Roots in 3–manifold topology

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Let $\mathcal{C}$ be some class of objects equipped with a set of *simplifying moves*. When we apply these to a given object $M \in \mathcal{C}$ as long as possible, we get a *root* of M.

Our main result is that under certain conditions the root of any object exists and is unique. We apply this result to different situations and get several new results and new proofs of known results. Among them there are a new proof of the Kneser–Milnor prime decomposition theorem for 3–manifolds and different versions of this theorem for cobordisms, knotted graphs, and orbifolds.

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1 Introduction

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Our main result is that under certain conditions the root of any object exists and is unique. We apply this result to different situations and get several new results and new proofs of known results. Among them there are a new proof of the Kneser–Milnor prime decomposition theorem for 3–manifolds and different versions of this theorem for cobordisms, knotted graphs, and orbifolds.

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2 Definition, existence and uniqueness of a root

Let Γ be an oriented graph and e an edge of Γ with initial vertex v and terminal vertex w. We will call the transition from v to w an *edge move* on v.

Definition 1 A vertex $R(v)$ of Γ is *a root* of v, if the following holds:

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(1) $R(v)$ can be obtained from v by edge moves.

(2) $R(v)$ admits no further edge moves.

Recall that a set A is *well ordered* if any subset of A has a least element. Basic examples are the set of non-negative integers $\mathbb{N}_0$ and its power $\mathbb{N}_0^k$ with lexicographical order.

Definition 2 Let Γ be an oriented graph with vertex set $V(\Gamma)$ and A a well ordered set. Then a map $c\colon V(\Gamma) \to A$ is called a *complexity function*, if for any edge e of Γ with vertices v, w and orientation from v to w we have $c(v) > c(w)$.

Definition 3 Let Γ be an oriented graph. Then two edges e and d of Γ with the same initial vertex v are called *elementary equivalent*, if their endpoints have a common root. They are called *equivalent* (notation: $e \sim d$), if there is a sequence of edges $e = e_1, e_2, \dots, e_n = d$ such that the edges e_i and e_{i+1} are elementary equivalent for all $i, 1 \le i < n$.

Definition 4 Let Γ be an oriented graph. We say that Γ possesses property (CF) if it admits a complexity function. Γ possesses property (EE) if any two edges of Γ with common initial vertex are equivalent.

It turns out that property (CF) guarantees existence, while property (EE) guarantees uniqueness of the root.

Theorem 1 *Let Γ be an oriented graph possessing properties (CF) and (EE). Then any vertex has a unique root.*

Proof **Existence** Let v be a vertex of Γ. Denote by X the set of all vertices of Γ which can be obtained from v by edge moves. By property (CF), there is a complexity function $c\colon V(\Gamma) \to A$. Since A is well ordered, the set $c(X)$ has a least element a_0. Then any vertex in $c^{-1}(a_0)$ is a root of v.

Uniqueness Assume that v is a least counterexample, ie, v has two different roots $u \ne w$ and $c(v) \le c(v')$ for any vertex v' having more than one root. Let e respectively d be the first edge of an oriented edge path from v toward u respectively w. By property (EE), we have a sequence $e = e_1, e_2, \dots, e_n = d$ such that the edges e_i and e_{i+1} are elementary equivalent for all $i, 1 \le i < n$. Hence, their endpoints v_i, v_{i+1} have a common root r_i. As $c(v_i) < c(v)$ for all i, that root is in fact unique. Thus $u = r_1 = \cdots = r_n = w$ which is a contradiction. □

The following sections are devoted to applications of Theorem 1.

3 A simple proof of the Kneser–Milnor prime decomposition theorem

Definition 5 Let S be a 2–sphere in the interior of a compact 3–manifold M. Then a *spherical reduction* (or *compression move*) of M along S consists in cutting M along S and attaching two balls to the two 2–spheres arising under the cut.

Let us consider the three types of spherical reductions. If S bounds a 3–ball in M, then the reduction of M along S is *trivial*, ie, it produces a copy of M and a 3–sphere. If S does not bound a 3–ball and separates M into two parts, then the reduction along S produces two 3–manifolds M_1, M_2 such that $M = M_1 \# M_2$, the connected sum of M_1, M_2. If S does not separate M, then the reduction along S produces a 3–manifold M_1 such that $M = M_1 \# S^2{\times}S^1$ or $M = M_1 \# S^2\tilde{\times}S^1$ (the latter is possible only if M is non-orientable).

Recall that a 3–manifold M is prime if it is not a connected sum of two 3–manifolds different from S^3. Also, M is irreducible, if it admits no non-trivial spherical reductions.

Theorem 2 (Kneser–Milnor prime decomposition [3; 6]) *Any closed orientable 3–manifold M is a connected sum of prime factors. The factors are determined uniquely up to homeomorphism.*

The same holds for compact 3–manifolds with boundary. For non-orientable 3–manifolds the Kneser–Milnor theorem is also true, but with the following modification: the factors are determined uniquely up to homeomorphism and replacement of the direct product $S^2{\times}S^1$ by the skew product $S^2\tilde{\times}S^1$ and vice versa. Note that these two products are the only 3–manifolds that are prime, but reducible.

To get into the situation of Theorem 1, we introduce an oriented graph Γ. The set of vertices of Γ is defined to be the set of all compact 3–manifolds considered up to homeomorphism and removing all connected components homeomorphic to S^3. Two manifolds M_1, M_2 are joined by an oriented edge from M_1 to M_2 if and only if the union of non-spherical components of M_2 can be obtained from M_1 by a non-trivial spherical reduction and removing spherical components.

Our next goal is to prove that any vertex of Γ has a unique root.

Remark 1 Our choice of vertices of Γ is important. If we did not neglect spherical components, then the root would not be unique. For example, the manifold $S^2{\times}S^1 \# S^2{\times}S^1$ would have two different roots: a 3–sphere and the union of two disjoint 3–spheres.

In order to construct a complexity function, we need Kneser's Lemma[1], see [3].

Lemma 1 *For any compact 3–manifold M there exists an integer constant C_0 such that any sequence of non-trivial spherical reductions consists of no more than C_0 moves.*

Corollary 1 Γ *possesses property (CF).*

Proof We define the complexity function $c\colon V(\Gamma) \to \mathbb{N}_0$ as follows: $c(M)$ is the maximal number of spherical reductions in any sequence of non-trivial spherical reductions of M. By Lemma 1, c is well defined and evidently it is compatible with the orientation on Γ. □

Lemma 2 Γ *possesses property (EE).*

Proof Let S_e, S_d be two non-trivial spheres in M and e, d be the corresponding edges of Γ. We prove the equivalence $e \sim d$ by induction on the number $m = \#(S_e \cap S_d)$ of curves in the intersection assuming that the spheres have been isotopically shifted so that m is minimal.

Base of induction Let $m = 0$, ie $S_e \cap S_d = \varnothing$. Denote by M_e, M_d the manifolds obtained by reducing M along S_e, S_d, respectively. Since $S_e \cap S_d = \varnothing$, S_d survives the reduction along S_e and thus may be considered as a sphere in M_e. Let N be obtained by reducing M_e along S_d. Of course, compression of M_d along S_e also gives N. We claim that any root R of N is a common root of M_e and M_d (and hence the edges e, d are elementary equivalent). Indeed, if the sphere S_d is non-trivial in M_e, then R is a root of M_e by the definition of a root. If S_d is trivial in M_e, then $N = M_e \cup S^3$ and the manifolds M_e, N determine the same vertex of Γ, so again R is a root of M_e. Symmetrically, R is a root of M_d whether or not S_e is trivial in M_d. Therefore, R is a root of M_e and M_d.

