(Y, f(x_0))$. That is, for some point $\widehat{y}_0 \in p^{-1}(f(x_0))$ we have $p_\#(\pi_1(\widehat{Y}, \widehat{y}_0)) = f_\#(\pi_1(X, x_0))$ [33, Theorem 2.3.6]. Then by [33, Theorem 2.4.5] there exists a lift $\widehat{f}\colon X \to \widehat{Y}$ of f such that $\widehat{f}(x_0) = \widehat{y}_0$. (Note that a lift $\widehat{f}$ is not necessary uniquely determined, because of possible ambiguity in choice of $\widehat{y}_0$.) So, $p_\#(\pi_1(\widehat{Y}, \widehat{f}(x_0))) = f_\#(\pi_1(X, x_0))$. From path connectivity of X it follows that the equality $p_\#(\pi_1(\widehat{Y},\widehat{f}(x))) = f_\#(\pi_1(X,x))$ holds for each point $x \in X$. Indeed, take a path $\gamma\colon (I,0,1) \to (X,x,x_0)$; we have

$$\begin{aligned} p_\#(\pi_1(\widehat{Y},\widehat{f}(x))) &= p_\# \Big([\widehat{f}(\gamma)] \cdot \pi_1(\widehat{Y},\widehat{f}(x_0)) \cdot [\widehat{f}(\gamma^{-1})] \Big) \\ &= [p_\# \widehat{f}(\gamma)] \cdot p_\#(\pi_1(\widehat{Y},\widehat{f}(x_0))) \cdot [p_\# \widehat{f}(\gamma^{-1})] \\ &= [f(\gamma)] \cdot f_\#(\pi_1(X,x_0)) \cdot [f(\gamma^{-1})] = f_\#(\pi_1(X,x)). \end{aligned}$$

(2) This is clear; see eg Massey [25, Corollary V.6.4].

(3) Take an arbitrary $x_0 \in X$. We have

$$(p_f)_\#(\pi_1(\widehat{Y},\widehat{f}^{(1)}(x_0))) = (p_f)_\#(\pi_1(\widehat{Y},\widehat{f}^{(2)}(x_0))).$$

Hence [25, Corollary V.6.4] there exists a covering transformation ρ of the covering $p_f\colon \widehat{Y}_f \to Y$ such that $\widehat{f}^{(1)}(x_0) = \rho \widehat{f}^{(2)}(x_0)$. Therefore $\widehat{f}^{(1)} = \rho \widehat{f}^{(2)}$ [33, Theorem 2.2.2].

We prove (4) and (5) simultaneously, using notation common for these two items: $(\widehat{Y}_f, p_f)$ is a Hopf covering for f, $\{f_t\}\colon f_0 = f \sim f_1$ a homotopy. Take an arbitrary point $x_0 \in X$. According to [33, Theorem 1.8.7], we have

$$(f_1)_\#(\pi_1(X,x_0)) = [\omega] \cdot f_\#(\pi_1(X,x_0)) \cdot [\omega^{-1}],$$

where $\omega\colon (I,0,1) \to (X, f_1(x_0), f(x_0))$ is defined by $\omega(t) = f_{1-t}(x_0)$. Then, putting $\widehat{\omega}(t) = \widehat{f}_{1-t}(x_0)$, we obtain

$$\begin{aligned} (p_f)_\#(\pi_1(\widehat{Y},\widehat{f}_1(x_0))) &= (p_f)_\#([\widehat{\omega}] \cdot \pi_1(\widehat{Y},\widehat{f}(x_0))) \cdot [\widehat{\omega}^{-1}]) \\ &= [\omega] \cdot (p_f)_\#(\pi_1(\widehat{Y},\widehat{f}(x_0))) \cdot [\omega^{-1}] \\ &= [\omega] \cdot f_\#(\pi_1(X,x_0)) \cdot [\omega^{-1}] = (f_1)_\#(\pi_1(X,x_0)). \end{aligned}$$

The necessary statement follows now as in the proof of (1). □

Returning to preimage classes, we obtain the following description of Nielsen classes and $\{f_t\}$–relation (see Hopf [17, Satz III] and Brooks [6, Theorem (3.4)] for roots):

Theorem 2 *Let $(\widehat{Y}, p)$ and $\widehat{f}$ be a Hopf covering and a Hopf lift for $f\colon X \to Y \supset B$. Let $\{f_t\}\colon X \to Y$ be a homotopy from $f_0 = f$ to f_1 and $\{\widehat{f}_t\}\colon X \to \widehat{Y}$ its lift such that $\widehat{f}_0 = \widehat{f}$. Then*

(1) *two preimage points $x_0, x_1 \in f^{-1}(B)$ are Nielsen equivalent if and only if the points $\widehat{f}(x_0)$, $\widehat{f}(x_1)$ lie in the same path component of the set $p^{-1}(B)$;*

(2) *Nielsen classes of* $f\colon X \to Y \supset B$ *are precisely nonempty sets of the form* $\hat{f}^{-1}(C)$*, where* C *is a path component of the set* $p^{-1}(B)$*;*

(3) *a point* $x_0 \in f_0^{-1}(B)$ *is* $\{f_t\}$*–related to a point* $x_1 \in f_1^{-1}(B)$ *if and only if the points* $\hat{f}_0(x_0)$*,* $\hat{f}_1(x_1)$ *are contained in the same path component of the set* $p^{-1}(B)$*;*

(4) *a preimage class* $A_0 \subset f_0^{-1}(B)$ *is* $\{f_t\}$*–related to a class* $A_1 \subset f_1^{-1}(B)$ *if and only if the sets* $\hat{f}_0(A_0)$ *and* $\hat{f}_1(A_1)$ *are contained in one path component of the set* $p^{-1}(B)$*;*

(5) *a preimage class* $A_0 \subset f_0^{-1}(B)$ *is* $\{f_t\}$*–related to a class* $A_1 \subset f_1^{-1}(B)$ *if and only if* A_0*,* A_1 *are* 0*– and* 1*–sections of some preimage class of* $F\colon X \times I \to Y \supset B$*, where* $F(x,t) = f_t(x)$*.*

Proof It is clear that (3) ⇒ (1) ⇒ (2) and (3) ⇒ (4) ⇒ (5). Let us prove (3).

Suppose the points x_0, x_1 are $\{f_t\}$–related, that is, for some paths $\alpha\colon (I,0,1) \to (X, x_0, x_1)$ and $\beta\colon (I,0,1) \to (B, f_0(x_0), f_1(x_1))$ we have $[\{f_t\alpha(t)\}] = [\beta]$. Let $\hat{\beta}\colon (I,0,1) \to (\hat{Y}, \hat{\beta}(0) = \hat{f}_0(x_0), \hat{\beta}(1))$ be the lift of β into $\hat{Y}$ beginning at $\hat{f}_0(x_0)$. Denote by C the path component of $p^{-1}(B)$ that contains $\hat{f}_0(x_0)$; we have $\beta(I) \subset C$. The path $\{\hat{f}_t\alpha(t)\}$ is homotopic to $\hat{\beta}$ and starts at the same point $\hat{f}_0(x_0)$. Hence $\hat{\beta}(1) = \hat{f}_1(x_1)$, what implies $\hat{f}_1(x_1) \in C$.