Inductive step Suppose that $m > 0$. Using an innermost circle argument, we find a disc $D \subset S_d$ such that $D \cap S_e = \partial D$ and compress S_e along D. By minimality of m, we get two non-trivial spheres S', S'', each disjoint to S_e and intersecting S_d in a smaller number of circles, see Figure 1. Taking one of them (say, S') and denoting the corresponding edge by e', we get $e \sim e'$ and $e' \sim d$ by the inductive assumption. Therefore, $e \sim d$. □

[1] Here is Kneser's original statement of the lemma: Zu jeder M^3 gehoert eine Zahl k mit der folgenden Eigenschaft: Nimmt man mit M^3 nacheinander $k+1$ Reduktionen vor, so ist mindestens eine davon trivial. Durch k (oder weniger) nicht triviale Reduktionen wird M^3 in eine irreduzible M^3 verwandelt.

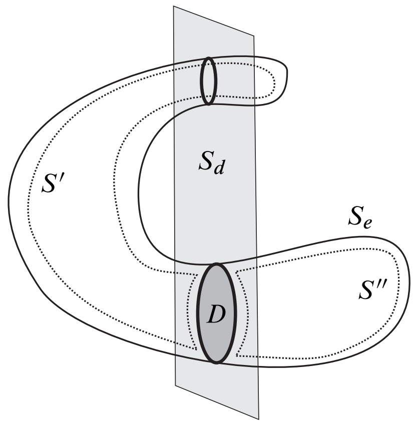

Figure 1: Removing intersections

Theorem 3 *For any 3–manifold M, the root $R(M)$ under spherical reduction exists and is unique up to homeomorphism and removal of spherical components.*

Proof This follows from Theorem 1. □

Note that $R(M)$ is the disjoint union of the irreducible factors of M, ie, of the prime factors other than $S^2\times S^1$ or $S^2\tilde{\times}S^1$. So the root appears more natural than the Kneser–Milnor decomposition and the same is true for other applications of Theorem 1, see the following sections. In order to get the full version of the prime decomposition theorem, additional efforts are needed. The advantage of this two-step method is that the first step (existence and uniqueness of roots) is more or less standard, while all individual features appear only at the second step.

In our situation, note that the number n of factors $S^2\times S^1$ is determined by M. Let M be orientable. Denote by $\Sigma(M)$ the subgroup $\Sigma(M)\subset H_2(M;Z_2)$ consisting of all spherical elements (that is, of elements of the form $f_*(\mu)$, where μ is the generator of $H_2(S^2;Z_2)$ and $f\colon S^2\to M$ is a map). Let us show that n coincides with the rank $r(M)$ of $\Sigma(M)$ over Z_2. This follows from the following observations.

(1) Spherical reductions along separating spheres preserve $\Sigma(M)$ and $r(M)$ while any spherical reduction along a non-separating sphere decreases $r(M)$ by one and kills one $S^2\times S^1$ factor.

(2) After performing all possible spherical reductions, we get a 3–manifold M', which is aspherical (according to the Sphere Theorem) and thus has $r(M')=0$.

We thus easily deduce the Kneser–Milnor theorem from Theorem 3: M is the connected sum of the connected components of $R(M)$ plus n factors $S^2 \times S^1$. The factors are determined uniquely up to homeomorphism.

4 Decomposition into boundary connected sum

By a *disc reduction* of a 3–manifold M we mean cutting M along a proper disc D. If ∂D bounds a disc in ∂M, then the reduction is called *trivial*. It makes little sense to consider disc reductions separately from spherical ones. Usually one uses them together or considers only irreducible manifolds. We start with the first approach. By an SD–reduction we will mean a spherical or a disc reduction.

As in Section 3, we begin by introducing an oriented graph Γ. The set of vertices of Γ is defined to be the set of all compact 3–manifolds, but considered up to homeomorphism and removing connected components homeomorphic to S^3 or D^3. Two vertices M_1, M_2 are joined by an edge oriented from M_1 to M_2 if M_2 can be obtained from M_1 by a non-trivial SD–reduction.

Lemma 3 *Γ possesses properties (CF) and (EE).*

Proof Let M be a 3–manifold. We assign to it two integer numbers $g^{(2)}(\partial M)$ and $s(M)$. The first number is equal to $\sum g^2(F)$, where $g(F)$ is the genus of a component $F \subset \partial M$, and the sum is taken over all components of ∂M. Note that *any* non-trivial disc reduction lowers $g^{(2)}(\partial M)$ while $g^{(1)}(\partial M)$, the sum of the genera of the components $F \subset \partial M$, stays unchanged when ∂D is separating. The second number $s(M)$ is the maximal number of non-trivial spherical reductions on M introduced in the previous section, where we denoted it by $c(M)$. Now we introduce the complexity function $c\colon V(\Gamma) \to \mathbb{N}^2$ by letting $c(M) = (g^{(2)}(\partial M), s(M))$, where the pairs are considered in lexicographical order. It is clear that c is compatible with the orientation of Γ.

For proving property (EE), we apply the same method as in the proof of Lemma 2. Let F_e, F_d be two surfaces such that each of them is either a non-trivial sphere or a non-trivial disc. We will use the same induction on the number m of curves in $F_e \cap F_d$.

Base of induction Let $m = 0$ and let M_e, M_d be the manifolds obtained by reducing M along F_e and F_d, respectively. We wish to prove that M_e and M_d have a common root. Let N be obtained by reducing M_e along F_d as well as by reducing M_d along F_e.

Suppose that either both $F_e \subset M_d$ and $F_d \subset M_e$ are non-trivial, or both are spheres, or one of them is a sphere and the other is a disc (in the latter case the disc is automatically non-trivial, since spherical reductions do not affect the property of a disc to be non-trivial). Then any root of N is a common root of M_e, M_d (the same proof as in Lemma 2 does work). That covers all cases when at least one of the surfaces is a sphere.

Suppose that both surfaces F_e, F_d are discs such that F_e is trivial in M_d. Then the disc $D \subset \partial M_d$ bounded by ∂F_e must contain at least one of the two copies F_d^+, F_d^- of F_d appeared under the cut. Let $S \subset \mathrm{Int} M_d$ be a sphere which runs along F_e and D such that $S \cap F_e = \varnothing$. The last condition guarantees us that S may be considered as a sphere sitting in M as well as in M_e. Now there are two possibilities.

(1) D contains only one copy (say, F_d^+) of F_d (in this case the disc $F_d \subset M_e$ also is trivial). Then the spherical reductions of M_e and M_d along S produce the same manifold N'. So any root of N' is a common root of M_e and M_d.

(2) D contains both copies F_d^+ and F_d^- of F_d (in this case the disc $F_d \subset M_e$ is non-trivial). Note that S survives the reduction along F_e and thus may be considered as a sphere in N. Denote by N' the manifold obtained from M_d by spherical reduction along S. It is easy to see that the spherical reduction of N along S produces a manifold homomorphic to the disjoint union of N' and a 3–ball. Since 3–balls are neglected, any root of N' is a common root of M_e and M_d.

$$\begin{tikzcd}
 & & M \arrow[dll, "F_e"'] \arrow[drr, "F_d"] & & \\
M_e \arrow[dr, "F_d"'] & & & & M_d \arrow[ddll, "S"] \\
 & N \arrow[dr, "S"'] & & & \\
 & & N' & &
\end{tikzcd}$$

The inductive step is performed exactly in the same way as in the proof of Lemma 2. The only difference is that in addition to an innermost circle argument we use an outermost arc argument for decreasing the number of arcs in the intersection of discs. □

Theorem 4 *For any 3–manifold M the SD–root $R(M)$ exists and is unique up to homeomorphism and removal of spherical and ball connected components.*

Proof This follows from Theorem 1 and Lemma 3. □

If a 3–manifold M is irreducible, then one can sharpen Theorem 4 by considering only D–reductions. Any such disc reduction can be realized by removing an open regular neighborhood of D in M and getting a submanifold of M. So any D–root of M is contained in M.

Theorem 5 *For any irreducible 3–manifold M, the D–root $R(M)$ exists and is unique up to isotopy and removal of ball connected components.*

Proof Given M, we begin by introducing an oriented graph $\Gamma = \Gamma(M)$. The set of vertices of Γ is defined to be the set of all compact 3–submanifolds of M, considered up to isotopy and removing connected components homeomorphic to D^3. Two vertices Q_1, Q_2 are joined by an edge oriented from Q_1 to Q_2 if Q_2 can be obtained from Q_1 by a non-trivial D–reduction. Property (CF) for Γ is evident: one can take the complexity function $c\colon V(\Gamma) \to \mathbb{N}$ by letting $c(Q) = g^{(2)}(\partial Q)$. Property (EE) can be proved exactly in the same way as in the proof of Lemma 3. The only difference is that cutting along a trivial disc produces the same manifold plus a ball (because of irreducibility) and thus preserves the corresponding vertex of Γ. Therefore, the conclusion of the theorem follows from Theorem 1. $\square$

Remark 2 This gives another proof of Bonahon's Theorem that the *characteristic compression body* (which can be defined as the complement to a D–root $R(M) \subset \mathrm{Int} M$) is unique up to isotopy, see [1].

In the previous section we have mentioned that there is only one closed orientable 3–manifold which is reducible and prime. On the contrary, the number of boundary reducible 3–manifolds which are ∂–prime (ie, prime with respect to boundary connected sums) is infinite. For example, if we take an irreducible boundary irreducible 3–manifold with $n > 1$ boundary components $C_1, \dots C_n$ and join each C_i, $i < n$, with C_{i+1} by a solid tube, we obtain a boundary reducible ∂–prime manifold. (In analogy to the closed case, the ∂–reductions take place along non-separating discs). Nevertheless, any compact 3–manifold has a unique decomposition into a boundary connected sum of ∂–prime factors. This theorem was first proved by A Swarup [8]. Below we show that for irreducible 3–manifolds it can easily be deduced from Theorem 5.

Theorem 6 *Any irreducible 3–manifold M is a boundary connected sum of ∂–prime factors. The factors are determined uniquely up to homeomorphism and – in the non-orientable case – replacement of the direct product $D^2 \times S^1$ by the skew product $D^2 \tilde{\times} S^1$ and vice versa.*

Proof Existence of a boundary prime decomposition is evident. Let us prove the uniqueness. Recall that any disc reduction can be realized by cutting out a regular neighborhood of a proper disc. We are allowed also to cast out 3–balls. It follows that we may think of the root $N = R(M)$ of M (which is well defined by Theorem 5) as sitting in M such that M can be obtained from N by adding disjoint 3–balls and attaching handles of index one. Therefore M can be presented as $M = N \cup H$ such that H is a union of disjoint handlebodies and $N \cap H$ is a collection of discs on ∂N.

One may easily achieve that the intersection of any connected component N_i with H consists of no more than one disc, see Figure 2.

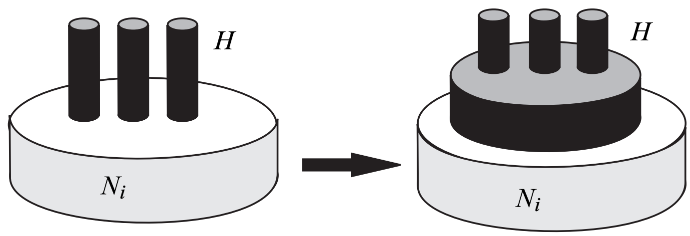

Figure 2: The intersection of N_i with H consists of no more than one disc.

Moreover, the presentation $M = N \cup H$ as above is unique up to isotopy.

We now wish to cut H to get from the root to a prime decomposition, see Figure 3:

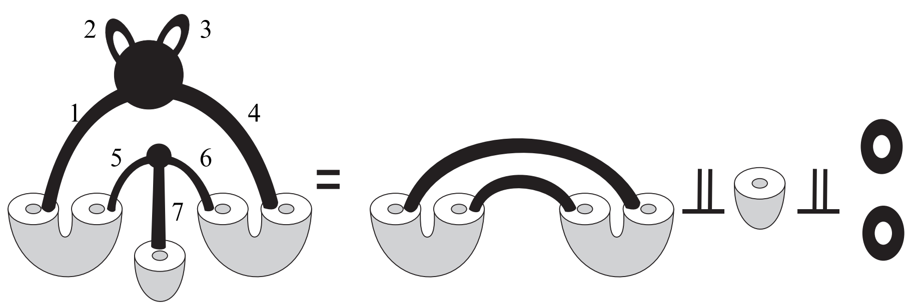

Figure 3: H consists of two balls and seven handles.

Temporarily shrink each connected component of N to a red point and each component of H to a core with exactly one non-red (say: green) vertex. Call the resulting ancillary graph G.

G *admits cutting* if there is a graph $\widehat{G}$ (again with vertices coloured green and red) such that $\widehat{G}$ has one more component than G and $G = \widehat{G}/x_1 \sim x_2$ for some green vertices x_1, x_2 of $\widehat{G}$.

Note that the cutting loci (and thus also the outcome of a maximal cutting procedure) are uniquely determined by the equivalence relation generated by the following rule: edges outgoing from the green vertex x are elementary equivalent if they lie in the same connected component of $G - \{x\}$.

Let M' be obtained from M by cutting along discs corresponding to such cuts of G.

It follows from the construction that the connected components of the resulting new manifold M' are the ∂–prime factors of M. $\square$

5 Roots of knotted graphs

Now we will consider pairs of the type (M, G), where M is a compact 3–manifold and G is an arbitrary graph (compact one-dimensional polyhedron) in M. Recall that a 2–sphere $S \subset M$ is *in general position* (with respect to G), if it does not pass through vertices of G and intersects edges transversely. It is *clean* if its intersection with G is empty.

Definition 6 Let S be a general position sphere in (M, G). Then the *reduction* (or *compression*) of (M, G) along S consists in cutting (M, G) along S and taking disjoint cones over $(S_\pm, S_\pm \cap G)$, where $S_\pm$ are the two copies of S appearing under the cut.

Equivalently, the reduction along S can be described as compressing S to a point and cutting the resulting singular manifold along that point.