To prove the converse, suppose that the points $\hat{f}_0(x_0)$, $\hat{f}_1(x_1)$ lie in the same path component C of $p^{-1}(B)$. Consider arbitrary paths $\alpha\colon (I,0,1) \to (X, x_0, x_1)$ and $\hat{\beta}\colon (I,0,1) \to (\hat{Y}, \hat{f}_0(x_0), \hat{f}_1(x_1))$ with $\hat{\beta}(I) \subset C$. Denoting $\beta = p\hat{\beta}$, we have $\beta(I) \subset B$. Then $\{\hat{f}_t\alpha(t)\} \cdot \hat{\beta}^{-1}$ is a loop at $\hat{f}_0(x_0)$. Therefore the path $p(\{\hat{f}_t\alpha(t)\} \cdot \hat{\beta}^{-1})$ is a loop at $f(x_0)$. By definition of Hopf covering, there exists a loop γ at x_0 in X such that

$$[f\gamma] = [p(\{\hat{f}_t\alpha(t)\} \cdot \hat{\beta}^{-1})] = [\{p\hat{f}_t\alpha(t)\} \cdot \beta^{-1}] = [\{f_t\alpha(t)\}] \cdot [\beta^{-1}].$$

Consequently,

$$[\{f_t((\gamma^{-1} \cdot \alpha)(t))\}] = [f\gamma]^{-1} \cdot [\{f_t\alpha(t)\}] = [\beta].$$

The last equality shows that x_0 is $\{f_t\}$–related to x_1. □

Note that (5) of Theorem 2 is taken in [8, Definition (1.6)] as a definition (the $\{f_t\}$–relation is called there "F–Nielsen relation").

From this theorem, we derive a number of simple corollaries. The first one is evident; it generalizes [6, Theorem (3.8)] (see Definition 5 for the notion of topological essentiality):

Corollary 1 *Let $f\colon X \to Y \supset B$ be a map, $(\widehat{Y}_f, p_f)$ and $\widehat{f}\colon X \to \widehat{Y}$ its Hopf covering and lift and $\widehat{B}$ a path component of $p^{-1}(B)$. Then $\widehat{f}^{-1}(\widehat{B})$ is a topologically essential preimage class if and only if $\widehat{f}_1^{-1}(\widehat{B}) \neq \varnothing$ for any homotopy $\{\widehat{f}_t\}$ beginning at $\widehat{f}_0 = \widehat{f}$.*

Let us underline once more that in general the Nielsen number $\mathrm{N_t}(f, B)$ may be infinite. We obtain now simple sufficient conditions for its finiteness; see Hopf [17, Satz II, IIa] and Brooks [6, Theorem (3.5), Corollary (3.6)] for roots.

Corollary 2 *Suppose (additionally to our usual conventions) that B is locally path connected. Then each Nielsen class of $f\colon X \to Y \supset B$ is both open and closed in $f^{-1}(B)$. Hence, if $f^{-1}(B)$ is compact, then the number of Nielsen classes is finite.*

Proof The second statement is clear. To prove the first, it suffices to show that each Nielsen class is an open subset of the preimage set. Let $x \in f^{-1}(B)$. Take an open neighbourhood $V_1 \subset Y$ of $f(x)$ such that every two paths in V_1 starting at $f(x)$ and having the same end points are homotopic in Y. Let $W \subset B \cap V_1$ be an open (in B) path connected neighbourhood of $f(x)$. Then $W = B \cap V_2$ for some open set $V_2 \subset Y$. Put $V = V_1 \cap V_2$. Take an open path connected neighbourhood $U \subset X$ of x such that $f(U) \subset V$. Suppose $y \in U \cap f^{-1}(B)$. Take paths $\alpha\colon (I, 0, 1) \to (U, x, y)$ and $\beta\colon (I, 0, 1) \to (W, f(x), f(y))$. Then $f\alpha\colon (I, 0, 1) \to (V, f(x), f(y))$ is homotopic (in Y) to β; that is, the points x, y belong to the same preimage class. □

Corollary 3 *Suppose (additionally to our usual conventions) that B is locally path connected and at least one of the following conditions holds:*

(1) *X is compact, and B is closed in Y;*

(2) *f is proper, and B is compact.*

Then the number of Nielsen classes is finite.

Proof In both cases, $f^{-1}(B)$ is compact. Application of Corollary 2 finishes the proof. □

Remark 4 See the statement and proof of [8, Theorem (1.3)] for a different list of conditions on spaces X, Y, B which also imply openness of Nielsen classes and finiteness of its number.

3 $\mathcal{R}$–sets

Take $f\colon X \to Y \supset B$, its Hopf covering $(\widehat{Y}, p)$, and a Hopf lift $\widehat{f}$. For each path component $\widehat{B}$ of $p^{-1}(B)$, call its preimage $\widehat{f}^{-1}(\widehat{B})$ an $\mathcal{R}$–set. Some $\mathcal{R}$–sets may be empty, but we nevertheless distinguish them through the path components $\widehat{B}$ which define them. That is, an $\mathcal{R}$–set $\widehat{f}^{-1}(\widehat{B})$ is considered to carry a label $\widehat{B}$. The collection of all $\mathcal{R}$–sets for a map f and its Hopf lift $\widehat{f}$ is denoted by $\mathcal{R}(f, \widehat{f})$.

Note (see Brown and Schirmer [7, Remark 4.5]) that the number of path components of $p^{-1}(B)$ and hence of $\mathcal{R}$–sets equals the Reidemeister preimage number $\mathrm{R}(f, B)$, which is often useful for computation of the Nielsen number (see Definition 5), and this is the reason of the presence of the letter $\mathcal{R}$ in the name of $\mathcal{R}$–sets. We will not go into details, but refer to Brooks [6], for example, for roots.

From Theorem 2 the following proposition holds:

Proposition 2 *Nonempty $\mathcal{R}$–sets (considered without labels) are exactly Nielsen classes.*

We have already defined $\{f_t\}$–relation between Nielsen classes. Now we will extend this relation to $\mathcal{R}$–sets.

Let $\{f_t\}\colon f_0 \sim f_1$ be a homotopy. By Proposition 1, a Hopf covering $(\widehat{Y}, p)$ constructed for f_0 is also a Hopf covering for f_1. Further, if $\widehat{f_0}$ is a Hopf lift for f_0, and $\{\widehat{f_t}\}$ is a lift of the homotopy $\{f_t\}$ starting at this $\widehat{f_0}$, then $\widehat{f_1}$ is a Hopf lift for f_1. The homotopy $\{f_t\}$ induces therefore a bijection between $\mathcal{R}$–sets of f_0 and f_1 by the following rule:

$$\mathcal{R}(f_0, \widehat{f_0}) \leftrightarrow \mathcal{R}(f_1, \widehat{f_1}), \quad \widehat{f_0}^{-1}(\hat{B}) \leftrightarrow \widehat{f_1}^{-1}(\hat{B}),$$

for each path component $\widehat{B}$ of $p^{-1}(B)$. From Theorem 2 it follows that for Nielsen classes (equivalently, nonempty $\mathcal{R}$–sets considered without labels) this is just an $\{f_t\}$–relation. Hence it gives a bijection between the sets of all topologically essential preimage classes of f_0 and f_1 (see Definition 5).