If (M', G') is obtained from (M, G) by reduction along S, we write $(M', G') = (M_S, G_S)$. The two cone points in M_S are called *stars*. They lie in G_S if and only if $S \cap G \neq \varnothing$.

It makes little sense to consider all possible spherical reductions, since then there would be no chance to get existence and uniqueness of a root, see below. In order to describe allowable reductions of the pair (M, G), we introduce two properties of spheres in (M, G).

Definition 7 Let S be a 2–sphere in (M, G). Then S is called

(1) *compressible* if there is a disc $D \subset M$ such that $D \cap S = \partial D$, $D \cap G = \varnothing$, and each of the two discs bounded by ∂D on S intersects G; otherwise S is *incompressible*;

(2) *admissible* if $S \cap G$ consists of no more than three transverse crossing points.

The following examples show that compressions along compressible and inadmissible spheres may produce different roots.

Example 1 Take the knot k in $M = S^2 \times S^1$ shown in Figure 4.

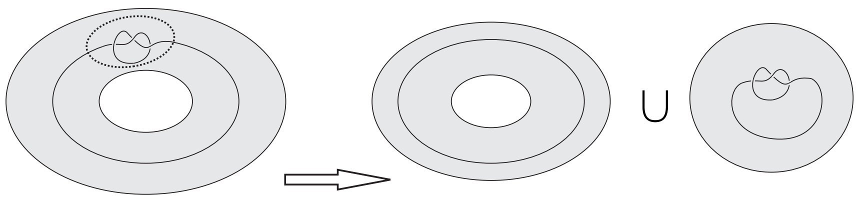

Figure 4: Example of a spherical reduction along a compressible sphere

k mainly follows $\{*\} \times S^1$, but has a trefoil in it. If we allowed to perform reduction along the dotted sphere S (which is compressible!), then (M, k) would split off a 3–sphere containing the trefoil knot. But note that k is in fact equivalent to $\{*\} \times S^1$. Indeed, by deforming some little arc of the trefoil all the way across $S^2 \times \{*\}$, we can change an overcrossing to an undercrossing so that the knot k comes undone. Thus, (M, k) would equal (M, k) plus a non-trivial summand and there would be no hope for uniqueness of the root.

Example 2 Let (M, G) be the standard circle with two parallel chords in S^3. As we see from Figure 5, compressions of (M, G) along two different spheres (admissible and non-admissible) produce two different roots.

Note that the existence of an admissible compressible sphere implies the existence of either a separating point of G or an $(S^2 \times S^1)$–summand of M.

Definition 8 A sphere S in (M, G) is called *trivial* if it bounds a ball $V \subset M$ such that the pair $(V, V \cap G)$ is homeomorphic to the pair $(\mathrm{Con}(S^2), \mathrm{Con}(X))$, where $X \subset S^2$ consists of ≤ 3 points and Con is the cone. An incompressible admissible non-trivial sphere is called *essential*.

Note that reduction of (M, G) along a trivial sphere produces a homeomorphic copy of (M, G) and a *trivial* pair (S^3, G), where G is either empty, or a simple arc, or an unknotted circle, or an unknotted (ie contained in a disc) theta–curve.

Let G be a knotted graph in a 3–manifold M. Following the same lines as in the previous sections, we begin by introducing an oriented graph Γ. The set $V(\Gamma)$ of vertices is defined to be the set of all pairs (M, G), considered up to homeomorphism of

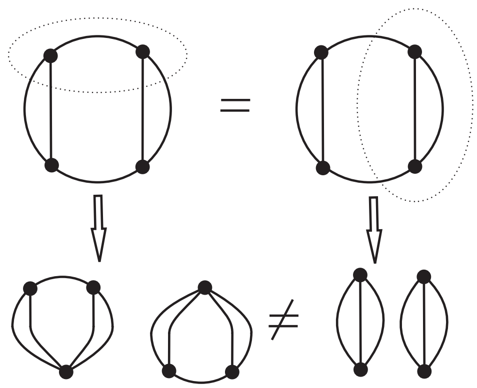

Figure 5: Example of a spherical reduction along an inadmissible sphere

pairs and removing connected components homeomorphic to trivial pairs. Two vertices $(M_1, G_1), (M_2, G_2)$ are joined by an edge oriented from (M_1, G_1) to (M_2, G_2) if (M_2, G_2) can be obtained from (M_1, G_1) by reduction along some essential sphere.

We need an analogue of Lemma 1 (Kneser's Lemma) for manifolds with knotted graphs.

Lemma 4 *Suppose that (M, G) contains no clean essential spheres, ie, that the manifold $M \setminus G$ is irreducible. Then there is a constant C_1 depending only on (M, G) such that any sequence of reductions along essential spheres consists of no more than C_1 moves.*

Proof We choose a triangulation T of (M, G) such that G is the union of (some of the) edges and vertices of T. Let $C_1 = 10t$, where t is the number of tetrahedra in T. Consider a sequence $S_1, \dots, S_n \subset (M, G)$ of $n > C_1$ disjoint spheres such that each sphere S_k is essential in the pair (M_k, G_k) obtained by reducing $(M_0, G_0) = (M, G)$ along $S_1, \dots, S_{k-1}$. It is easy to see that the spheres are essential in (M, G) and not parallel one to another.

We claim that the union $F = S_1 \cup \dots \cup S_n$ can be shifted into normal position by an isotopy of the pair (M, G). To prove that, we adjust to our situation two types of moves for the standard normalization procedure of arbitrary incompressible surface in an irreducible 3–manifold.

Tube compression Let D be a disc in (M, G) such that $D \cap F = \partial D$ and D does not intersect the edges of T (in practice D lies either in a face or in the interior of a tetrahedron). Since F is a collection of incompressible spheres, compression along D produces a copy F' of F and a clean sphere S'. By irreducibility of $M \setminus G$, S' bounds a clean ball, which helps us to construct an isotopy of (M, G) taking F to F'.

Isotopy through an edge Let D be a disc in a tetrahedron Δ such that $D \cap F$ is an arc in ∂D and ∂D intersects an edge e of Δ along the complementary arc of ∂D. If e were in G, by incompressibility there could be no further intersection of G and the component of F containing the arc of ∂D. As each clean sphere bounds a ball this component would be a trivial sphere in (M, G). Thus, e is not in G, so we can use D to construct an isotopy of (M, G) removing two points in $F \cap e$.

Performing those moves as long as possible, we transform F into normal position.

Now we notice that the normal surface F decomposes M into pieces (called *chambers*) and crosses each tetrahedron of T along triangle and quadrilateral pieces (called *patches*). Let us call a patch *black*, if it does not lie between two parallel patches of the same type. Each tetrahedron contains at most 10 black patches: at most 8 triangle patches and at most 2 quadrilateral ones. Since $n > 10t$, at least one of the spheres is *white*, ie contains no black patches. Let C be a chamber such that ∂C contains a white sphere and a non-white sphere. Then C crosses each tetrahedron along some number of prisms of the type $P \times I$, where P is a triangle or a quadrilateral.

Since the patches $P \times \{0, 1\}$ belong to different spheres, C is $S^2 \times I$.

This contradicts our assumption that the spheres be not parallel. □

Lemma 5 *Let a pair (M_S, G_S) be obtained from a pair (M, G) by reduction along an incompressible sphere S such that $S \cap G \neq \varnothing$. Then $M \setminus G$ is reducible if and only if so is $M_S \setminus G_S$.*

Proof Let S_1 be a clean essential sphere in M. Suppose that $S \cap S_1 \neq \varnothing$. Using an innermost circle argument, we find a disc $D_1 \subset S_1$ such that $D_1 \cap S = \partial D_1$. Since S is incompressible, ∂D_1 bounds a clean disc $D \subset S$. Now we may construct a new clean sphere $S_1' \subset M$ such that $S_1' \cap S$ consists of a smaller number of circles than $S_1 \cap S$. Indeed, if the sphere $D \cup D_1$ is essential, we take S_1' to be a copy of $D \cup D_1$ shifted away from S. If $D \cup D_1$ bounds a clean ball, we use this ball for constructing an isotopy of D_1 to the other side of S. This isotopy takes S_1 to S_1'.

Doing so for as long as possible, we get a clean essential sphere $S_1 \subset M$ such that $S \cap S_1 = \varnothing$. It follows that then S_1, considered as a sphere in M_S, is essential.

The proof of the lemma in the reverse direction is evident. □

Lemma 6 *For any pair (M, G) there exists a constant C such that any sequence of reductions along essential spheres consists of no more than C moves.*

Proof Let $S_1, \dots S_n \subset (M, G)$ be such a sequence of essential spheres. For any $k, 1 \le k \le n$, we denote by (M_k, G_k) the pair obtained from $(M_0, G_0) = (M, G)$ by reductions along the spheres $S_1, \dots, S_k$. We may assume that the last pair (M_n, G_n) admits no further reductions along clean essential spheres. Otherwise we extend the sequence of reductions by new reductions along clean essential spheres until we get a pair with irreducible graph complement. Denote also by (M'_k, G'_k) the pair obtained from (M_k, G_k) by additional reductions along all remaining clean spheres from the sequence $S_1, \dots S_n$. Note that (M'_n, G'_n) is obtained from (M'_k, G'_k) by reductions along dirty spheres and that $M'_n \setminus G'_n$ is irreducible. By Lemma 5 $M'_k \setminus G'_k$ also is irreducible and hence contains no clean essential spheres.

It is convenient to locate the set X of *clean stars* (the images of the cone points under reductions of all clean essential spheres from $S_1, \dots S_n$). Then X consists of no more than $2C_0$ points, where $C_0 = C_0(M, G)$ is the constant from Lemma 1 for a compact 3–manifold whose interior is $M \setminus G$. We may think of X as being contained in all (M'_k, G'_k).

Let us decompose the set $S_1, \dots S_n$ into three subsets U, V, W as follows:

(1) $S_k \in U$ if S_k is clean.

(2) $S_k \in V$ if S_k, considered as a sphere in (M'_{k-1}, G'_{k-1}), is an essential sphere (necessarily dirty).

(3) $S_k \in W$ if S_k is a trivial dirty sphere in (M'_{k-1}, G'_{k-1}).

Now we estimate the numbers $\#U, \#V, \#W$ of spheres in U, V, W. Of course, $\#U \le C_0$ and $\#V \le C_1$, where C_0 is as above and $C_1 = C_1(M'_0, G'_0)$ is the constant from Lemma 4. Let us prove that $\#W \le 2C_0$. Indeed, the reduction along each sphere $S_k \in W$ transforms (M'_{k-1}, G'_{k-1}) into a copy of (M'_{k-1}, G'_{k-1}) and a trivial pair $(S^3_{k-1}, \Gamma_{k-1})$ containing at least one clean star. Since no star can appear in two different trivial pairs and since the total number of clean stars does not exceed $2C_0$, we get $\#W \le 2C_0$. Combining these estimates, we get $n \le C = 3C_0 + C_1$. □

Lemma 7 *The graph $\Gamma = \Gamma(M, G)$ possesses properties (CF) and (EE).*

Proof We define the complexity function $c \colon V(\Gamma) \to \mathbb{N}_0$ just as in the proof of Corollary 1: $c(M, G)$ is the maximal number of reductions in any sequence of essential spherical reductions of (M, G). By Lemma 6, this is well defined. Evidently, c is compatible with the orientation.

The proof of property (EE) is similar to the proof of the same property for the case $G = \varnothing$, see Lemma 2. Let S_e, S_d be two essential spheres in (M, G) corresponding to edges e, d of Γ. We prove the equivalence $e \sim d$ by induction on the number $m = \#(S_e \cap S_d)$ of curves in the intersection assuming that the spheres have been shifted by isotopy of (M, G) so that m is minimal. The base of the induction, when $m = 0$, is evident, since reductions along disjoint spheres commute and thus, just as in the proof of Lemma 2. produce knotted graphs having a common root.

Let us prove the inductive step. Suppose that S_d contains a disc D such $D \cap S_e = \partial D$. Assume that D is clean, ie $D \cap G = \varnothing$. Then we compress S_e along D. We get two spheres S', S'', each disjoint with S_e and intersecting S_d in a smaller number of circles. Since S_e is incompressible and m is minimal, at least one of them (say S') must be clean and essential. As $S_e \sim S'$ and $S' \sim S_d$ by the inductive assumption, we get $S_e \sim S_d$.

Now we may assume that S_d (and, by symmetry, S_e) contain no innermost clean discs. Since S_d contains at least two innermost discs and $\#(S_d \cap G) \leq 3$, there is an innermost disc $D \subset S_d$ crossing G at exactly one point. Its boundary decomposes S_e into two discs D', D'', both crossing G. Since $\#(S_e \cap G) \leq 3$, at least one of them (let D') crosses G at exactly one point. Then the sphere $S'_e = D \cup D'$ is admissible, incompressible, and non-trivial. Moreover, it is actually disjoint with S_e and crosses S_d in less than m circles. Using the inductive assumption, we get $S_e \sim S_d$ again. □

Theorem 7 *For any knotted graph (M, G) the root $R(M, G)$ exists and is unique up to homeomorphism and removal of trivial components.*

Proof This follows from Theorem 1 and Lemma 7. □

According to our definition of the graph Γ for the case of knotted graphs, its vertices and hence roots of knotted graphs are defined only modulo removing trivial pairs. One of the advantages of roots introduced in that manner is the flexibility of their construction: each next reduction can be performed along *any* essential sphere. We pay for that by the non-uniqueness: roots of (M, G) can differ by their trivial connected components. This is natural, but might seem to be inconvenient. We improve this by introducing *efficient roots*, which are free from that shortcoming (the idea is borrowed from Petronio [7]).

Definition 9 A system $\mathcal{S} = S_1 \cup \dots \cup S_n$ of disjoint incompressible spheres in (M, G) is called *efficient* if the following holds:

(1) reductions along all the spheres give a root of (M, G);

(2) any sphere S_k, $1 \leq k \leq n$, is essential in the pair $(M_{\mathcal{S} \setminus S_k}, G_{\mathcal{S} \setminus S_k})$ obtained from (M, G) by reductions along all spheres S_i, $1 \leq i \leq n$, except S_k.

Efficient systems certainly exist; to get one, one may construct a system satisfying (1) and merely throw away one after another all spheres not satisfying (2). Having an efficient system, one can get another one by the following moves.