We will need the following lemma.

Lemma 1 *Suppose that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\varphi}\big\downarrow & & \big\| \\ X' & \xrightarrow{f'} & Y \end{array}$$

commutes; let $(\widehat{Y}_f, p_f)$, $(\widehat{Y}_{f'}, p_{f'})$ *and* $\widehat{f}$, $\widehat{f}'$ *be Hopf coverings and Hopf lifts for* f, f'. *Then*

(1) *there exists a unique covering* $q\colon \widehat{Y}_f \to \widehat{Y}_{f'}$ *such that* $q\widehat{f} = \widehat{f}'\varphi$ *and* $p_f = p_{f'}q$;

(2) *a map* $\mathcal{R}(f, \widehat{f}) \to \mathcal{R}(f', \widehat{f}')$, $\widehat{f}^{-1}(\widehat{B}) \mapsto (\widehat{f}')^{-1}(q\widehat{B})$, *where* $\widehat{B} \subset \widehat{Y}_f$ *is a path component of* $p_f^{-1}(B)$, *is well-defined; this map brings a nonempty* $\mathcal{R}$*–set (considered without label), that is, a preimage class* A *of the problem* $f\colon X \to Y \supset B$, *to that (nonempty)* $\mathcal{R}$*–set, that is, preimage class* A' *of* $f'\colon X' \to Y \supset B$, *which contains* $\varphi(A)$.

Proof (1) Take an arbitrary point $x_0 \in X$. We have

$$(p_f)_\#(\pi_1(\widehat{Y}_f, \widehat{f}(x_0))) = f_\#(\pi_1(X, x_0)) = (f'\varphi)_\#(\pi_1(X, x_0))$$
$$\subset f'_\#(\pi_1(X', \varphi(x_0))) = (p_{f'})_\#(\pi_1(\widehat{Y}_{f'}, \widehat{f}'\varphi(x_0))).$$

Hence there exists a unique covering $q\colon (\widehat{Y}_f, \widehat{f}(x_0)) \to (\widehat{Y}_{f'}, \widehat{f}'\varphi(x_0))$ such that $p_f = p_{f'}q$. To prove the equality $q\widehat{f} = \widehat{f}'\varphi$, note that the two maps $q\widehat{f}, \widehat{f}'\varphi\colon X \to \widehat{Y}_{f'}$ are lifts of the same map f and coincide on x_0.

(2) follows from $\varphi(\widehat{f}^{-1}\widehat{B}) \subset (\widehat{f}')^{-1}(q\widehat{B})$. □

4 Nielsen number

Definition 5 [8, Definition (1.8)] A preimage class A_0 of $f\colon X \to Y \supset B$ is called topologically essential if for each homotopy $\{f_t\}\colon X \to Y$ beginning at $f_0 = f$ there is a Nielsen class A_1 of $f_1\colon X \to Y \supset B$ which is $\{f_t\}$–related to A_0; that is, the class A_0 can not "disappear" under homotopies, or there is no homotopy which can "delete" the class A_0. Otherwise the class A_0 is called inessential. The number of topologically essential preimage classes is called the topological Nielsen number of the given preimage problem, or of the map f with respect to B, and it is denoted by $\mathrm{N_t}(f\colon X \to Y \supset B)$ or shortly $\mathrm{N_t}(f, B)$; it is an integer or infinity.

In the present paper we often omit the word "topologically", since we do not consider the other type of essentiality (algebraic). Note that some authors use the word "geometrical" instead of "topological".

It follows from the definition of essential class (see also [8, Theorem (1.9)]) that:

Proposition 3 *The Nielsen number* $\mathrm{N_t}(f, B)$ *is a homotopy invariant of a map* f *and* $\mathrm{N_t}(f, B) \leqslant \mathrm{MP}(f, B)$.

The next theorem implies stronger invariance of the Nielsen number.

Theorem 3 *Suppose that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & (Y, B, Y - B) \\ \varphi\big\downarrow & & \psi\big\downarrow \\ X' & \xrightarrow{f'} & (Y', B', Y' - B') \end{array}$$

commutes up to homotopy and ψ is a homotopy equivalence. If $\varphi_\#\colon \pi_1(X, x_0) \to \pi_1(X', \varphi(x_0))$ is surjective, then $\mathrm{N_t}(f\colon X \to Y \supset B) \leqslant \mathrm{N_t}(f'\colon X' \to Y' \supset B')$. If moreover φ has a right homotopy inverse, then $\mathrm{N_t}(f\colon X \to Y \supset B) = \mathrm{N_t}(f'\colon X' \to Y' \supset B')$.

To prove this, we need two lemmas. The first unites and generalizes [5, Lemmas 3, 3′] and will be used to prove Theorem 1.

Lemma 2 *Suppose that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \varphi\big\downarrow & & \big\| \\ X' & \xrightarrow{f'} & Y \end{array}$$

commutes.

(1) *If $\varphi_\#\colon \pi_1(X, x_0) \to \pi_1(X', \varphi(x_0))$ is surjective, then*

  (1.1) *$(\widehat{Y}_{f'}, p_{f'})$ and $\widehat{f}'\varphi$ can be taken as Hopf covering and lift for f;*

  (1.2) *the map $\mathcal{R}(f, \widehat{f}'\varphi) \to \mathcal{R}(f', \widehat{f}')$ (defined in (2) of Lemma 1) is injective;*

  (1.3) *it maps essential classes of f to essential classes of f'; in particular, $\mathrm{N_t}(f, B) \leqslant \mathrm{N_t}(f', B)$.*

(2) *Suppose moreover that φ has a right homotopy inverse. If for a map $g \sim f$ the problem $g\colon X \to Y \supset B$ has only essential preimage classes, then the map $g' = g\chi$ is homotopic to f' and the problem $g'\colon X' \to Y \supset B$ also has only essential preimage classes; in particular, $\mathrm{N_t}(f, B) = \mathrm{N_t}(f', B)$.*

Proof (1.1) This follows from

$$(p_{f'})_\#(\pi_1(\widehat{Y}_{f'}, \widehat{f}'\varphi(x_0))) = f'_\#(\pi_1(X', \varphi(x_0)))$$
$$(f'\varphi)_\#(\pi_1(X, x_0)) = f_\#(\pi_1(X, x_0)).$$

(1.2) This holds because the map under consideration is given by

$$(\widehat{f})^{-1}(\widehat{B}) = \varphi^{-1}((\widehat{f}')^{-1}(\widehat{B})) \mapsto (\widehat{f}')^{-1}(\widehat{B}).$$

(1.3) If a preimage class A of $f\colon X \to Y \supset B$ is taken to the class $A' \supset \varphi(A)$ of $f'\colon X' \to Y \supset B$ which can be "deleted" by a homotopy $\{f'_t\}$, then the class A can be "deleted" by the homotopy $\{f'_t\varphi\}$.

(2) Denote by χ the right homotopy inverse for φ, ie, $\varphi\chi \sim \mathrm{id}_{X'}$. It is evident that

$$g' = g\chi \sim f\chi = f'\varphi\chi \sim f'.$$

By (1.1) and Proposition 1, we can take the same Hopf covering $(\widehat{Y}, p)$ for maps f, f', $f\chi$ simultaneously. Let $\widehat{f}'$ be a Hopf lift for f', then $\widehat{f} = \widehat{f}'\varphi$ and $\widehat{f}\chi$ are Hopf lifts for f and $f\chi$. Lift the homotopy $f\chi \sim f'$ starting at $\widehat{f}\chi$; let $\widetilde{f}'$ be its final map. By Proposition 1 it is a Hopf lift for f'. Note that

$$\widetilde{f}' \sim \widehat{f}\chi = \widehat{f}'\varphi\chi \sim \widehat{f}'.$$

This homotopy and the above equalities define (see Section 3) maps in the following sequence:

$$(*)\quad \mathcal{R}(f', \widehat{f}') \leftrightarrow \mathcal{R}(f', \widetilde{f}') \leftrightarrow \mathcal{R}(f\chi, \widehat{f}\chi) \to \mathcal{R}(f, \widehat{f}) = \mathcal{R}(f'\varphi, \widehat{f}'\varphi) \to \mathcal{R}(f', \widehat{f}')$$

Going through this sequence, we obtain

$$(\widehat{f}')^{-1}(\widehat{B}) \mapsto (\widetilde{f}')^{-1}(\widehat{B}) \mapsto (\widehat{f}\chi)^{-1}(\widehat{B}) \mapsto \widehat{f}^{-1}(\widehat{B}) = (\widehat{f}'\varphi)^{-1}(\widehat{B}) \mapsto (\widehat{f}')^{-1}(\widehat{B}),$$

where $\widehat{B}$ is an arbitrary path component of $p^{-1}(B)$; this map is the identity map $\mathcal{R}(f', \widehat{f}') \to \mathcal{R}(f', \widehat{f}')$. In particular, it gives a bijection of the set of essential Nielsen classes of f' onto itself.

Now suppose $g \sim f$ has only essential preimage classes. The following diagram commutes:

$$\begin{array}{ccc} \mathcal{R}(f\chi, \widehat{f}\chi) & \longrightarrow & \mathcal{R}(f, \widehat{f}) \\ \downarrow & & \downarrow \\ \mathcal{R}(g\chi, \widehat{g}\chi) & \longrightarrow & \mathcal{R}(g, \widehat{g}) \end{array} \qquad \begin{array}{ccc} (\widehat{f}\chi)^{-1}(\widehat{B}) & \longrightarrow & \widehat{f}^{-1}(\widehat{B}) \\ \downarrow & & \downarrow \\ (\widehat{g}\chi)^{-1}(\widehat{B}) & \longrightarrow & \widehat{g}^{-1}(\widehat{B}) \end{array}$$

where the vertical maps are bijections defined by homotopies $\{f_t\}\colon f \sim g$ and $\{f_t\chi\}\colon f\chi \sim g\chi$ (see Section 3), and the horizontal ones are those of (1.2) of the present Lemma.

Suppose there exists a nonempty nonessential class of $g\chi$. Going in the diagram up to $\mathcal{R}(f\chi, \widehat{f}\chi)$ and then to the left side of $(*)$, we obtain a nonessential class of f'.

In contrast to it, going in the diagram right and up and then to the right part of $(*)$, we obtain a (nonempty) essential class of f'. But, as noted above, going through $(*)$ gives an identity map of $\mathcal{R}(f', \widehat{f'})$. The contradiction proves the statement. $\square$

The condition of Item (2) of Lemma 2 means that the space X' is dominated by the space X. Recall the definition from [19, Chapter I, Exercise R, p 32]:

Definition 6 A space X' is dominated by a space X (or X' is a homotopy retract of X) if there exist maps $\chi\colon X' \to X$ and $\varphi\colon X \to X'$ such that $\varphi\chi \sim \mathrm{id}_{X'}$.

Lemma 3 *If* $(Y, B, Y - B) \xrightarrow{\psi} (Y', B', Y' - B')$ *is a homotopy equivalence, then* $\mathrm{N_t}(f\colon X \to Y \supset B) = \mathrm{N_t}(\psi f\colon X \to Y' \supset B')$ *for each map* $f\colon X \to Y$.

Proof Let $\theta\colon (Y', B', Y' - B') \to (Y, B, Y - B)$ be a homotopy inverse for ψ (that is, $\theta\psi$ and $\psi\theta$ are homotopic to id_Y and $\mathrm{id}_{Y'}$ respectively, by homotopies of maps of corresponding triples). Let $p_f\colon \widehat{Y} \to Y$, $p_{\psi f}\colon \widehat{Y}' \to Y'$ be Hopf coverings for $f\colon X \to Y$, $\psi f\colon X \to Y'$, and let $\widehat{f}$, $\widehat{\psi f}$ be Hopf lifts for f, ψf. The proof is in 6 steps.

Step 1 There exist lifts $\widehat{\psi}\colon \widehat{Y} \to \widehat{Y}'$, $\widehat{\theta}\colon \widehat{Y}' \to \widehat{Y}$ of maps ψ, θ, and we may assume that $\widehat{\psi f} = \widehat{\psi}\,\widehat{f}$.

From

$$(p_{\psi f})_\#(\pi_1(\widehat{Y}', \widehat{\psi f}(x_0))) = (\psi f)_\#(\pi_1(X, x_0)) = (\psi p_f)_\#(\pi_1(\widehat{Y}, \widehat{f}(x_0)))$$

we conclude that there exists a lift $\widehat{\psi}$ of ψ such that $\widehat{\psi f}(x_0) = \widehat{\psi}\,\widehat{f}(x_0)$ and hence $\widehat{\psi f} = \widehat{\psi}\,\widehat{f}$.

Now we prove existence of $\widehat{\theta}$. Since $f \sim \theta\psi f$, from Proposition 1 it follows that $(\widehat{Y}_f, p_f)$ is a Hopf covering for $\theta\psi f$. Therefore it suffices to apply to ψf, $\theta\psi f$ what was just proved for f, ψf.

Step 2 $\widehat{\psi}$, $\widehat{\theta}$ are maps of triples:

$$(\widehat{Y}, p_f^{-1}(B), \widehat{Y} - p_f^{-1}(B)) \rightleftarrows (\widehat{Y}', p_{\psi f}^{-1}(B'), \widehat{Y}' - p_{\psi f}^{-1}(B')).$$

This is easy; for $\widehat{\psi}$, we have

$$\widehat{\psi}^{-1}(p_{\psi f}^{-1}(B')) = p_f^{-1}(\psi^{-1}(B')) = p_f^{-1}(B),$$

and similarly for $\widehat{\theta}$.

Step 3 $\widehat{\psi}$, $\widehat{\theta}$ map sets of path components of $p_f^{-1}(B)$, $p_{\psi f}^{-1}(B')$ bijectively.