(1) Let $a \subset (M, G)$ be a clean simple arc which joins a sphere S_i with a clean sphere $S_j, i \neq j$, and has no common points with $\mathcal{S}$ except its ends. Then the boundary ∂N of a regular neighborhood $N(S_i \cup a \cup S_j)$ consists of a copy of S_i, a copy of S_j, and the (interior) connected sum $S_i \# S_j$ of S_i and S_j. The move consists in replacing S_i by $S_i \# S_j$.

(2) The same, but with the following modifications:
 i) a is a simple subarc of G such that all vertices of G contained in a have valence two, and
 ii) S_j crosses G in two points.

Both moves are called *spherical slidings*, see Figure 6. Note that spherical slidings do not affect the corresponding root.

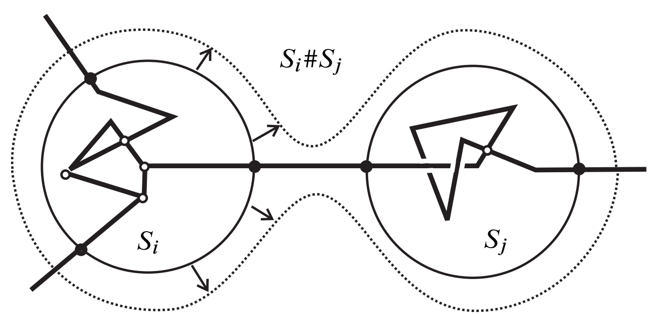

Figure 6: Spherical sliding

Definition 10 Two efficient system in (M, G) are *equivalent* if one system can be transformed into the other by a sequence of spherical slidings and an isotopy of (M, G).

The following theorem can be proved by a modification of the proof of property (EE) in Lemma 7. See [5] for details.

Theorem 8 *Any two efficient systems in (M, G) are equivalent.*

Definition 11 A root of (M, G) is *efficient*, if it can be obtained by reducing (M, G) along all spheres of an efficient system.

Theorem 9 *For any (M,G) the efficient root exists and is unique up to homeomorphism.*

Proof This is evident, since spherical slidings of an efficient system do not affect the corresponding root. □

Remark 3 Theorem 9 easily implies the Schubert Theorem on the uniqueness of prime knot decomposition in S^3 as well as the corresponding theorem for knots in any irreducible 3–manifold. Indeed, the connected components of the efficient root of (M,K) are exactly the prime factors of the knot K.

6 Colored knotted graphs and orbifolds

Let $\mathcal{C}$ be a set of colors. (For example think of $\mathcal{C}=\mathbb{N}$.) By a *coloring* of a graph G we mean a map $\varphi\colon E(G)\to\mathcal{C}$, where $E(G)$ is the set of all edges of G.

Definition 12 Let G_φ be a colored graph in a 3–manifold M. Then the pair (M,G_φ) is called *admissible*, if there is no incompressible sphere in (M,G_φ) which crosses G_φ transversely in two points of different colors.

It follows from the definition that if (M,G_φ) is admissible, then G_φ has no valence two vertices incident to edges of different colors. We define reductions along admissible spheres, trivial pairs, roots, efficient systems, spherical slidings, and efficient roots just in the same way as for the uncolored case.

Theorem 10 *For any admissible pair (M,G_φ) the root exists and is unique up to color preserving homeomorphisms and removing trivial pairs. Moreover, any two efficient systems in (M,G_φ) are equivalent and thus the efficient root is unique up to color preserving homeomorphisms.*

Proof The proof is literally the same as for the uncolored case. There is only one place where one should take into account colorings: the last paragraph of the proof of Lemma 7. Indeed, in this paragraph there appears an incompressible sphere S'_e that crosses G in two points. We need to know that these points have the same colors, and exactly for that purpose one has imposed the restriction that the pair (M,G_φ) must be admissible. □

Further generalizations of the above result consist in specifying sets of *allowed* single colors, pairs of colors, and triples of colors. The idea is to define admissibility of spheres according to whether their intersection with G_φ belongs to one of the specified sets and allow reductions only along those spheres. Again, all proofs, in particular, the proof of the corresponding version of Theorem 10, are literally the same with only one exception where we need that (M, G_φ) is admissible. We naturally obtain a generalized version of the *orbifold splitting theorem* proved recently by C Petronio [7].

Recall that a 3–orbifold can be described as a pair (M, G_φ), where all vertices of G_φ have valence smaller than or equal to 3, all univalent vertices are in ∂M, and G_φ is colored by the set $\mathcal{C}$ of all integer numbers greater than 1. We specify the following sets of colors:

We allow no single colors, ie, we do not perform reductions along spheres crossing G_φ at a single point. The set of allowed pairs consists of pairs $(n, n), n \geq 2$. The allowed triples are the following: $(2, 2, n), n \geq 2$, and $(2, 3, k), 3 \leq k \leq 5$. See [7] for background. An orbifold (M, G_φ) is called admissible, if it is admissible in the above sense, ie, if there is no incompressible sphere in (M, G_φ) which crosses G_φ transversely in two points of different colors.

In view of the previous discussion, the following theorem is an easy consequence of Theorem 10.

Theorem 11 *For any admissible orbifold (M, G_φ) the root exists and is unique up to orbifold homeomorphisms and removing trivial pairs. Moreover, any two efficient systems in (M, G_φ) are equivalent and thus the efficient root is unique up to orbifold homeomorphism.*

7 Roots versus prime decompositions

In Section 3 we have seen that the existence and uniqueness of roots of 3–manifolds with respect to spherical reductions (that is, *spherical roots*) is very close to the existence and uniqueness of prime decompositions with respect to connected sums. Indeed, to get the disjoint union of prime factors one should merely perform as long as possible spherical reductions along *separating* essential spheres. The same is true for prime decompositions of irreducible manifolds with respect to boundary connected sums and for prime decompositions of knotted graphs and orbifolds with respect to taking connected sums (which are inverse operations to reductions along separating essential spheres).

In contrast to that, the uniqueness of prime factors requires additional arguments. For decompositions of 3–manifolds into connected sums and boundary connected sums such arguments are given in Sections 3 and 4. The uniqueness of prime decompositions of knotted graphs is not known yet.

Conjecture 1 Any knotted graph is a connected sum of prime factors. The factors are determined uniquely up to homeomorphism.

The uniqueness of prime decompositions of orbifolds is also unsettled. The main result of [7] does not solve the problem, since it works only for orbifolds without non-separating 2–suborbifolds. So the following conjecture remains unsettled.

Conjecture 2 Any 3–dimensional orbifold is a connected sum of prime factors. The factors are determined uniquely up to homeomorphism.

8 Annular roots of manifolds

In addition to spherical and and disc (S– and D–) reductions from Sections 3 and 4 we introduce A–reductions (reductions along annuli).

Definition 13 Let A be an annulus in M such that its boundary circles lie in different components of ∂M. Then we cut M along A and attach two plates $D^2_1\times I, D^2_2\times I$ by identifying their base annuli $\partial D^2_1\times I, \partial D^2_2\times I$ with the two copies of A, which appear under cutting.

A reduction along an annulus $A \subset M$ is called *trivial*, if A is compressible, and non-trivial otherwise. Note that incompressible annuli having boundary circles in different components of ∂M are automatically essential, ie, not only incompressible, but also boundary incompressible. It makes little sense to consider annular reductions separately from spherical and disc ones. We will use them together calling them SDA*–reductions*. As above, we begin by introducing an oriented graph Γ. The set of vertices of Γ is defined to be the set of all compact 3–manifolds, but considered up to homeomorphism and removing connected components homeomorphic to S^3 or D^3. Two vertices M_1, M_2 are joined by an edge oriented from M_1 to M_2 if M_2 can be obtained from M_1 by a non-trivial SDA–reduction.

Our next goal is to prove that Γ possesses properties (CF) and (EE). The inductive proof of property (EE) is based on the following lemma, which helps us to settle the base of induction.

Lemma 8 *Let a 3–manifold N be obtained from a 3–manifold M by annular reduction along a compressible annulus $A \subset M$. Then N contains a sphere S such that the spherical reduction N_S of N can be obtained from M by cutting along two disjoint proper discs.*

Proof Let D be a compressing disc for A. Denote by U a closed regular neighborhood of $A \cup D$ in M. Then the relative boundary $\partial_{rel} U = \mathrm{Cl}(\partial N \cap \mathrm{Int} M)$ consists of a parallel copy of A and two proper discs D', D''. Denote by S a 2–sphere in N composed from a copy of D and a core disc of one of the attached plates, see Figure 7. Then it is easy to see that the manifold N_S obtained from N by spherical reduction along S is homeomorphic to the result of cutting M along D' and D''. □

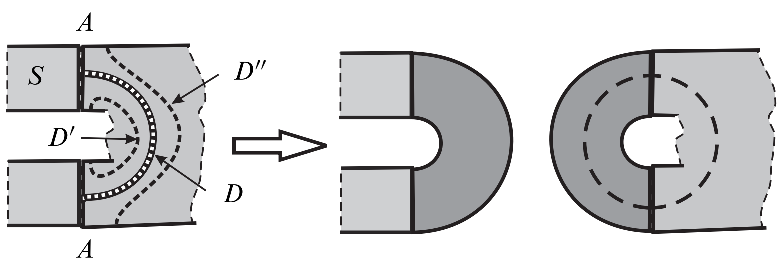

Figure 7: Reduction along compressible annulus

Lemma 9 Γ *possesses properties (CF) and (EE).*

Proof Property (CF) is easy:
The complexity function $c(M) = (g^{(2)}(\partial M), s(M))$ introduced in the proof of Lemma 3 works.

To prove property (EE), consider two surfaces in M corresponding to edges e, d of Γ. Each surface is either a non-trivial sphere or disc, or an incompressible annulus having boundary circles in different components of ∂M. We prove the equivalence $e \sim d$ by induction on the number $\#(F_e \cap F_d)$ of curves in the intersection assuming that the surfaces have been isotopically shifted so that this number is minimal.

Base of induction Let $m = 0$, ie $F_e \cap F_d = \varnothing$. Denote by M_e, M_d the manifolds obtained by reducing M along F_e, F_d, respectively. Since $F_e \cap F_d = \varnothing$, F_d survives the reduction along F_e and thus may be considered as a surface in F_e. Let N be obtained by reduction of F_e along F_d. Of course, reduction of M_d along F_e also gives N. We claim that there is a root R of N which is a common root of M_e and M_d (and hence the edges e, d are elementary equivalent).

Indeed, if both surfaces $F_e \subset M_d$, $F_d \subset M_e$ are non-trivial, then any root of N is a common root of M_e and M_d. If one of them is a trivial sphere or a trivial disc, then the same tricks as in the proofs of Lemmas 2 and 3 do work.

Suppose that one of the surfaces (let F_e) is a trivial (ie compressible) annulus. Then we apply Lemma 8 and get a manifold N' such that any root of N' is a common root of M_e and M_d.

Now we suppose that both annuli F_e, F_d are trivial. Then we apply Lemma 8 twice: construct two manifolds N', N'' such that each can be obtained from M_e (respectively, M_d) by cutting along two disjoint discs. Since both N', N'' are spherical reductions of N, they have a common root by Lemma 2.

Inductive assumption Any two edges e, d with $\#(F_e \cap F_d) \leq m$ are equivalent.

Inductive step Suppose that $\#(F_e \cap F_d) \leq m+1$. We may assume that $F_e \cap F_d$ contains no trivial circles and trivial arcs. Otherwise we could apply an innermost circle or an outermost arc argument just as in the proof of Lemmas 2 and 3. It follows that F_e and F_d are annuli such that $F_e \cap F_d$ consists either of non-trivial circles (which are parallel to the core circles of the annuli) or of non-trivial arcs (which join different boundary circles of the annuli).

First we suppose that $F_e \cap F_d$ consists of non-trivial circles of F_e and F_d. Then one can find two different components A, B of ∂M such that a circle of ∂F_e is in A and a circle of ∂F_d is in B. Denote by s the first circle of $F_e \cap F_d$ we meet at our radial way along F_e from the circle $\partial F_e \cap A$ to the other boundary circle of F_e. Let F'_e be the subannulus of F_e bounded by $\partial F_e \cap A$ and s, and F'_d the subannulus of F_d bounded by s and $\partial F_d \cap B$. Then the annulus $F'_e \cup F'_d$ is essential and is isotopic to an annulus X such that $\#(X \cap F_e) < m$ and $\#(X \cap F_d) = 0$, see Figure 8 (to get a real picture, multiply by S^1). It follows from the inductive assumption that $e \sim x \sim d$, where x is the edge corresponding to the annulus X.

Now we suppose that $F_e \cap F_d$ consists of more than one radial segments, each having endpoints in different components of ∂F_e and different components of ∂F_d. Let $s_1, s_2 \subset F_e \cap F_d \subset F_e$ be two neighboring segments. Denote by D the quadrilateral part of F_e between them.

Case 1 First we assume that F_d crosses F_e at s_1, s_2 in opposite directions. This means that each part of $F_d \setminus (s_1 \cup s_2)$ approaches D from the same side. Then we cut F_d along s_1, s_2 and attach to it two parallel copies of D lying on different sides of F_e. We get a new surface F'_d consisting of two disjoint annuli, at least one of which (denote it by X) is essential, see Figure 9 to the left. The real picture showing the behavior of the annuli in a neighborhood of D can be obtained by multiplying by I.

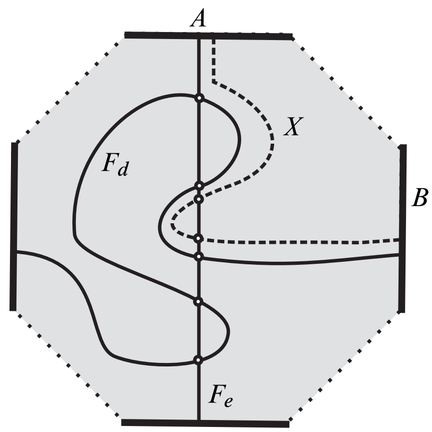

Figure 8: X is disjoint with F_d and crosses F_e in a smaller number of circles.

Since $\#(X \cap F_e) \leq m-2$ and, after a small isotopy of X, $\#(X \cap F_d) = \varnothing$, we get $e \sim x \sim d$, where x is the edge corresponding to the annulus X.