Let $\{h_t\}\colon (Y,B,Y-B)\to(Y,B,Y-B)$ be a homotopy joining $h_0=\theta\psi$ to $h_1=\mathrm{id}_Y$. Lift the homotopy

$$\{H_t=h_t p_f\}\colon (\widehat{Y}, p_f^{-1}(B), \widehat{Y}-p_f^{-1}(B))\to(Y,B,Y-B),$$

which joins $H_0=\theta\psi p_f$ to $H_1=p_f$, to $\widehat{Y}$, starting at $\widehat{H}_0=\widehat{\theta}\widehat{\psi}$. The lift is a map of triples

$$\{\widehat{H}_t\}\colon (\widehat{Y}, p_f^{-1}(B), \widehat{Y}-p_f^{-1}(B))\to(\widehat{Y}, p_f^{-1}(B), \widehat{Y}-p_f^{-1}(B)),$$

and $\widehat{H}_1$ is a covering transformation of p_f. Similarly, $\widehat{\psi}\widehat{\theta}$ is homotopic by homotopy of maps of triples

$$(\widehat{Y}', p_{\psi f}^{-1}(B'), \widehat{Y}'-p_{\psi f}^{-1}(B'))\to(\widehat{Y}', p_{\psi f}^{-1}(B'), \widehat{Y}'-p_{\psi f}^{-1}(B'))$$

to a covering transformation of $p_{\psi f}$. This implies the required statement.

Step 4 Preimage classes of the problem $f\colon X\to Y\supset B$ coincide with those of $\psi f\colon X\to Y'\supset B'$.

Firstly, since $\psi^{-1}(B')=B$, we have $f^{-1}(B)=(\psi f)^{-1}(B')$. Secondly, from Step 2 it follows that two points belong to the same Nielsen class of the problem $f\colon X\to Y\supset B$ iff they belong to one Nielsen class of $\psi f\colon X\to Y'\supset B'$.

Step 5 If a preimage class A of $f\colon X\to Y\supset B$ is inessential, then it is inessential as a preimage class of $\psi f\colon X\to Y'\supset B'$.

In fact, if a homotopy $\{f_t\}$, with $f_0=f$, "deletes" A as a preimage class of $f\colon X\to Y\supset B$, then the homotopy $\{\psi f_t\}$ starts at ψf and "deletes" A as a preimage class of $\psi f\colon X\to Y'\supset B'$.

Step 6 Previous step shows that $\mathrm{N_t}(f,B)\geqslant\mathrm{N_t}(\psi f,B')$. Taking in this inequality ψf, $\theta\psi f$ in place of f, ψf, we obtain $\mathrm{N_t}(\psi f,B')\geqslant\mathrm{N_t}(\theta\psi f,B)$. But $\theta\psi f\sim f$ implies $\mathrm{N_t}(\theta\psi f,B)=\mathrm{N_t}(f,B)$, and the Lemma is proved. □

Now we prove Theorem 3.

Proof Lemma 3 implies that $\mathrm{N_t}(f,B)=\mathrm{N_t}(\psi f,B')$. Since $\psi f\sim f'\varphi$, the last number equals $\mathrm{N_t}(f'\varphi,B')$. By (1.3) of Lemma 2 (applied to the triangle of maps $f'\varphi$, f' and φ) this is less than or equal to $\mathrm{N_t}(f',B')$, hence $\mathrm{N_t}(f,B)\leqslant\mathrm{N_t}(f',B')$ and the first statement is proved.

If φ has a right homotopy inverse, then by (2) of Lemma 2 we have $\mathrm{N_t}(f'\varphi,B')=\mathrm{N_t}(f',B')$; hence $\mathrm{N_t}(f,B)=\mathrm{N_t}(f',B')$. □

Corollary 4 *Suppose that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & (Y, Y-B, B) \\ {\scriptstyle\varphi}\big\downarrow & & {\scriptstyle\psi}\big\downarrow \\ X' & \xrightarrow{f'} & (Y', Y'-B', B') \end{array}$$

commutes up to homotopy and φ, ψ are homotopy equivalences. Then we have

$$\mathrm{N_t}(f\colon X \to Y \supset B) = \mathrm{N_t}(f'\colon X' \to Y' \supset B').$$

For roots the following corollary is stated (without proof) in a slightly stronger form in [6, Theorem (3.10)].

Corollary 5 *Suppose that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & (Y, Y-B, B) \\ {\scriptstyle\varphi}\big\downarrow & & {\scriptstyle\psi}\big\downarrow \\ X' & \xrightarrow{f'} & (Y', Y'-B', B') \end{array}$$

commutes and φ, ψ are homeomorphisms. Then

$$\mathrm{N_t}(f\colon X \to Y \supset B) = \mathrm{N_t}(f'\colon X' \to Y' \supset B').$$

5 Other lemmas needed for proof of Theorem 1

Our proof of Theorem 1 imitates those of Brooks [5, Theorems 1,2]. We also need several lemmas.

From [9, Chapter 3, Section 21, 2A, Exercises 3,4] and [9, Chapter 1, Section 6, 3C, Corollary] it follows that:

Lemma 4 *Each local coefficient system defined on the 2–skeleton $X^{(2)}$ of a polyhedron X extends (uniquely up to isomorphism) to X.*

Lemma 5 [5, Lemma 4] *Suppose $\mathcal{C}$ is a family of mutually disjoint closed subsets of a topological space Z, and let $D = \bigcup_{C\in\mathcal{C}} C$. Suppose also that each set $C \in \mathcal{C}$ is both closed and open in D, and D is closed in Z. Then the inclusions*

$$i_C\colon (Z, Z-D) \hookrightarrow (Z, Z-C), \quad C \in \mathcal{C},$$

induce an isomorphism

$$\Big(\sum_{C\in\mathcal{C}} i_{C*m}\Big)\colon H_m(Z,Z-D)\cong\sum_{C\in\mathcal{C}} H_m(Z,Z-C)$$

for each $m\geqslant 0$.

Lemma 6 *Suppose that* Y *is connected, locally path connected and semilocally simply connected space; its closed subspace* B *is locally path connected and* $\pi_1(Y,Y-B)=0$. *Then for each path* $\alpha\colon(I,0,1)\to(Y,B,Y-B)$ *there exists a path* $\beta\sim\alpha$ *such that* $\beta([0,\frac{1}{2}])\subset B$ *and* $\beta((\frac{1}{2},1])\subset Y-B$.

That is, not only each path in Y with end points in $Y-B$ can be pushed off from B (as it is since B can be bypassed in Y; see Remark 1), but also a path with one end point in B and another in $Y-B$ can be "half-pushed off" from B.

Proof Put $t_1=\max\{t\in I\,|\,\alpha([0,t])\subset B\}$. It is clear that $t_1<1$. Let $S=\{t\in I\,|\,t>t_1,\alpha(t)\in B\}$. Consider two cases.

Case 1 Suppose t_1 is not a limit point of S (in particular, S is empty). Then there exists $t_2\in I$ such that $t_2>t_1$ and $\alpha((t_1,t_2])\subset Y-B$. Let $\alpha_1(t)=\alpha(tt_1)$, $\alpha_2(t)=\alpha(tt_2+(1-t)t_1)$, and $\alpha_3(t)=\alpha(t+(1-t)t_2)$, for $t\in I$. Take a path $\beta_3\sim\alpha_3$ such that $\beta_3(I)\subset Y-B$. The path $\beta=\alpha_1\cdot(\alpha_2\cdot\beta_3)$ has necessary properties.