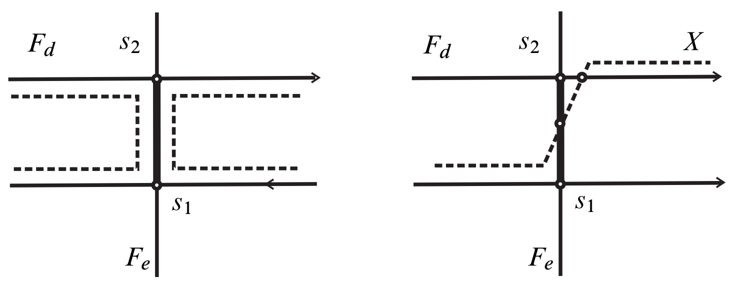

Figure 9: X crosses each annulus in a smaller number of segments.

Case 2 We assume now that at all segments F_d crosses F_e at s_1, s_2 in the same direction (say, from left to right). Then s_1, s_2 decompose F_d into two strips L_1, L_2 such that L_1 approaches s_1 from the left side of F_e and s_2 from the right side. Then the annulus $L_1 \cup D$ is isotopic to an annulus X such that $\#(X \cap F_e) \leq m-1$ and $\#(X \cap F_d) = 1$, see Figure 9 to the right. Since X crosses F_e one or more times in the same direction, it is essential. Therefore, we get $e \sim x \sim d$ again.

Case 3 Suppose M is not homeomorphic to $S^1 \times S^1 \times I$ and F_e and F_d are annuli such that $F_e \cap F_d$ consists of one radial segment. Denote by F'_d the relative boundary $\partial_{rel}(N) = \mathrm{Cl}(\partial N \cap \mathrm{Int} M)$ of a regular neighborhood $N = N(F_e \cup F_d)$ in M. Then F'_d is an annulus having boundary circles in different components of ∂M.

Case 3.1 If F'_d is incompressible, then we put $X = F'_d$.

Case 3.2 If F'_d admits a compressing disc D, then the relative boundary of a regular neighborhood $N = N(F'_d \cup D)$ consists of a parallel copy of F'_d and two proper discs D_1, D_2. If at least one of these discs (say, D_1) is essential, then we put $X = D_1$.

Case 3.3 Suppose that the discs D_1, D_2 are not essential. Then the circles $\partial D_1, \partial D_2$ bound discs D'_1, D'_2 contained in the corresponding components of ∂M. We claim that at least one of the spheres $S_1 = D_1 \cup D'_1, S_2 = D_2 \cup D'_2$ (denote it by X) must be essential. Indeed, if both bound balls, then M is homeomorphic to $S^1 \times S^1 \times I$, contrary to our assumption.

In all three cases 3.1–3.3, X is disjoint to F_e as well as to F_d. Therefore, $e \sim x \sim d$, where x is the edge corresponding to the annulus X.

Case 4 This is the last logical possibility. Suppose that $M = S^1 \times S^1 \times I$. Then $e \sim d$ since reducing $M = S^1 \times S^1 \times I$ along any incompressible annulus having boundary circles in different components of M produces the same manifold $S^2 \times I$. □

Theorem 12 *For any 3–manifold M the SDA–root $R(M)$ exists and is unique up to homeomorphism and removal of spherical and ball connected components.*

Proof This follows from Theorem 1 and Lemma 9. □

It turns out that the condition on boundary circles of annuli to lie in different components of ∂M is essential. Below we present an example of a 3–manifold M with two incompressible boundary incompressible annuli $A, B \subset M$ such that ∂M is connected and reductions of M along A and along B lead us to two different 3–manifolds admitting no further essential reduction, ie to two different "roots".

Example Let Q be the complement space of the figure-eight knot. We assume that the torus ∂Q is equipped with a coordinate system such that the slope of the meridian is (1,0). Choose two pairs (p,q), (m,n) of coprime integers such that $|q|, |n| \geq 2$ and $|p| \neq |m|$. Let a and b be corresponding curves in ∂Q. Then the manifolds $Q_{p,q}$ and $Q_{m,n}$ obtained by Dehn filling of Q are not homeomorphic. By Thurston [9], they are hyperbolic.

Consider the thick torus $X = S^1 \times S^1 \times I$ and locate its exterior meridian $\mu = S^1 \times \{*\} \times \{1\}$ and interior longitude $\lambda = \{*\} \times S^1 \times \{0\}$. Then we attach to X two copies Q', Q'' of Q as follows. The first copy Q' is attached to X by identifying an annular regular neighborhood $N(a)$ of a in ∂Q with an annular regular neighborhood $N(\mu)$ of μ in ∂X. The second copy Q'' is attached by identifying $N(b)$ with $N(\lambda)$. Denote by M the resulting manifold $Q' \cup X \cup Q''$.

Since Q is hyperbolic, M contains only two incompressible boundary incompressible annuli A and B, where A is the common image of $N(a)$ and $N(\mu)$, and B is the common image of $N(b)$ and $N(\lambda)$. It is easy to see that reduction of M along A gives us a disjoint union of a punctured $Q'_{p,q}$ and a punctured Q'' while the reduction along B leaves us with a punctured Q' and a punctured $Q''_{m,n}$. After filling the punctures (by reductions along spheres surrounding them), we get two different manifolds, homeomorphic to $Q_{p,q} \cup Q$ and $Q_{m,n} \cup Q$. Since their connected components (ie $Q_{p,q}$, $Q_{m,n}$, Q) are hyperbolic, they are irreducible, boundary irreducible and contain no essential annuli. Hence $Q_{p,q} \cup Q$ and $Q_{m,n} \cup Q$ are different roots of M.

9 Other roots

Roots of cobordisms Recall that a *3–cobordism* is a triple $(M, \partial_- M, \partial_+ M)$, where M is a compact 3–manifold and $\partial_- M$, $\partial_+ M$ are unions of connected components of ∂M such that $\partial_- M \cap \partial_+ M = \varnothing$ and $\partial_- M \cup \partial_+ M = \partial M$. One can define S– and D–reductions on cobordisms just in the same way as for manifolds. The A–reduction on cobordisms differs from the one for manifolds only in that one boundary circle of A must lie in $\partial_- M$ while the other in $\partial_+ M$.

Theorem 13 *For any compact 3–cobordism $(M, \partial_- M, \partial_+ M)$ its root exists and is unique up to homeomorphism of cobordisms and removing disjoint 3–spheres and balls.*

The proof of this theorem is the same as the proof of Theorem 12. We point out that considering roots of cobordisms was motivated by the paper [2] of S Gadgil, which is interesting although the proof of his main theorem contains a serious gap. We found the gap after proving Theorem 13, which clarifies the situation with Gadgil's construction.

Roots of virtual links Recall that a virtual link can be defined as a link $L \subset F \times I$, where F is a closed orientable surface. Virtual links are considered up to isotopy and destabilization operations, which, in our terminology, correspond to reduction along annuli. Each annulus must be disjoint to L and have one boundary circle in $F \times \{0\}$, the other in $F \times \{1\}$. We also allow spherical reductions. The proof of the following theorem is the same as the proof of Theorem 12.

Theorem 14 *For any virtual link its root exists and is unique up to homeomorphism of cobordisms and removing disjoint 3–spheres and balls.*

This theorem is equivalent to the main theorem of Kuperberg [4].

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