Case 2 Suppose t_1 is a limit point of S. Let $U_1\subset Y$ be a path connected open neighbourhood of $\alpha(t_1)$ such that each loop at $\alpha(t_1)$ in U_1 is homotopic (in Y) to a constant path. Let $V\subset B\cap U_1$ be a path connected open (in B) neighbourhood of $\alpha(t_1)$. Then $V=B\cap U_2$ for some open set $U_2\subset Y$. Denote $U=U_1\cap U_2$. There exist $t_2,t_3\in I$ such that $t_1<t_2<t_3$, $\alpha([t_1,t_2])\subset U$, $\alpha(t_2)\in B$, and $\alpha((t_2,t_3])\subset Y-B$. Denote the "pieces" $\alpha_1(t)=\alpha(tt_1)$, $\alpha_2(t)=\alpha(tt_2+(1-t)t_1)$, $\alpha_3(t)=\alpha(tt_3+(1-t)t_2)$, and $\alpha_4(t)=\alpha(t+(1-t)t_3)$, for $t\in I$. Take a path $\beta_2\colon(I,0,1)\to(V,\alpha(t_1),\alpha(t_2))$; we have $\alpha_2\sim\beta_2$ (homotopy in Y). Since B can be bypassed in Y, the path α_4 is homotopic to some path β_4 with $\beta_4(I)\subset Y-B$. Then $\beta=(\alpha_1\cdot\beta_2)\cdot(\alpha_3\cdot\beta_4)$ is the required path. The Lemma is proved. □

Just as Brooks in his proofs of [5, Lemmas 5, 5′], we will use Seifert–van Kampen theorem and relative Hurewicz theorem in our proof of Lemma 8. The next Lemma explains comprehensively, why the spaces, to which the theorems will be applied here, are indeed path connected.

Lemma 7 *Suppose that spaces* Y, B *satisfy the conditions of Lemma 6. Let* $p\colon\widetilde{Y}\to Y$ *be a (arbitrary) covering. Let* $\mathcal{C}$ *be an arbitrary family of path components of* $p^{-1}(B)$ *and* $D=\bigcup_{C\in\mathcal{C}}C$. *Then* $\widetilde{Y}-D$ *is path connected.*

Proof Take arbitrary points $\widetilde{y}_0, \widetilde{y}_1 \in \widetilde{Y} - D$. To prove the existence of a path joining them, we consider three cases.

Case 1 Suppose $\widetilde{y}_0, \widetilde{y}_1 \in \widetilde{Y} - p^{-1}(B)$. Take a path $\widetilde{\alpha}\colon (I, 0, 1) \to (\widetilde{Y}, \widetilde{y}_0, \widetilde{y}_1)$. The path $\alpha = p(\widetilde{\alpha})$ in Y joins the points $p(\widetilde{y}_0), p(\widetilde{y}_1) \in Y - B$. Since B can be bypassed in Y, there exists a path β such that $\beta(I) \subset Y - B$ and $\alpha \sim \beta$. Lift of this homotopy to $\widetilde{Y}$ that starts at $\widetilde{\alpha}$ gives a path over β in $\widetilde{Y} - p^{-1}(B) \subset \widetilde{Y} - D$ which joins $\widetilde{y}_0$ to $\widetilde{y}_1$.

Case 2 Suppose $\widetilde{y}_0 \in p^{-1}(B) - D$ and $\widetilde{y}_1 \in \widetilde{Y} - p^{-1}(B)$. Join $\widetilde{y}_0$ to $\widetilde{y}_1$ with a path $\widetilde{\alpha}$ in $\widetilde{Y}$. The path $\alpha = p(\widetilde{\alpha})$ in Y begins at $p(\widetilde{y}_0) \in B$ and ends at $p(\widetilde{y}_1) \in Y - B$. By Lemma 6, there exists a path β such that $\beta\big([0, \frac{1}{2}]\big) \subset B$, $\beta\big((\frac{1}{2}, 1]\big) \subset Y - B$, and $\alpha \sim \beta$. The lift of this homotopy to $\widetilde{Y}$ starting at $\widetilde{\alpha}$ gives a path $\widetilde{\beta}$ in $\widetilde{Y}$ between $\widetilde{y}_0$, $\widetilde{y}_1$. Denote by $\widetilde{B}^0$ the path component of $p^{-1}(B)$ that contains $\widetilde{y}_0$ and by B^0 its image under p, ie, the path component of B which contains $p(\widetilde{y}_0)$. Since $p|_{\widetilde{B}^0}\colon \widetilde{B}^0 \to B^0$ is a covering, we have $\widetilde{\beta}\big([0, \frac{1}{2}]\big) \subset \widetilde{B}^0 \subset \widetilde{Y} - D$ and $\widetilde{\beta}\big((\frac{1}{2}, 1]\big) \subset \widetilde{Y} - p^{-1}(B) \subset \widetilde{Y} - D$. So, $\widetilde{\beta}(I) \subset \widetilde{Y} - D$.

Case 3 If $\widetilde{y}_0, \widetilde{y}_1 \in p^{-1}(B) - D$, take a point $\widetilde{y}_2 \in \widetilde{Y} - p^{-1}(B)$ and apply Case 2 to the pairs of points $\widetilde{y}_0$, $\widetilde{y}_2$ and $\widetilde{y}_1$, $\widetilde{y}_2$. □

Now we generalize and unify the lemmas [5, Lemmas 5, 5′]. The proof below contains nothing essentially new in comparison with those of Brooks, except reference to Lemma 7; and we have slightly changed the sequence of exploited ideas. We give the proof nevertheless, for completeness.

Lemma 8 *Suppose that the conditions of Lemma 6 are fulfilled, and for some integer $n \geqslant 3$ we have $\pi_m(Y, Y - B) = 0$ for all $1 \leqslant m \leqslant n-1$. Let $f\colon X \to Y$ be a map of connected, locally path connected space X; and let $p\colon \widehat{Y} \to Y$ be its Hopf covering. Let $\mathcal{C}$ be an arbitrary family of path components of $p^{-1}(B)$ and $D = \bigcup_{C \in \mathcal{C}} C$. Then $\pi_m(\widehat{Y}, \widehat{Y} - D, \widehat{y}') = 0$, $\widehat{y}' \in \widehat{Y} - D$, for all $1 \leqslant m \leqslant n-1$, and the inclusions*

$$i_C\colon (\widehat{Y}, \widehat{Y} - D) \hookrightarrow (\widehat{Y}, \widehat{Y} - C), \quad C \in \mathcal{C},$$

induce an isomorphism

$$\pi_n(\widehat{Y}, \widehat{Y} - D, \widehat{y}') \cong \sum_{C \in \mathcal{C}} \pi_n(\widehat{Y}, \widehat{Y} - C, \widehat{y}').$$

Proof Let $q\colon \widetilde{Y} \to \widehat{Y}$ be a universal covering. Note that Lemma 7 implies path connectedness of the spaces $\widetilde{Y} - q^{-1}(D)$, $\widetilde{Y} - q^{-1}(p^{-1}(B) - D)$, and $\widetilde{Y} - (pq)^{-1}(B)$. We will for brevity omit the basic points in notation of their homotopy groups.

The composition $pq\colon \widetilde{Y} \to Y$ induces an isomorphism

$$\pi_m(\widetilde{Y}, \widetilde{Y} - (pq)^{-1}(B)) \cong \pi_m(Y, Y - B)$$

for each $m > 0$ (for $m = 1$ just a bijection) [33, Theorem 7.2.8]. So, we have $\pi_m(\widetilde{Y}, \widetilde{Y} - (pq)^{-1}(B)) = 0$ for all $1 \leqslant m \leqslant n-1$. In particular,

$$\pi_1(\widetilde{Y}, \widetilde{Y} - (pq)^{-1}(B)) = \pi_2(\widetilde{Y}, \widetilde{Y} - (pq)^{-1}(B)) = 0,$$

so by exactness of the homotopy sequence, $\pi_1(\widetilde{Y} - (pq)^{-1}(B)) \cong \pi_1(\widetilde{Y}) = 0$.

Applying the relative Hurewicz theorem [33, Theorem 7.5.4] to the pair of spaces $(\widetilde{Y}, \widetilde{Y} - (pq)^{-1}(B))$, we obtain $H_m(\widetilde{Y}, \widetilde{Y} - (pq)^{-1}(B)) = 0$ for all $1 \leqslant m \leqslant n-1$.

Represent

$$(**) \qquad \widetilde{Y} = (\widetilde{Y} - q^{-1}(D)) \cup (\widetilde{Y} - q^{-1}(p^{-1}(B) - D)).$$

Note that

$$(\widetilde{Y} - q^{-1}(D)) \cap (\widetilde{Y} - q^{-1}(p^{-1}(B) - D)) = \widetilde{Y} - (pq)^{-1}(B).$$

Writing the Mayer–Vietoris exact sequence for the pairs $(\widetilde{Y}, \widetilde{Y} - q^{-1}(D))$ and $(\widetilde{Y}, \widetilde{Y} - q^{-1}(p^{-1}(B) - D))$,

$$\ldots \to H_i(\widetilde{Y}, \widetilde{Y} - (pq)^{-1}(B)) \to H_i(\widetilde{Y}, \widetilde{Y} - q^{-1}(D)) \oplus H_i(\widetilde{Y}, \widetilde{Y} - q^{-1}(p^{-1}(B) - D)) \\ \to H_i(\widetilde{Y}, \widetilde{Y}) \to H_{i-1}(\widetilde{Y}, \widetilde{Y} - (pq)^{-1}(B)) \to \ldots$$

we obtain $H_m(\widetilde{Y}, \widetilde{Y} - q^{-1}(D)) = 0$ for all $1 \leqslant m \leqslant n-1$.

Apply Seifert–van Kampen theorem to the representation $(**)$ of simply connected space $\widetilde{Y}$. Since the intersection $\widetilde{Y} - (pq)^{-1}(B)$ of the two subspaces was proved above to be simply connected, the two subspaces are also simply connected.

The relative Hurewicz theorem applied to the pair $(\widetilde{Y}, \widetilde{Y} - q^{-1}(D))$ of simply connected spaces gives $\pi_m(\widetilde{Y}, \widetilde{Y} - q^{-1}(D)) = 0$, and therefore $\pi_m(\widehat{Y}, \widehat{Y} - D) = 0$ for all $1 \leqslant m \leqslant n-1$ [33, Theorem 7.2.8]. The first statement is proved.

The inclusions $j_C\colon (\widetilde{Y}, \widetilde{Y} - q^{-1}(D)) \hookrightarrow (\widetilde{Y}, \widetilde{Y} - q^{-1}(C))$, $C \in \mathcal{C}$, induce by Lemma 5 an isomorphism

$$H_n(\widetilde{Y}, \widetilde{Y} - q^{-1}(D)) \cong \sum_{C \in \mathcal{C}} H_n(\widetilde{Y}, \widetilde{Y} - q^{-1}(C)).$$

By naturality of the Hurewicz isomorphism and commutativity of diagrams

$$\begin{array}{ccc} (\widetilde{Y}, \widetilde{Y} - q^{-1}(D)) & \xrightarrow{j_C} & (\widetilde{Y}, \widetilde{Y} - q^{-1}(C)) \\ {\scriptstyle q}\downarrow & & {\scriptstyle q}\downarrow \\ (\widehat{Y}, \widehat{Y} - D) & \xrightarrow{i_C} & (\widehat{Y}, \widehat{Y} - C), \end{array}$$

the map from left to right of the sequence

$$\pi_n(\widehat{Y}, \widehat{Y} - D) \cong \pi_n(\widetilde{Y}, \widetilde{Y} - q^{-1}(D)) \cong \sum_{C \in \mathcal{C}} \pi_n(\widetilde{Y}, \widetilde{Y} - q^{-1}(C)) \cong \sum_{C \in \mathcal{C}} \pi_n(\widehat{Y}, \widehat{Y} - C),$$

where the central map is induced by the family $\{j_C\}$ and the outer maps by q, is induced by the family $\{i_C\}$. This finishes the proof. □

6 Proof of Theorem 1

For theory of obstructions to deformations, see Blakers and Massey [1, (4.4)], Hu [19, Chapter VI, Exercise E; 18, 1] and Schirmer [31, **2**]. We refer below to the paper [31] which contains a good summary of results (but only in the case of simply connected subspace; with evident changes they hold true in general case, for coefficients in local systems of groups).

Proof By (2) of Lemma 2 we may assume that X is itself a polyhedron of dimension less than or equal to n. Moreover, since $\pi_m(Y, Y - B) = 0$ for all $1 \leqslant m \leqslant n-1$, we assume that $f^{-1}(B) \cap X^{(n-1)} = \varnothing$ [31, p 57]. Hence we consider only the case of $\dim X = n \geqslant 3$.

Let $p\colon \widehat{Y} \to Y$ and $\widehat{f}\colon X \to \widehat{Y}$ be a Hopf covering and a Hopf lift for f. Let $\mathcal{C}$ be the family of all those path components $\widehat{B}$ of $p^{-1}(B)$ for which $\widehat{f}^{-1}(\widehat{B})$ is either empty or an inessential preimage class. Denote $D = \bigcup_{C \in \mathcal{C}} C$. For $C \in \mathcal{C}$, the pullback under $\widehat{f}$ of the local coefficient system $\{\pi_n(\widehat{Y}, \widehat{Y} - C, y')\}$ on $\widehat{Y} - C$ is a local system on $X^{(n-1)}$; it extends uniquely up to isomorphism to a local system Γ_C on X by Lemma 4. Similarly, the local system $\{\pi_n(\widehat{Y}, \widehat{Y} - D, y')\}$ on $\widehat{Y} - D$ gives a local system Γ_D on X.

For each $C \in \mathcal{C}$ and $\widehat{y}' \in \widehat{Y} - D$ the homomorphism

$$\pi_n(\widehat{Y}, \widehat{Y} - D, \widehat{y}') \to \pi_n(\widehat{Y}, \widehat{Y} - C, \widehat{y}')$$

induced by the inclusion

$$i_C\colon (\widehat{Y}, \widehat{Y} - D) \hookrightarrow (\widehat{Y}, \widehat{Y} - C)$$

gives in its turn a homomorphism $k_C\colon \Gamma_D \to \Gamma_C$. By Lemma 8,

$$\Big(\sum_{C\in\mathcal{C}} k_C\Big)\colon \Gamma_D \to \sum_{C\in\mathcal{C}} \Gamma_C$$

is an isomorphism, therefore

$$\Big(\sum_{C\in\mathcal{C}} k_{C*}\Big)\colon H^n(X,\Gamma_D) \to \sum_{C\in\mathcal{C}} H^n(X,\Gamma_C)$$

is also an isomorphism.

For $C\in\mathcal{C}$, let $\omega_C \in H^n(X,\Gamma_C)$ be the first obstruction to deforming the map $\widehat{f}\colon X \to \widehat{Y}$ into $\widehat{Y}-C$. Let $\omega_D \in H^n(X,\Gamma_D)$ be the first obstruction to deforming the map $\widehat{f}\colon X\to\widehat{Y}$ into $\widehat{Y}-D$. Then $\omega_C = k_{C*}(\omega_D)$. From our definition of the family $\mathcal{C}$ and Corollary 1 it follows that for each $C\in\mathcal{C}$ the map $\widehat{f}$ can be deformed into $\widehat{Y}-C$, hence $\omega_C = 0$; therefore

$$\Big(\sum_{C\in\mathcal{C}} k_{C*}\Big)(\omega_D) = \sum_{C\in\mathcal{C}} \omega_C = 0.$$

Since $\big(\sum_{C\in\mathcal{C}} k_{C*}\big)$ is an isomorphism, we obtain $\omega_D = 0$. So, there exists a map $\widehat{g}\sim\widehat{f}$ such that $\widehat{g}(X)\subset\widehat{Y}-D$ (see [31, p 57]). The map $g = p\widehat{g}$ is the desired one. □

We give a simple corollary of our theorem.

Corollary 6 *Suppose that X is a finite-dimensional connected polyhedron, Y is a connected triangulated topological manifold without boundary, B is a finite nonempty subpolyhedron of Y, and $\dim X = \dim Y - \dim B \geqslant 3$. Then for each map $f\colon X\to Y$ there exists a map $g\sim f$ such that each Nielsen preimage class of $g\colon X\to Y\supset B$ is topologically essential; in particular, $\mathrm{N_t}(f,B) = \mathrm{MP_{cl}}(f,B)$.*

It follows easily from Theorem 1 and the next easy Lemma.

Lemma 9 *Suppose that Y is a connected triangulated topological manifold without boundary, and a subset B of Y is a finite subpolyhedron such that $\dim B < \dim Y$ and $Y-B$ is connected. Then $\pi_k(Y, Y-B) = 0$ for all $1\leqslant k\leqslant \dim Y - \dim B - 1$.*

Proof Let $1\leqslant k\leqslant \dim Y - \dim B - 1$. It suffices to prove that for an arbitrary map $f\colon (D^k, \partial D^k, x_0)\to (Y, Y-B, y_0)$ there exists a homotopy $\{f_t\}$ of the form $(D^k,\partial D^k)\times I \to (Y, Y-B)$, $f_0 = f$, such that $f_1(D^k)\subset Y-B$ (see eg Postnikov

[29, p 342]). Since B is compact, there exists an integer r such that $f(\partial D^k)$ is contained in those (closed) simplices of the r–th barycentric subdivision of the given triangulation of Y, which do not intersect B. We replace the original triangulation of Y by its r–th barycentric subdivision, saving the same symbol Y. There exists a simplicial approximation g for f such that g and f are homotopic by a homotopy of the form $(D^k, \partial D^k) \times I \to (Y, Y - B)$ (see eg Spanier [33, Theorem 3.4.8]). Applying [30, Theorem 5.3] to subpolyhedra B, $g(D^k)$, $g(\partial D^k)$ of the manifold Y, we obtain an isotopy $F\colon Y \times I \to Y \times I$ of Y constant on $g(\partial D^k)$ such that $F_1(g(D^k) - g(\partial D^k)) \cap B = \varnothing$ and hence $F_1 g(D^k) \subset Y - B$ (as usual, $F_t\colon Y \to Y$ is the t–level map of the isotopy Y, that is, the composition $y \mapsto F(y,t) = (y', t) \mapsto y' = F_t(y)$). Then $\{g_t = F_t g\}\colon (D^k, \partial D^k) \to (Y, Y - B)$ is a homotopy from $g_0 = g$ to a map $g_1\colon (D^k, \partial D^k) \to (Y, Y - B)$ such that $g_1(D^k) \subset Y - B$. This proves the Lemma. □

7 Some other settings

Here we recall how another, at first sight more general, setting [10; 11] can be reduced to the preimage problem.

Let $f_1, \dots, f_r\colon X \to Y$ be continuous maps, $B \subset Y$ a nonempty subset. The common preimage set is

$$\operatorname{Pr}(f_1, \dots, f_r, B) = \{x \in X \mid f_1(x) = \dots = f_r(x) \in B\}.$$

In particular, if B consists of just one point, this is set of common roots; for $B = Y$ this is the coincidence set $\operatorname{Coin}(f_1, \dots, f_r)$ (the possibility to reduce this setting to the preimage problem was noted in [31; 8]). This setting is equivalent to the following preimage problem:

$$\Delta\{f_k\}_{k=1}^r\colon X \to Y^r \supset \Delta B.$$

Indeed, $\operatorname{Pr}(f_1, \dots, f_r, B) = (\Delta\{f_k\}_{k=1}^r)^{-1}(\Delta B)$, and r–tuples of homotopies of maps $f_1, \dots, f_r$ are in one-to-one correspondence with homotopies of the diagonal product $\Delta\{f_k\}_{k=1}^r$. Hence the problem of estimating the numbers

$$\operatorname{MP}(f_1, \dots, f_r, B) = \min_{g_1 \sim f_1, \dots, g_r \sim f_r} |\operatorname{Pr}(g_1, \dots, g_r, B)|$$

$$\text{and} \quad \operatorname{MP}_{\mathrm{cl}}(f_1, \dots, f_r, B) = \min_{g_1 \sim f_1, \dots, g_r \sim f_r} |\{\text{Nielsen classes of } \operatorname{Pr}(g_1, \dots, g_r, B)\}|$$

(where Nielsen classes for this general case are *defined* as those of the corresponding preimage problem; note that for coincidences of two maps this definition agrees with the standard one) is equivalent to finding estimates for $\operatorname{MP}(\Delta\{f_k\}_{k=1}^r, \Delta B)$ and $\operatorname{MP}_{\mathrm{cl}}(\Delta\{f_k\}_{k=1}^r, \Delta B)$.

Consequently, we can carry appropriate invariants and theorems from the preimage problem over this general setting.

Acknowledgements The author is grateful to Professor S A Bogatyi and Professor A V Zarelua for useful discussions, and to Professor D L Gonçalves for sending important materials. The author is obliged to the referee for his (her) remarks which inspired to improve this article much, and thanks the editors Professor M Scharlemann and R Weidmann for their attentive attitude.

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Received: 30 March 2006 Revised: 4 March 2007